THE GRADED CENTERS OF DERIVED DISCRETE ALGEBRAS

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Abstract. We describe in the paper the graded centers of the derived categories of the derived discrete algebras. In particular, we prove that if A is a derived discrete algebra, then the reduced part of the graded center of the derived category of A is nontrivial if and only if A has infinite global dimension. Moreover, we show that the nilpotent part of the graded center is controlled by the objects for which the Auslander–Reiten translation coincides with a power of the suspension functor.

Throughout the paper $\mathbb{F}$ denotes a fixed algebraically closed field. All considered categories are $\mathbb{F}$-categories and the considered algebras and modules are finite dimensional over $\mathbb{F}$. By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$, we denote the sets of integers, nonnegative integers, and positive integers, respectively. For $i, j \in \mathbb{Z}$, $[i, j] := \{k \in \mathbb{Z} \mid i \leq k \leq j\}$ (in particular, $[i, j] = \varnothing$ if $i > j$). Moreover, $[i, \infty) := \{k \in \mathbb{Z} \mid i \leq k\}$ and $(-\infty, j] := \{k \in \mathbb{Z} \mid k \leq j\}$.

Introduction

For a triangulated category $\mathcal{T}$ with the suspension functor Σ one defines the graded center $\mathfrak{Z}(\mathcal{T})$ as the graded ring which in degree $p \in \mathbb{Z}$ consists of all natural transformations $\mathrm{Id} \to \Sigma^p$ which commute with Σ up to $(-1)^p$. In a series of papers [7, 9, 10] by Linckelmann, Kessar, and Stancu, this notion has been proved useful in many situations, when studying representations of finite groups. Moreover, Krause and Ye [8] studied the graded centers for some classes of triangulated categories appearing in the representation theory of finite dimensional algebras.

An important homological invariant of an algebra A is the derived category $\mathcal{D}^b(A)$ of its module category. This category has a structure of a triangulated category, thus it is natural to study its graded center, which we denote by $\mathfrak{Z}(A)$. The algebras with the easiest to understand derived categories are the derived discrete algebras described by Vossieck [12]. Our aim in this paper is to calculate the graded centers for the derived discrete algebras.

2000 *Mathematics Subject Classification.* 16G20, 18E30.

Key words and phrases. derived discrete algebras, the graded center of a triangulated category.

The precise description of the graded centers in the non-trivial cases can be found in Section 6. We formulate here only some consequences of this description. For a graded commutative ring R we denote by R_{nil} the ideal of the nilpotent elements of R and we put $R_{\mathrm{red}} := R/R_{\mathrm{nil}}$. Moreover, given an algebra A we denote by τ the Auslander–Reiten translation in $\mathcal{D}^b(A)$.

Theorem. *Let A be a derived discrete algebra and $R := \mathfrak{Z}(A)$.*

(1) $R_{\mathrm{red}} = \mathbb{F}$ *if and only if* $\operatorname{gl.dim} A < \infty$.
(2) $R_{\mathrm{nil}} \neq 0$ *if and only if there exists an object X in $\mathcal{D}^b(A)$ such that $\tau X \simeq \Sigma^p X$ for some $p \in \mathbb{Z}$.*

The paper is organized as follows. In Section 1 we recall the theorem of Krause and Ye stating that, when calculating $\mathfrak{Z}(A)$ for an algebra A, we may replace the derived category $\mathcal{D}^b(A)$ by the homotopy category $\mathcal{K}^b(\operatorname{proj} A)$ of perfect complexes over A. Next, in Section 2 we collect information about the derived discrete algebras. As a consequence it follows that we may concentrate in our calculations on the one-cycle gentle algebras not satisfying the clock condition. In Section 3 we describe $\mathcal{K}^b(\operatorname{proj} A)$ for a given one-cycle gentle algebra A not satisfying the clock condition of finite global dimension. This description is used in Section 4 in order to calculate the graded centers for the one-cycle gentle algebras not satisfying the clock condition of finite global dimension. Next, in Section 5 we study the case of the one-cycle gentle algebras not satisfying the clock condition of infinite global dimension. Finally, in Section 6 we summarize our calculations.

For background on the representation theory of algebras (including the language of quivers) we refer to [1].

The article was written while the author was staying at the University of Bielefeld as an Alexander von Humboldt Foundation fellow. The author also acknowledges the support from the Research Grant No. N N201 269135 of the Polish Ministry of Science and High Education.

