

# ON THE ACTION OF THE GROUP OF ISOMETRIES ON A LOCALLY COMPACT METRIC SPACE: CLOSED-OPEN PARTITIONS AND CLOSED ORBITS

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ABSTRACT. In the present work we study the dynamic behavior of the orbits of the natural action of the group  $G$  of isometries on a locally compact metric space  $X$  using suitable closed-open subsets of  $X$ . Precisely, we study the dynamic behavior of an orbit even in cases where  $G$  is not locally compact with respect to the compact-open topology. In case  $G$  is locally compact we decompose the space  $X$  into closed-open invariant disjoint sets that are related to various limit behaviors of the orbits. We also provide a simple example of a locally compact separable and complete metric space  $X$  with discrete group of isometries  $G$  such that the natural action of  $G$  on  $X$  has closed and non-closed orbits.

## 1. INTRODUCTION

The group of isometries and their actions play an important role in many branches of Mathematics (especially in Geometry). This class of actions is rich, as a recent result of Abels, Noskov and the author in [2] shows. In [2] it is shown that if  $Y$  is a locally compact  $\sigma$ -compact metrizable space then a locally compact group  $\Gamma$  acts properly on  $Y$  if and only if there exists a  $\Gamma$ -invariant proper compatible metric on  $Y$  (recall that a metric on  $Y$  is called proper or Heine-Borel if every ball has compact closure in  $Y$ ). So, in this case, we can consider such a group as a closed subgroup of the group of isometries of a proper metric space (modulo the kernel of the action). The first result concerning the local compactness of the group of isometries of a locally compact metric space is the van Dantzig - van der Waerden theorem in 1928 (see [7] and [10, Theorem 4.7]) which says that the group  $G$  of isometries of

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a connected, locally compact metric space  $X$  is locally compact (with respect to the compact-open topology) and acts properly on  $X$ . In [12] (see also [14]) this result is generalized for the case of a locally compact metric space which has quasi-compact (i.e. compact but not necessarily Hausdorff) space of connected components (or quasi-components). In particular it is shown that the group of isometries of  $X$  is locally compact but may fail to act properly on  $X$  even for the case that  $X$  has only two connected components. A crucial point in obtaining this result is making use of suitable closed-open subsets of  $X$  (for more details see the next section). At the same time Gao and Kechris in [8, Theorem 5.4 and Corollary 6.2] (see also [6]) showed a stronger result: that the group of isometries  $G$  of a locally compact separable metric space  $X$  with finitely many pseudo-components (which are also closed-open subsets of  $X$ , see [8, Proposition 5.3]) is locally compact and in case  $X$  is locally compact, separable and pseudo-connected then  $G$  acts properly on  $X$  (for definitions and more details see [8, p. 32] and Section 3 below). Important examples of locally compact, separable and pseudo-connected spaces are the proper (Heine-Borel) spaces. Comparing the results of [12] and [8] we would like to mention that the assumption about the quasi-compactness of the space of connected components of  $X$  in [12] is purely topological hence the result in [12] applies to any metric that induces the topology of  $X$ . Obviously the assumption in [8] about finitely many pseudo-components depends on the choice of the metric on  $X$  but the result is stronger since a locally compact metric space with quasi-compact space of connected components has finitely many pseudo-components.

The purpose of this paper is to show that the closed-open subsets of  $X$  used in [12] and [8] also give information for the space  $X$  and the dynamic behavior of the orbits of the natural action of  $G$  on  $X$ , even for the case that  $G$  is not a locally compact group. In what follows,  $X$  will denote a locally compact metric space with a fixed metric  $d$  and  $G := Iso(X, d)$  will denote the group of (surjective) isometries of  $X$  endowed with the compact-open topology. The natural action of  $G$  on  $X$  is the action with  $(g, x) \mapsto g(x)$ ,  $g \in G$ ,  $x \in X$ . The main results in this work are stated below:

In Section 3 (see Propositions 3.1 and Corollary 3.4 below) we show the following:

**Proposition.** *Let  $x, y \in X$  and a net  $\{g_i\}$  in  $G$  with  $g_i x \rightarrow y$ . Then there exist a subnet  $\{g_j\}$  of  $\{g_i\}$ , a closed-open subset  $A$  of  $X$  that contains  $x$  and a map  $f : A \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $A$ ,  $f(x) = y$  and  $f(A)$  is an open subset of  $X$ .*

*The same result also holds if we replace  $A$  with the pseudo-component  $C_x$  that contains  $x$ . In this case  $f(C_x) = C_{f(x)}$ .*

The previous proposition gives as corollaries the van Dantzig - van der Waerden theorem (see Corollary 3.2 below) and the results of Gao and Kechris in [8, Theorem 5.4 and Corollary 6.2] (see Corollary 3.5 below). In Section 4 we give some applications in case  $G$  is locally compact and there exist closed orbits for the action of  $G$  on  $X$ . We also give a simple example of a locally compact separable and complete metric space with a discrete group of isometries such that the natural action of  $G$  on  $X$  has closed and non-closed orbits (see Example 4.4 below). In Section 5 we show that the closed-open subsets of  $X$  used in [12] leads to a decomposition of the space  $X$  into closed-open  $G$ -invariant disjoint sets that are related to the limit behavior of the orbits (see Theorem 5.1 below): Let

$$L(x) = \{y \in X \mid \text{there exists a net } \{g_i\} \text{ in } G \\ \text{with } g_i \rightarrow \infty \text{ and } \lim g_i x = y\},$$

denote the limit set of  $x \in X$ , where  $g_i \rightarrow \infty$  means that the net  $\{g_i\}$  has no cluster point in  $G$ .

**Theorem.** *Assume that  $G$  is locally compact and not compact and let*

$$CL = \{x \in X \mid L(x) \text{ is not empty and compact}\}, \\ NCL = \{x \in X \mid L(x) \text{ is not compact}\} \text{ and} \\ P = \{x \in X \mid L(x) \text{ is the empty set}\}.$$

*Then the sets  $CL$ ,  $NCL$  and  $P$  are closed-open  $G$ -invariant disjoint, their union is  $X$  and each one of them is a union of pseudo-components.*

## 2. PRELIMINARIES

A continuous action of a topological group  $\Gamma$  on a topological space  $Y$  is a continuous map  $\Gamma \times Y \rightarrow Y$  with  $(g, x) \mapsto gx$ ,  $g \in \Gamma$ ,  $x \in Y$  such that  $(1, x) \mapsto x$ , for every  $x \in Y$  where 1 denotes the unit element of  $\Gamma$ , and  $h(gx) = (hg)x$  for every  $h, g \in \Gamma$  and  $x \in Y$ . For  $U \subseteq Y$  let  $\Gamma U$  denote the set  $\{gx \mid g \in \Gamma, x \in U\}$ . Especially, if  $U = \{x\}$  then the set  $\Gamma x := \Gamma\{x\}$  is called the orbit of  $x \in Y$  under  $\Gamma$ . If  $\Gamma U = U$  we say that  $U$  is  $\Gamma$ -invariant. The subgroup  $\Gamma_x := \{g \in \Gamma \mid gx = x\}$  of  $\Gamma$  is called the isotropy group of  $x \in Y$ .

