TELEMAN'S CASSON-TYPE INSTANTON INVARIANT IS ZERO

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ABSTRACT. Teleman's Casson-type instanton invariant is defined by a count of a zero-dimensional moduli space of flat instantons on negative definite four-manifolds with $b_2 \equiv 0 \ (mod \ 4)$ and $b_1 = 1$. We use a moduli space of $PU(2)$ Seiberg-Witten monopoles to exhibit an oriented one-dimensional cobordism of the instanton moduli space to the empty space.

INTRODUCTION

In a recent work of Andrei Teleman [18] low energy instanton moduli spaces defined over smooth, closed, oriented, negative definite four-manifolds X appeared. All these moduli spaces are compact and do not contain reducibles. Among these are certain 'Casson-type' moduli spaces defined for manifolds with second Betti number divisible by four. If in addition the first Betti-number is one the expected dimension of the moduli space is zero, and a 'Casson-type' invariant can be defined by an algebraic count of the elements in the moduli space, provided the latter is regular. This type of invariant has been suggested by Teleman, although an explicit definition of the invariant is missing in [18].

If non-empty, the 'Casson-type' moduli spaces are related to interesting topological properties of X. For instance, no element of a basis of $H^2(X; \mathbb{Z})$ diagonalising the intersection form (such a basis exists by Donaldson's theorem [2]) is representable by a sphere. By Hopf's theorem on the cokernel of the Hurewicz homomorphism $\pi_2(X) \to H_2(X; \mathbb{Z})$ the second homology group of the fundamental group of X has to be sufficiently non-trivial. Furthermore, Andrew Lobb and the author have shown that non-emptiness also gives an obstruction to connected sum decompositions into pieces with $b_2(X) \neq 0 \ (mod \ 4)$, ruling out natural appearing candidates for a non-empty moduli space. We also have studied a method to obtain manifolds with non-empty Casson-type moduli space by a construction related to immersed two-knots in the four-sphere. We are confident that this method will eventually yield a non-empty moduli space [12].

In this article we will give an explicit definition of the Casson-type invariant. We will adopt the holonomy perturbations as appearing in [10] in order to obtain generic regularity for the moduli space. Natural orientations for the moduli space, obtained by the determinant line bundle of the (Fredholm-) deformation operator parametrised by the configuration space, then yields the Casson-type invariant as algebraic count of the perturbed moduli space. That this signed count is independent of the chosen perturbation then follows from the fact that the parametrised moduli space doesn't contain reducibles, and the fact that the orientation was chosen in the given natural way from the ambient configuration space containing the moduli space.

We will then show that this Casson-type invariant is always zero. The proof uses the idea of the 'cobordism-program' for the proof of Witten's conjecture on the relationship of Donaldson's polynomial invariant and the Seiberg-Witten invariants. This program, suggested independently by Pidstrygach and Tyurin [14] and Okonek and Teleman [13], was carried on by Feehan and Leness who have proved Witten's conjecture in many special cases first [7], and apparently now in full [8]. The idea is to use $PU(2)$ monopole moduli spaces containing both the instanton moduli space defining the Donaldson-invariant, and certain Seiberg-Witten moduli space as fixed point subspaces of a natural circle-action. The S^1-quotient of this moduli space then yields a cobordism between a bundle over the instanton moduli space and bundles over the Seiberg-Witten moduli spaces. There are many very difficult technical problems involved, mainly because the moduli space needs not to be compact and parts of the instanton moduli space and some Seiberg-Witten moduli spaces may lie in lower strata of the Uhlenbeck-compactification.

In our situation the Casson-type instanton moduli space is compact. We will show that a compact two-dimensional $PU(2)$ monopole moduli space can be found and that the S^1 quotient yields an oriented cobordism between the Casson-type moduli space and the empty space. This implies that the algebraic count is zero, so the Casson-type invariant vanishes.

Technically our approach to transversality is slightly different from Feehan and Leness's. Whereas in their situation the instanton moduli spaces, regarded as subspace of the $PU(2)$ monopole moduli space, can be made regular by the generic metrics theorem of Freed and Uhlenbeck, this cannot be done here because it consists of flat connections. We therefore use a sort of holonomy perturbations. Whereas Feehan and Leness have used a version of holonomy perturbations at least in some early work as well [5], they have imposed different cutoff-behaviours which make the holonomy perturbation vanish at the locus of the instantons and in the compactification. This is not suitable for our situation (and our approach would not suited for their approach). Feehan and Leness's approach need perturbations that allow to apply glueing theory in the compactification in the perspective of their proof of Witten's conjecture.

From the point of view of investigating invariants of smooth four-manifold our vanishing result is rather negative. However, it appears that this result has relevance to Teleman's classification program on complex surfaces of class VII [19, 20], and in this perspective, it is rather a positive result.

1. Casson-invariant for negative definite four-manifolds

In this section we define Teleman's Casson-type instanton invariant via gauge theory of anti-selfdual connections in appropriate bundles. We shall first define the configuration space and fix our notation conventions, then define the moduli space and discuss some of its properties. Then a suitable space of holonomy perturbations is introduced in order to get a regular moduli space. After introducing preferred

orientations for the zero-dimensional moduli space we define the invariant. The book of Donaldson and Kronheimer [4] can be seen as a general reference here.

1.1. **The configuration space.** Let X be a smooth closed oriented Riemannian four-manifold. Let further $E \to X$ be a Hermitian rank-2 bundle. We suppose a smooth connection a on the determinant line $\det(E)$ is fixed, and define the space $\mathscr{A}_a(E)$ to be the affine space of unitary connections in E which induce the fixed connection a in the determinant line bundle, and which are of Sobolev class L_l^2 with $l \geq 3$. This choice of l ensures that there is a Sobolev embedding $L_l^2 \hookrightarrow C^0$ which we will use further down. We define the 'gauge group' $\mathscr{G}$ to be the group of automorphisms of E with determinant one and of Sobolev class L_{l+1}^2. It acts by the formula $(u, \nabla_A) \mapsto u \circ \nabla_A \circ u^{-1}$ if we denote by ∇_A the linear connection A and $u \in \mathscr{G}$. The quotient $\mathscr{B} = \mathscr{A}_a(E)/\mathscr{G}$ is called the configuration space. We shall denote by $\Gamma_A \subseteq \mathscr{G}$ the stabiliser of the connection A under the gauge group $\mathscr{G}$. It is the centraliser of the holonomy group associated to A. The centre $Z = \mathbb{Z}/2$ of $SU(2)$, seen as constant gauge transformations on X, is in the stabiliser Γ_A for any connection. A connection A is called irreducible if $\Gamma_A = Z$, otherwise reducible. A connection A on E is reducible if and only if there is a proper A-invariant subbundle of E.

There is an equivalent viewpoint of the space $\mathscr{A}_a(E)$ and the configuration space $\mathscr{B}$ which is the following: Denote by $P \to X$ a principal bundle with structure group $U(2)$ and by $\mathrm{ad}(P) \to X$ the $PU(2)$ bundle associated to P by the adjoint representation. Let $\mathscr{A}(\mathrm{ad}(P))$ denote the space of $PU(2)$ connections in $\mathrm{ad}(P)$ of class L_l^2. The group of automorphisms of P of determinant one and Sobolev class L_{l+1}^2 acts on $\mathscr{A}(\mathrm{ad}(P))$. The space $\mathscr{A}(\mathrm{ad}(P))$ is naturally isomorphic to the space $\mathscr{A}_a(E)$ above if the bundle P is taken as the unitary frame bundle P_E of E, and this is equivariant with respect to the mentioned automorphism groups. In this case the bundle $\mathrm{ad}(P)$ is naturally identified with the bundle $\mathfrak{su}(E)$ of traceless skew-adjoint endomorphisms of E. Unless we want to make explicit our viewpoint we shall simply write $\mathscr{A}$ for the spaces of connections we have in mind.

Principal $U(2)$ bundles or Hermitian vector bundles of rank 2 on a manifold of dimension 4 are classified by their first and second Chern-classes. However, it is a common convention in gauge theory, notably in our main references [4], [7], [10] to encode this information in the determinant line bundle $w \to X$ (or its first Chern class) of E respectively P, and in the 'instanton number'

$$\kappa = -\frac{1}{4}\langle p_1(\mathfrak{su}(E)), [X]\rangle = \langle c_2(E) - \frac{1}{4}c_1(E)^2, [X]\rangle \,. \tag{1}$$

In notation we will stick to the common convention and write $\mathscr{B}_\kappa^w$ for the configuration space $\mathscr{B}$ if we want to make explicit the underlying bundle.

The subspace of $\mathscr{A}$ of irreducible connections is as usually denoted by $\mathscr{A}^*$. There are slices for the action of $\mathscr{G}$ on $\mathscr{A}^*$ giving $\mathscr{B}^* = \mathscr{A}^*/\mathscr{G}$ the structure of a Banach manifold.

1.2. **The moduli space.** We denote by F_A the curvature of the connection $A \in \mathscr{A}(\mathrm{ad}(P_E)) = \mathscr{A}(\mathfrak{su}(E))$ and F_A^+ its self-dual part. Recall that it is equivariant with respect to the action of the gauge group $\mathscr{G}$. The anti-selfduality equation for $A \in \mathscr{A}$ is

$$F_A^+ = 0. \tag{2}$$

The moduli space of anti-selfdual connections in $\mathrm{ad}(P)$ is defined to be the space

$$M_\kappa^w := \left\{ [A] \in \mathscr{B}_\kappa^w \,|\, F_A^+ = 0 \right\}.$$

We refer to elements of this moduli space or of particular representatives as *instantons*.

There is an elliptic deformation complex associated to an instanton $[A]$:

$$0 \to L_{l+1}^2(X; \mathfrak{su}(E)) \xrightarrow{-d_A} L_l^2(X; \Lambda^1 \otimes \mathfrak{su}(E)) \xrightarrow{d_A^+} L_{l-1}^2(X; \Lambda_+^2 \otimes \mathfrak{su}(E)) \to 0 .$$

Here the second map is the derivative of the gauge-group action on A and the third is the derivative of the map $\mathscr{A} \to L_{l-1}^2(X; \mathfrak{su}(E))$ given by the left hand side of the equation (2). The cohomology groups of this complex are as usually denoted by H_A^0, H_A^1 and H_A^2. An instanton A is irreducible if and only if H_A^0 is zero. It is called *regular* if H_A^2 vanishes, and the moduli space M_κ^w is regular if each instanton in it is regular. If an instanton $[A]$ is both irreducible and regular, then the moduli space M has the structure of a smooth manifold in a neighbourhood of $[A]$. Its dimension is given by minus the index of the above complex or by the index of the 'deformation operator'

$$\delta_A := -d_A^* \oplus d_A^+ : L_l^2(X; \Lambda^1 \otimes \mathfrak{su}(E)) \to L_{l-1}^2(X; \mathfrak{su}(E)) \oplus \Lambda_+^2 \otimes \mathfrak{su}(E)). \qquad (3)$$

This index is given by the number

$$
\begin{aligned}
d &:= 4\kappa + 3\left(b_1(X) - b_2^+(X) - 1\right) \\
&= -2\left\langle p_1(\mathfrak{su}(E)), [X] \right\rangle + 3\left(b_1(X) - b_2^+(X) - 1\right).
\end{aligned}
\qquad (4)
$$

We call this number also the *expected dimension* of the moduli space M_κ^w.

In general the moduli space M_κ^w need not to be compact. However, there is the natural 'Uhlenbeck-compactification' of it. In fact sequences of instantons can have sub-sequences whose curvatures become more and more concentrated at points of the manifold - this phenomenon is called *bubbling*. We define an ideal instanton of instanton number κ to be a pair $([A], \mathbf{x})$, where $\mathbf{x}$ is an element of the n-fold symmetric product $Sym^n(X) = X^n/S_n$ (an unordered n-tuple of points in X) for some $n \geq 0$ and $[A]$ is an instanton in the moduli space $M_{\kappa-n}^w$. There is a topology on the space of ideal instantons

$$\mathrm{I}M_\kappa^w := \bigcup_{n=0}^\infty M_{\kappa-n}^w \times \ \mathrm{Sym}^n(X)$$

which is compact. In this topology each stratum admits its previously defined topology, but different strata are related by the notion of 'weak convergence' [4, §4.4.]. The closure $\overline{M}_\kappa^w \subseteq \mathrm{I}M_\kappa^w$ is therefore compact.

1.3. Moduli spaces over negative definite four-manifolds.
We restrict now our attention to smooth, closed Riemannian four-manifolds X with $b_2^+(X) = 0$ and $b_2(X) \geq 1$. According to the theorem of Donaldson's [2] the intersection form of such a four-manifold is equivalent to the diagonal one. Let $\{e_i\} \subseteq H^2(X; \mathbb{Z})$ be a set of elements inducing a basis of $H^2(X; \mathbb{Z})/Torsion$ which diagonalises the intersection form. Let $E \to X$ now have first Chern-class $c_1(E) = \sum e_i$. The following lemma ensures that the moduli space M associated to E will not contain reducibles.

Lemma 1.1. [18] *Suppose the Hermitian rank-2 bundle $E \to X$ has first Chern class $w := c_1(E) = \sum e_i$ and its second Chern class is strictly negative, $\langle c_2(E), [X] \rangle < 0$. Then $E \to X$ does not admit a topological decomposition $E = L \oplus K$ into the sum of two complex line bundles L and K.*

   Proof: Suppose $E = L \oplus K$ and $c_1(L) = \sum l_i\, e_i$. Then

$$\langle c_2(E), [X] \rangle = \langle c_1(L)(c_1(E) - c_1(L)), [X] \rangle = \sum (l_i^2 - l_i) \geq 0 \ .$$

$\square$

Corollary 1.2. *Let $E \to X$ be as in the previous lemma. Then the associated moduli space M_κ^w does not admit reducibles.*

   For a connection $A \in \mathscr{A}(\mathfrak{su}(E))$ Chern-Weil theory gives the following formula:

$$\frac{1}{8\pi^2}(\|F_A^-\|_{L^2(X)}^2 - \|F_A^+\|_{L^2(X)}^2) = -\frac{1}{4}\langle p_1(\mathfrak{su}(E)), [X] \rangle$$
$$= \langle c_2(E) - \frac{1}{4}c_1(E)^2, [X] \rangle \tag{5}$$

In particular, for anti-selfdual connections the left hand side of this equation is always positive.

