An affine crystallographic semigroup acting on hyperbolic space

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1. Introduction.

Let $G_n = \mathrm{Aff}(\mathbb{R}^n)$ be the group of affine transformations of $\mathbb{R}^n$. The group G_n is the semidirect product $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with its group of translations. A semigroup S of G_n is said to act *properly discontinuously* on $\mathbb{R}^n$ if for every compact subset K of $\mathbb{R}^n$ the set $\{g \in S : gK \cap K \neq \emptyset\}$ is finite. If a discrete group consists of isometries, then it acts properly on $\mathbb{R}^n$. But this is not true for an arbitrary discrete subgroup of G_n, e.g. for an infinite discrete subgroup of $GL_n(\mathbb{R})$. A semigroup S of G_n is called *crystallographic* if S acts properly discontinuously on $\mathbb{R}^n$ and there exists a compact subset K_0 of $\mathbb{R}^n$ such that $\bigcup_{s\in S} sK_0 = \mathbb{R}^n$.

If the signature of a non degenerate quadratic form B on $\mathbb{R}^n$ is $(n-1,1)$ the form B is called *hyperbolic*. Let $O(B)$ (resp. $SO(B)$) denote the orthogonal (resp. special orthogonal) group of B. Define the subgroup G_B of G_n as the subgroup leaving the form

B invariant. It is clear that G_B is the semidirect product $O(B) \ltimes \mathbb{R}^n$.

It turns out that a crystallographic affine semigroup leaving a positive definite quadratic form invariant (i.e. Euclidean semigroup) is a group [GS 1], [GS 2]. The major step in the proof of this claim was to prove that the Zariski closure of an Euclidean crystallographic semigroup is a virtually solvable group. This questions came from our works on the Auslander conjecture. It also became clear that we need to have more information about affine crystallographic semigroup. The motivation questions here are the following:

Question *(H.Abels, G.Margulis, G.Soifer). Is the Zariski closure of a crystallographic affine semigroup leaving a hyperbolic form invariant a virtually solvable group ?*

Our interest to these questions came from our work on the Auslander conjecture. The main result of this paper is the following

Main Theorem *Let S be a crystallographic semigroup , $S \in Aff(\mathbb{R}^n), n \leq 3$. Then the Zariski closure of S is a virtually solvable group.*

Thus we have prove

Corollary A. *The Zariski closure of a crystallographic semigroup S of $Aff(\mathbb{R}^n), n \leq 3$ leaving a hyperbolic form invariant is a virtually solvable group .*

Based on this fact it is not difficult to show that

Corollary B *A crystallographic semigroup S of $Aff(\mathbb{R}^n), n \leq 3$ leaving a hyperbolic form invariant is a group.*

It is not clear if the same statement can be proved for a hyperbolic semigroup acts on $\mathbb{R}^n, n \geq 4$.

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2. Preliminaries.

In order to make the exposition as self -contained as possible in this section we collect the information needed in the proofs.

In this section we introduce the terminology we will use throughout the whole paper and recall terminology and results from [A], [AMS 1], [AMS 2], [AMS 3], [AMS 4] and [BG]. We will prove some basic lemmas about the geometry and dynamics of action of an affine transformation under the assumption that its linear part is hyperbolic.

2.0. Let V be a finite dimensional vector space over a local field k with absolute value $|\cdot|$ and let $P = \mathbb{P}(V)$ be the projective space based on V. Let $g \in GL(V)$ and let $\chi_g(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i) \in k[\lambda]$ be the characteristic polynomial of the linear transformation g. Set $\Omega(g) = \{\lambda_i : |\lambda_i| = \max_{1 \leq j \leq n} |\lambda_j|\}$. Put $\chi_1(\lambda) = \prod_{\lambda_i \in \Omega(g)}(\lambda - \lambda_i)$ and $\chi_2(\lambda) = \prod_{\lambda_i \notin \Omega(g)}(\lambda - \lambda_i)$. Since the absolute value of an element is invariant under Galois automorphism then χ_1 and χ_2 belong to $k[\lambda]$. Therefore $\chi_1(g) \in GL(V)$ and $\chi_2(g) \in GL(V)$. Let us define by $V(g)$ (resp. $W(g)$) the subspace of V corresponding to $\ker(\chi_1(g))$ (resp. $\ker(\chi_2(g))$). We will often use for an element $g \in GL(V)$ the following notation $V(g) = V^+(g)$, $W(g) = W^-(g)$, $V(g^{-1}) = V^-(g)$ and $W(g^{-1}) = W^+(g)$. Let $\lambda_-(g) = \max\{|\lambda| : \lambda$ is an eigenvalue of g of absolute value less than 1 $\}$. Let $\lambda_+(g) = \min\{|\lambda| : \lambda$ is an eigenvalue of g of absolute value more then 1$\}$. Put $\lambda(g) = \max\{\lambda_+^{-1}(g), \lambda_-(g)\}$. It is clear that $\lambda(g) = \lambda(g^{-1})$

2.1. Recall that $g \in GL(V)$ is called *proximal* if $\dim(V^+(g)) = 1$. A proximal element g has a unique eigenvalue of maximal absolute value hence this eigenvalue has algebraic and geometric multiplicity one. For $S \subseteq GL(V)$ set $\Omega_0(S) = \{g \in S : g \text{ and } g^{-1} \text{ are proximal}\}$. A semisimple element $g \in \Omega_0(GL(V))$ is called *dipole*.