*In what follows,  $X$  will denote a locally compact metric space with a fixed metric  $d$  and  $G := Iso(X, d)$  will denote the group of (surjective) isometries of  $X$  endowed with the compact-open topology. The natural action of  $G$  on  $X$  is the action with  $(g, x) \mapsto g(x)$ ,  $g \in G$ ,  $x \in X$ .*

If we endow  $G$  with the topology of pointwise convergence then  $G$  is a topological group (see [5, Ch. X, §3.5 Corollary]). On  $G$  there is also the topology of uniform convergence on compact subsets which is the same as the compact–open topology. In the case of a group of isometries these topologies coincide with the topology of pointwise convergence, and the natural action of  $G$  on  $X$  with  $(g, x) \mapsto g(x)$  is continuous (see [5, Ch. X, §2.4 Theorem 1 and §3.4 Corollary 1]).

We recall that in [4, Ch. III, §4.1 Definition 1] a continuous action of a topological group  $\Gamma$  on a topological space  $Y$  is said to be *proper* (or *Bourbaki proper*) if the map

$$\Gamma \times Y \rightarrow Y \times Y \text{ with } (g, x) \mapsto (x, gx), \text{ for } g \in \Gamma \text{ and } x \in Y$$

is proper, i.e. it is continuous, closed and the inverse image of a singleton is a compact set.

To simplify the proofs we shall use the following equivalent definition for properness: a continuous action is proper if the extended limit sets  $J(x)$  are empty for every  $x \in Y$ , where

$$J(x) = \{y \in Y \mid \text{there exist nets } \{x_i\} \text{ in } Y \text{ and } \{g_i\} \text{ in } \Gamma \\ \text{with } g_i \rightarrow \infty, \lim x_i = x \text{ and } \lim g_i x_i = y\},$$

where  $g_i \rightarrow \infty$  means that the net  $\{g_i\}$  has no cluster point in  $G$ . It is easy to see that in the special case of actions by isometries  $J(x) = L(x)$  holds for every  $x \in Y$ , where

$$L(x) = \{y \in Y \mid \text{there exists a net } \{g_i\} \text{ in } \Gamma \\ \text{with } g_i \rightarrow \infty \text{ and } \lim g_i x = y\},$$

denotes the limit set of  $x \in Y$  under the action of  $\Gamma$  on  $Y$ . Hence an action by isometries is proper if and only if  $L(x)$  is the empty set for every  $x \in Y$ . The limit and the generalized limit sets for locally compact spaces and groups are closed and  $\Gamma$ -invariant (see [3]). The following example shows that even in case that  $X$  has two connected components the action of  $G$  on  $X$  may not be proper (see also [14]).

**Example 2.1.** Let  $X = L_1 \cup L_2 \subset \mathbb{R}^2$  where  $L_1 = \{(0, t) \mid t \in \mathbb{R}\}$  and  $L_2 = \{(2, t) \mid t \in \mathbb{R}\}$ . We consider the metric  $d = \min\{d_E, 1\}$  where  $d_E$  is the usual Euclidean metric on  $\mathbb{R}^2$ . With this metric  $X$  is a locally compact separable space. Since for a point  $x \in X$  (actually for every  $x \in X$ ) the isotropy group  $G_x$  contains an isomorphic copy of the reals the action of  $G$  on  $X$  is not proper.

Let  $F$  be a subset of  $G$ . We define  $K(F)$  to be the set

$$K(F) := \{x \in X \mid \text{the set } Fx \text{ has compact closure in } X\}.$$

These sets played a crucial role in [12] where it is shown that they are closed-open subsets of  $X$  (see [12, Lemma 3.1], [14]). Actually we can find the same definition and result in Bourbaki (see [5, Ch. X, Exercise 13, p. 323]) but as we mentioned in [2] there is a mistake in part d) of this exercise (for a counterexample see [2] or [11]).

### 3. THE GENERAL CASE

In this section *the only assumption is that  $X$  is locally compact without assuming the same for  $G$* . We study the following problem: Assume that there is a pair of points  $x, y \in X$  and a net  $\{g_i \mid i \in I\}$  in  $G$  such that  $g_i x \rightarrow y$ . What can we say about the convergence of  $\{g_i\}$ ?

**Proposition 3.1.** *Let  $x, y \in X$  and a net  $\{g_i \mid i \in I\}$  in  $G$  with  $g_i x \rightarrow y$ . Then there exist a subnet  $\{g_j \mid j \in J\}$  of  $\{g_i \mid i \in I\}$ , a closed-open subset  $A$  of  $X$  that contains  $x$  and a map  $f : A \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $A$ ,  $f(x) = y$  and  $f(A)$  is an open subset of  $X$ . Moreover, if  $\{g_i\}$  is a sequence and  $X$  is a second countable space (in which case the limit sets can be described using sequences) then  $f(A)$  is closed.*