Proposition 1.3. *Suppose the negative definite four-manifold X has non-zero second Betti-number divisible by four and first Betti-number $b_1(X) = 1$. Then the moduli space M_0^w, associated to the bundle $E \to X$ with $c_1(E) = w$ and $\langle c_2(E), [X] \rangle = -1/4\ b_2(X)$, is compact, does not contain reducibles, consists of flat connections in $\mathfrak{su}(E)$ and is of expected dimension zero.*

   Proof: Equation (5) implies that all the lower strata of the Uhlenbeck-compactification of M_0^w are empty, so M_0^w must already be compact. The remaining claims follow from the above lemma and the dimension-formula (4). $\square$

1.4. **Holonomy perturbations.** As the instanton moduli space we are studying consists of flat connections we cannot achieve transversality by perturbing the metric. A convenient choice of perturbation for this situation consists in holonomy perturbations, as used for instance in [2], [15], [17], [10]. This is also the approach we shall choose, and so we will follow Kronheimer's exposition closely - here and in the section on $PU(2)$ monopoles.

   Let B be a closed 4-ball in X. Suppose a submersion $q : S^1 \times B \to X$ which satisfies

$$q(1, \_) = id_B$$

is given. Therefore each map $q_x : S^1 \to X$, defined by $q_x(z) = q(z, x)$, parametrises a path in X which is centered at x. Given a smooth connection $A \in \mathscr{A}$ the expression $\mathrm{Hol}_{q_x}(A)$ denotes the holonomy of the connection A around the loop q_x. Therefore $\mathrm{Hol}_{q_x}(A)$ is an element of the group $\mathrm{Aut}(E)_x$. By letting variate $x \in B$ a unitary automorphism $\mathrm{Hol}_q(A)$ of E over B is given, i.e. a section of the bundle $U(E)$ over B. The bundle $U(E)$ will be considered as a subbundle of the vector bundle $\mathfrak{gl}(E)$. Now let $\omega \in \Omega_*^2(X; \mathbb{C})$ be a complex-valued self-dual two-form with compact support inside the ball B. Multiplying it with the section $\mathrm{Hol}(A)$ of $\mathfrak{gl}(E)$ and extending it by zero onto X a section

$$\omega \otimes \mathrm{Hol}_q(A) \in \Omega_+^2(\mathfrak{gl}(E)) \ ,$$

is given. It defines, after applying the projection $\pi : \mathfrak{gl}(E) \to \mathfrak{su}(E)$, a section

$$V_{q,\omega}(A) \in \Omega_+^2(\mathfrak{su}(E)) \ .$$

There is an extension of this map to our configuration space of connections of class L_l^2 which admits uniform bound on its derivatives, once a reference connection $A_0 \in \mathscr{A}$ is fixed [10], Proposition :

Proposition 1.4. [10] *For fixed q and ω the map $V_{q,\omega}$ extends to a smooth map of Banach manifolds*

$$V_{q,\omega} : \mathscr{A} \to L_l^2(X, \Lambda_+^2 \otimes \mathfrak{su}(E)) \ .$$

Furthermore there are uniform bounds on the derivatives of this map: There are constants K_n, depending only on q and A_0, such that the n-th derivative

$$D^n V_{q,\omega}|_A L_l^2(X, \Lambda^1 \otimes \mathfrak{su}(E))^n \to L_l^2(X, \Lambda_+^2 \otimes \mathfrak{su}(E))$$

satisfies

$$\|D^n V_{q,\omega}|_A(a_1, \ldots, a_n)_{L_{l,A_0}^2}\| \le K_n \|\omega\|_{C^l} \prod_{i=1}^n \|a_i\|_{L_{l,A_0}^2} \ .$$

Eventually a collection of submersions $q_i : S^1 \times B_i \to X$, $i \in \mathbb{N}$ as above will be chosen. Let $K_{n,i}$ be corresponding constants as guaranteed by the proposition above. Let C_i be a sequence of numbers such that

$$C_i \ge \sup\{K_{n,i}|1 \le n \le i\} \ .$$

Assume that ω_i is a sequence of self-dual two-forms with each ω_i having support inside B_i, and that the series

$$\sum_i C_i \|\omega_i\|_{C^l}$$

is convergent. Topologise the space of maps from $\mathscr{A}$ to $L_l^2(X, \Lambda_+^2 \otimes \mathfrak{su}(E))$ with the C^n semi-norms on bounded subsets. It follows from the above proposition and the choice of the constants C_i that the series

$$\sum_i V_{q_i,\omega_i}$$

converges in the C^n-topology for each $n \in \mathbb{N}$. Therefore the series defines a smooth map of Banach manifolds

$$V_{\boldsymbol{\omega}} : \mathscr{A} \to L_l^2(X, \Lambda_+^2 \otimes \mathfrak{su}(E)) \ .$$

We define the perturbation space correspondingly:

Definition 1.5. *Fix the maps q_i, and the constants C_i as above. The space $\mathscr{W}$ is defined to be the Banach space consisting of all sequences $\boldsymbol{\omega} = (\omega_i)_{i \in \mathbb{N}}$, with ω_i an element of the Banach space $\Omega_+^2(B_i)$, such that the sum*

$$\sum_i C_i \|\omega_i\|_{C^l}$$

converges. We then define $V_{\boldsymbol{\omega}}$ to be the series $\sum V_{q_i,\omega_i}$.

Remark. *The space of smooth self-dual two-forms with compact support in B_i indeed is a Banach space with respect to convenient norms, see for instance Lemma 8.2 of [9]. The C^l-norm is a weaker norm.*

The dependence of $V_{\boldsymbol{\omega}}$ on $\boldsymbol{\omega}$ is linear, and the map $\mathscr{W} \times \mathscr{A} \to L_l^2(X; \Lambda_+^2 \otimes \mathfrak{su}(E))$, $(\boldsymbol{\omega}, A) \mapsto V_{\boldsymbol{\omega}}(A)$ is a smooth map of Banach-manifolds. Given $\boldsymbol{\omega} \in \mathscr{W}$ the perturbed anti-selfduality equation for $A \in \mathscr{A}$ is now

$$F_A^+ + V_{\boldsymbol{\omega}}(A) = 0 \tag{6}$$

Correspondingly we define the moduli space perturbed by $\boldsymbol{\omega}$ to be the space

$$M_\kappa^w(\boldsymbol{\omega}) := \{[A] \in B_\kappa^w | F_A^+ + V_{\boldsymbol{\omega}}(A) = 0\} \ .$$

There is an elliptic deformation complex associated to an instanton $[A]$ in the perturbed moduli space $M_\kappa^w(\boldsymbol{\omega})$ also:

$$0 \to L_{l+1}^2(X; \mathfrak{su}(E)) \xrightarrow{-d_A} L_l^2(X; \Lambda^1 \otimes \mathfrak{su}(E)) \xrightarrow{d_{A,\boldsymbol{\omega}}^+} L_{l-1}^2(X; \Lambda_+^2 \otimes \mathfrak{su}(E)) \to 0 \ ,$$

where now $d_{A,\boldsymbol{\omega}}^+ = d_A^+ + dV_{\boldsymbol{\omega}}|_A$ is the derivative of the map $\mathscr{A} \to L_{l-1}^2(X; \mathfrak{su}(E))$, $A \mapsto F_A^+ + V_{\boldsymbol{\omega}}(A)$ at the instanton A. The deformation operator of the $\boldsymbol{\omega}$-perturbed equations is $\delta_{A,\boldsymbol{\omega}} := -d_A^* \oplus d_{A,\boldsymbol{\omega}}^+$.

Note that $d_{A,\boldsymbol{\omega}}^+$ differs from d_A^+ only by the addition of a compact operator $L_l^2(X; \Lambda^1 \otimes \mathfrak{su}(E)) \to L_{l-1}^2(X; \Lambda_+^2 \otimes \mathfrak{su}(E))$; the index of this elliptic complex is therefore the same than that of the unperturbed anti-selfduality equations above. The cohomology groups of this complex are denoted by H_A^0, $H_{A,\boldsymbol{\omega}}^1$ and $H_{A,\boldsymbol{\omega}}^2$. Again, an instanton is called *regular* if the cohomology group $H_{A,\boldsymbol{\omega}}^2$ vanishes. Likewise, local models for the moduli space $M(\boldsymbol{\omega})$ show that in a neighbourhood of a point $[A]$ which is both irreducible and regular the moduli space admits the structure of a smooth manifold of the expected dimension.

1.5. **Compactness.** The existence of an Uhlenbeck-type compactification of the perturbed moduli space $M_\kappa^w(\boldsymbol{\omega})$ was proved in [10]:

Proposition 1.6. *Let A_n be a sequence of connections in the Hermitian bundle $E \to X$ representing points $[A_n]$ in the moduli space $M_\kappa^w(\boldsymbol{\omega})$. Then there is a point $\mathbf{x} \in \mathrm{Sym}^n(X)$, a connection A' in a bundle $E' \to X$ representing an element of a moduli space $M_{\kappa-n}^w(\boldsymbol{\omega})$ with the following property: After passing to a subsequence, there are bundle isomorphisms*

$$h_n : E|_{X \setminus \mathbf{x}} \to E'|_{X \setminus \mathbf{x}} \ ,$$

such that $(h_n)_*(A_n)$ converges to A' in $L_1^p(K)$ for all compact subset $K \subseteq X \setminus \mathbf{x}$.

Here $p \geq 2$ is chosen large enough for a certain bootstrapping argument to work. That we have a weaker notion of convergence than in [4, §4.4] is due to the fact that changes of the connection A have effect globally on the section $V_{\boldsymbol{\omega}}(A)$.

An ideal instanton of instanton number κ is a pair $([A], \mathbf{x})$ where $\mathbf{x} \in Sym^n(X)$ and $[A]$ is an element of the perturbed moduli space $M_{\kappa-n}^w(\boldsymbol{\omega})$. The space of $\boldsymbol{\omega}$-perturbed ideal instantons of instanton number κ is defined to be

$$\mathrm{IM}_\kappa^w(\boldsymbol{\omega}) := \bigcup_{n=0}^\infty M_{\kappa-n}^w(\boldsymbol{\omega}) \times \mathrm{Sym}^n(X) \ ,$$

with the notion of convergence between the different strata as in Proposition 1.6, and with each stratum having its original topology. It follows from the above Proposition that the closure of the moduli space $M_\kappa^w(\boldsymbol{\omega})$ inside the space of ideal instantons $\mathrm{IM}_\kappa^w(\boldsymbol{\omega})$ is compact.

1.6. **Transversality.** Under an additional condition on the set of submersions $q_i :$ $S^1 \times B_i \to X$ transversality can be achieved for the moduli space. The condition is as follows:

Condition 1.7. *For any point $x \in X$ the set of loops*

$$\left\{ q_i|_{S^1 \times \{x\}} | i \in \mathbb{N},\ x \in int(B_i) \right\}$$

ought to be C^1-dense in the space of smooth loops based at x.

Theorem 1.8. [10] *Suppose the submersions $q_i : S^1 \times B_i \to X$ satisfy the above condition. Then the smooth map of Banach manifolds*

$$g : \mathscr{W} \times \mathscr{A}^* \to L^2_{l-1}(X; \Lambda^2_+ \otimes \mathfrak{su}(E))$$
$$(\boldsymbol{\omega}, A) \mapsto F_A^+ + V_{\boldsymbol{\omega}}(A)$$

is transverse to zero.

The key-point in the proof is that for an irreducible connection $A \in \mathscr{A}^*$ and point $x \in X$ the holonomy-sections $\mathrm{Hol}_{q_i}(A)$, associated to submersions $q_i : S^1 \times B_i \to X$ such that x is contained in the interiour of B_i, span $\mathfrak{gl}(E)_x$. Furthermore, after exhibiting a basis out of these sections, this basis continues to be a basis of $\mathfrak{gl}(E)$ in a neighbourhood of x. It is at this point that the inclusion $L^2_l \hookrightarrow C^0$ is used.

Let us denote by $\mathscr{M} := g^{-1}(0)/\mathscr{G}$ the *parametrised moduli space*. Applying the Sard-Smale theorem to the projection $\mathscr{M} \to \mathscr{W}$ yields the following result in the standard way:

Corollary 1.9. *For a residual set of perturbations $\boldsymbol{\omega} \in \mathscr{W}$ the moduli space $M^{w,*}_\kappa(\boldsymbol{\omega})$ is regular for all $w \in H^2(X; \mathbb{Z})$ and instanton numbers κ. It therefore admits the structure of a smooth manifold of the expected dimension d given by (4).*

As usually, a 'residual' subset of a complete metric space is a countable intersection of open and dense sets. By Baire's theorem such a set is dense itself.

1.7. **Orientations.** As in Donaldson's first applications of gauge theory to 4-manifold topology [3] the moduli space is given an orientation by a choice of orientation for a real determinant line bundle of a family of Fredholm operators.

The determinant line of a Fredholm operator $T : V \to W$ is given by $\det(\ker(T)) \otimes \det(\mathrm{coker}(T))^*$, where $\det(F)$ denots the maximal exteriour power of a finite dimensional vector space F, and $\det(F)^*$ its dual. For a family of Fredholm operators $(T_c : V \to W)_{c \in C}$, parametrised continously by a topological space C, there is a line bundle $\det(T)$ on the space C whose fibre over the point $c \in C$ is given by the line $\det(T_c)$, and whose topology is given as described in the next paragraph.