1. The graded center of a triangulated category

Let $\mathcal{T}$ be a triangulated category with the suspension functor Σ. By the graded center $\mathfrak{Z}(\mathcal{T})$ of $\mathcal{T}$ we mean the $\mathbb{Z}$-graded abelian group $\bigoplus_{p\in\mathbb{Z}} \mathfrak{Z}_p(\mathcal{T})$, where, for $p \in \mathbb{Z}$, $\mathfrak{Z}_p(\mathcal{T})$ consists of the natural transformations $\eta : \mathrm{Id} \to \Sigma^p$ such that $\eta_{\Sigma X} = (-1)^p \cdot \Sigma\eta_X$ for each object X of $\mathcal{T}$. If $\eta' \in \mathfrak{Z}_p(\mathcal{T})$ and $\eta'' \in \mathfrak{Z}_q(\mathcal{T})$ for $p, q \in \mathbb{Z}$, then we define the product $\eta' \circ \eta''$ of η' and η'' by $(\eta' \circ \eta'')_X := \Sigma\eta'_X \circ \eta''_X$ for an object X of $\mathcal{T}$. In this way we give $\mathfrak{Z}(\mathcal{T})$ a structure of a graded commutative ring, where a $\mathbb{Z}$-graded ring $R = \bigoplus_{p\in\mathbb{Z}} R_p$ is called graded commutative if $r_1 \cdot r_2 = (-1)^{p\cdot q} \cdot r_2 \cdot r_1$ for all $r_1 \in R_p$, $r_2 \in R_q$, $p, q \in \mathbb{Z}$. For completeness, one may also study "a commutative version" of the graded center, i.e. the ring $\mathfrak{Z}'(\mathcal{T}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{Z}'_p(\mathcal{T})$, such that, for $p \in \mathbb{Z}$, $\mathfrak{Z}'_p(\mathcal{T})$ consists of the natural transformations $\eta : \mathrm{Id} \to \Sigma^p$ such that $\eta_{\Sigma X} = \Sigma \eta_X$ for each object X of $\mathcal{T}$. Obviously, $\mathfrak{Z}'(\mathcal{T})$ is a commutative ring graded by $\mathbb{Z}$.

Let $\mathcal{A}$ be an additive category. By $\mathcal{K}^b(\mathcal{A})$ we denote the bounded homotopy category of $\mathcal{A}$ defined as follows. The objects of $\mathcal{K}^b(\mathcal{A})$ are the differential complexes $X = (X^p, d_X^p)$ of objects of $\mathcal{A}$ such that $X^p = 0$ for all but finite $p \in \mathbb{Z}$. If X and Y are objects of $\mathcal{K}^b(\mathcal{A})$, then $\mathrm{Hom}_{\mathcal{K}^b(\mathcal{A})}(X, Y)$ consists of the equivalence classes of the morphisms $X \to Y$ of complexes modulo the null-homotopic maps. Recall, that if X and Y are complexes, then a morphism $f : X \to Y$ of complexes is given by morphisms $f^p : X^p \to Y^p$, $p \in \mathbb{Z}$, in $\mathcal{A}$ such that $f^{p+1} \circ d_X^p = d_Y^p \circ f^p$ for each $p \in \mathbb{Z}$. Moreover, if X and Y are complexes, then a morphism $f : X \to Y$ is called null-homotopic, if there exist morphisms $h^p : X^p \to Y^{p-1}$, $p \in \mathbb{Z}$, in $\mathcal{A}$, such that $f^p = d_Y^{p-1} \circ h^p + h^{p+1} \circ d_X^p$. It is known (see for example [6, I.3.2]) that $\mathcal{K}^b(\mathcal{A})$ has a structure of a triangulated category with the suspension functor Σ given by the shift of complexes, i.e. if $p \in \mathbb{Z}$, then $(\Sigma X)^p := X^{p+1}$ and $d_{\Sigma X}^p := -d_X^{p+1}$ for an object X of $\mathcal{K}^b(\mathcal{A})$ and $(\Sigma f)^p := f^{p+1}$ for a morphism f in $\mathcal{K}^b(\mathcal{A})$.