*Proof.* Since  $d(x, g_i^{-1}y) = d(g_i x, y) \rightarrow 0$  it follows that  $g_i^{-1}y \rightarrow x$ . By the local compactness of  $X$  there exists an index  $i_0 \in I$  such that, if  $F := \{g_i \mid i \geq i_0\}$  then  $x \in K(F)$  and  $y \in K(F^{-1})$ , where  $F^{-1} := \{g_i^{-1} \mid i \geq i_0\}$ . Set  $A := K(F)$ . By [12, Lemma 3.1]  $A$  is a closed-open subset of  $X$ . If  $g_i|_A$  denotes the restriction of each  $g_i$  on  $A$ , then the Arzela-Ascoli theorem implies that the set  $\{g_i|_A : A \rightarrow X \mid i \geq i_0\}$  has compact closure in  $C(A, X)$  (this the set of all continuous maps from  $A$  to  $X$ ). Thus, there exists a subnet  $\{g_j \mid j \in J\}$  of  $\{g_i \mid i \in I\}$  and a map  $f : A \rightarrow X$  with  $f(x) = y$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $A$ . We show that  $f(A)$  is open: Let  $z \in f(A)$ . That is, there is  $w \in A$  such that  $f(w) = z$ . It is enough to show that if  $\{z_l\} \subset X$  is a net such that  $z_l \rightarrow z$  then  $z_l \in A$  eventually for every  $l$ . Since  $w \in A$  then  $g_j w \rightarrow f(w) = z$ . Hence  $g_j^{-1}z \rightarrow w$ . As before there exists an index  $j_0$  such that, if  $F_1 := \{g_j \mid j \geq j_0\}$  then  $w \in K(F_1)$  and  $z \in K(F_1^{-1})$ . Again by the Arzela-Ascoli theorem there exist a subnet  $\{g_k\}$  of  $\{g_j\}$  and a map  $h : K(F_1^{-1}) \rightarrow X$  which preserves the distance such that  $g_k^{-1} \rightarrow h$  pointwise on  $K(F_1^{-1})$  and  $h(z) = w$ . Since  $K(F_1^{-1})$  is open and  $z \in K(F_1^{-1})$  we may assume that  $z_l \in K(F_1^{-1})$  eventually for every  $l$ . Hence  $h(z_l) \rightarrow h(z) = w \in A$  as  $l \rightarrow \infty$  and for each  $l$ ,  $g_k^{-1}z_l \rightarrow h(z_l)$  as  $k \rightarrow \infty$ . Therefore  $h(z_l) \in A$  eventually for every  $l$ . Fix a point  $h(z_l) \in A$ . Then  $g_k(h(z_l)) \rightarrow f(h(z_l))$ . Thus  $d(z_l, f(h(z_l))) \leq d(g_k g_k^{-1}z_l, g_k h(z_l)) +$

$d(g_k h(z_l), f(h(z_l))) = d(g_k^{-1} z_l, h(z_l)) + d(g_k h(z_l), f(h(z_l))) \rightarrow 0$  as  $k \rightarrow \infty$ . Hence  $z_l = f(h(z_l)) \in A$  eventually for every  $l$ .

Note that up to this point we have only used the property that the sets  $K(F)$  are open. If  $\{g_i \mid i \in \mathbb{N}\}$  is a sequence we can set  $F_2 := \{g_i \mid i \in \mathbb{N}\}$  and  $A := K(F_2)$ . Then  $A$  is a non-empty closed-open subset of  $X$  and, as before, there exists a subsequence  $\{g_{i_n} \mid n \in \mathbb{N}\}$  of  $\{g_i\}$  (here we use that  $X$  is second countable) and a map  $f : A \rightarrow X$  with  $f(x) = y$  which preserves the distance such that  $g_{i_n} \rightarrow f$  pointwise on  $A$ . We will use now the property that the sets  $K(F)$  are closed to show that  $f(A)$  is also a closed subset of  $X$ . If we set  $F_3 := \{g_{i_n} \mid n \in \mathbb{N}\}$  then it is easy to verify that  $f(A) \subseteq K(F_3^{-1})$  and there exist a subsequence  $\{g_{i_{n_l}} \mid l \in \mathbb{N}\}$  of  $\{g_{i_n}\}$  and a map  $h : K(F_3^{-1}) \rightarrow X$  which preserves the distance such that  $g_{i_{n_l}}^{-1} \rightarrow h$  pointwise on  $K(F_3^{-1})$ . Take a sequence  $\{f(a_k) \mid k \in \mathbb{N}\}$ ,  $a_k \in A$  such that  $f(a_k) \rightarrow b$  for some  $b \in X$ . We will show that  $b \in f(A)$ . Fix  $k \in \mathbb{N}$ . Since  $f(A) \subseteq K(F_3^{-1})$  and  $K(F_3^{-1})$  is closed then  $f(a_k) \in K(F_3^{-1})$ ,  $g_{i_{n_l}}^{-1} f(a_k) \rightarrow h(f(a_k))$  as  $l \rightarrow \infty$  and  $b \in K(F_3^{-1})$ . The latter implies that  $g_{i_{n_l}}^{-1} b \rightarrow h(b)$ . Note that  $d(g_{i_{n_l}}^{-1} b, h(b)) = d(b, g_{i_{n_l}} h(b)) \rightarrow 0$  so  $g_{i_{n_l}} h(b) \rightarrow b$ . We will show that  $h(b) \in A$  and  $g_{i_{n_l}} h(b) \rightarrow f(h(b))$ , hence  $b = f(h(b)) \in f(A)$  and the proof is finished. Indeed, observe that  $d(g_{i_{n_l}}^{-1} f(a_k), a_k) = d(f(a_k), g_{i_{n_l}} a_k) \rightarrow 0$  as  $l \rightarrow \infty$ . Therefore  $h(f(a_k)) = a_k$ . Thus  $a_k = h(f(a_k)) \rightarrow h(b)$  as  $k \rightarrow \infty$ . But  $a_k \in A$  and  $A$  is a closed subset of  $X$  hence  $h(b) \in A$ . So  $g_{i_{n_l}} h(b) \rightarrow f(h(b))$ .  $\square$

Note that  $f(A)$  may not be  $G$ -invariant, see for instance Example 2.1. As an application of Proposition 3.1 we can prove the van Dantzig - van der Waerden theorem in a short and elegant way comparing to the proof in the original work of van Dantzig and van der Waerden [7] or to the lengthy one in [10, Theorem 4.7, pp. 46–49]:

**Corollary 3.2.** *(The van Dantzig - van der Waerden theorem) The group  $G$  of isometries of a connected, locally compact metric space  $X$  is locally compact (with respect to the compact-open topology) and  $G$  acts properly on  $X$ .*

*Proof.* It is enough to show that  $G$  acts properly on  $X$  (i.e.  $L(x) = \emptyset$  for every  $x \in X$ , see Section 2) because in this case for every pair of points  $x, y \in X$  there exist open neighborhoods  $U_x, U_y$  of  $x, y$  respectively such that the set  $\{g \in G \mid gU_x \cap U_y \neq \emptyset\}$  has compact closure in  $G$  (see e.g. [3]). Let  $x, y \in X$  and a net  $\{g_i\}$  in  $G$  such that  $g_i x \rightarrow y$ . Proposition 3.1 implies that there exist a subnet  $\{g_j\}$  of  $\{g_i\}$ , a closed-open subset  $A$  of  $X$  and a map  $f : A \rightarrow X$  which preserves the distance

such that  $g_j \rightarrow f$  pointwise on  $A$ . Since  $X$  is connected it follows that  $A = X$ . Note that  $d(x, g_i^{-1}y) = d(g_i x, y) \rightarrow 0$  hence we can repeat the same procedure as before and find a subnet  $\{g_k\}$  of  $\{g_i\}$  and maps  $f, h : X \rightarrow X$  which preserve the distance such that  $g_k \rightarrow f$  and  $g_k^{-1} \rightarrow h$  pointwise on  $X$ . Obviously  $h$  is the inverse of  $f$ . This shows that  $f \in G$ . Hence,  $L(x) = \emptyset$  and since  $x \in X$  was arbitrary the action is proper.  $\square$