If T_{c_0} has trivial cokernel then the family of kernels $\ker(T_c)$ admits the structure of a vector bundle in a natural way over a neighbourhood of c_0, so $\det(T)$ admits a natural topology when restricted to that neighbourhood. If T_{c_0} has non-trivial cokernel we pick a subspace $J \subseteq W$ that surjects onto the cokernel of T_{c_0}. The space J also surjects onto the cokernels of T_c for c out of a neighbourhood of c_0, and there is a natural exact sequence

$$0 \to \ker T_c \to T_c^{-1}(J) \to J \to \mathrm{coker} T_c \to 0 \ ,$$

where the third of the five maps involved is induced by T_c. It is an algebraic fact (see for instance [11, section 20.2] or [4]) that this implies the existence of a natural isomorphism

$$\det(\ker(T_c)) \otimes \det(\operatorname{coker}(T_c))^* \cong \det(T_c^{-1}(J)) \otimes \det(J)^* \ .$$

Now in a neighbourhood of c_0 the family $T_c^{-1}(J)$ forms naturally a vector bundle, and so there is a natural structure of real line bundle of the right hand member of this isomorphism, over this neighbourhood. It is possible to *define* the topology of $\det(T)$ on this neighbourhood by that of $\det(T_c^{-1}(J)) \otimes \det(J)^*$. Indeed, any choice $K \subseteq W$ with $J \subseteq K$ yields the same topology by the above construction. One can also see that there are then continous transition functions on overlaps.

In our situation, the determinant line bundle formed by the family of Fredholm operators

$$\delta_{A,\boldsymbol{\omega}} = -d_A^* \oplus d_{A,\boldsymbol{\omega}}^+ : L_l^2(X; \Lambda^1 \otimes \mathfrak{su}(E)) \to L_{l-1}^2(X; \mathfrak{su}(E) \oplus \Lambda_+^2 \otimes \mathfrak{su}(E))$$

defined on the space of connections $\mathscr{A}$ is relevant for orientations. The restriction to $\mathscr{A}^*$ descends to the quotient $\mathscr{B}^*$. We denote by $\Lambda_{\boldsymbol{\omega}}$ this line bundle. Its restriction to the regular part of the moduli space $M^*(\boldsymbol{\omega})$ is equal to its orientation line bundle. In fact, at these points, the cokernels of the deformation operator vanish and the kernels $H^1_{A,\boldsymbol{\omega}}$ are precisely the tangent spaces to the moduli space $M^*(\boldsymbol{\omega})$. In particular, a regular moduli space $M^*(\boldsymbol{\omega})$ is orientable if $\Lambda_{\boldsymbol{\omega}}$ is orientable, and a trivialisation of $\Lambda_{\boldsymbol{\omega}}$ provides a preferred orientation for the moduli space $M^*(\boldsymbol{\omega})$.

Now the space of perturbations $\mathscr{W}$ is contractible, so the line bundles $\Lambda_{\boldsymbol{\omega}}$ on $\mathscr{B}^*$ corresponding to different perturbations $\boldsymbol{\omega} \in \mathscr{W}$ are canonically isomorphic. It is a theorem of Donaldson's that the line bundle $\Lambda := \Lambda_0 \to \mathscr{B}_\kappa^{*w}$ is indeed trivial, and an orientation is determined by a 'homology orientation' of X (that is, an orientation of the real vector space $\mathscr{H}^1(X; \mathbb{R}) \oplus \mathscr{H}_+^2(X; \mathbb{R})$ of harmonic one-forms and self-dual two-forms), see [3, 4]. We denote by the letter o a choice of trivialisation of the line bundle Λ.

1.8. **Definition of the invariant.** Let us now return to the particular situation of Proposition 1.3. If the compact moduli space M_0^w were regular (and zero-dimensional) we would define an integer by a signed count of its finite number of elements (each regular point is isolated). In general, we will consider the following perturbation of the moduli space M_0^w:

Proposition 1.10. *Suppose the C^0-norm of the perturbation $\boldsymbol{\omega} \in \mathscr{W}$ is sufficiently small. Then the moduli space $M_0^w(\boldsymbol{\omega})$ is compact. If further $\boldsymbol{\omega}$ is chosen among a residual subset of $\mathscr{W}$ such that the conclusion of Corollary 1.9 holds, then $M_0^w(\boldsymbol{\omega})$ is a compact zero-dimensional manifold.*

Proof: The claim on compactness is an easy consequence of the Chern-Weil formula (5) and the structure of the compactification of the moduli space, Proposition (1.6). □

Definition 1.11. *Let $\boldsymbol{\omega} \in \mathscr{W}$ be a perturbation such that we are in the situation of the proceeding Proposition. Suppose an orientation o of the determinant line bundle $\Lambda \to \mathscr{B}^*$ is chosen, and, therefore an orietation for the moduli space $M_0^w(\boldsymbol{\omega})$,*

according to the proceeding subsection. Then we define the number $n_o(\boldsymbol{\omega})$ as the signed count of the moduli space $M_0^w(\boldsymbol{\omega})$,

$$n_o(\boldsymbol{\omega}) := \#M_0^w(\boldsymbol{\omega}) \ .$$

Here an instanton $[A] \in M_0^w(\boldsymbol{\omega})$ is counted with $+1$ if the orientation of the determinant line $\det(\delta_{A,\boldsymbol{\omega}})$ at $[A]$ determined by o coincides with the preferred orientation

$$\det(\delta_{A,\boldsymbol{\omega}}) = \det(\ker(\delta_{A,\boldsymbol{\omega}})) \otimes \det(\mathrm{coker}(\delta_{A,\boldsymbol{\omega}}))^* = \mathbb{R}$$

determined by the trivial kernel and cokernel of $\delta_{A,\boldsymbol{\omega}}$ at the irreducible and regular point $[A]$, and with -1 in the opposite case.

It is worth to notice (see for instance the Appendix A of [16]) that the relative sign between two instantons $[A_0]$ and $[A_1]$ can be computed from the number of crossings $\mu = \sum_t \dim(\ker(\delta_{A_t,\boldsymbol{\omega}}))$ of a generic path $t \mapsto A_t$. It is $(-1)^\mu$.

Proposition 1.12. *For any two choices of perturbation $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathscr{W}$, such that the conclusion of Proposition 1.10 holds, we have*

$$n_o(\boldsymbol{\omega}) = n_o(\boldsymbol{\omega}') \ .$$

This number is therefore independent of the underlying Riemannian metric on X and the chosen perturbation. It only depends on the topology of the smooth manifold X and the cohomology class w which determines the bundle $E \to X$ which defines the moduli space, and a choice of orientation o. It is therefore convenient to denote this number by $n_{w,o}(X)$.

2. Moduli spaces of $PU(2)$ Seiberg-Witten monopoles

Here we shall recall the $PU(2)$-monopole equations and their moduli space associated to the data of a $Spin^c$-structure $\mathfrak{s}$ and a Hermitian bundle $E \to X$ of rank 2 on a Riemannian four-manifold X. We shall define the configuration space and the moduli space and recall how to get a uniform bound on the spinor component of a solution to the monopole equations. We then show how the equations are perturbed, sketch the Uhlenbeck-compactification for the perturbed moduli space and show how to obtain transversality. Furthermore, we shall show how to define a preferred orientation on the irreducible part of the moduli space. At least in slightly different situations these results are already well-known [7], [19].

2.1. **The configuration space.** Let X be a closed oriented Riemannian four-manifold with a $Spin^c$ structure $\mathfrak{s}$ on it. The $Spin^c$ structure consists of two Hermitian rank 2 vector bundles $S_{\mathfrak{s}}^{\pm}$ with identified determinant line bundles and a Clifford multiplication

$$\gamma : \Lambda^1(T^*X) \to \mathrm{Hom}_{\mathbb{C}}(S_{\mathfrak{s}}^+, S_{\mathfrak{s}}^-) \ .$$

Let's furthermore suppose we are given a Hermitian vector bundle E with determinant line bundle $w = \det(E)$ on X. We can then form the 'twisted' spinor bundles

$$W_{\mathfrak{s},E}^{\pm} := S_{\mathfrak{s}}^{\pm} \otimes E.$$

Clifford multiplication extends by tensoring with the identity on E.

We continue to denote by $\mathscr{A}$ the space of connections in E inducing a fixed connection in the determinant line w of class Sobolev class L_l^2. We define our pre-configuration space to be the product

$$\mathscr{C} := \mathscr{A} \times L_l^2(X; S^+ \otimes E) \ .$$

According to our above notation convention we shall denote $\mathscr{C}_{\kappa,\mathfrak{s}}^w$ if we want to emphasise that this configuration space is associated to the topological data consisting of a $Spin^c$ structure $\mathfrak{s}$ and a bundle $E \to X$ with instanton number κ as in the formula (1) and determinant line bundle $w := \det(E)$.

The gauge group $\mathscr{G}$ from above acts naturally on $\mathscr{C}$ by the action on $\mathscr{A}$ as above and by $(u, \Psi) \mapsto (\mathrm{id} \otimes u)(\Psi)$ on the spinor section. The quotient

$$\mathscr{B}_{\kappa,\mathfrak{s}}^w := \mathscr{C}_{\kappa,\mathfrak{s}}^w / \mathscr{G} \ .$$

is called the configuration space of $PU(2)$ monopoles. By $\Gamma_{(A,\Psi)} \subseteq \mathscr{G}$ we shall denote the stabiliser of the pair (A, Ψ) under the gauge group action. We shall define by $\mathscr{C}^*$ and by $\mathscr{B}_{\mathfrak{s}}^*$ the spaces corresponding to *trivial* stabiliser. Furthermore, we shall define by $\mathscr{C}^{**}$ the subspace of pairs (A, Ψ) with A irreducible, $\Gamma_A = \mathbb{Z}/2$, and with $\Psi \neq 0$. Certainly $\mathscr{C}^{**} \subseteq \mathscr{C}^*$ but the converse is not true.

2.2. A quadratic map on the spinor component.
The bundle $S^+ \otimes E$ is modelled on $\mathbb{C}_+^2 \times \mathbb{C}^2$. We define the orthogonal projections

$$P : \ \mathfrak{gl}(\mathbb{C}^2 \otimes \mathbb{C}^2) \to \mathfrak{sl}(\mathbb{C}^2) \otimes \mathfrak{sl}(\mathbb{C}^2)$$

as being the tensor product of the two orthogonal projections $\mathfrak{gl}(\mathbb{C}^2) \to \mathfrak{sl}(\mathbb{C}^2)$. We then define the map

$$\mu : \mathbb{C}_+^2 \otimes \mathbb{C}_2 \oplus \mathbb{C}_+^2 \otimes \mathbb{C}_2 \to \mathfrak{sl}(\mathbb{C}^2) \otimes \mathfrak{sl}(\mathbb{C}^2)$$
$$(\Psi, \Phi) \mapsto P(\Psi\Phi^*) \ ,$$

where $(\Psi\Phi^*) \in \mathfrak{gl}(\mathbb{C}_+^2 \otimes \mathbb{C}^2)$ is defined to be the endomorphism $\Xi \mapsto \Psi(\Phi, \Xi)$, where (Φ, Ξ) stands for the (standard) inner product of the two elements. Instead of $\mu(\Psi, \Psi)$ we shall just write $\mu(\Psi)$, where then μ is a quadratic map. Both the bilinear and the quadratic map are equivariant with respect to the structure groups of S^+ and E and so defines well-defined maps

$$\mu : S^+ \otimes E \oplus S^+ \otimes E \to \mathfrak{sl}(S^+) \otimes \mathfrak{sl}(E) \ ,$$

respectively

$$\mu : S^+ \otimes E \to \mathfrak{su}(S^+) \otimes_{\mathbb{R}} \mathfrak{su}(E) \ .$$

Note that on the level of bundles this map is also equivariant with respect to the action of the gauge group $\mathscr{G}$.

2.3. The moduli space, some of its properties.
Suppose we are given a $Spin^c$ connection ∇ on $S^+ \oplus S^-$. Composing the Clifford multiplication with the tensor product connection $\nabla \otimes \nabla_A$ yields the 'twisted' Dirac operator

$$\not{D}_A^+ := \gamma \circ (\nabla \otimes \nabla_A) : L_l^2(X; S^+ \otimes E) \to L_{l-1}^2(X; S^- \otimes E) \ .$$

As the particular choice of $Spin^c$ connection will be kept fixed in what follows it is suppressed from notation. The $PU(2)$ monopole equations associated to a pair $(A, \Psi) \in \mathscr{C}$ are given by

$$\begin{aligned} \not{D}_A^+ \Psi &= 0 \\ \gamma(F_A^+) - \mu(\Psi) &= 0 \ . \end{aligned} \tag{7}$$

Solutions of this equations will also be called 'monopoles'. The left-hand side of these equations can be considered[1] as a map $g : \mathscr{C} \to L_{l-1}^2(X; S^- \otimes E \oplus \Lambda_+^2(X) \otimes \mathfrak{su}(E))$. As such, it is equivariant with respect to the action of the gauge group. The moduli space of $PU(2)$ monopoles is defined to be the space

$$M_{\kappa,\mathfrak{s}}^w := \left\{ [A, \Psi] \in \mathscr{B}_{\kappa,\mathfrak{s}}^w \,|\, (7) \text{ holds} \right\} / \mathscr{G}. \tag{8}$$

Again, there is an elliptic deformation complex associated to a monopole (A, Ψ):

$$0 \to L_{l+1}^2(X; \mathfrak{su}(E)) \xrightarrow{d_{(A,\Psi)}^0} L_l^2(X; S^+ \otimes E \oplus \Lambda^1 \otimes \mathfrak{su}(E))$$

$$\xrightarrow{d_{(A,\Psi)}^1} L_{l-1}^2(X; S^- \otimes E \oplus \Lambda_+^2 \otimes \mathfrak{su}(E)) \to 0 .$$

Here $d_{(A,\Psi)}^0$ is the derivative of the gauge-group action $u \mapsto u(A, \Psi)$ at the identitiy, and $d_{(A,\Psi)}^1 = dg|_{(A,\Psi)}$, the derivative of the monopole map g at a solution (A, Ψ). As in instanton theory, the kernel of the operator $d_{(A,\Psi)}^{0,*}$ provides a slice of the action of the gauge group $\mathscr{G}$ on $\mathscr{C}$ in a neighbourhood of the orbit through (A, Ψ). Again, a configuration (A, Ψ) is called irreducible if the zeroth cohomology space of the above complex vanishes, $H_{(A,\Psi)}^0 = 0$, and regular if $H_{(A,\Psi)}^2 = 0$.