Let g be a semisimple element in $GL(\mathbb{R}^n)$. Then the space $\mathbb{R}^n$ can be decomposed into the direct sum of three subspaces $A^+(g),\ A^-(g),\ A^0(g)$ determined by the condition that all eigenvalues of the restriction $g \mid A^+(g)$ (resp. $g \mid A^-(g),\ g \mid A^0(g)$) have absolute value more than 1 (resp. less than 1, equal to 1). It is clear $B^+(g) = A^+(g) \oplus A^0(g)$ and $B^-(g) = A^-(g) \oplus A^0(g)$, then obviously $B^+(g) \cap B^-(g) = A^0(g)$. Put $Cr(g) = B^+(g) \cup B^-(g)$. A semisimple element $g \in GL_n(\mathbb{R})$ is called *hyperbolic* if $\dim(A^0(g)) = \min\{dim A^0(t) | \, t \text{ is a semisimple element, } t \in GL_n(\mathbb{R})\}$.

2.2. Let $\| \cdot \|$ and d denote the norm and metric on $\mathbb{R}^n$ corresponding to the standard inner product on $\mathbb{R}^n$. Let $P = \mathbb{P}(\mathbb{R}^n)$ be the projective space corresponds to $\mathbb{R}^n$. Let $\pi : \mathbb{R}^n/\{0\} \to P$ be a natural projection. For a subset non-zero subset X from $\mathbb{R}^n$ we put $\pi(X) = \pi(X/\{0\})$.

The metric $\| \cdot \|$ on $\mathbb{R}^n$ induces the metric $\widehat{d}$ on the projective space $P = \mathbb{P}(V)$. Thus for any point $p \in P$ and a subset $A \subseteq P$ we can define

$$\widehat{d}(p, A) = \min_{a \in A} \widehat{d}(p, a).$$

Let B be a subset in P. We define

$$\widehat{d}(A, B) = \min_{a \in A, b \in B} \widehat{d}(a, b).$$

For two subsets $X \subseteq \mathbb{R}^n, X \neq \{0\}$ and $Y \subseteq \mathbb{R}^n, Y \neq \{0\}$ we put $\widehat{d}(X, Y) = \widehat{d}(\pi(X), \pi(Y))$. A hyperbolic element g is called *ε-hyperbolic* if

$$\widehat{d}(A^+(g), B^-(g)) \geq \varepsilon$$

and

$$\widehat{d}(A^-(g), B^+(g)) \geq \varepsilon.$$

Two different hyperbolic elements g_1 and g_2 are called *transversal* if $A^{\pm}(g_1)\bigcap B^{\mp}(g_2) = \{0\}$ and $A^{\pm}(g_2)\bigcap B^{\mp}(g_1) = \{0\}$. Two transversal hyperbolic elements g_1 and g_2 are called ε- *transversal* if

$$\min_{1\leq i\neq j\leq 2}\{\widehat{d}(A^+(g_i), B^-(g_j)), \widehat{d}(A^-(g_i), B^+(g_j))\} \geq \varepsilon.$$

Let $l : G_n \to GL_n(\mathbb{R})$ be the natural homomorphism (see [A]). Recall that $l(g)$ is called the *linear part* of an affine transformation g. Let $X \subseteq G_n$ then the set $l(X) = \{l(x), x \in X\}$ is called the *linear part* of X. It is clear that $G_B = \{x \in G_n \mid l(x) \in O(B)\}$ and $l(G_B) = O(B)$. An affine transformation is called *dipole* (corr. *hyperbolic*, $\varepsilon -$ *hyperbolic*)if $l(g)$ is dipole (corr. $l(g)$ is hyperbolic, $l(g)$ is ε- hyperbolic). Two affine transformations g_1 and g_2 are called *transversal* (corr. $\varepsilon -$ *transversal*) if the linear parts $l(g_1)$ and $l(g_2)$ are transversal (corr. ε -transversal).

2.3. Consider a hyperbolic element $g \in G_n$ without fixed points. Then there exists a g–invariant line L_g and the restriction of g to L_g is the translation by a non- zero vector t_g. Let us note that all such lines are parallel and t_g does not depend on the choice of L_g and $l(g)t_g = t_g$. We will assume that we fixed once and for all some point in the affine space $\mathbb{R}^n$ as an origin point and the line L_g we define as a closest to the origin g -invariant line. Define an affine subspace $C^0(g)$ by $D^0(g) = L_g + A^0(g)$. Let as remind the following useful observation [A]: if a semigroup S acts proper discontinuous and g is a hyperbolic affine transformation $g \in S$, then g acts without fixed points then $t_g \neq 0$ and $l(g)t_g = t_g$. Therefore, if a semigroup $S, S \subseteq G_n$ acts proper discontinuous then the linear part $l(g)$ of every hyperbolic element $g \in S$ has an eigenvalue 1. Actually every element from S of infinite order has an eigenvalue 1 [A]

2.4. For a non-zero vector $v, v \in \mathbb{R}^n$ by L_v we denote a line parallel to v and passes through the point of origin. Let S be a semigroup, $S \subseteq G_n$ and let K be a compact subset of $\mathbb{R}^n$. We consider the set of norm one vectors $X_\infty(S, K)$ defined as follows: $v \in X_\infty(S, K)$ if there exist a constant $C = C(S, K)$, a sequence of points $\{p_i\}_{i\in\mathbb{N}} \subseteq K$ and a sequence of elements $\{s_i\}_{i\in\mathbb{N}} \subseteq S$ such that $d(s_ip_i, p_i) \to \infty$ and $d(L_v, s_ip_i) \leq C$ as $i \to \infty$