A question which arises from Proposition 3.1 is whether there is any difference if one replaces  $A$  with the pseudo-component that contains the point  $x \in X$ . We answer this question in the affirmative in Corollary 3.4. Before we present these result we need some formulation that we can also find in [8, p. 32]:

An important notion in the definition of the pseudo-component of a point  $x \in X$  is the *radius of compactness*  $\rho(x)$  of  $x$ :

$$\rho(x) := \sup\{r > 0 \mid \text{the open ball } B(x, r) \text{ has compact closure}\}$$

where  $B(x, r)$  denotes the open ball centered at  $x \in X$  with radius  $r > 0$ . It is easy to see that if  $g \in G$  then  $\rho(gx) = \rho(x)$ . We define an equivalence relation  $\mathcal{E}$  on  $X$  as follows: Firstly we define a directed graph  $\mathcal{R}$  on  $X$  by

$$x\mathcal{R}y \quad \text{if and only if} \quad d(x, y) < \rho(x).$$

Let  $\mathcal{R}^*$  be the transitive closure of  $\mathcal{R}$ , i.e.

$$x\mathcal{R}^*y \quad \text{if and only if for some} \quad u_0 = x, u_1, \dots, u_n = y$$

we have  $u_i\mathcal{R}u_{i+1}$  for every  $i < n$ . Finally, define the following equivalence relation  $\mathcal{E}$  on  $X$

$$x\mathcal{E}y \quad \text{if and only if} \quad x = y \text{ or } (x\mathcal{R}^*y \text{ and } y\mathcal{R}^*x).$$

We call the  $\mathcal{E}$ -equivalence class of  $x \in X$  the *pseudo-component* of  $x$ , and we denote it by  $C_x$ . We call  $X$  *pseudo-connected* if it has only one pseudo-component. It follows that pseudo-components are closed-open subsets of  $X$  (see [8, Proposition 5.3]). An immediate consequence of the definitions is that  $gC_x = C_{gx}$  for every  $g \in G$ .

The following example shows that in many cases the closed-open set  $A$  in Proposition 3.1 may contain strictly the pseudo-component that contains the point  $x \in X$ .

**Example 3.3.** Let  $X = L_1 \cup L_2 \cup L_3 \subset \mathbb{R}^2$  where  $L_1 = \{(0, t) \mid t \in \mathbb{R}\}$ ,  $L_2 = \{(2, t) \mid t \in \mathbb{R}\}$  and  $L_3 = \{(4, t) \mid t \in \mathbb{R}\}$  endowed with the metric  $d = \min\{d_E, 1\}$  where  $d_E$  is the usual Euclidean metric on  $\mathbb{R}^2$ . With this metric  $X$  is a locally compact separable and complete metric space with finitely many pseudo-components. Let  $x := (0, 0)$  and let

$g_n : X \rightarrow X$ ,  $n \in \mathbb{N}$  with  $g_n(t, a) = (t, a)$  if  $a = 0$  or  $2$  and  $t \in \mathbb{R}$  and  $g_n(4, t) = (4, t + n)$ ,  $t \in \mathbb{R}$ . Obviously  $g_n x = x$  for every  $n \in \mathbb{N}$ , the map  $g_n$  restricted to  $L_1 \cup L_2$  is the identity and the pseudo-component  $C_x$  of  $x$  is the set  $L_1$ . Hence, if we take as  $A := L_1 \cup L_2$  then  $A$  contains strictly  $C_x$ .

**Corollary 3.4.** *Assume that  $X$  is locally compact (perhaps with infinitely many pseudo-components). Let  $x, y \in X$  and a net  $\{g_i\}$  in  $G$  such that  $g_i x \rightarrow y$ . Then there exist a subnet  $\{g_j\}$  of  $\{g_i\}$  and a map  $f : C_x \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $C_x$ ,  $f(x) = y$  and  $f(C_x) = C_{f(x)}$ .*

*Proof.* With a slight modification of the technical Lemma 5.5 in [8] in order to use nets instead of sequences we have the following: Let  $x, y \in X$  and  $\{g_i\}$  be a net in  $G$  with  $g_i x \rightarrow y$ . Then for  $F := \{g_i\}$  the set  $F(z)$  has compact closure in  $X$  for every  $z \in C_x$ . For  $A := K(F)$  Proposition 3.1 implies that there exist a subnet  $\{g_j\}$  of  $\{g_i\}$  and a map  $f : C_x \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $C_x$  and  $f(x) = y$ . Since  $g_j^{-1} f(x) \rightarrow x$  then there exist a subnet  $\{g_k\}$  of  $\{g_j\}$  and a map  $h : C_{f(x)} \rightarrow X$  which preserves the distance such that  $g_k^{-1} \rightarrow h$  pointwise on  $C_{f(x)}$  and  $h(y) = x$ . Take a point  $z \in C_x$ . Since  $g_k z \rightarrow f(z)$  and the pseudo-component  $C_{f(x)}$  is a closed-open subset of  $X$  then  $f(z) \in C_{f(x)}$  (so  $f(C_x) \subseteq C_{f(x)}$ ) and  $g_k z \in C_{f(x)}$  eventually for every  $k$ . Hence  $z = g_k^{-1}(g_k z) \rightarrow h(f(z))$ . In the same way we can deduce that  $f(h(w)) = w$  for every  $w \in C_{f(x)}$ , thus  $f(C_x) = C_{f(x)}$ .  $\square$

As an implication of the previous corollary we can take the results of Gao and Kechris [8, Theorem 5.4 and Corollary 6.2]:

**Corollary 3.5.** *(The Gao - Kechris theorem) The isometry group of a locally compact metric space with only finitely many pseudo-components is locally compact. In case  $X$  is locally compact and pseudo-connected then  $G$  acts properly on  $X$ .*

*Proof.* Let  $C_1, C_2, \dots, C_n$  denote the pseudo-components of  $X$  and take points  $x_1 \in C_1, x_2 \in C_2, \dots, x_n \in C_n$  and open balls  $B(x_m, r) \subseteq C_m$ ,  $m = 1, 2, \dots, n$ ,  $r > 0$  such that all  $B(x_m, r)$  have compact closures. We

will show that the set  $V := \bigcap_{m=1}^n \{g \in G \mid gx_m \in B(x_m, r)\}$  is an (open)

neighborhood of the identity in  $G$  with compact closure. Indeed take a net  $\{g_i\} \subseteq V$ . Since each  $B(x_m, r)$  has compact closure there exist a subnet  $\{g_j\}$  of  $\{g_i\}$  and points  $y_1 \in C_1, y_2 \in C_2, \dots, y_n \in C_n$  such that  $g_j x_m \rightarrow y_m$ , as  $j \rightarrow \infty$ , for every  $m = 1, 2, \dots, n$ . Corollary 3.4 implies that there exist a subnet  $\{g_l\}$  of  $\{g_j\}$  and maps  $f_m : C_m \rightarrow C_m$  which

preserve the distance such that  $g_l \rightarrow f_m$  on  $C_m$  and  $f_m(C_m) = C_m$  for all  $m$ . This shows that  $\{g_k\}$  converges to a *surjection* of  $X$  which actually gives that  $\{g_k\}$  converges to an isometry of  $X$ .