Now assume that $\mathcal{A}$ is an abelian category. Then for an object X of $\mathcal{K}^b(\mathcal{A})$ and $p \in \mathbb{Z}$ we define the p-th cohomology group $H^p(X)$ of X by $H^p(X) := \mathrm{Ker}\, d_X^p / \mathrm{Im}\, d_X^{p-1}$. If $f \in \mathrm{Hom}_{\mathcal{K}^b(\mathcal{A})}(X, Y)$ and $p \in \mathbb{Z}$, then f induces the map $H^p(f) : H^p(X) \to H^p(Y)$. If $f \in \mathrm{Hom}_{\mathcal{K}^b(\mathcal{A})}(X, Y)$, then f is called a quasi-isomorphism provided $H^p(f)$ is an isomorphism for each $p \in \mathbb{Z}$. The derived category $\mathcal{D}^b(\mathcal{A})$ of $\mathcal{A}$ is by definition the localization of $\mathcal{K}^b(\mathcal{A})$ with respect to the quasi-isomorphisms [11]. It follows that $\mathcal{D}^b(\mathcal{A})$ has a structure of a triangulated category with the suspension functor induced by the suspension functor in $\mathcal{K}^b(\mathcal{A})$.

The following theorem proved in [8] will be a useful tool in our calculations.

Theorem 1.1 (Krause/Ye)**.** *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then $\mathfrak{Z}(\mathcal{D}^b(\mathcal{A}))$ is positively graded. Moreover, if $\mathcal{P}$ is the full subcategory of $\mathcal{A}$ consisting of the projective objects, then $\mathfrak{Z}(\mathcal{D}^b(\mathcal{A})) \simeq \mathfrak{Z}(\mathcal{K}^b(\mathcal{P}))$.*

We also have the following variant of the above theorem, whose proof is almost identical.

Theorem 1.2. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then $\mathfrak{Z}'(\mathcal{D}^b(\mathcal{A}))$ is positively graded. Moreover, if $\mathcal{P}$ is the full subcategory of $\mathcal{A}$ consisting of the projective objects, then $\mathfrak{Z}'(\mathcal{D}^b(\mathcal{A})) \simeq \mathfrak{Z}'(\mathcal{K}^b(\mathcal{P}))$.*

We will apply the above theorems in the situation when $\mathcal{A}$ is the category $\mathrm{mod}\, A$ of modules over an algebra A and, consequently, $\mathcal{P}$ is the full subcategory $\mathrm{proj}\, A$ of $\mathrm{mod}\, A$ formed by the projective modules. In the above situation we will write just $\mathfrak{Z}(A)$ ($\mathfrak{Z}'(A)$) instead of

$\mathfrak{Z}(\mathcal{D}^b(\operatorname{mod} A))$ and $\mathfrak{Z}(\mathcal{K}^b(\operatorname{proj} A))$ ($\mathfrak{Z}'(\mathcal{D}^b(\operatorname{mod} A))$ and $\mathfrak{Z}'(\mathcal{K}^b(\operatorname{proj} A))$, respectively).

2. Derived discrete algebras

We begin this section with the definition of gentle algebras. Let $(Q, \mathfrak{R})$ be a pair consisting of a finite connected quiver (i.e. directed graph) $Q = (Q_0, Q_1)$, where Q_0 and Q_1 are the sets of vertices and arrows in Q, respectively, and a set $\mathfrak{R}$ of paths in Q. We say that $(Q, \mathfrak{R})$ is a gentle quiver if the following conditions are satisfied:

(1) for each $x \in Q_0$ there are at most two $\alpha \in Q_1$ such that $s\alpha = x$ (i.e. α starts at x) and at most two $\beta \in Q_1$ such that $t\beta = x$ (i.e. β terminates at x),
(2) $\mathfrak{R}$ consists of paths of length 2,
(3) for each $\alpha \in Q_1$ there is at most one $\beta \in Q_1$ such that $s\beta = t\alpha$ and $\beta\alpha \notin \mathfrak{R}$, and at most one $\gamma \in Q_1$ such that $t\gamma = s\alpha$ and $\alpha\gamma \notin \mathfrak{R}$,
(4) for each $\alpha \in Q_1$ there is at most one $\beta \in Q_1$ such that $s\beta = t\alpha$ and $\beta\alpha \in \mathfrak{R}$, and at most one $\gamma \in Q_1$ such that $t\gamma = s\alpha$ and $\alpha\gamma \in \mathfrak{R}$.

If, in addition, the number of arrows in Q equals the number of vertices in Q, then we say that $(Q, \mathfrak{R})$ is a one-cycle gentle quiver.