Lemma 2.5. *Let S be a crystallographic semigroup $S \subseteq G_n$. Then*

1. *For every two compact subset K_1 and K_2 in $\mathbb{R}^n$ $X_\infty(S, K_1) = X_\infty(S, K_2)$. Therefore we will write $X_\infty(S)$ instead of $X_\infty(S, K)$.*

2. *For every $v \in X_\infty$ and $s \in S$ we have $sv/\|sv\| \in X_\infty$.*

3. *$X_\infty(S) = S^{n-1} = \{v \in \mathbb{R}^n, \|v\| = 1\}$.*

Proof. The proof is straightforward. □

2.6. Now we will recall an important definition first introduced by G.Margulis [GM 1] for $n = 3$, generalized in [AMS 2] in the case when the signature a quadratic form is $(k+1, k)$ and for an arbitrary quadratic form in [AMS 3]. We will follow along [AMS 3]. Let B be a quadratic form of signature (p, q), $p \geq q, p + q = n$. Let v be a vector in $\mathbb{R}^n$ $v = x_1v_1 + \cdots + x_pv_p + y_1w_1 + \cdots + y_qw_q$, where $v_1, v_2, \ldots, v_p$, $w_1, w_2, \ldots, w_q$ is a basis of $\mathbb{R}^n$. We can and will assume that

$$B(v, v) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2.$$

Consider the set Ψ of all maximal B-isotropic subspaces. Let X be the subspace spanned by $\{v_1, v_2, \ldots, v_p\}$ and Y be the subspace spanned by $\{w_1, w_2, \ldots, w_q\}$. It is clear that $\mathbb{R}^n = X \oplus Y$. Define the cone

$$\mathbb{C}_B = \{v \in \mathbb{R}^n | B(v, v) < 0\}.$$

Clear that $Y \subset \mathbb{C}_B$. We have the two projections

$$\pi_X : \mathbb{R}^n \longrightarrow X \text{ and } \pi_{\mathrm{Y}} : \mathrm{\mathbb{R}^n} \longrightarrow \mathrm{Y}$$

along Y and X, respectively The restriction of π_Y to $V \in \Psi$ is a linear isomorphism $V \longrightarrow Y$. Hence if we fix an orientation on Y we have also fixed an orientation on each $V \in \Psi$. For $V \in \Psi$ let us denote the B–orthogonal complement of V by $V^\perp = \{z \in \mathbb{R}^n \ ; \ B(z, V) = 0\}$. We have $V \subset V^\perp$ since V is B–isotropic. We also have

$$\dim V^\perp = \dim V + (p - q) = p.$$

The restriction of π_X to $V^\perp$ is a linear isomorphism $V^\perp \longrightarrow X$. Hence if we fix an orientation on X we have also fixed an orientation on $V^\perp$ for each $V \in \Psi$. Thus we have orientations on both V and $V^\perp$ and we have naturally induced an orientation on any subspace W, such that $V^\perp = W \oplus V$. If $V_1 \in \Psi$ and $V_2 \in \Psi$ are transversal, then $W = V_1^\perp \cap V_2^\perp$ is a subspace which is transversal to both V_1 and V_2; therefore $W \oplus V_1 = V_1^\perp$ and $W \oplus V_2 = V_2^\perp$. So there are two orientations ω_1 and ω_2 on W, where ω_i is defined if we consider W as a subspace in $V_i^\perp$. We have [AMS 3, lemma 2.1]

Lemma 2.7. *The orientations defined above on W are the same if q is even and are opposite if q is odd.*

2.8. Assume now that B has signature $(k+1, k)$. Let g be a hyperbolic element without fixed points, $g \in G_B$. Then, $B^+(g) = (A^+(g))^\perp$ and $B^-(g) = (A^-(g))^\perp$, $\dim A^+(g) = \dim A^-(g) = k$ and $\dim A^0(g) = 1$. We define (see [AMS2])an orientation on the space $A^0(g)$ induced by an orientation on $B^+(g)$. Let $v_0(g)$ be the corresponding unite vector. Thus $C^0(g)$ is an g-invariant line and the restriction $g \mid C^0(g)$ is a translation by a non-zero vector $t_g, t_g \in A^0(g)$. Remark, $B(t_g, t_g) > 0$ since $t_g \in A^0(g)$. It is easy to check that if p is an arbitrary point in $\mathbb{R}$ and $t_p = gp - p$ then $B(t_p, t_p) = B(t_g, t_g)$ and $B(t_p, v_0(g)) = B(t_g, v_0(g))$. Recall that for ε-hyperbolic element g there exist two non-zero constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ such that for any vector $v \in A^0(g)$ we have $c_1(\varepsilon)B(v, v) \leq \|v\| \leq c_2(\varepsilon)B(v, v)$. As in [AMS 3] define a sing $\alpha(g)$ of a hyperbolic affine transformation

g by

$$\alpha(g) = B(t_q, v_0(g))/B(t_g, t_g)^{1/2}.$$

It is clear that: $\alpha(g) = \pm 1$ and for arbitrary point $p \in \mathbb{R}^n$ we have

$$\alpha(g) = B(t_p, v_0(g))/B(t_p, t_p)^{1/2}.$$

Let us recall the following important observation [A]: if a semigroup $S \subseteq \mathbb{R}^n$ contains two hyperbolic transversal elements g_1 and g_2 such that $\alpha(g_1)\alpha(g_2) < 0$ then S does not act proper on $\mathbb{R}^n$.