Assume that  $X$  is pseudo-connected. In order to show that  $G$  acts properly on  $X$  it is enough to show that the limit set  $L(x)$  is empty for every  $x \in X$  (see Section 2). Let  $x, y \in X$  and a net  $\{g_i\}$  in  $G$  such that  $g_i x \rightarrow y$ . Corollary 3.4 implies that there exist a subnet  $\{g_j\}$  of  $\{g_i\}$  and a map  $f : C_x \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $C_x$ ,  $f(x) = y$  and  $f(C_x) = C_{f(x)}$ . Since  $X$  is pseudo-connected then  $C_x = C_{f(x)} = X$ . Hence  $f \in G$  thus  $L(x)$  is empty.  $\square$

*Remark 3.6.* Note that in Corollary 3.5 we do not require that  $X$  is separable like in [8, Theorem 5.4 and Corollary 6.2]. This is *not* a real improvement since we can show that *a locally compact metric space  $X$  with countably many pseudo-components is separable*. For a proof we can imitate the proof of Lemma 3 in [10, Appendix 2] (actually this is a result of Sierpinski, see [13]): We define a relation  $\mathcal{S}$  on  $X$  by  $x\mathcal{S}y$  if and only if there exist separable open balls  $B(x, r_1)$  and  $B(y, r_2)$  with  $y \in B(x, r_1)$  and  $x \in B(y, r_2)$ . For every  $A \subseteq X$  we denote by  $\mathcal{S}A := \{y \in X \mid y\mathcal{S}x \text{ for some } x \in A\}$ . If  $A = \{x\}$  is a singleton we write  $\mathcal{S}x$  instead of  $\mathcal{S}\{x\}$ . Set  $\mathcal{S}^{n+1}x := \mathcal{S}\mathcal{S}^n x$  for every  $n \in \mathbb{N}$  and  $U(x) := \bigcup_{n=1}^{+\infty} \mathcal{S}^n x$ . Then by [10, Lemma 3 in Appendix 2] each  $U(x)$  is a separable closed-open subset of  $X$  and if  $U(x) \cap U(y) \neq \emptyset$  then  $U(x) = U(y)$ . By construction every  $U(x)$  contains the pseudo-component of  $x \in X$ . Therefore  $X$  is separable.

*Remark 3.7.* Proposition 3.1 and Corollary 3.4 point out a natural generalization of the notion of properness for locally compact metric spaces with groups of isometries which are not closed in the space of all continuous selfmaps of  $X$  endowed with the compact-open topology: In particular, it will be interesting to study actions with the property “if  $x, y \in X$  and there is a net  $\{g_i\}$  in  $G$  such that  $g_i x \rightarrow y$  then there exist a subnet  $\{g_j\}$  of  $\{g_i\}$  and a map  $f : X \rightarrow X$  *not necessarily surjective*, which preserves the distance and such that  $g_i \rightarrow f$  pointwise on  $X$ ”. That is, let’s say, if  $g_i \rightarrow \infty$  then this happens in a “strong” way.

#### 4. CLOSED ORBITS

In this section we assume that *both*  $X$  and  $G$  are *locally compact* and we will discuss some implications of the existence of closed orbits. In the previous section we saw that if there is a pair of points  $x, y \in X$  and

a net  $\{g_i\}$  in  $G$  such that  $g_i x \rightarrow y$  then there exist a subnet  $\{g_j\}$  of  $\{g_i\}$ , a closed-open subset  $A$  of  $X$  that contains  $x$  and a map  $f : A \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $A$ ,  $f(x) = y$  and  $f(A)$  is an open subset of  $X$ . A question which arises naturally is: When is the map  $f$  a restriction of some element of  $G$  on  $A$ ? An answer can be given using the following general proposition but before we see that we need again some formulation: Given a continuous action of a locally compact group  $\Gamma$  on a locally compact space  $Y$  we can define a homomorphism  $\phi : \Gamma \rightarrow H(Y)$  with  $\phi(g)(x) := gx$ ,  $g \in \Gamma$ ,  $x \in Y$  where  $H(Y)$  denotes the group of homeomorphisms of  $Y$  endowed with the compact-open topology. This homomorphism is always continuous (see e.g. [3] or [15, Lemma 10.4 (c)]). If  $g|_A$  denotes the restriction of  $g \in \Gamma$  on a subset  $A$  of  $Y$  we define  $\phi : \Gamma \rightarrow C(A, Y)$  with  $\phi(g)(x) := gx$ ,  $g \in \Gamma$ ,  $x \in A$  where  $C(A, Y)$  denotes the space of all continuous maps from  $A$  to  $Y$  endowed with the compact-open topology. Note that  $\phi$  is a continuous map.

**Proposition 4.1.** *Let  $Y$  be a locally compact space,  $A$  be an open or closed subset of  $Y$  and  $\Gamma$  be a locally compact  $\sigma$ -compact group which acts continuously on  $Y$ . If there exists a point  $x \in A$  with closed orbit such that  $\phi(\Gamma_x)$  is closed in  $C(A, Y)$  then  $\phi(\Gamma)$  is closed in  $C(A, Y)$ .*