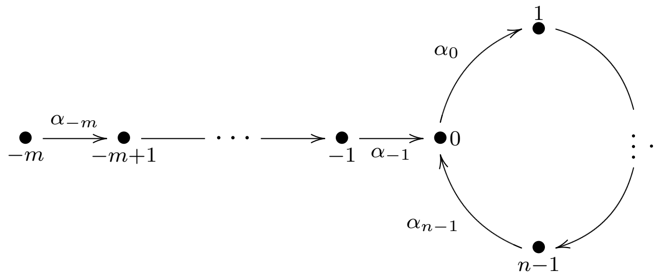
Let $(Q, \mathfrak{R})$ be a one-cycle gentle quiver. We say that $\alpha \in Q_1$ is a cycle arrow if the quiver $(Q_0, Q_1 \setminus \{\alpha\})$ is connected. Let Q_1' and Q_1'' be the sets of clockwise and anti-clockwise oriented cycle arrows, respectively (we leave to the reader to formulate the formal definition of these notions). We say that $(Q, \mathfrak{R})$ satisfies the clock condition if the number of $\alpha\beta \in \mathfrak{R}$ such that $\alpha, \beta \in Q_1'$ equals the number of $\alpha\beta \in \mathfrak{R}$ such that $\alpha, \beta \in Q_1''$. Note that this condition is obviously independent on the choice of orientation.

An algebra A is called gentle one-cycle not satisfying the clock condition if $A \simeq \mathbb{F}Q/\langle\mathfrak{R}\rangle$ for a one-cycle gentle quiver $(Q, \mathfrak{R})$ not satisfying the clock condition. Here, for a quiver Q we denote by $\mathbb{F}Q$ the path algebra of Q. Moreover, if Q is a quiver and $\mathfrak{R}$ is a set of paths in Q, then $\langle\mathfrak{R}\rangle$ denotes the ideal in $\mathbb{F}Q$ generated by $\mathfrak{R}$.

Let A be an algebra. For a complex X of A-modules we define its cohomology dimension vector $\mathbf{hdim}\, X \in \mathbb{N}^{\mathbb{Z}}$ by $(\mathbf{hdim}\, X)_p := \dim_{\mathbb{F}} H^p(X)$, $p \in \mathbb{Z}$. We say that A is derived discrete if for each $\mathbf{h} \in \mathbb{N}^{\mathbb{Z}}$ the indecomposable objects in $\mathcal{D}^b(\operatorname{mod} A)$ with the cohomology dimension vector $\mathbf{h}$ form only a finite number of the isomorphism classes.

Examples of derived discrete algebras are the hereditary algebras of Dynkin type. Vossieck proved [12] that a connected algebra A is derived discrete if and only if either A is derived equivalent to a hereditary algebra of Dynkin type (i.e. $\mathcal{D}^b(\operatorname{mod} A)$ is equivalent as a triangulated category to $\mathcal{D}^b(\operatorname{mod} B)$ for a hereditary algebra B of Dynkin type) or A is Morita equivalent to a one-cycle gentle algebra which does not satisfy the clock condition. One easily verifies that $\mathfrak{Z}(A) = \mathbb{F}$ and $\mathfrak{Z}'(A) = \mathbb{F}$ if A is a hereditary algebra of Dynkin type (see also [8]). Consequently, we concentrate in our paper on describing the graded centers for the one-cycle gentle algebras which does not satisfy the clock condition.

For $n \in \mathbb{N}_+$ and $m \in \mathbb{N}$ let $\Delta(n, m)$ be the following quiver



Next, for $(r, n) \in \mathbb{N}_+^2$ such that $r \in [1, n]$ we put

$$\mathfrak{R}(r, n) := \{\alpha_{i+1}\alpha_i \mid i \in [n - r, n - 2]\} \cup \{\alpha_0\alpha_{n-1}\}.$$

Let Ω denote the set of triples $(r, n, m) \in \mathbb{N}^3$ such that $n \in \mathbb{N}_+$ and $r \in [1, n]$. For $(r, n, m) \in \Omega$ we put $\Lambda(r, n, m) := \mathbb{F}\Delta(n, m)/\langle \mathfrak{R}(r, n) \rangle$. It is easy to check that $(\Delta(n, m), \mathfrak{R}(r, n))$ is a one-cycle gentle quiver not satisfying the clock condition for each $(r, n, m) \in \Omega$. It was proved in [5] that if A is a one-cycle gentle algebra not satisfying the clock condition, then there exists $(r, n, m) \in \Omega$ such that A and $\Lambda(r, n, m)$ are derived equivalent. Consequently, it is sufficient to describe the graded centers for the algebras of the above form.