2.9. Now we explain the main ideas of the proof of theorem 1. First under the assumption that the Zariski closure of S is not a solvable group we construct two hyperbolic transversal elements g_1 and g_2 in S. By lemma 1, $X_\infty(S) = S^{n-1}$. Briefly speaking lemma 1 says, there are all possible directions in $X_\infty(S)$. Since direction determines the sign of a hyperbolic element we transform them to two transversal hyperbolic transformations with different sign, i.e. $\alpha(g_1)\alpha(g_2) < 0$. This gives us necessarily contradiction since we know that in this case S does not acts properly discontinuous on $\mathbb{R}^n$. It will reduce the proof to the case when the Zariski closure $l(S)$ is a group with common fixed vector and then will lead to the proof.

3. Main results.

First we will recall some known facts on hyperbolic elements in G_B [AMS 3], [AMS 5]. Since $B^\pm(g) = (A^\pm(g))^\perp$ it is easy to see, that if g and h are two ε -hyperbolic elements such that $\widehat{d}(A^+(g), A^+(h)) \geq \varepsilon$ then there exists a constant $c(\varepsilon)$ such that

$\widehat{d}(A^+(g), B^+(h)) \geq c(\varepsilon)$. Thus we can conclude

Lemma 3.1 *: Let g and h be two ε -hyperbolic elements such that*

$$\min\{\widehat{d}(A^+(g), A^-(h)), \widehat{d}(A^-(g), A^+(h))\} \geq \varepsilon.$$

Then there exists a constant $c(\varepsilon)$ such that two ε- hyperbolic elements g and h are $c(\varepsilon)$-transversal

Remark 3.2 *Let B be a hyperbolic form of a signature $(2,1)$, and let g be an element $g, g \in G_B$. Then g is hyperbolic if and only g is dipole. $A^{\pm}(g) = V^{\pm}(g)$ and $U^{\pm}(g) = W^{\pm}(g)$.*

We recall some results from [AMS 2].

Let X be a subset of P. Put $B(X, r) = \{p \in P, \widehat{d}(p, X) \leq r\}$.

Lemma 3.3. *There exist an $\lambda(\varepsilon) < 1$ and $c(\varepsilon)$ such that for any two ε–hyperbolic ε–transversal elements g and h with $\lambda(g) < (\varepsilon)$, $\lambda(h) < \lambda(\varepsilon)$ we have*

(1) the element gh is $\varepsilon/2$–hyperbolic and is $\varepsilon/2$–transversal to both g and h;

(2) $A^+(gh) \subseteq U(A^+(g), c(\varepsilon)\lambda(g))$;

(3) $A^-(gh) \subseteq U(A^-(h), c(\varepsilon)\lambda(h))$;

(4) $\lambda(gh) \leq c(\varepsilon)\lambda(g)\lambda(h)$.

Let n=3 and B is a quadratic form of signature $(2,1)$. First we will prove the following

Proposition 3.4. *Let S be a crystallographic semigroup , $S \subseteq Aff(\mathbb{R}^3)$. Then $l(S)$ the linear part S is not Zariski dense in $O(B)$.*

We will divided the proof into a sequence of lemmas.

Assume that the linear part $l(S)$ is Zariski dense in $G(B)$. Then since the set of hyperbolic elements is Zariski dense in S [AMS 1], it is easy to show that there exists a pair of hyperbolic transversal elements g and h in S such that $A^+(g) \neq A^+(h)$ and

$A^-(g) \neq A^-(h)$. Fix these elements and consider two subspaces $D(g) = A^+(g) \oplus A^-(g)$ and $D(h) = A^+(h) \oplus A^-(h)$ of $\mathbb{R}^3$. It follows from lemma 4 that for any $\delta > 0$ there exists $N, N \in \mathbb{Z}, N > 0$ such that $\widehat{d}(A^+(g^m h^n), A^+(g)) \leq \delta$, $\widehat{d}(A^+(h^n g^m), A^+(h)) \leq \delta$, $\widehat{d}(A^-(g^m h^n), A^-(h)) \leq \delta$ and $\widehat{d}(A^-(h^n g^m), A^-(g)) \leq \delta$ for $n > N, m > N$. Any two elements in $\mathbb{C}_B$ with the same B -norm are conjugate by an element from $O(B)$. We can and will conjugate S such that $D(g) \bigcap D(h) = Y$. Therefore we have

Lemma 3.5. *If the Zariski closure of the linear part $l(S)$ is $O(B)$ then there are two ε- hyperbolic ε-transversal elements g and h in S such that $D(g) \bigcap D(h) = Y$.*

From now on we fix two elements g and h from S which satisfy lemma 3.5.