*Proof.* Since  $\Gamma$  is locally compact and  $\sigma$ -compact and  $\Gamma(x)$  is closed in  $Y$ , the map  $\varphi : \Gamma/\Gamma_x \rightarrow \Gamma(x)$  with  $\varphi(g\Gamma_x) := \Gamma x$ ,  $g \in \Gamma$  is a homeomorphism (see [15, Theorem 10.10 (c)]). Let  $\{g_i | i \in I\}$  be a net in  $\Gamma$  such that  $\phi(g_i) \rightarrow h$  for some  $h \in C(A, Y)$ . Since the orbit  $\Gamma(x)$  is closed, there exists  $\gamma \in \Gamma$  such that  $\gamma x = h(x)$  so  $g_i \Gamma_x \rightarrow \gamma \Gamma_x$ . The quotient map  $\Gamma \rightarrow \Gamma/\Gamma_x$  is open and  $\Gamma$  is locally compact hence there exist an open neighborhood  $V$  of  $v$  with compact closure and nets  $\{f_i\}$  in  $V$ ,  $\{v_i\}$  in  $\Gamma_x$  such that  $g_i = f_i v_i$  eventually for every  $i \in I$ . Thus, there exist a subnet  $\{f_j\}$  of the net  $\{f_i\}$  and  $f \in \Gamma$  such that  $f_j \rightarrow f$ . The set  $A$  is locally compact, hence the composition map  $T : C(Y, Y) \times C(A, Y) \rightarrow C(A, Y)$  with  $T(f_1, f_2) = f_1 \circ f_2$ ,  $f_1 \in C(Y, Y)$ ,  $f_2 \in C(A, Y)$  is continuous (see [15, Lemma 9.4 (c)]). Thus,  $\phi(v_j) = \phi(f_j^{-1}) \circ \phi(g_j) \rightarrow \phi(f^{-1}) \circ h$ . Since  $\phi(\Gamma_x)$  is closed in  $C(A, Y)$  there exists  $g \in \Gamma_x$  such that  $\phi(f^{-1}) \circ h = \phi(g)$  from which follows that  $h = \phi(fg)$ . Hence  $\phi(\Gamma)$  is closed in  $C(A, Y)$ .  $\square$

Note that in our case, if  $X$  is a second countable locally compact metric space with locally compact group of isometries  $G$  then  $G$  is  $\sigma$ -compact (see [5, Ch. X, §3.3 Corollary]). Following the proof of the previous proposition, if  $x, y \in Y$  and  $\{g_i\}$  is a net in  $\Gamma$  such that

$g_i x \rightarrow y$  then  $g_i = f_i v_i$  for some nets  $\{f_i\}$  in a compact subset of  $\Gamma$  and  $\{v_i\}$  in  $\Gamma_x$ , so we have the following:

**Corollary 4.2.** *Let  $Y$  be a locally compact space and  $\Gamma$  be a locally compact  $\sigma$ -compact group which acts continuously on  $Y$ . If  $x$  is a point of  $Y$  with closed orbit then the limit set  $L(x)$  is not empty if and only if the isotropy group  $\Gamma_x$  of  $x$  is not compact.*

If we assume that  $X$  is second countable (hence separable) and complete and its group of isometries  $G$  is locally compact then  $G$  is second countable (see [5, Ch. X, §3.3 Corollary]). So, in this case, both  $X$  and  $G$  are *Polish* spaces. In [9] Glimm showed that for the case of an action of a locally compact separable group on a locally compact separable space the existence of a *Borel section* (or selection) is actually equivalent to the fact that each orbit is locally closed. Recall that a subset  $S$  of a topological space  $Y$  is called a *section* (or selection) for a continuous action of a topological group  $\Gamma$  on  $Y$  if  $S$  meets every orbit in exactly one point. In our case if  $x, y$  is a pair of points of  $X$  and  $\{g_i\}$  is a net in  $G$  such that  $g_i x \rightarrow y$  then  $g_i^{-1} y \rightarrow x$ , since  $g_i$  preserves the metric. Thus, for isometric actions, locally closed orbits are closed and vice versa. A question which arises naturally is the following:

**Question 4.3.** If  $X$  and  $G$  are locally compact do there exist always closed orbits?

Note that we are considering the full group of isometries of  $X$  because if we ask the same question for the action of a closed subgroup of  $G$  on  $X$  then the answer is negative (see [11]). The following simple example shows that the answer is also negative for the action of  $G$  on  $X$ :

**Example 4.4.** Let  $X = R \cup Q \subset \mathbb{R}^2$  where  $R = \{(t, 0) \mid t \in \mathbb{R}\}$  and  $Q = \{(q, 1) \mid q \in \mathbb{Q}\}$ . For every pair of points  $w_1 = (x_1, y_1)$ ,  $w_2 = (x_2, y_2) \in X$  define

$$d(w_1, w_2) := \begin{cases} |x_1 - x_2|, & \text{if } w_1, w_2 \in R \\ |x_1 - x_2| + 1, & \text{if only one of } w_1, w_2 \text{ is not in } R \\ |x_1 - x_2| + 2, & \text{if } w_1, w_2 \in Q. \end{cases}$$

It is easy to verify that  $d$  is a metric on  $X$  and  $X$  with this metric is a locally compact, separable and complete space. The group of isometries  $G$  is generated by the horizontal translations by rationals and by the horizontal reflections with centers of the form  $(x, y) \in X$  with  $x \in \mathbb{Q}$  (if we want to have only translations we may take  $Q := \{(q, 1) \mid q \in \mathbb{Q} + \sqrt{2}\mathbb{N}\}$ ). Hence  $G$  is a discrete group (so it is locally compact). If  $w = (x, 1) \in Q$  then the orbit  $G(w) = Q$  and if  $w = (x, 0) \in R$  then the

orbit  $G(w) = \{(x, 0) \in R \mid x \in \mathbb{Q}\}$  so it is not closed in  $X$ . Moreover if  $w \in Q$  then  $L(x) = \emptyset$  and if  $w \in R$  then  $x \in L(x) = R$ .

In this direction Gao and Kechris in [8, p. 35] asked the following question which still remains open:

**Question 4.5.** (Gao - Kechris) Let  $(X, d)$  be a locally compact complete metric space with finitely many pseudo-components (or connected components). Is it true that the action of  $G := Iso(X, d)$  on  $X$  has closed orbits?

Based on this we ask the following question.

**Question 4.6.** Let  $(X, d)$  be a locally compact and complete metric with only two connected components, one compact and one not compact. If the action of  $G := Iso(X, d)$  on the non-compact component is proper is it true that the orbits of points in the compact component are closed?

The last question is of great interest in case of a metric space having only one end in its Freudenthal (end-point) compactification.

As we saw in Example 4.4 the set of points of  $X$  with closed orbits may not be the whole space  $X$ , so it is natural to ask the following:

**Question 4.7.** Let  $X$  be a locally compact metric space. Is the set of points of  $X$  with closed orbits closed or open? Does it contain entire pseudo-components?