The deformation operator

$$D_{(A,\Psi)} = d_{(A,\Psi)}^{0,*} \oplus d_{(A,\Psi)}^1$$

is elliptic. Its kernel is given by the cohomology space $H_{(A,\Psi)}^1$ of the above complex and its cokernel is given by the sum of cohomology spaces $H_{(A,\Psi)}^0 \oplus H_{(A,\Psi)}^2$. If this cokernel vanishes then the slice theorem together with implicit function theorems show that the moduli space has the structure of a smooth manifold in a neighbourhood of $[A, \Psi]$, of dimension given by the index of the elliptic operator.

The deformation operator is homotopic to the operator $D_{(A,0)}$ which has the simple form

$$D_{(A,0)} = -d_A^* \oplus d_A^+ \oplus \slashed{D}_A^+ ,$$

as a map $L_l^2(X; \Lambda^1 \otimes \mathfrak{su}(E) \oplus S^+ \otimes E) \to L_{l-1}^2(X; (\Lambda^0 \oplus \Lambda_+^2) \otimes \mathfrak{su}(E) \oplus S^- \otimes E)$. It is therefore equal to the sum of the Instanton deformation operator $\delta_A = -d_A^* \oplus d_A$ and the Dirac operator $\slashed{D}_A^+$. The expected dimension of the moduli space $M_{\kappa,\mathfrak{s}}^w$ is therefore given by the index of this elliptic operator,

$$\text{ex-dim}(M_{\kappa,\mathfrak{s}}^w) = \text{ind}(\delta_A) \oplus 2\,\text{ind}_{\mathbb{C}}(\slashed{D}_A^+) ,$$

where the index of δ_A has been in equation (4) above and the index of the Dirac operator will be computed in the next section.

We terminate the discussion here by sketching the Uhlenbeck compactification for $PU(2)$ monopoles, for details see [5], [19]. The important fact is that the quadratic map μ satisfies a pointwise properness condition,

$$(\mu(\Psi)\Psi, \Psi) \geq c^2\, |\Psi|^4$$

for a real constant $c > 0$. This then yields, via the Weitzenböck formula for the Dirac operator D_A^+, to an 'a-priori bound' for the spinor Ψ of a monopole $[A, \Psi] \in M_{\kappa,\mathfrak{s}}^w$:

$$\|\Psi\|_\infty^2 \leq K/c^2 . \tag{9}$$

---

[1]Here and later we will feel free to use the isomorphism $\gamma : \Lambda_+^2 \to \mathfrak{su}(S^+)$ without making it explicit in the notation

The positive constant K on the right-hand side only depends on fixed geometric data - the Riemannian metric and the fixed $Spin^c$ connection. This C^0 bound implies that there is an Uhlenbeck compactification of the moduli space $M_{\kappa,\mathfrak{s}}^w$ by a space of 'ideal monopoles',

$$\mathrm{IM}_{\kappa,\mathfrak{s}}^w := \bigcup_{n=0}^{\infty} M_{\kappa-n,\mathfrak{s}}^w \times \mathrm{Sym}^n(X) \ ,$$

which is given a compact topology as in the instanton situation, and where each stratum has its previously defined topology. The closure $\overline{M}_{\kappa,\mathfrak{s}}^w \subseteq \mathrm{IM}_{\kappa,\mathfrak{s}}^w$ is compact.

2.4. **Index computations.** The index of the elliptic operator $\not{D}_A^+$ can be computed from the Atiyah-Singer index theorem, and is given by

$$\mathrm{ind}_{\mathbb{C}}(\not{D}_A^+) = \langle \mathrm{ch}(E)e^{\frac{1}{2}c_1(S_{\mathfrak{s}}^+)}\hat{\mathrm{A}}(TX), [X] \rangle \ .$$

This formula can be found in [16] §6.4. In our situation we thus obtain

$$\mathrm{ind}_{\mathbb{C}}(\not{D}_A^+) = 2\,\mathrm{ind}_{\mathbb{C}}(\not{D}_{\mathfrak{s}}^+) + \frac{1}{2}\langle c_1(S_{\mathfrak{s}}^+)c_1(E) + c_1(E)^2, [X] \rangle - \langle c_2(E), [X] \rangle \ , \quad (10)$$

where we have denoted by $\not{D}_{\mathfrak{s}}^+$ the Dirac operator determined by the $Spin^c$ structure $\mathfrak{s}$ and the fixed $Spin^c$ connection that we have suppressed in the notation.

For our application in mind we have to keep E fixed so as to define the Casson-type instanton moduli space M_0^w on a negative definite four-manifold, in particular $c_2(E) = -b_2(X)/4$, $c_1(E) = w$. However we are free to choose the $Spin^c$ structure $\mathfrak{s}$ and so $c_1(S_{\mathfrak{s}}^+)$ can be chosen among the characteristic vectors of the intersection form on $H^2(X;\mathbb{Z})$.

Proposition 2.1. *Let X be negative definite. If we choose a $Spin^c$ structure $\mathfrak{s}$ with first Chern-class equal to*

$$c_1(S_{\mathfrak{s}}^+) = -\sum_{i=1}^{k} e_i + \sum_{i=k+1}^{b_2(X)} e_i$$

with $k = 1 + 3\frac{b_2(X)}{4}$ then the complex index of the coupled Dirac operator $\not{D}_A^+$ equals one, $ind_{\mathbb{C}}(\not{D}_A^+) = 1$.

Indeed, the index of the untwisted Dirac-operator $\not{D}_{\mathfrak{s}}^+$ is given by $\mathrm{ind}_{\mathbb{C}}(\not{D}_{\mathfrak{s}}^+) = \frac{1}{8}\langle c_1(S_{\mathfrak{s}}^+)^2, [X] \rangle - \frac{1}{8}\mathrm{sign}(X) = 0$. An elementary computation then yields the claimed result from formula (10). □

If we wish to make explicit that the coupled Dirac operator $\not{D}_A^+$ depends on the $Spin^c$ structure $\mathfrak{s}$ and the Hermitian bundle $E \to X$, then we shall write $\mathrm{ind}(\not{D}_A^+) =: \mathrm{ind}(\not{D}_{\mathfrak{s}}^+, E)$. Similarly, we shall write $\mathrm{ind}(\delta_A) =: \mathrm{ind}(\delta, E)$ for the deformation operator δ_A of the instanton moduli space.

Proposition 2.2. *Suppose a $Spin^c$ structure $\mathfrak{s}$ is chosen so that the index of the Dirac operator $\mathrm{ind}(\not{D}_{\mathfrak{s}}^+, E)$ is one, where $E \to X$ is a Hermitian bundle defining the Casson type moduli space. Then the expected dimension of the moduli space of $PU(2)$ monopoles $M_{0,\mathfrak{s}}^w$ is 2. Furthermore, the lower strata of the space of ideal monopoles $IM_{0,\mathfrak{s}}^w = \bigcup_{n=0}^{\infty} M_{-n,\mathfrak{s}}^w$ are all of strictly negative expected dimension.*

Proof: From the formulae (4) and (10) the expected dimension of the moduli space $M^w_{-n,\mathfrak{s}}$ equals

$$\mathrm{ind}(\delta, E_{-n}) + 2\,\mathrm{ind}(\slashed{D}^+_{\mathfrak{s}}, E_{-n}) = \mathrm{ind}(\delta, E) - 8n + 2\left(\mathrm{ind}(\slashed{D}^+_{\mathfrak{s}}, E) + n\right)$$
$$= \mathrm{ind}(\delta, E) + 2\,\mathrm{ind}(\slashed{D}^+_{\mathfrak{s}}) - 6n$$
$$= \text{ex-dim}(M^w_{0,\mathfrak{s}}) - 6n.$$

Under the assumption made this is strictly negative for $n > 0$. $\square$

2.5. A circle action on the configuration space of $PU(2)$ monopoles.

We will now introduce an action of the circle S^1 on the configuration space $\mathscr{B}^w_{\kappa,\mathfrak{s}}$ of $PU(2)$ monopoles and will describe its fixed point set. On the pre-configuration space this action $S^1 \times \mathscr{C} \to \mathscr{C}$ is simply given by scalar multiplication on the spinor,

$$(z, (A, \Psi)) \mapsto (A, z\Psi).$$

This action descends to an action $\rho : S^1 \times \mathscr{B}^w_{\kappa,\mathfrak{s}} \to \mathscr{B}^w_{\kappa,\mathfrak{s}}$ on the configuration space. However, the latter is not effective - in fact it is the two-fold covering of an effective action $\rho^{1/2}$ due to the fact that the stabiliser Γ_A of a connection A in the gauge groupe $\mathscr{G}$ always contains the centre $Z(SU(2)) = \pm\mathrm{id}$. We therefore have $\rho(z, [A, \Psi]) = [A, z\Psi]$ and $\rho^{1/2}(z, [A, \Psi]) = [A, z^{1/2}\Psi]$, where $z^{1/2}$ is an arbitrary square-root of z. The following proposition can be found in [21], [6].

Proposition 2.3. *A configuration $[A, \Psi] \in \mathscr{B}^w_{\kappa,\mathfrak{s}}$ belongs to the fixed point set of ρ respectively $\rho^{1/2}$ if and only if for any representative $(A, \Psi) \in \mathscr{C}$ we have one of the following:*

  (1) *There is an A-parallel decomposition of E into the sum of two line-bundles, $E = K \oplus L$, and Ψ is a non-vanishing section of either $S^+ \otimes K$ or $S^+ \otimes L$.*
  (2) *The spinor vanishes, $\Psi \equiv 0$.*

On the complement of the fixed point set the action $\rho^{1/2}$ is free.

Remark. *On negative four-manifolds with the bundle $E \to X$ chosen as in Lemma 1.1 there are no fixed points of the first type in the above Proposition.*

2.6. Holonomy perturbations.

If we wish to obtain transversality for moduli spaces of $PU(2)$ monopoles we have to perturb the Dirac equation in the $PU(2)$ monopole equation, too. This is in contrast to the classical abelian Seiberg-Witten theory. There are different approaches in this situation [7], [19], some using holonomy perturbations, some not. We will simply adopt the holonomy perturbations of the previous section to our situation.

To a smooth connection $A \in \mathscr{A}$, a submersion $q : S^1 \times B \to X$ as in section 1.4 and a complex-valued one-form $\alpha \in \Omega^1(X; \mathbb{C})$ with support in B we can associate a section

$$V_{q,\alpha}(A) := \alpha \otimes \mathrm{Hol}_q(A) \in \Omega^1(X; \mathfrak{gl}(E)).$$

There is then a result completely analogue to that of Proposition 1.4, the discussion following it, and the definition of the perturbation space $\mathscr{W}$:

Definition 2.4. *Let $\mathscr{V}$ be defined to be the Banach space of sequences $\alpha = (\alpha_i)_{i \in \mathbb{N}}$, with each α_i an element of the Banach space $\Omega^1(B_i; \mathbb{C})$, such that the sum*

$$\sum_i C_i \|\alpha_i\|_{C^l(B_i)}$$

is finite. Here the constants C_i are defined as in section 1.4. The map $V_{\boldsymbol{\alpha}}$ is defined to be the series $\sum V_{q_i, \alpha_i}$ which converges in the C^∞ - topology of maps $\mathscr{A} \to L_l^2(X; \Lambda^1 \otimes \mathfrak{gl}(E))$.

The depence of $V_{\boldsymbol{\alpha}}$ on $\boldsymbol{\alpha}$ is linear and we obtain a smooth map of Banach manifolds

$$V : \mathscr{V} \times \mathscr{A} \to L_l^2(X; \Lambda^1 \otimes \mathfrak{gl}(E)) \ ,$$

given by $(\boldsymbol{\alpha}, A) \mapsto V_{\boldsymbol{\alpha}}(A)$. This map is also equivariant with respect to the action of the gauge group $\mathscr{G}$ on $\mathscr{A}$ and $\mathfrak{gl}(E)$.