Lemma 3.6. *There exists a sequence $\{g_i\}_{i \in \mathbb{N}}$ of $\varepsilon/2$- hyperbolic elements from S such that the sequence $\{A^+(g_n)\}_{i \in \mathbb{N}}$ converge to $A^-(g)$. The same true for $A^-(h)$.*

Proof. . Recall that g and h are two ε- hyperbolic, ε- transversal elements from a semigroup S. Assume that there exists a constat c such that for every hyperbolic element $x \in S$ we have $\widehat{d}(A^+(x), A^-(g)) \geq c$. Let L_g be the unique g- invariant line and let K_0 be a compact subset of $\mathbb{R}^n$ such that $SK_0 = \mathbb{R}^n$ and $L_g \bigcap K_0 \neq \varnothing$. Then there are two infinite subsets $\{x_n, n \in \mathbb{N}\} \subseteq K_0$ and $\{g_n, n \in \mathbb{N}\} \subseteq S$ such that

(1) $g_n x_n \in L_g$ for all n

(2) $\|g_n x_n - x_n\| \to \infty$ when $n \to \infty$

(3) $g_n x_n - x_n / B(g_n x_n - x_n, g_n x_n - x_n)^{1/2} \to -v_0(g)$ when $n \to \infty$

Since the Zariski closure of $l(S) = O(B)$ by [AMS 1] we have that there exist a finite subset $H = \{s_1, \ldots, s_m\} \subseteq S$ and positive real ε_1 such that for every every element $s \in S$ there exists an suitable element s_i from T such that ss_i is ε_1- hyperbolic. Therefore we can assume that for every n element $g_n s_1$ is ε_1- hyperbolic. A projective space P is compact and hence we can assume that the sequence $\{A^-(g_n s_1)\}_{n \in \mathbb{N}}$ converge to a point A^- in P. Let us show that there exist a positive δ and an element $s_0 \in S$ such that

(1) for all positive integer n element $g_n s_1 s_0$ is δ- hyperbolic

(2) there exists a positive number N such that $\widehat{d}(A^-(g_n s_1 s_0), A^+(g)) > \delta$ for all $n > N$.

If $A^- = A^+(h)$ then we can take $s_0 = 1$. Assume that $\widehat{d}(A^-, A^+(h)) > \delta_1 \geq 0$. By Lemma 4 (3) there exists N_1 such that $\widehat{d}(A^+(h^m g^m), A^+(h)) < \min(\delta_1/8, \varepsilon/8)$ and $\widehat{d}(A^-(h^m g^m), A^-(g)) < \min(c/8, \delta_1/8, \varepsilon/8)$ for $m > N_1$. Therefore $\widehat{d}(A^-, A^+(h^m g^m)) > \delta_1/2$. Thus there exists positive integer N_1 such that for all $n > N_1$ we have $\widehat{d}(A^-(g_n s_1), A^+(h^m g^m)) > \delta/4/$. According to our assumption for all n we have $\widehat{d}(A^-(g_n s_1), A^-(g)) > c$. We thus conclude that for n and m bigger then $N = \max(N_1, N_2)$ and $\delta_2 = \min(\delta_1/4, c/4)$, $g_n s_1$ and $h^m g^m$ are δ_2- transversal. Therefore by (3), Lemma 3.4, there exists M_1 such that for $m_0 > M_1$ and all $n > N$ element $g_n s_1 h^{m_0} g^{m_0}$ is $\delta_2/2$-hyperbolic and $\widehat{(d)}(A^-(g_n s_1 h^{m_0} g^{m_0}), A^-(g)) < \delta_2/8$. We can conclude that for $s_0 = h^{m_0} g^{m_0}$, every $n > N$ and $\delta = \min(\varepsilon/2, c/2)$ we have $\widehat{d}(A^-(g_n s_1 s_0), A^+(g)) > \delta$. Put $K_1 = s_1^{-1} s_0^{-1} K_0 \bigcup K_0$, $\widehat{g}_n = g_n s_1 s_0$, $y_n = s_1^{-1} s_0^{-1} x_n, y_n \in K_1$. Then

(1) for every $n \in \mathbb{N}$ element $\widehat{g_n}$ is δ-hyperbolic, g_n and g are δ- transversal

(2) $\|\widehat{g_n} y_n - y_n\| \to \infty$ when $n \to \infty$

(3) $\widehat{g_n} y_n - y_n / B(\widehat{g_n} y_n - y_n, \widehat{g_n} y_n - y_n)^{1/2} \to -v_0(g)$

Put $h_n(m) = \widehat{g_n} g^m$. By our assumption from (1) and lemma 3.4 follows that there exists positive integer M_1 such that element $h_n(m)$ is $\delta/2$-hyperbolic and $\widehat{d}(A^-(h_n(m)), A^-(g)) \leq \lambda(g)^{m/2}$ for all $n, m > M_1$. Consider the intersection $A^+(h_n(m)) \bigcap A^-(g)$. Denote by $w_0(h_n(m))$ the unique vector, $w_0(h_n(m)) \in A^+(h_n(m)) \bigcap A^-(g)$ such that, $B(w_0(h_n(m)), v_0(g)) = 1$. Since $\widehat{d}(A^-(h_n(m)), A^-(g)) \leq \lambda(g)^{m/2}$ for all $n, m > M_1$ there exists a constant $c_2(\delta)$ such that $\|v_0(h_n(m)) - w_0(h_n(m))\| \leq c_2(\delta)\lambda(g)^{m/2}$. Therefore we can chose a positive number m_0 such that for every anisotropic vector w we will have $|B(w, v_0(h_n(m))) - B(w, w_0(h_n(m)))| \leq 1/8|B(w,w)|^{1/2}$. Put $h_n = h_n(m_0)$. Set $K_2 = g^{-m_0} K_1$. We thus have a sequence $\{h_n\}_{n \in \mathbb{N}}$ and a subset $\{x_n, n \in \mathbb{N}\} \subseteq K_2$ such that $h_n x_n - x_n / B(h_n x_n - x_n, h_n x_n - x_n)^{1/2} \to -v_0(g)$ when $n \to \infty$. Denote $w_n = h_n x_n - x_n$.