In the following example we give (a partial) negative answer to this question. Namely the set of points of  $X$  with closed orbits may not be open or may not contains entire pseudo-components:

**Example 4.8.** This example is based on the same idea as Example 4.4. Let  $X = D \cup S \subset \mathbb{R}^2$  where  $D$  is the closed unit disk and  $S$  is an orbit of a point on a circle with center the origin and radius 2 under an irrational rotation  $2e^{2\pi a}$ ,  $a \notin \mathbb{Q}$ . The distance of two points in the unit disk is the usual Euclidean one. To measure the distance from a point of  $x \in S$  to a point of  $y \in D$  we firstly move on the radius connecting  $x$  with the origin until we meet the circle with center the origin and radius the distance from  $y$  to the origin. Then we move on this circle in the shortest way until we meet the point  $y$ . In a similar way we measure the distance of two points  $x, y \in S$ : Firstly we move on the radius connecting  $x$  with the origin until we meet the unit circle and then we move on this circle and follow back the radius connecting  $y$  with the origin in the shortest way until we meet the point  $y$ . The space  $X$  endowed with this metric is locally compact, separable and complete

and its group of isometries  $G$  is discrete (hence locally compact) like in Example 4.4. Moreover, all the points of the closed unit disk except the origin do not have closed orbits and the origin is a fixed point. The action of  $G$  on  $S$  is proper and the orbits coincide with  $S$ . Note that the pseudo-component (and the connected component) that contains the origin is the closed unit disk hence the set of points of  $X$  with closed orbit is not closed and does not contain entire pseudo-components.

The only thing that remains to be clarified is whether the set of points of a space  $X$  with closed orbits is a closed subset of  $X$ . In this direction we know that if there is a pair of points  $x, y \in X$  and a net  $\{g_i\}$  in  $G$  such that  $g_i x \rightarrow y$  then, by Corollary 3.4, there exist a subnet  $\{g_j\}$  of  $\{g_i\}$  and a map  $f : C_x \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $C_x$ . So, if  $X$  is separable then by Proposition 4.1, we know that if there exists some point  $z \in C_x$  such that  $\phi(\Gamma_z)$  is closed in  $C(C_x, X)$  then  $\phi(\Gamma)$  is closed in  $C(C_x, X)$ . From this we can deduce that the set of points of  $X$  with closed orbits is closed. The general question remains open as well as the following generalized one:

**Question 4.9.** Is there any locally compact, separable and complete metric space such that  $G$  is locally compact and *every* orbit is not closed?

## 5. CLOSED-OPEN $G$ -INVARIANT PARTITIONS

In [12, Theorem] we showed that in case  $X$  has quasi-compact (i.e. compact but not necessarily Hausdorff) space of connected components (or quasi-components) the group  $G$  is locally compact. This is an application of the fact that the sets  $K(F)$ , defined in Section 2, are closed and open. In this section we will see another application of this property of  $K(F)$  concerning the structure of  $X$ . We will show that if  $G$  is locally compact then there is a decomposition of  $X$  into closed-open  $G$ -invariant sets that are related to various limit behaviors of the orbits: To be more precise, let

$$\begin{aligned} CL &= \{x \in X \mid L(x) \text{ is not empty and compact}\}, \\ NCL &= \{x \in X \mid L(x) \text{ is not compact}\} \text{ and} \\ P &= \{x \in X \mid L(x) \text{ is the empty set}\}. \end{aligned}$$

**Theorem 5.1.** *Let  $(X, d)$  be a locally compact metric space and  $G := \text{Iso}(X, d)$  the group of isometries of  $X$ . Assume that  $G$  is locally compact and not compact. Then*

- (i) *The closure of each orbit is a minimal set.*

- (ii) *If the closure of an orbit is compact then it is stable. Hence each  $x \in CL$  has a stable orbit closure.*
- (iii) *The sets  $CL, NCL$  and  $P$  are closed-open  $G$ -invariant disjoint, their union is  $X$  and each one of them is a union of pseudo-components.*

*Proof.* (i) Recall that a non-empty  $G$ -invariant closed subset  $A$  of  $X$  is called *minimal* if it has no  $G$ -invariant closed proper subsets. Equivalently,  $A = \overline{G(x)}$  for every  $x \in A$ . Let  $y \in X$  and  $x \in \overline{G(y)}$ . Obviously  $\overline{G(x)} \subseteq \overline{G(y)}$  and there exists a sequence  $\{g_n\}$  in  $G$  such that  $g_n y \rightarrow x$ . Since  $d(g_n y, x) = d(y, g_n^{-1} x)$  it follows that  $y \in \overline{G(x)}$ , thus  $\overline{G(y)} \subseteq \overline{G(x)}$ . Note also that since  $L(y)$  is a  $G$ -invariant closed subset of  $X$  then whenever  $L(y) \neq \emptyset$  we have that  $L(y) = \overline{G(y)}$ . So, if  $G$  is not compact then  $CL = \{x \in X \mid \overline{G(x)} \text{ is compact}\}$  and  $NCL = \{x \in X \mid \overline{G(x)} \text{ is not compact and } L(x) \neq \emptyset\}$ .

(ii) Assume that  $\overline{G(x)}$  is compact for some  $x \in X$ . We will show that  $\overline{G(x)}$  is stable, that is, for every open set  $U \subseteq X$  with  $\overline{G(x)} \subseteq U$  there exists a  $G$ -invariant open set  $V$  such that  $\overline{G(x)} \subseteq V \subseteq U$ . Since  $X$  is locally compact and  $\overline{G(x)}$  is compact then  $\overline{G(x)}$  has a neighborhood base consisting of compact sets, let's say  $\mathcal{W}$  (see [4, Ch. I, §9.7 Proposition 10]). There is a natural direction defined on  $\mathcal{W}$ :  $W_1 \leq W_2$  if and only if  $W_2 \subseteq W_1$  for  $W_1, W_2 \in \mathcal{W}$ . We argue by contradiction: Assume that for every  $W \in \mathcal{W}$  there exist  $x_w \in W$  and a point  $g_w \in G$  such that  $g_w x_w \notin U$ . It is not hard to see that there exist a point  $y \in \overline{G(x)}$  and a subnet  $\{x_i\}$  of the net  $\{x_w \mid w \in \mathcal{W}\}$  such that  $x_i \rightarrow y$ . Since  $g_i y \in \overline{G(x)}$  and  $\overline{G(x)}$  is compact there exist a subnet  $\{g_j y\}$  of  $\{g_i y\}$  and a point  $z \in \overline{G(x)}$  such that  $g_j y \rightarrow z$ . Note that  $d(g_j x_j, z) \leq d(g_j x_j, g_j y) + d(g_j y, z) = d(x_j, y) + d(g_j y, z) \rightarrow 0$  which is a contradiction since we have assumed that  $g_i x_i \notin U$  for every index  $i$ .