The perturbed $PU(2)$ monopole equations associated to the $Spin^c$ structure $\mathfrak{s}$ and the Hermitian bundle $E \to X$, specified by the instanton number κ and the line bundle w, and the perturbations $(\boldsymbol{\omega}, \boldsymbol{\alpha}) \in \mathscr{V} \times \mathscr{W}$ are then given by the equations

$$\begin{aligned}
\not{D}_A^+ \Psi + h(\Psi)\, \gamma(V_{\boldsymbol{\alpha}}(A))\Psi &= 0 \\
\gamma(F_A^+) - \mu(\Psi) + h(\Psi)\, \gamma(V_{\boldsymbol{\omega}}(A)) &= 0 \ .
\end{aligned} \tag{11}$$

Here $h(\Psi)$ is a smooth real-valued function on $L_l^2(X; S^+ \otimes E)$ which is such that it takes values in the interval $[0, 1]$ and is equal to 1 for $\|\Psi\|_\infty \leq K/c^2$, the constant of the a-priori bound to the unperturbed equations (9), and which is 0 for $\|\Psi\|_\infty \geq K/c^2 + 1$. Due to this trick that the author learned from Kim Frøyshov we obtain an a-priori bound for a solution (A, Ψ) of the *perturbed* monopole equations (11):

$$\|\Psi\|_\infty \leq K/c^2 + 1 \ . \tag{12}$$

The moduli space of $PU(2)$ monopoles perturbed by $(\boldsymbol{\omega}, \boldsymbol{\alpha}) \in \mathscr{W} \times \mathscr{V}$ is defined to be the space

$$M_{\kappa, \mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha}) := \big\{ [A, \Psi] \in \mathscr{B}_{\kappa, \mathfrak{s}}^w \,|\, (11) \text{ holds } \big\} \ .$$

Again, there is an elliptic deformation complex analogue to the one in section 2.3. Let (A, Ψ) be a solution to the perturbed monopole equations (11). Then we have the elliptic complex

$$\begin{aligned}
0 \to L_{l+1}^2(X; \mathfrak{su}(E)) &\xrightarrow{d_{(A,\Psi)}^0} L_l^2(X; S^+ \otimes E \oplus \Lambda^1 \otimes \mathfrak{su}(E)) \\
&\xrightarrow{d_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1} L_{l-1}^2(X; S^- \otimes E \oplus \Lambda_+^2 \otimes \mathfrak{su}(E)) \to 0 \ .
\end{aligned} \tag{13}$$

The deformation operator is defined to be

$$D_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})} := d_{(A,\Psi)}^{0,*} \oplus d_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1 \ . \tag{14}$$

Note also that this deformation operator for the perturbed monopole equations and that for the unperturbed monopole equations only differ by addition of compact operators. In particular, their index is equal. As usually, the cohomology spaces of the above complex are denoted by $H_{(A,\Psi)}^0$, $H_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1$ and $H_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^2$.

2.7. **Compactification.** The a-priori bound (12) together with Chern-Weil theory implies that the L^2-norms of the curvatures are bounded on the perturbed moduli space. There is therefore an Uhlenbeck-type compactification analogue to that of the perturbed monopole moduli spaces $M^w_{\mathfrak{s},\kappa}(\boldsymbol{\omega}, \boldsymbol{\alpha})$. A proof of the following proposition follows the pattern of the corresponding proofs in [10, 18, 5].

Proposition 2.5. *Let (A_n, Ψ_n) be a sequence of elements, where A_n are connections in the Hermitian bundle $E \to X$ and Ψ_n are sections of the bundle $S^+ \otimes E$, representing points $[A_n, \Psi_n]$ in the moduli space $M^w_{\mathfrak{s},\kappa}(\boldsymbol{\omega}, \boldsymbol{\alpha})$. Then there is a point $\mathbf{x} \in \mathrm{Sym}^n(X)$, a connection A' in a bundle $E' \to X$ and a section Ψ' of the bundle $S^+ \otimes E'$, representing an element $[A', \Psi']$ of a moduli space $M^w_{\mathfrak{s},\kappa-n}(\boldsymbol{\omega}, \boldsymbol{\alpha})$ with the following property: After passing to a subsequence, there are bundle isomorphisms*

$$h_n : E|_{X \setminus \mathbf{x}} \to E'|_{X \setminus \mathbf{x}} \ ,$$

such that $(h_n)_*(A_n, \Psi_n)$ converges to (A', Ψ') in $L^p_1(K)$ for all compact subset $K \subseteq X \setminus \mathbf{x}$.

An ideal monopole of instanton number κ is a pair $([A, \Psi], \mathbf{x})$ where $\mathbf{x} \in \mathrm{Sym}^n(X)$ and $[A, \Psi]$ is an element of the perturbed moduli space $M^w_{\mathfrak{s},\kappa-n}(\boldsymbol{\omega}, \boldsymbol{\alpha})$. The space of $(\boldsymbol{\omega}, \boldsymbol{\alpha})$-perturbed ideal monopoles of instanton number κ is defined to be

$$\mathrm{IM}^w_\kappa(\boldsymbol{\omega}) := \bigcup_{n=0}^{\infty} M^w_{\kappa-n}(\boldsymbol{\omega}) \times \mathrm{Sym}^n(X) \ ,$$

with the notion of convergence between the different strata as in Proposition 2.5, and with each stratum having its original topology. It follows from the above Proposition that the closure of the moduli space $M^w_{\mathfrak{s},\kappa}(\boldsymbol{\omega}, \boldsymbol{\alpha})$ inside the space of ideal instantons $\mathrm{IM}^w_{\mathfrak{s},\kappa}(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is compact.

2.8. **Transversality.** We will now show that the perturbation space $\mathscr{V} \times \mathscr{W}$ is sufficient to obtain generic regularity for the moduli space of perturbed $PU(2)$ monopoles. Recall that the space $\mathscr{C}^{**}$ consists of pairs (A, Ψ) with irreducible connection A and non-vanishing spinor Ψ.

Theorem 2.6. *Suppose again the condition 1.7 holds. Then the smooth map of Banach manifolds*

$$\mathscr{F} : \mathscr{V} \times \mathscr{W} \times \mathscr{C}^{**} \to L^2_{l-1}(X; S^- \otimes E \oplus \Lambda^2_+ \otimes \mathfrak{su}(E)) \ ,$$

given by the left-hand side of the perturbed $PU(2)$ monopole equations (11), is transverse to zero.

Proof: The proof is a generalisation of that of the transversality theorem in [10] to monopoles. Suppose we have an element $(\boldsymbol{\omega}, \boldsymbol{\alpha}, A, \Psi) \in \mathscr{W} \times \mathscr{V} \times \mathscr{C}^{**}$ such that $\mathscr{F}((\boldsymbol{\omega}, \boldsymbol{\alpha}, A, \Psi)) = 0$. Let us denote by $\slashed{D}^+_{A,\boldsymbol{\alpha}} := \slashed{D}^+_A + h(\Psi)\gamma(V_{\boldsymbol{\alpha}}(A))$ the 'perturbed Dirac operator' in (11). We will show that the derivative

$$P := d\mathscr{F}|_{(\boldsymbol{\omega}, \boldsymbol{\alpha}, A, \Psi)} : \mathscr{V} \times \mathscr{W} \times L^2_l(X; S^+ \otimes E \oplus \Lambda^1 \otimes \mathfrak{su}(E))$$
$$\to L^2_{l-1}(X; S^- \otimes E \oplus \Lambda^2_+ \otimes \mathfrak{su}(E))$$

is surjective. This derivative is given by the explicit expression

$$P(\boldsymbol{\nu}, \boldsymbol{\beta}, a, \Phi) = (\slashed{D}^+_{A,\boldsymbol{\alpha}}\Phi + \gamma(a + h\,dV_{\boldsymbol{\alpha}}|_A(a) + h\,V_{\boldsymbol{\beta}}(A) + dh|_\Psi(\Phi)\,V_{\boldsymbol{\alpha}}(A))\,\Psi,$$
$$d^+_{A,\boldsymbol{\omega}}a + \gamma^{-1}(\mu(\Psi, \Phi) + \mu(\Phi, \Psi)) + h\,V_{\boldsymbol{\nu}}(A) + dh|_\Psi(\Phi)\,V_{\boldsymbol{\omega}}(A)) \ ,$$

where $d_{A,\boldsymbol{\omega}}^+$ now denotes the operator $d_A^+ + h(\Psi)\, dV_{\boldsymbol{\omega}}|_A$. Note that by the assumption that we consider a solution to the perturbed monopole equation it follows that $h(\Psi) \neq 0$.

Instead of proving directly that P is surjective we shall consider the operators

$$P_k' : \mathscr{W} \times \mathscr{V} \times T_k \to L_{k-1}^2 \ ,$$

where T_k is the slice of the gauge-group action given by

$$T_k = \ker(d_{A,\Psi}^{0,*}) \subseteq L_k^2(X; S^+ \otimes E \oplus \Lambda^1 \otimes \mathfrak{su}(E)) \ ,$$

and where P_k' has the same formal expression as P above.

As a first step we show that $P_1' : \mathscr{W} \times \mathscr{V} \times T_1 \to L^2$ is surjective. Suppose $(b, \Sigma) \in L^2(X; S^- \otimes E \oplus \Lambda_+^2 \otimes \mathfrak{su}(E))$ is L^2-orthogonal to the image of P_1'. First we shall variate $\boldsymbol{\nu} \in \mathscr{W}$ alone. We therfore have

$$0 = \langle P_1'(\boldsymbol{\nu}), (\Sigma, b) \rangle_{L^2} = \langle h\, V_{\boldsymbol{\nu}}(A), b \rangle_{L^2}$$

for all $\boldsymbol{\nu} \in \mathscr{W}$. Now $\boldsymbol{\nu} \mapsto V_{\boldsymbol{\nu}}(A)$ has L^2-dense image (see the proof of the above Theorem 1.8 in [10]) , so $b = 0$.

Next we would like to variate the spinor Φ alone, i.e. to consider $P_1'(\Phi)$ for an arbitrary L_1^2 section in $S^+ \otimes E$. This, however, is not possible because only the slice T_1 is envolved in the definition of P_1' - it is not clear whether for a general spinor Φ there is a solution $(0, \Phi) \in T_1$. If it were possible, the argument would continue like this: We would have

$$0 = \langle P_1'(\Phi), (\Sigma, 0) \rangle_{L^2} = \langle \not{D}_{A,\boldsymbol{\alpha}}^+ \Phi, \Sigma \rangle_{L^2}$$

for all $\Phi \in L_1^2(X; S^+ \otimes E)$. As a consequence, we would get $\not{D}_{A,\boldsymbol{\alpha}}^- \Sigma = 0$ in the distributional sense, where $\not{D}_{A,\boldsymbol{\alpha}}^-$ is the formal L^2-adjoint of $\not{D}_{A,\boldsymbol{\alpha}}^+$. By elliptic regularity of the operator $\not{D}_A^-$ we would then see that Σ is actually of Sobolev class L_l^2, so $\not{D}_{A,\boldsymbol{\alpha}}^- \Sigma = 0$ would hold in the usual sense. Now the point is that we still can conlude that Σ satisfies the Dirac equation $\not{D}_{A,\boldsymbol{\alpha}}^- \Sigma = 0$. Here is the argument: First, there is an elliptic deformation complex as in (13) for any Sobolev index k, in particular for $k = 1$. Applying Hodge theory to this elliptic complex gives the topological decomposition:

$$L_1^2(X; S^+ \otimes E \oplus \Lambda^1 \otimes \mathfrak{su}(E)) = \ker(d_{(A,\Psi)}^{0,*}) \oplus \operatorname{im}(d_{(A,\Psi)}^0) = T_1 \oplus \operatorname{im}(d_{(A,\Psi)}^0).$$

Second, we observe that $\operatorname{im}(d_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1) = \operatorname{im}(d_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1|_{T_1})$ because $d_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1 \circ d_{(A,\Psi)}^0 = 0$. Third, the restriction of P_1' to the slice T_1 is precisely equal to $d_{\boldsymbol{\omega},\boldsymbol{\alpha}}^1|_{T_1}$. As a consequence, for any $\Phi \in L_1^2(X; S^+ \otimes E)$ there is an element $(a, \Phi') \in T_1$ of the slice such that $P_1'(a, \Phi') = (\not{D}_{A,\boldsymbol{\alpha}}^+ \Phi, \dots)$ (we are only interested in the spinor component of P_1' in the argument to show that Σ satisfies the Dirac equation).

By assumption $\Psi \neq 0$, so if $\Psi(x) \neq 0$ then $\Psi(y)$ is non-zero for all y in a neighbourhood U of x. We shall now variate $\boldsymbol{\beta}$ alone, so that we obtain

$$0 = \langle P_1'(\boldsymbol{\beta}), (\Sigma, 0) \rangle_{L^2} = \langle h\, \gamma(V_{\boldsymbol{\beta}}(A))\Psi, \Sigma \rangle_{L^2} \tag{15}$$

for all $\boldsymbol{\beta} \in \mathscr{V}$. Note that the map $\Lambda^1 \otimes \mathfrak{gl}(E) \to S^- \otimes E$ given by $e \mapsto \gamma(e)\Psi$ is pointwise surjective at any point where $\Psi \neq 0$. By the condition 1.7 and the fact that A is irreducible there is a finite number of submersions $q_i : S^1 \times B_i \to X$ such that $x \in \operatorname{int}(B_i)$ and such that the holonomys $\operatorname{Hol}_{q_i}(A)(x) \in \mathfrak{gl}(E)_x$ span $\mathfrak{gl}(E)_x$. Furthermore, the holonomy sections $\operatorname{Hol}_{q_i}(A)$ on B_i continue to span $\mathfrak{gl}(E)$ in a neighbourhood of x, because l was chosen so that we have a Sobolev inclusion

$L_l^2 \hookrightarrow C^0$, and so the holonmy sections are continous. Multiplying the finite number of holonmy sections $\mathrm{Hol}_{q_i}(A)$ with convenient one-forms $\beta_i \in \Omega^1(B_i; \mathbb{C})$ supported in small enough neighbourhoods of x we obtain a perturbation $\boldsymbol{\beta} = (\beta_i) \in \mathscr{V}$, where all but these finite number of one-forms are zero, such that the equation (15) implies that Σ is zero in a neighbourhood U' of x. As we also have $\not{D}_{A,\boldsymbol{\alpha}}^- \Sigma = 0$ the unique continuation principle [1] for solutions to the perturbed Dirac-equation implies that $\Sigma = 0$ on the whole of X.