Since $B(w_0(h_n(m)), v_0(g)) = 1$ and $|B(v_0(h_n(m)) - w_0(h_n(m)), v_0(h_n(m)) - w_0(h_n(m)))| \leq 1/8$ we thus have

$$B(w_n, v_0(h_n))/B(w_n, w_n)^{1/2} < B(-v_0(g), v_0(h_n) + 1/8 < -B(v_0(g), w_0(g)) + 1/4 < 0.$$

Then $\alpha(g_n) < 0$ for sufficiently big n. Therefore S contains two hyperbolic transversal elements with different sign. Contradiction, since S acts properly discontinuously and therefore all elements from S have the same sign. This proves lemma. □

Put $a_1 = A^+(g), a_2 = A^+(h)$, $a_3 = A^-(g)$, $a_4 = A^-(h)$

Corollary 3.7. *For each point $a_i, 1 \leq i \leq 4$ there exist two sequences $L_i^{(1)} = \{x_t^{(i)}\}_{t\in\mathbb{N}}$ and $L_i^{(2)} = \{y_t^{(i)}\}_{t\in\mathbb{N}}$ and positive number δ such that*
(1) x_t and y_t are δ- hyperbolic for every $t \in \mathbb{N}$,
(2) $\lim_{t\to\infty} A^+(x_t^{(i)}) = \lim_{t\to\infty} A^+(y_t^{(i)}) = a_i$,
(3) $\lim_{t\to\infty} A^-(x_t^{(i)}) = b_i$, $\lim_{t\to\infty} A^-(y_t^{(i)}) = c_i$ and $d(b_i, c_i) > \delta$.

Proof. Let x and y be two ε-hyperbolic elements such that each pair x and g, y and g, x and y^{-1} are ε-transversal for some ε. Put $x_t^{(1)} = g^t x^t$ and $y_t^{(1)} = g^t y^t$. It is clear that this sequences fulfill *(1)*,*(2)* and *(3)* for a_1. Take h instead of g and we get sequence suite for a_2. From lemma 3.6 and arguments above follows that such sequences exist for points a_3 and a_4. □

3.8. We will use here notations and definitions from ***2.6***. Let $V, V \subseteq \mathbb{R}^n$ be a maximal B- isotropic subspace and let v be a vector from V such that V is spanned by v and $\pi_Y(v) = w_1$. Let v_0 be a vector from $V^\perp \cap X$ such that $B(v_0, v_0) = 1$ and the basis $\pi_X(v), v_0$ has the same orientation as v_1, v_2. Let W be a maximal B- isotropic subspace, $W \neq V$. Then $\dim(V^\perp \bigcap W^\perp) = 1$. There exists the unique vector $w_0(W)$, $w_0(W) \in V^\perp \bigcap W^\perp$ such that $w_0(W) = v_0 + \alpha(W)v$. Set $\Phi_V^+ = \{W \in \Phi | \, \alpha(W) > 0\}$ and $\Phi_V^- = \{W \in \Phi | \, \alpha(W) < 0\}$. From lemma 2.7 and the chose of hyperbolic elements g and h

immediately follows that $\Phi^{\pm}_{a_1} = \Phi^{\mp}_{a_3}$ and $\Phi^{\pm}_{a_2} = \Phi^{\mp}_{a_4}$.

Remark Base on corollary 3.7 we come to the following important observation. Let $x \in S$ be a hyperbolic element. Then there is a constant $c = c(S)$ and two hyperbolic elements g_1 and h_1 such that x and g_1 (resp. x and g_2) are c-transversal and $A^+(x) \in \Phi^+_{A^+(g_1)}$ (resp. $A^+(x) \in \Phi^-_{A^+(g_2)}$) or $A^+(x) \in \Phi^-_{A^+(g_1)}$ (resp. $A^+(x) \in \Phi^-_{A^+(g_2)}$)

Now we will prove Proposition 3.5

Proof. Assume that $l(S)$ is Zariski dense in G_B. Let K_0 be a compact subset of $\mathbb{R}^n$ such that $SK_0 = \mathbb{R}^n$ and let L be a line parallel to w_1 such that $L \bigcap K_0 \neq \varnothing$. Then there are two infinite subsets $\{l_n, n \in \mathbb{N}\} \subseteq L$ and $\{g_n, n \in \mathbb{N}\} \subseteq S$ such that

(1) $g_n l_n \in K_0$ for all n,

(2) $\|g_n l_n - l_n\| \to \infty$ when $n \to \infty$,

(3) $g_n l_n - l_n / |B(g_n l_n - l_n, g_n l_n - l_n)|^{1/2} \to w_1$ when $n \to \infty$.