(iii) Obviously the sets  $CL, NCL$  and  $P$  are  $G$ -invariant, disjoint and their union is  $X$ . Item (i) implies that  $CL = K(G)$  hence, by [12, Lemma 3.1],  $CL$  is closed and open. Since  $J(x) = L(x)$  for every  $x \in X$  (because we have an action by isometries, see Section 2),  $P = \{x \in X \mid J(x) = \emptyset\}$ . Take a point  $x$  in the complement of  $P$ . Then  $J(x) \neq \emptyset$  and since  $J(x) = L(x)$  we have that  $J(x) = L(x) = \overline{G(x)}$ . Hence  $X \setminus P = \{x \in X \mid x \in J(x)\}$  and it is well known (see e.g. [3]) that this is a closed subset of  $X$ , so  $P$  is open. We claim that  $P$  is also closed. Let  $\{x_n \mid n \in \mathbb{N}\}$  be a sequence of points of  $P$  such that  $x_n \rightarrow x$  for some  $x \in X$ . We argue by contradiction: If  $x \notin P$  then  $x \in J(x) = L(x)$  hence there exists a net  $\{g_i \mid i \in I\}$  in  $G$  with  $g_i \rightarrow \infty$  and  $g_i x \rightarrow x$ . Fix a positive real number  $r > 0$  such that the ball centered at  $x$  with radius

$2r$  has compact closure and fix also a point  $x_{n_0}$  such that  $d(x_{n_0}, x) < r$ . Note that  $d(g_i x_{n_0}, x) \leq d(g_i x_{n_0}, g_i x) + d(g_i x, x) = d(x_{n_0}, x) + d(g_i x, x)$ . Since  $g_i x \rightarrow x$  the net  $\{g_i x_{n_0} \mid i \in I\}$  is eventually in the open ball  $B(x, 2r)$  hence it has a convergent subnet. This implies that  $L(x_{n_0}) \neq \emptyset$  (since  $g_i \rightarrow \infty$ ) thus  $x_{n_0} \notin P$  which is a contradiction.

In Corollary 3.4 we saw that if  $x, y \in X$  and  $\{g_i\}$  is a net in  $G$  such that  $g_i x \rightarrow y$  then there exist a subnet  $\{g_j\}$  of  $\{g_i\}$  and a map  $f : C_x \rightarrow X$  which preserves the distance such that  $g_j \rightarrow f$  pointwise on  $C_x$ . This shows that the set  $P$  is a union of pseudo-components. If we work as in the proof of Corollary 3.4 and take  $F := G$  then it is easy to see that the set  $CL$  has also the same property.  $\square$

A question which arises from the previous theorem is the following: Can the sets  $CL, NCL$  and  $P$  may coexist in any combination? We answer this question in the affirmative using the following simple examples. Note that the Arzela-Ascoli theorem implies that  $CL = X$  if and only if the group  $G$  is compact. If  $X$  is connected the van Dantzig - van der Waerden theorem implies that  $P = X$  and in Example 2.1 we have that  $NCL = X$ .

**Example 5.2.** ( $CL \neq \emptyset, NCL \neq \emptyset$  and  $P = \emptyset$ ). Let  $X = \{(0, 0)\} \cup L_1 \cup L_2 \subset \mathbb{R}^2$  where  $L_1 = \{(2, t) \mid t \in \mathbb{R}\}$  and  $L_2 = \{(4, t) \mid t \in \mathbb{R}\}$ . We consider the metric  $d = \min\{d_E, 1\}$  where  $d_E$  is the usual Euclidean metric on  $\mathbb{R}^2$ . As in Example 2.1 it is easy to see that  $CL = \{(0, 0)\}$ ,  $NCL = L_1 \cup L_2$  and  $P = \emptyset$ .

**Example 5.3.** ( $CL \neq \emptyset, NCL = \emptyset$  and  $P \neq \emptyset$ ). If we take  $X = \{(0, 0)\} \cup L_1$ , where  $L_1 = \{(2, t) \mid t \in \mathbb{R}\}$ , and the metric as in the previous example then  $CL = \{(0, 0)\}$ ,  $NCL = \emptyset$  and  $P = L_1$ .

**Example 5.4.** ( $CL = \emptyset, NCL \neq \emptyset$  and  $P \neq \emptyset$ ). In Example 4.4 we have that  $CL = \emptyset, NCL = R$  and  $P = Q$ .

**Example 5.5.** ( $CL \neq \emptyset, NCL \neq \emptyset$  and  $P \neq \emptyset$ ). This example is a modification of the Example 4.4. Firstly we replace the metric  $d$  in Example 4.4 by the bounded metric  $d' = \frac{d}{1+d}$  (note that  $d'$  and  $d$  give the same group of isometries). Then we add the point  $(3, 0)$  to  $X$  and finally we endow the set  $Y := X \cup \{(3, 0)\}$  with a new metric  $d^*$  requiring that  $d^*|_{X \times X} = d'|_{X \times X}$  and  $d^*((3, 0), w) = 1$  for every  $w \in X$ . If  $G$  denotes the group of isometries of  $Y$  with respect to  $d^*$  it is easy to see that  $CL = \{(3, 0)\}$ ,  $NCL = R$  and  $P = Q$ .

In case  $P$  is not empty we have a very interesting result concerning its structure. This result is an application of a theorem of Abels in [1]. Namely, in [1], Abels proved that if a non-compact locally compact

group  $G$  with compact space of connected components acts properly on a locally compact space  $Y$  such that the orbit space  $G \backslash Y$  is paracompact then  $Y$  is homeomorphic to a product of the form  $\mathbb{R}^n \times M$  for some  $n \in \mathbb{N}$  where  $M$  is a closed subset of  $X$ . Actually  $n$  is the same  $n$  if we write the group  $G$  as a homeomorphic image of the product  $\mathbb{R}^n \times K$  where  $K$  is a maximal compact subgroup of  $G$  in the Malcev-Iwasawa's decomposition theorem for  $G$  (see [15, Theorem 32.5]). If we apply this theorem to our case we have the following:

**Proposition 5.6.** *Let  $(X, d)$  be a locally compact metric space and  $G := \text{Iso}(X, d)$  the group of isometries of  $X$ . Assume that  $G$  is locally compact, not compact with compact space of connected components (or the connected component of the identity of  $G$  is not compact). Then  $P$ , if it is not empty, is homeomorphic to a product of the form  $\mathbb{R}^n \times M$  for some  $n \in \mathbb{N}$  where  $M$  is a closed subset of  $P$ .*

*Proof.* The proof is an immediate consequence of the previous mentioned theorem of Abels in [1] taking into account that if  $G_1$  denotes the connected component of the identity of  $G$  then  $G_1$  is a closed subgroup of  $G$ . Hence  $G_1$  acts properly on  $P$  and the orbit space  $G_1 \backslash X$  is metrizable (see [3]).  $\square$

*Remark 5.7.* As a final remark we would like to point out that the results of this paper also hold for the natural action of a locally compact, pointwise equicontinuous group of homeomorphisms  $\Gamma$  on a locally compact uniform space  $Y$ .

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