Therefore we have shown that P_1' is surjective. Suppose now that $P_1'(\boldsymbol{\beta}, \boldsymbol{\nu}, b, \Phi)$ lies in the Sobolev class L_{l-1}^2. The additional hypothesis that $(b, \Phi) \in T_1$ now imply by elliptic regularity that (b, Φ) is of Sobolev class L_l^2. Obviously we then have $P_l'(\boldsymbol{\beta}, \boldsymbol{\nu}, b, \Phi) = P_1'(\boldsymbol{\beta}, \boldsymbol{\nu}, b, \Phi)$, so that P_l' and in particular P is in fact surjective onto L_{l-1}^2. $\qquad\square$

Corollary 2.7. *For a residual set of perturbations $(\boldsymbol{\omega}, \boldsymbol{\alpha}) \in \mathscr{V} \times \mathscr{W}$ the subspace $M_{\mathfrak{s}, \kappa}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is regular for all $w \in H^2(X; \mathbb{Z})$ and instanton numbers κ. It therefore admits the structure of a smooth manifold of the expected dimension.*

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.8. *Suppose the connection A is irreducible and that the Dirac-operator D_A^+ has non-negative index. Then there is an open and dense subset of elements $\boldsymbol{\alpha} \in \mathscr{V}$ such that the perturbed Dirac-operator*

$$\not{D}_{A,\boldsymbol{\alpha}}^+ := \not{D}_A^+ + \gamma(V_{\boldsymbol{\alpha}}(A)) : L_l^2(X; S^+ \otimes E) \to L_{l-1}^2(X; S^- \otimes E)$$

is surjective.

Proof: Let us consider the map

$$g : \mathscr{V} \times L_l^2(X; S^+ \otimes E) \setminus \{0\} \to L_{l-1}^2(X; S^- \otimes E)$$
$$(\boldsymbol{\alpha}, \Psi) \mapsto \not{D}_A^+ \Psi + \gamma(V_{\boldsymbol{\alpha}}(A))\Psi \ .$$

As in the proof of the last theorem we see that 0 is a regular value of this map. Let $\mathscr{M}$ be the zero-set $g^{-1}(0) \subseteq \mathscr{V} \times L_l^2(X; S^+ \otimes E) \setminus \{0\}$. The projection onto the first factor $\pi : \mathscr{M} \to \mathscr{V}$ is then a Fredholm map of the same index as that of the Dirac operator $\not{D}_A^+$. In fact, for $(\boldsymbol{\alpha}, \Psi) \in \mathscr{M}$ the kernel and cokernel of $d\pi_{(\boldsymbol{\alpha}, \Psi)}$ and of $\frac{\partial g}{\partial \Psi}|_{(\boldsymbol{\alpha}, \Psi)}$ are naturally isomorphic. Now by the Sard-Smale theorem there is a residual subset of $\mathscr{V}$ consisting of regular values for π. Note that we simply have

$$\left. \frac{\partial g}{\partial \Psi} \right|_{(\boldsymbol{\alpha}, \Psi)} = \not{D}_{A,\boldsymbol{\alpha}}^+ \ .$$

Therefore, if $\boldsymbol{\alpha}$ is a regular value for π, the perturbed Dirac-operator $\not{D}_{A,\boldsymbol{\alpha}}^+$ is surjective. The dependence of the bounded operator $\not{D}_{A,\boldsymbol{\alpha}}^+ : L_l^2(X; S^+ \otimes E) \to L_{l-1}^2(X; S^- \otimes E)$ on $\boldsymbol{\alpha}$ is continous. Therefore the residual set of values $\boldsymbol{\alpha}$ for which this operator is surjective is also open. $\qquad\square$

2.9. **Orientations.** In analogy to the instanton situation we will determine natural orientations for the subspace of the $PU(2)$ moduli space $M_{\kappa,\mathfrak{s}}^w(\boldsymbol{\omega},\boldsymbol{\alpha})$ consisting of regular points with *trivial* stabiliser under the gauge group, and thus forming a manifold of the expected dimension. Again, this is done by considering the determinant line bundle of the family of Fredholm operators given by the deformation operators parametrised by the configuration space $\mathscr{B}_{\kappa,\mathfrak{s}}^{w,*}$.

Let

$$D_{(A,\Psi),\boldsymbol{\omega},\boldsymbol{\alpha}} := d_{(A,\Psi)}^{0,*} \oplus d_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1$$

be the deformation operator associated to the 'elliptic complex' (13). To be more precise, (13) is only a complex if (A,Ψ) satisfies the perturbed monopole equations. However, the operator $D_{(A,\Psi),\boldsymbol{\omega},\boldsymbol{\alpha}}$ is Fredholm for any $(A,\Psi) \in \mathscr{C}$. We obtain a homotopy of Fredholm operators by the formula $t \mapsto D_{(A,t\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}$. The operator $D_{(A,0),\boldsymbol{\omega},\boldsymbol{\alpha}}$ is simply given by the direct sum $\delta_{A,\boldsymbol{\omega}} \oplus \slashed{D}_{A,\boldsymbol{\alpha}}^+$. Now let

$$\Theta_{\boldsymbol{\omega},\boldsymbol{\alpha}}(t) \to \mathscr{B}_{\kappa,\mathfrak{s}}^{w,*}$$

be the quotient of the determinant line bundle of the operator $D_{(A,t\Psi),\boldsymbol{\omega},\boldsymbol{\alpha}}$ by the gauge group $\mathscr{G}$. We have got the following Lemma:

Lemma 2.9.   (1) *Suppose the moduli space $M_{\kappa,\mathfrak{s}}^{w,**}(\boldsymbol{\omega},\boldsymbol{\alpha}) \subseteq \mathscr{B}_{\kappa,\mathfrak{s}}^{w,**}$ is regular. Then the restriction of the determinant line bundle $\Theta_{\boldsymbol{\omega},\boldsymbol{\alpha}}(1)$ to it is equal to the the orientation bundle of $M_{\kappa,\mathfrak{s}}^{w,**}(\boldsymbol{\omega},\boldsymbol{\alpha})$.*
    (2) *The restriction of the determinant line bundle $\Theta_{\boldsymbol{\omega},\boldsymbol{\alpha}}(0)$ to the subspace $\mathscr{B}_{\kappa,\mathfrak{s}}^{w,**}$ is given by*

$$\Theta_{\boldsymbol{\omega},\boldsymbol{\alpha}}(0)|_{\mathscr{B}^{**}} = \pi^* \Lambda_{\boldsymbol{\omega}} \otimes \Omega_{\boldsymbol{\alpha}} \ ,$$

where $\Lambda_{\boldsymbol{\omega}} \to \mathscr{B}_{\kappa}^{w,*}$ is the determinant line bundle of the instanton deformation operator of section 1.7, the map $\pi : \mathscr{B}_{\kappa,\mathfrak{s}}^{w,**} \to \mathscr{B}_{\kappa}^{w,*}$ is the projection onto the connection component, and where $\Omega_{\boldsymbol{\alpha}}$ is the determinant line bundle of the family of Dirac operators $\slashed{D}_{A,\boldsymbol{\alpha}}^+$.

Note that the line bundles $\Theta_{\boldsymbol{\omega},\boldsymbol{\alpha}}(1)$ and $\Theta_{\boldsymbol{\omega},\boldsymbol{\alpha}}(0)$ are naturally isomorphic. Note further that the line bundle $\Omega_{\boldsymbol{\alpha}}$ is trivial and admits a natural trivialisation because it is the (real) determinant line bundle of the complex line bundle on $\mathscr{B}_{\kappa,\mathfrak{s}}^{w,*}$ formed by the complex determinant line bundle of the family of Dirac operators $\slashed{D}_{A,\boldsymbol{\alpha}}^+$. Recall that the letter o designated a choice of trivialisation of the determinant line bundle $\Lambda_0 \to \mathscr{B}_{\kappa}^{w,*}$ in section 1.7, and that there is also a natural isomorphism between Λ_0 and $\Lambda_{\boldsymbol{\omega}}$. We therefore get the following

Corollary 2.10. *Suppose the moduli space $M_{\kappa,\mathfrak{s}}^{w,**}(\boldsymbol{\omega},\boldsymbol{\alpha})$ is regular. Then it is orientable. Furthermore, a choice of trivialisation o of the trivial line bundle Λ_0 determines a natural orientation of $M_{\kappa,\mathfrak{s}}^{w,**}(\boldsymbol{\omega},\boldsymbol{\alpha})$.*

2.10. **Local models around the instantons.** We will recall some theory of local models in general here, and then apply the results to the neighbourhood of the instantons $[A] \in M_0^w(\boldsymbol{\omega})$ inside the $PU(2)$ monopole moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega},\boldsymbol{\alpha})$, where the instantons are considered as $PU(2)$ monopoles $[A,0]$ with vanishing spinor. We actually need some equivariant version for local models here, and we will emphasise why this works as well. Furthermore, we shall make use of the elliptic deformation complex (13). To simplify the notations we shall continue to write H^0, H^1, H^2 if we mean actually the *harmonic representatives* of the cohomology spaces H^0, H^1

and H^2 of that complex.

The following result is standard, see [4] or [5].

Proposition 2.11. *Suppose (A, Ψ) is an element of the pre-configuration space $\mathscr{C}$. Let $T_{(A,\Psi)}$ be the slice of to the gauge-group action $\mathscr{G}$ given by $T_{(A,\Psi)} := \ker(d^{0,*}_{A,\Psi})$. Let $\pi : T_{(A,\Psi)} \to \mathscr{B}_{\mathfrak{s}}$ be the projection map given by $(a, \Phi) \mapsto [A + a, \Psi + \Phi]$. The induced map*

$$T_{(A,\Psi)}/\Gamma_{(A,\Psi)} \to \mathscr{B}_{\mathfrak{s}} \ ,$$

yields a homeomorphism of a neighbourhood of $[0] \in T_{(A,\Psi)}/\Gamma_{(A,\Psi)}$ onto a neighbourhood of $[A, \Psi]$ in the configuration space $\mathscr{B}_{\mathfrak{s}}$ of $PU(2)$ monopoles.

As we have seen above a point $[A, 0] \in \mathscr{B}_{\mathfrak{s}}$ is always a fixed point of the circle action on $\mathscr{B}_{\mathfrak{s}}$. On the slice $T_{(A,0)}$ we have a circle action $S^1 \times T_{(A,0)} \to T_{(A,0)}$ given by $(z, (a, \Phi)) \mapsto (a, z\Phi)$. The actions of $\Gamma_{(A,0)}$ and S^1 commute and they factor through an obviuos action of the group $\Gamma_{(A,0)} \times_{\mathbb{Z}/2} S^1$. To simplify the notation we will now write Γ instead of $\Gamma_{(A,0)}$. The projection $\pi : T_{(A,0)} \to \mathscr{B}_{\mathfrak{s}}$ is then equivariant with respect to the action of $\Gamma \times_{\mathbb{Z}/2} S^1$ on the slice and the S^1-action ρ on $\mathscr{B}_{\mathfrak{s}}$ of section 2.5. Therefore we get

Proposition 2.12. *The map*

$$T_{(A,0)}/\Gamma \times_{\mathbb{Z}/2} S^1 \to \mathscr{B}_{\mathfrak{s}}/S^1$$

induced from the projection π yields a homeomorphism of a neighbourhood of $[0]$ in $T_{(A,0)}/\Gamma \times_{\mathbb{Z}/2} S^1$ onto a neighbourhood of $[A, 0]$ in the S^1-quotient $\mathscr{B}_{\mathfrak{s}}/S^1$.

We will now use this result to describe a neighbourhood of a point $[A, \Psi]$ in the moduli space $M^w_{\kappa,\mathfrak{s}}$. Let $\mathscr{F}_{\boldsymbol{\omega},\boldsymbol{\alpha}}$ be the map given by the left-hand side of the $PU(2)$ monopole equations (11) where the perturbations $(\boldsymbol{\omega}, \boldsymbol{\alpha})$ are kept fixed. We consider the restriction of this map to the slice $T_{(A,\Psi)}$:

$$
\begin{aligned}
f : T_{(A,\Psi)} &\to L^2_{l-1}(X; S^- \otimes E \oplus \Lambda^2_+ \otimes \mathfrak{su}(E)) \\
(a, \Phi) &\mapsto \mathscr{F}_{\boldsymbol{\omega},\boldsymbol{\alpha}}(A + a, \Psi + \Phi)
\end{aligned}
\tag{16}
$$

This map is equivariant with respect to the natural action of $\Gamma_{(A,\Psi)}$ on both spaces. Furthermore, it is equivariant with respect to the natural $\Gamma \times_{\mathbb{Z}/2} S^1$ actions in case we consider a fixed point of the circle action of the form $[A, 0]$. From the above propositions we therefore get:

Proposition 2.13. *The projection π of Proposition 2.11 induces a homeomorphism of a neighbourhood of the origin in $f^{-1}(0)/\Gamma_{A,\Psi}$ onto a neighbourhood of $[A, \Psi]$ in $M^w_{\kappa,\mathfrak{s}}$. Furthermore, it induces a homeomorphism of the origin in $f^{-1}(0)/\Gamma \times_{\mathbb{Z}/2} S^1$ onto a neighbourhood of $[A, 0]$ in $M^w_{\kappa,\mathfrak{s}}/S^1$.*

Next it can be shown that this description of the neighbourhood of $[A, \Psi]$ in the moduli space by the zero-set of the map f modulo stabiliser can be cut down to the zero-set of a map $h : H^1_{\boldsymbol{\omega},\boldsymbol{\alpha}} \to H^2_{\boldsymbol{\omega},\boldsymbol{\alpha}}$ between the finite-dimensional cohomology spaces of the elliptic complex (13) associated to (A, Ψ), modulo stabiliser. See the discussion in [4, section 4.2.4 and 4.2.5]. We will now make this local description explicit around fixed points $[A, 0]$ of the circle action on the moduli-space.