Using the same arguments as we used in the proof of lemma 3.7 we can additionally assume that there exists a positive δ_1 such that g_n is δ_1-hyperbolic for all $n \in \mathbb{N}$. A projective space P is compact hence we can assume that there are two points $p_1 \in P$ and $p_2 \in P$ such that $\lim_{n\to\infty} \pi(A^+(g_n)) = p_1$ and $\lim_{n\to\infty} \pi(A^-(g_n)) = p_2$. Remark that $\widehat{d}(p_1, p_2) \geq \delta_1$. Let V_1 (resp. V_2) be a subspace of $\mathbb{R}^3$ such that $\pi(V_1) = p_1$ (resp. $\pi(V_2) = p_2$). It is easy the show that there exists a hyperbolic element $s_0 \in S$ such that $\{\pi(A^+(s_0)), \pi(A^-(s_0))\} \subseteq P \setminus \{p_1, p_2\}$. Thus there exists a positive number N such that for $n > N$ elements g_n and s_0 are transversal. By the lemma 3.3 there exist positive integers M and δ_2 such that for $m > M$

(1) element $s^m g_n$ is δ_2- hyperbolic for all $n \in \mathbb{N}$,

(2) $\lambda(s^m g_n) < \lambda^{m/2}(s_0)$,

(3) $\|s^m g_n l_n - l_n\| \to \infty$ when $n \to \infty$,

(4) $s^m g_n l_n - l_n / |B(s^m g_n l_n - l_n, s^m g_n l_n - l_n)|^{1/2} \to w_1$ when $n \to \infty$.

Fix m and consider the sequence $\{s^m g_n\}_{n\in\mathbb{N}}$. We can assume since P is compact that $\{A^+(s^m g_n)\}_{n\in\mathbb{N}}$ and $\{A^-(s^m g_n)\}_{n\in\mathbb{N}}$ converge. Denote $A^+_m = \lim_{n\to\infty} A^+(s^m g_n)$ and $A^-_m = \lim_{n\to\infty} A^-(s^m g_n)$. As above we will assume that the sequence $\{A^-_m\}_{n\in\mathbb{N}}$ converges. Put $A^- = \lim_{m\to\infty} A^-_m$. Then there exists a positive number δ_3, such that
$\max\{\min\{\widehat{d}(A^-, A^+(g)), \widehat{d}(A^-, A^-(g)\}, \{\min\{\widehat{d}(A^-, A^+(h)), \widehat{d}(A^-, A^-(h)\}\} > \delta_3$. Therefore we can assume that $\min\{\widehat{d}(A^-, A^+(g)), \widehat{d}(A^-, A^-(g)\} > \delta_3$. Since $\Phi^{\pm}(A^+(g)) = \Phi^{\mp(A^-(g))}$ let us first assume that $A^- \in \Phi^+(A^+(g))$. We have $B(w_1, w_1) = -1$. Then from ***3.8*** we conclude that there exists a non-zero positive number c such that
$B(w_1, w_0(A^+(g))) < -c$ and $B(w_1, w_0(A^-(g))) > c$. For each point $A^+(g)$ and $A^-(g)$ there are two sequences which fulfill the properties of the corollary 3.7. Hence there exist two sequences $L_1 = \{x_t, t \in \mathbb{N}\}$ and $L_2 = \{y_t, t \in \mathbb{N}\}$ such that *(1)* $\lim_{t\to\infty} A^+(x_t) = A^+(g)$, $\lim_{t\to\infty} A^+(y_t) = A^-(g)$,
(2) $\lim_{t\to\infty} A^-(x_t) = A$ and $\widehat{d}(A, A^+) > 0$,
(3) $\lim_{t\to\infty} A^-(y_t) = B$ and $\widehat{d}(B, A^+) > 0$.
From the above by the lemma 3.3 it follows that for every ε there exists a positive integer M such that for all $m > M, r, t > M$ we have $\|v_1(m,n) - w_0(A^+(g))\| < \varepsilon$ and $\|v_2(m,n) - w_0(A^+(g))\| < \varepsilon$, where $v_1(m,n) = v_0(x^m_r s^m_0 g_t)$, $v_1(m,n) = v_0(y^m_r s^m_0 g_t)$. Thus there exists a positive number M_1 such that for $m > M_1, r, t > M_1$ we have $|B(v_1(m,n), w_1) - B(w_0(A^+(g)), w_1)| \leq c/8$ and $|B(v_2(m,n), w_1) - B(w_0(A^-(g)), w_1)| \leq c/8$. Fix $m_0 > M$ and define $h^+_t = x^{m_0}_{m_0} s^{m_0}_0 g_t$ and $h^-_t = y^{m_0}_{m_0} s^{m_0}_0 g_t$. It is easy to see, that $h^+_t l_t - l_t / |B(h^+_t l_t - l_n, h^+_t l_t - l_t)|^{1/2} \to w_1$ and $h^-_t l_t - l_t / |B(h^-_t l_t - l_t, h^-_t l_t - l_t)|^{1/2} \to w_1$ when $t \to \infty$. Then there exists a positive integer N such that for $t > N$ we have $B(h^-_t l_t - l_t, w_0(A^+(g))) > c/2$ and $B(h^-_t l_t - l_t, w_0(A^-(g))) < -c/2)$. It follows that for $t_0 > N$ we have $B(h^-_{t_0} l_{t_0} - l_{t_0}) < -c/4 < 0$ and $B(h^+_{t_0} l_{t_0} - l_{t_0}) > c/4 > 0$. Thus there are two transversal hyperbolic elements in S with the opposite signature. Hence S does not act properly discontinuous on $\mathbb{R}^n$ [A, Proposition 1]. Contradiction. $\square$

Proof. (***Main Theorem***)Conversely, suppose that the Zariski closure of S is not virtually solvable. Let S_0 be a semisimple part of the Zariski closure of $G = l(S)$. By [GS 2] this group is not compact. Therefore we can assume that S_0 is a non-compact simple group. Hence, S_0 is isomorphic to $SL_2(\mathbb{R})$. Recall (see ***2.3***), that very element from the Zariski closure of $\lambda(S)$ has an eigenvalue 1. Therefore G is a subgroup of $SL_3(\mathbb{R})$. Thus we have a representation $\rho : SL_2(\mathbb{R}) \to SL_3(\mathbb{R})$. There are two possible cases: or $\rho(SL_2(\mathbb{R})) = SO(2,1)$ and $G = SO(2,1)$ or ρ is the direct sum of the standard an trivial representation of $SL_2(\mathbb{R})$. It follows from Proposition 3.5 that the first case is impossible. Then we have two possibilities:

(a) there is a one dimensional subspace V such that $l(s)v = v$ for every $s \in S$

(b) there is a $l(S)$ - invariant subspace V, $\dim V = 2$.