Let Q be the derivative of f at 0. It equals the restriction of $d^1_{\boldsymbol{\omega},\boldsymbol{\alpha}}$ of the elliptic complex (13) to the slice $T_{(A,0)}$. This map Q is a $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant Fredholm

map. Note that $\Gamma \times_{\mathbb{Z}/2} S^1$ acts isometrically on both $T_{(A,0)}$ and the image space of Q. Now there is a topological decomposition of the slice as

$$T_{(A,0)} = H^1_{(\boldsymbol{\omega},\boldsymbol{\alpha})} \oplus T' \ ,$$

where $H^1_{(\boldsymbol{\omega},\boldsymbol{\alpha})}$ is the kernel of Q and where T' is a $\Gamma \times_{\mathbb{Z}/2} S^1$-invariant complement of it. Simply take the L^2_l-orthogonal complement of $H^1_{(\boldsymbol{\omega},\boldsymbol{\alpha})}$ inside the slice $T_{(A,0)}$. There is also a topological decomposition of the target space as

$$L^2_{l-1}(X; S^- \otimes E \oplus \Lambda^2_+ \otimes \mathfrak{su}(E)) = H^2_{(\boldsymbol{\omega},\boldsymbol{\alpha})} \oplus \operatorname{im}(Q) \ ,$$

where the harmonic space $H^2_{(\boldsymbol{\omega},\boldsymbol{\alpha})}$ is given by $\ker((d^+_{A,\boldsymbol{\omega}})^*) \oplus \ker(\slashed{D}^-_{A,\boldsymbol{\alpha}})$, which is equally a $\Gamma \times_{\mathbb{Z}/2} S^1$-invariant subspace. Therefore Q is a map

$$Q : H^1_{(\boldsymbol{\omega},\boldsymbol{\alpha})} \oplus T' \to H^2_{(\boldsymbol{\omega},\boldsymbol{\alpha})} \oplus \operatorname{im}(Q) \ ,$$

and the restriction $Q' : T' \to \operatorname{im}(Q)$ is an equivariant isomorphism of Hilbert spaces.

Let $p : H^2_{(\boldsymbol{\omega},\boldsymbol{\alpha})} \oplus \operatorname{im}(Q) \to \operatorname{im}(Q)$ be the orthogonal projection.

Proposition 2.14. *There is a $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant diffeomorphism g of a neighbourhood of the origin in the slice $T_{(A,0)}$ such that*

$$p \circ f \circ g = p \circ Q \ .$$

Proof: To simplify the notation we shall only write H^1 for the harmonic space $H^1_{\boldsymbol{\omega},\boldsymbol{\alpha}}$. We define the map

$$G : H^1 \oplus T' \to H^1 \oplus T'$$

by the formula $G(h,t) := (h, Q'^{-1} \circ p \circ f(h,t))$, where $h \in H^1, t \in T'$. This is a $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant map. Its derivative at $(0,0)$ is easily seen to be the identity. Therefore G is a diffeomorphism of a neighbourhood of $(0,0)$ onto a neighbourhood of $(0,0)$. Let g be its inverse, which is necessarily equivariant. However, the equation $G \circ g(h,t) = (h,t)$ is simply equivalent to

$$p \circ f \circ g(h,t) = Q'(t) = p \circ Q(h,t),$$

which is the equality we sought to prove. $\qquad\square$

Corollary 2.15. *Suppose g is a $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant diffeomorphism as in the last proposition. Then we have*

$$f \circ g(h,t) = (\alpha(h,t), Q'(t)) \ ,$$

where α is a $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant map which has vanishing derivative at 0. As a consequence, up to a $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant diffeomorphism, the zero-set $f^{-1}(0)$ is given by the zero-set of the $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant map $\alpha(-,0) : H^1 \to H^2$.

This finite dimensional model is pretty standard. Our emphasis lies on the point that all this is true $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariantly.

The following statement is a simple consequence of the implicit function theorem and makes the charts of the moduli space $M^{w,**}_{\mathfrak{s},\kappa}$ a little more explicit:

Lemma 2.16. *Suppose $[A, \Psi]$ is a regular monopole. Then there is a smooth map $h : U \to T_{(A,\Psi)}$, defined on a neighbourhood U of $0 \in H^1_{(A,\Psi)}$ which yields a parametrisation of the zero-locus $f^{-1}(0) \subseteq T_{(A,\Psi)}$ of the map (16) around 0. Furthermore, h is of the form*

$$h(a, \Phi) = (a, \Phi) + q(a, \Phi) \ ,$$

where the derivative of the map q vanishes at the origin. If $\Psi = 0$ the map h is S^1-equivariant.

If (A, Ψ) has trivial stabiliser the composition $\pi \circ h_{(A,\Psi)}$ yields a smooth parametrisation of the moduli space $M^{w,**}_{\mathfrak{s},\kappa}$ in a neighbourhood of $[A, \Psi]$, where $h_{(A,\Psi)}$ denotes the map $(a, \Phi) \mapsto (A, \Psi) + h(a, \Phi)$.

3. The main Theorem and its proof

Our main result is the vanishing of the 'Casson-invariant' defined in section 1.8.

Theorem 3.1. *For any smooth negative definite four-manifold X with $b_1(X) = 1$ and $b_2(X) \equiv 0 \mod 4$ the Casson-invariant $n_{w,o}(X)$ is zero.*

3.1. **Sketch of proof.** Before proving the theorem in the following section we shall first give a sketch of it, omitting the discussion of regularity.

Suppose the Casson-invariant is defined by the count $n_{w,o}(X) = \#M^w_0$. We will prove this theorem by constructing a suitable compact moduli space $M^w_{0,\mathfrak{s}}$ of $PU(2)$ monopoles containing the instanton moduli space M^w_0 as a subspace, and such that the subspace $M^{w,**}_{0,\mathfrak{s}}$ is a smooth 2-dimensional manifold.

The circle action of section 2.5 restricts to a circle action on the moduli space $M^w_{0,\mathfrak{s}}$. The instanton moduli space M^w_0 is precisely equal to the fixed-point set of this circle action; the action is free on the complement $M^{w,**}_{0,\mathfrak{s}}$. For a generic perturbation $\boldsymbol{\alpha} \in \mathscr{V}$ the S^1-equivariant local model of $M^w_{0,\mathfrak{s}}$ around an instanton $[A]$, see section 2.10, is particularly easy to describe. As a consequence, the quotient of $M^{w,**}_{0,\mathfrak{s}}$ by the circle action is a smooth 1-dimensional manifold whose ends can be identified with the instanton moduli space M^w_0. Therefore the instanton moduli space M^w_0 is smoothly cobordant to the empty space, and so $n_{w,o}(X) \equiv 0 \mod 2$.

As a final step we discuss orientations. A choice of trivialisation of the determinant line bundle corresponding to the deformation operator $-d^*_A \oplus d^+_A$, determining the orientation of the instanton moduli space M^w_0, canonically determines an orientation of the $PU(2)$ monopole moduli space, and of the quotient $M^{w,**}_{0,\mathfrak{s}}/S^1$. We will then show that the orientations of the ends of $M^{w,**}_{0,\mathfrak{s}}/S^1$ coincides with the orientation of M^w_0. Therefore M^w_0 is smoothly orientedly cobordant to the empty space, and so $n_{w,o}(X) = 0$.

3.2. **The one-dimensional cobordism.** We will choose a $Spin^c$ structure $\mathfrak{s}$ according to Proposition 2.1. The complex index of the Dirac-operator D^+_A is then equal to one. By the discussion in section 2.3 and the expected dimension 0 of the instanton moduli space M^w_0 in Proposition 1.3, the expected dimension of the $PU(2)$ monopole moduli space $M^w_{0,\mathfrak{s}}$ is 2.

Next, we shall make a convenient genericity assumption. Note that countable intersections of residual sets are residual. By Theorem 1.8, Theorem 2.6 and Lemma 2.8 there is therefore a residual set of perturbation parameters $(\boldsymbol{\omega}, \boldsymbol{\alpha}) \in \mathscr{V} \times \mathscr{W}$ such that the following condition holds:

Condition 3.2. *1. The moduli space of instantons $M_0^w(\boldsymbol{\omega})$ consists of regular points only. It is therefore a compact zero-dimensional space and consists of a finite number of points.*

2. The moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is compact and the subspace $M_{0,\mathfrak{s}}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is a smooth two-dimensional manifold.

3. For each of the finite number of instantons $[A]$ occuring in $M_0^w(\boldsymbol{\omega})$ the perturbed Dirac-operator $D_{A,\boldsymbol{\alpha}}^+$ has trivial cokernel.

The Casson-invariant is the signed count $n_{w,o}(X) := \#M_0^w(\boldsymbol{\omega})$ according to Proposition 1.12.

Note that the circle action ρ of section 2.5 restricts to a circle action on the monopole moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})$. According to Proposition 2.3 and the it following remark, the fixed point set of this restriction equals the instanton moduli space,

$$M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})^{S^1} \cong M_0^w(\boldsymbol{\omega}) \ ,$$

where an instanton $[A]$ is considered as $PU(2)$ monopole $[A, 0]$. Note also that the complement of the fixed-point set of the circle action consists of monopoles with irreducible connection and non-vanishing spinor,

$$M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha}) \setminus M_0^w(\boldsymbol{\omega}) = M_{0,\mathfrak{s}}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha}) \ .$$

This complement $M_{0,\mathfrak{s}}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is a 2-dimensional smooth S^1-space, and it has a natural compactification (inside the entire moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})$) with ends given by the instanton moduli space $M_0^w(\boldsymbol{\omega})$. However, we want to be sure that each instanton only corresponds to one 'end' of the moduli space $M_{0,\mathfrak{s}}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})$. For this we have to study the local structure of the moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})$ in the neighbourhood of an instanton $[A, 0]$.

By Proposition 2.13 and Corollary 2.15 a neighbourhood of an instanton $[A, 0]$ in the monopole moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})$ or its S^1-quotient is described be a $\Gamma \times_{\mathbb{Z}/2} S^1$-equivariant map

$$\alpha(-, 0) : H^1_{(A,0),(\boldsymbol{\omega},\boldsymbol{\alpha})} \to H^2_{(A,0),(\boldsymbol{\omega},\boldsymbol{\alpha})} \ ,$$

where the spaces $H^i_{(A,0),(\boldsymbol{\omega},\boldsymbol{\alpha})}$, $i = 1, 2$ are the cohomology spaces (or harmonic spaces) of the elliptic deformation complex (13) associated to the solution $(A, 0)$ of the perturbed monopole equations (11), and are given by

$$H^1_{(A,0),(\boldsymbol{\omega},\boldsymbol{\alpha})} = \ker(\delta_{A,\boldsymbol{\omega}}) \oplus \ker(\not{D}_{A,\boldsymbol{\alpha}}^+)$$
$$H^2_{(A,0),(\boldsymbol{\omega},\boldsymbol{\alpha})} = \mathrm{coker}(d_{A,\boldsymbol{\omega}}^+) \oplus \mathrm{coker}(\not{D}_{A,\boldsymbol{\alpha}}^+) \ .$$

Under the above genericity assumption these spaces are given by

$$H^1_{(A,0)} = \ker(\not{D}_{A,\boldsymbol{\alpha}}^+) \cong \mathbb{C}$$
$$H^2_{(A,0)} = 0 \ ,$$

where we have omitted the dependence on the perturbation.

The S^1-action on $H^1_{(A,0)} \subseteq T_{(A,0)}$ corresponding via Proposition 2.12 to the action ρ on the configuration space $\mathscr{B}_{0,\mathfrak{s}}^w$ is such that the S^1-equivariant identification $\ker(\not{D}_{A,\boldsymbol{\alpha}}^+) \cong \mathbb{C}$ corresponds to the standard action of S^1 on $\mathbb{C}$.

Note that the stabiliser $\Gamma_{(A,0)}$ is isomorphic to $\mathbb{Z}/2$. Proposition 2.13 now implies that a neighbourhood of $[A, 0]$ in the moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is homeomorphic to

a neighbourhood of 0 in the quotient $\mathbb{C}/(\mathbb{Z}/2)$ - it can be taken as a cone over the real projective space $\mathbb{RP}^1$. It also implies that a neighbourhood of $[A, 0]$ in the S^1-quotient $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})/S^1$ is homeomorphic to a neighbourhood of 0 in the quotient $\mathbb{C}/S^1 \cong [0, \infty) \subseteq \mathbb{R}$. Thus we get

Proposition 3.3. *The S^1-quotient of the monopole moduli space $M_{0,\mathfrak{s}}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is a smooth one-dimensional manifold with boundary. Its boundary can be identified with the instanton moduli space $M_0^w(\boldsymbol{\omega})$.*

Corollary 3.4. *The instanton moduli space $M_0^w(\boldsymbol{\omega})$ is cobordant to the empty space. As a consequence, we must have $n_{w,o}(X) \equiv 0 \mod 2$.*

3.3. **Consideration of orientations.** We shall first agree that the boundary of an oriented manifold is oriented by the 'outward normal first' convention. In this way, for instance, the one-sphere S^1, seen as the boundary of the unit disc in $\mathbb{C}$ with its complex orientation, has its orientation 'counterclock-wise'.

We shall continue to suppose that perturbations $(\boldsymbol{\omega}, \boldsymbol{\alpha}) \in \mathscr{V} \times \mathscr{W}$ are chosen such that the Condition 3.2 holds. However, we shall not make this explicit in the notation of the harmonic spaces $H_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})}^1 = \ker(D_{(A,\Psi),(\boldsymbol{\omega},\boldsymbol{\alpha})})$ anymore and simply write $H_{(A,\Psi)}^1$ instead, and likewise for the deformation operators.

The zero-dimensional moduli space $M_0^w(\boldsymbol{\omega})$ is oriented by a choice of trivialisation o of the determinant line bundle $\Lambda_0 \to \mathscr{B}_0^w$, see section 1.7.