(a). Fix a hyperbolic element g. Let L_g be the unique g-invariant line and let $v_0(g)$ be a unique Euclidean norm one vector such that for a point $p, p \in L_g$ we have $gp - p = \alpha^2 v_0(g)$. Since there exists a compact subset K_0 such that $SK_0 = \mathbb{R}^3$. Following along the same arguments we used in the proof of lemma 3.6 we come to the conclusion that there exist a positive number ε, a set of ε- hyperbolic elements $g_n, g_n \in S$ and a subset $\{p_n, p_n \in K_0, n \in \mathbb{N}\}$ such that

(1) $g_n p_n \in L_g$,

(2) $g_n p_n - p_n / \|g_n p_n - p_n\| \to -v_0(g)$ when $n \to \infty$,

(3) $\|g_n p_n - p_n\| \to \infty$ when $n \to \infty$.

For every element $s \in S$ we have $l(s)v_0 = v_0$. Therefore $sg_n p_n - p_n / \|sg_n p_n - p_n\| \to -v_0(g)$ when $n \to \infty$ and for $q_n = s^{-1}p_n$ we have $g_n s q_n - q_n / \|g_n s q_n - q_n\| \to -v_0(g)$ when $n \to \infty$. Remark that $\{q_n, n \in \mathbb{N}\}$ is a subset of a compact set $s^{-1}K_0$. Thus, we can assume that the sequence $\{A^+(g_n)\}_{n \in \mathbb{N}}$ (resp. $\{A^-(g_n)\}_{n \in \mathbb{N}}$) converge to A^+ (resp. A^-) and $\widehat{d}(A^+, A^\pm(g)) > 0$ (resp. $\widehat{d}(A^-, A^\pm(g)) > 0$. Hence there exists a ε- hyperbolic element h transversal to g such that for a point p from the unique h invariant line L_h

we have $hp - p = -\beta^2 v_0(g)$. Then using the same arguments as in [A], we conclude that there exist an infinite sets N and M such that $h^m g^n K_0 \bigcap K_0 \neq \emptyset$ for all $m \in M$, $n \in N$. This is impossible because S acts properly discontinuously.

(b). There is natural affine space $A = \mathbb{R}^3/V$, projection $\pi_V : \mathbb{R}^3 \to A$ and induced homomorphism $\rho_V : \mathrm{Aff}(\mathbb{R}^3) \to \mathrm{Aff}(A)$. Remark, that $dim A = 1$. It is clear that there exists a compact subset K on the line A such that $\rho_V(S)K = A$. There exist a positive number δ and sequences $\{g_t\}_{t\in\mathbb{N}}$ and $\{h_t\}_{t\in\mathbb{N}}$ of δ- hyperbolic elements such that for a point $k \in K$

(1) $|g_t k - k| \to \infty$, $|h_t k - k| \to \infty$ when $t \to \infty$,

(2) $(g_t k - k)(h_t k - k) < 0$ for all $t \in \mathbb{N}$,

(3) g_t and h_t are δ-transversal for all $t \in \mathbb{N}$.

This follows by the same methods as in proofs of lemma 3.6 and proposition 3.5 From the other hand representation ρ of the linear part $l(S)$ determine the following representation of S

$$s \mapsto \begin{pmatrix} a_{11} & a_{12} & a_{13} & * \\ a_{21} & a_{22} & a_{23} & * \\ 0 & 0 & 1 & \alpha_s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\rho(l(s)) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

It is easy to see that $\rho_V(s) = \alpha_s$. Hence by *(1)*,*(2)* *(3)* above there exist two δ-hyperbolic, δ -transversal elements $\widehat{g}$ and $\widehat{h}$ such that $\alpha_{\widehat{g}}\alpha_{\widehat{h}} < 0$. Let $L_{\widehat{g}}$ (resp. $L_{\widehat{h}}$) be the unique $\widehat{g}$ -invariant (resp $\widehat{h}$ -invariant) line. Fix a point p and an Euclidian distance d in $\mathbb{R}^3$. Then by [AMS 2], lemma there exists a positive $c = c(\widehat{g}, \widehat{h})$ such that

$$d(p, L_{\widehat{h}^m \widehat{g}^n}) \leq c[d(p, L_{\widehat{g}}) + d(p, L_{\widehat{h}})]$$

for all positive numbers m and n. Since $\alpha_s^n = n\alpha_s$ and $\alpha_{\widehat{g}}\alpha_{\widehat{h}} < 0$. there are infinite sets of positive numbers N and M such that $|\alpha_{\widehat{g}^n} + \alpha_{\widehat{h}^m}| \leq c/2$ for $n \in N, m \in M$. Consider the ball $U(p, 2c)$. Then for $n \in N, m \in M$ the set $\{\widehat{h}^m \widehat{g}^n U(p, 2c) \bigcap U(p, 2c)\}$ is non-empty. Contradiction, which proves the theorem □

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