According to Corollary 2.10, the regular subspace $M_{\mathfrak{s},0}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})$ of the $PU(2)$ monopole moduli space is naturally oriented by a choice of trivialisation of the same determinant line bundle $\Lambda_0 \to \mathscr{B}_0^w$, and we shall choose the same trivialisation o as above.

The circle S^1 acts freely and smoothly on $M_{\mathfrak{s},0}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})$. An orientation of the S^1 quotient $M_{\mathfrak{s},0}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})/S^1$ is fixed by the following convention. Let $[A, \Psi]$ belong to $M_{\mathfrak{s},0}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})$, and let $[[A, \Psi]]$ be the corresponding element in the S^1 quotient. We require that the direct sum of tangent spaces

$$T_{[A,\Psi]}M^{**} = T_1 S^1.[A, \Psi] \oplus T_{[[A,\Psi]]}M^{**}/S^1 \tag{17}$$

is a direct sum of oriented vector spaces, where $S^1.[A, \Psi]$ denotes the S^1 orbit through $[A, \Psi]$. In other words, an oriented basis of $T_{[A,\Psi]}M^{**}$ is obtained by completing an oriented basis of the orbit with an oriented basis of $T_{[[A,\Psi]]}M^{**}/S^1$.

On the other hand, according to Proposition 3.3, the instanton moduli space $M_0^w(\boldsymbol{\omega})$ can be seen as the boundary of the 1-dimensional manifold $M_{\mathfrak{s},0}^w(\boldsymbol{\omega}, \boldsymbol{\alpha})/S^1$. The open submanifold $M_{\mathfrak{s},0}^{w,**}(\boldsymbol{\omega}, \boldsymbol{\alpha})/S^1$ was given an orientation in the last paragraph. This induces an orientation on the boundary $M_0^w(\boldsymbol{\omega})$. The proof of Theorem 3.1 will be complete if we show that the two orientations on $M_0^w(\boldsymbol{\omega})$ coincide, because then the oriented moduli space $M_0^w(\boldsymbol{\omega})$ is orientedly cobordant to the empty space and therefore the Casson-type invariant $n_{w,o}(X)$ must be zero.

The harmonic space $H_{(A,0)}^1 = \ker(\slashed{D}_{A,\boldsymbol{\alpha}}^+)$ is oriented by the trivialisation o of Λ_0, again by Corollary 2.10. If the instanton $[A] \in M_0^w(\boldsymbol{\omega})$ has orientation $+1$ the space $H_{(A,0)}^1$ is oriented by its natural complex orientation, otherwise it is oriented with the opposite of the complex orientation.

Now let us suppose that the monopole $[A', \Psi']$ is regular and has trivial stabiliser, and that further $(A', \Psi') \in T_{(A,0)}$ is close enough to $(A, 0)$ so that it is in the range of the parametrisation $h_{(A,0)}$ of Lemma 2.16, and let $h_{(A,0)}(\Phi) = (A', \Psi')$ for $\Phi \in H^1_{(A,0)}$. The tangent space $T_\Phi H^1_{(A,0)}$ of $H^1_{(A,0)}$ at Φ is canonically identified with $H^1_{(A,0)}$. Let us denote explicitly the following two cases:

(1) If $H^1_{(A,0)}$ admits its complex orientation the tangent space $T_\Phi H^1_{(A,0)}$ has an oriented basis consistent of

$$\left(\frac{\Phi}{\|\Phi\|}, \frac{i\,\Phi}{\|\Phi\|} \right) \ .$$

(2) If $H^1_{(A,0)}$ admits the opposite of the complex orientation the tangent space $T_\Phi H^1_{(A,0)}$ has an oriented basis consistent of

$$\left(-\frac{\Phi}{\|\Phi\|}, \frac{i\,\Phi}{\|\Phi\|} \right) \ .$$

Lemma 3.5. *Let $h : U \to T_{(A,0)}$ be a parametrisation of the zero-locus of the 'monopole map read through the slice' (16) as in Lemma 2.16. After possibly restricting h to a smaller neighbourhood U' of $0 \in H^1_{(A,0)}$, the composition*

$$\pi \circ h_{(A,0)}|_{U' \setminus \{0\}} : U' \setminus \{0\} \to M^{w,**}_{\mathfrak{s},0}$$

is a local diffeomorphism that is two-to-one and that is orientation-preserving.

Although this lemma seems obvious our proof is slightly technical. We defer it to the end of this section.

We will now terminate the proof of Theorem 3.1 assuming Lemma 3.5. Without loss of generality we may assume that the neighbourhood U equals the neighbourhood U' of the last lemma.

As h is S^1-equivariant the vector

$$d_\Phi(\pi \circ h_{(A,0)}) \left(\frac{i\,\Phi}{\|\Phi\|} \right) \ \in \ T_{[A',\Psi']} M^{w,**}_{\mathfrak{s},0}$$

yields a positive basis of the tangent space to the S^1-orbit $S^1.[A', \Psi']$ at $[A', \Psi']$.

Suppose the instanton $[A] \in M^w_0(\boldsymbol{\omega})$ counts as $+1$. By Lemma 3.5 the two vectors

$$\left(d_\Phi(\pi \circ h_{(A,0)}) \left(\frac{\Phi}{\|\Phi\|} \right), d_\Phi(\pi \circ h_{(A,0)}) \left(\frac{i\,\Phi}{\|\Phi\|} \right) \right)$$

form a positive basis of the tangent space $T_{[A',\Psi']} M^{**}$. By our orientation convention (17) a positive basis of the tangent space $T_{[[A',\Psi']]} M^{**}/S^1$ to the quotient M^{**}/S^1 is given by the vector

$$-d_\Phi(\pi \circ h_{(A,0)}) \left(\frac{\Phi}{\|\Phi\|} \right) \ .$$

This vector points 'outwards' towards the boundary instanton $[A]$. Therefore the orientation of $[A]$ as boundary point of the one-dimensional manifold with boundary $M^w_{\mathfrak{s},0}/S^1$ is positive.

Now suppose the instanton $[A] \in M_0^w(\boldsymbol{\omega})$ counts as -1. Now the two vectors

$$\left(-d_\Phi(\pi \circ h_{(A,0)}) \left(\frac{\Phi}{\|\Phi\|} \right) , d_\Phi(\pi \circ h_{(A,0)}) \left(\frac{i\,\Phi}{\|\Phi\|} \right) \right)$$

form a positive basis of the tangent space $T_{[A',\Psi']}M^{**}$ and a positive basis of the tangent space $T_{[[A',\Psi']]}M^{**}/S^1$ to the quotient M^{**}/S^1 is given by the vector

$$d_\Phi(\pi \circ h_{(A,0)}) \left(\frac{\Phi}{\|\Phi\|} \right) \ .$$

This vector points 'inwards' away from the boundary instanton $[A]$. Therefore the orientation of $[A]$ as boundary point of the one-dimensional manifold with boundary $M_{\mathfrak{s},0}^w/S^1$ is negative. $\qquad\square$

3.4. **Proof of Lemma 3.5.** By previous results we will only have to check that the map $\pi \circ h_{(A,0)}|_{U\setminus\{0\}}$ indeed has an orientation-preserving derivative for sufficiently small neighbourhoods U of $0 \in H_{(A,0)}^1$.

Suppose $h_{(A,0)}(\Phi) = (A',\Psi') \in (A,0) + T_{(A,0)}$. Let $h_{(A',\Psi')} : H_{(A',\Psi')}^1 \to (A',\Psi') + T_{(A',\Psi')}$ be a parametrisation of $M_{\mathfrak{s},0}^{w,**}$ around $[A',\Psi']$ as in Lemma 2.16. In order to prove the theorem we will read a convenient restriction of the map $\pi \circ h_{(A,0)}$ to a neighbourhood V of $\Phi \in H_{(A,0)}^1$ through the chart $(\pi \circ h_{(A',\Psi')})^{-1}$. This map

$$k := (\pi \circ h_{(A',\Psi')})^{-1} \circ (\pi \circ h_{(A,0)}|_V) : V \subseteq H_{(A,0)}^1 \to H_{(A',\Psi')}^1 \qquad (18)$$

can be described in an alternative way.

The spaces $H_{(A,0)}^1$ and $H_{(A',\Psi')}^1$ lie in the different slices $T_{(A,0)}$ and $T_{(A',\Psi')}$ to the action of the gauge group $\mathscr{G}$ on $\mathscr{C}$. We wish to define a 'gauge fixing changing' map

$$g : (A,0) + T_{(A,0)} \to (A',\Psi') + T_{(A',\Psi')} \ ,$$

at least in a neighbourhood of (A',Ψ'), with the following two properties:
  (1) $g((A',\Psi')) = (A',\Psi')$,
  (2) $[g((A',\Psi') + (a,\Sigma))] = [(A',\Psi') + (a,\Sigma)]$.

Lemma 3.6. *There exists a smooth gauge fixing changing map g satisfying the above two properties such that its derivative $d_{(A',\Psi')}g : T_{(A,0)} \to T_{(A',\Psi')}$ at (A',Ψ') has the following shape*

$$(d_{(A',\Psi')}g)(a,\Sigma) = (a,\Sigma) + c(a,\Sigma) \ ,$$

where the norm of the linear map c can be made as small as we wish by choosing (A',Ψ') close enough to $(A,0)$.

Proof:

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We apply the implicit function theorem to the map

$$F : L_{l+1}^2(X;\mathfrak{su}(E)) \times T_{(A,0)} \to L_{l-1}^2(X;\mathfrak{su}(E))$$

$$(\zeta,a,\Sigma) \mapsto d_{(A',\Psi')}^{0,*}\left(\exp(\zeta)(A'+a,\Psi'+\Sigma)\right) \ ,$$

where $\exp : L^2_{l+1}(X; \mathfrak{su}(E)) \to \mathscr{G}$ denotes the exponential from the Lia-algebra of the gauge group to the gauge group. Note that the partial derivative of F at $(0,0)$ in the first variable is given by

$$\left.\frac{\partial F}{\partial \zeta}\right|_{(0,0)} (\zeta) = d^{0,*}_{(A',\Psi')} d^0_{(A',\Psi')}(\zeta)$$

As (A', Ψ') is irreducible by assumption the Laplacian $\Delta^0_{(A',\Psi')} = d^{0,*}_{(A',\Psi')} d^0_{(A',\Psi')}$ is an isomorphism. By the implicit function theorem there is a map γ from a neighbourhood of 0 in $T_{(A,0)}$ to a neighbourhood of 0 in $L^2_{l+1}(X; \mathfrak{su}(E))$ such that

$$d^{0,*}_{(A',\Psi')}(\exp(\gamma(a, \Sigma))(A' + a, \Psi' + \Sigma)) = 0 .$$

The required map g is then given by $g((A', \Psi') + (a, \Sigma)) := \exp(\gamma(a, \Sigma))(A' + a, \Psi' + \Sigma)$. For the derivative we have

$$(d_{(A',\Psi')}g)(a, \Sigma) = (a, \Sigma) - d^0_{(A',\Psi')}\left(\Delta^0_{(A',\Psi')}\right)^{-1} d^{0,*}_{(A',\Psi')}(a, \Sigma) .$$

Notice that the operators $d^0_{(A',\Psi')}$ and $\Delta^0_{(A',\Psi')}$ variate continously with (A', Ψ'). But for $(a, \Sigma) \in T_{(A,0)} = \ker(d^{0,*}_{(A,0)})$ we have

$$d^{0,*}_{(A',\Psi')}(a, \Sigma) = (d^{0,*}_{(A',\Psi')} - d^{0,*}_{(A,0)})(a, \Sigma) ,$$

and the operator $d^{0,*}_{(A',\Psi')} - d^{0,*}_{(A,0)}$ can be made as small as we wish by chosing (A', Ψ') close enough to $(A, 0)$. $\qquad \square$

Now notice that the map k of (18), at least when restricted to a sufficiently small neighbourhood of $\Phi \in H^1_{(A,0)}$, is given by

$$k(0, \Phi + \Sigma) = h^{-1}_{(A',\Psi')}((g \circ h_{(A,0)})(0, \Sigma)) .$$

By the Lemma 3.6 and Lemma 2.16 the derivative of k at Φ is given by the form

$$d_\Phi k (\Sigma) = \Sigma + c_{(A',\Psi')}(\Sigma) ,$$

where the linear map $c_{(A',\Psi')}$ can be made as small as we like by chosing (A', Ψ') close enough to $(A, 0)$. In particular, for $\|c_{(A',\Psi')}\| < 1$ the map $d_\Phi k$ is an isomorphism.

We will compare the derivative $d_\Phi k : H^1_{(A,0)} \to H^1_{(A',\Psi')}$ to an orientation preserving map. Let R be a right-inverse to the deformation operator $D_{(A,0)}$. We define linear maps

$$P_{(A',\Psi')} : H^1_{(A',\Psi')} \to H^1_{(A,0)}$$
$$(a, \Sigma) \mapsto (a, \Sigma) + R (D_{(A,0)} - D_{(A',\Psi')})(a, \Sigma)$$

which obviously are isomorphisms for (A', Ψ') close enough to $(A, 0)$. In a small enough neighbourhood of $(A, 0)$ in the configuration space $\mathscr{C}$, consisting of regular configurations with at most finite stabilisers only, the maps $P_{(A',\Psi')}$ yield a local trivialisation of the bundle of kernels of the deformation operators $D_{(\omega,\alpha)}$. In particular, the maps $P_{(A',\Psi')}$ are all orientation-preserving with orientations determined by the (lift of the) determinant line bundle $\Theta_{(\omega,\alpha)}(1)$ formed by the family of deformation operators.

We now see that the difference of the two isomorphisms

$$P^{-1}_{(A',\Psi')}, d_\Phi k \ : \ H^1_{(A,0)} \to H^1_{(A',\Psi')}$$

can be made small enough by choosing (A',Ψ') close enough to $(A,0)$. Then both of these maps must be orientation-preserving.   □

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