We end this section with the following remark. Let $(Q, \mathfrak{R})$ be a one-cycle gentle quiver not satisfying the clock condition. Then one can determine $(r, n, m) \in \Omega$ such that $\mathbb{F}Q/\langle \mathfrak{R} \rangle$ and $\Lambda(r, n, m)$ are derived equivalent using the invariant introduced by Avella-Alaminos and Geiss [2].

3. Category

Throughout this section we fix $(r, n, m) \in \Omega$ such that $r < n$ and we put $\Lambda := \Lambda(r, n, m)$. We remark that the condition $r < n$ implies that $\operatorname{gl.dim} \Lambda < \infty$. In this section we describe a quiver Γ and relations such that the full subcategory of the indecomposable objects in $\mathcal{K}^b(\operatorname{proj} \Lambda)$ is equivalent to the path category of Γ modulo the given relations. This description follows from calculations made in [5] (we also refer to [3, 4]).

First, for $i \in [0, r - 1]$ we put $I_i := \mathbb{Z}^2$,

$$I_i' := \{(a, b) \in \mathbb{Z}^2 \mid a \leq b + \delta_{i,0} \cdot m\},$$

and

$$I_i'' := \{(a, b) \in \mathbb{Z}^2 \mid a + \delta_{i,0} \cdot n \leq b\},$$

where $\delta_{x,y}$ is the Kronecker delta. Then the vertices of Γ are $X_v^{(i)}$ for $i \in [0, r-1]$ and $v \in I_i'$, $Y_v^{(i)}$ for $i \in [0, r-1]$ and $v \in I_i''$, and $Z_v^{(i)}$ for $i \in [0, r-1]$ and $v \in I_i$.

Now we describe the arrows in Γ. Moreover, in order to be able to describe the relations in a compact way we associate to each arrow an additional data, which we call the degree of an arrow.

First fix $i \in [0, r-1]$ and $v = (a, b) \in I_i'$. We put

$$\mathcal{I}_v'^{(i)} := [a, b + \delta_{i,0} \cdot m] \times [b, \infty), \qquad \mathcal{X}_v^{(i)} := [a, b + \delta_{i,0} \cdot m] \times \mathbb{Z},$$

and

$$\mathcal{X}_v'^{(i)} := (-\infty, a + \delta_{i,r-1} \cdot m] \times [a, b + \delta_{i,0} \cdot m].$$

For $u \in \mathcal{I}_v'^{(i)}$, $u \neq v$, there is an arrow $f_{v,u}'^{(i)} : X_v^{(i)} \to X_u^{(i)}$ of degree 0. Next, for $u \in \mathcal{X}_v^{(i)}$ there is an arrow $g_{v,u}'^{(i)} : X_v^{(i)} \to Z_u^{(i)}$ of degree 1. Finally, for $u \in \mathcal{X}_v'^{(i)}$ there is an arrow $e_{v,u}'^{(i)} : X_v^{(i)} \to X_u^{(i+1)}$ of degree 2, where we always change the upper index modulo r.

Now fix $i \in [0, r-1]$ and $v := (a, b) \in I_i''$. We put

$$\mathcal{I}_v''^{(i)} := [a, b - \delta_{i,0} \cdot n] \times [b, \infty), \qquad \mathcal{Y}_v^{(i)} := \mathbb{Z} \times [a, b - \delta_{i,0} \cdot n],$$

and

$$\mathcal{Y}_v''^{(i)} := (-\infty, a - \delta_{i,r-1} \cdot n] \times [a, b - \delta_{i,0} \cdot n].$$

For $u \in \mathcal{I}_v'^{(i)}$, $u \neq v$, there is an arrow $f_{v,u}''^{(i)} : Y_v^{(i)} \to Y_u^{(i)}$ of degree 0. Next, for $u \in \mathcal{Y}_v^{(i)}$ there is an arrow $g_{v,u}''^{(i)} : Y_v^{(i)} \to Z_u^{(i)}$ of degree 1. Finally, for $u \in \mathcal{Y}_v''^{(i)}$ there is an arrow $e_{v,u}''^{(i)} : Y_v^{(i)} \to Y_u^{(i+1)}$ of degree 2.

Finally fix $i \in [0, r-1]$ and $v := (a, b) \in I_i$. We put

$$\begin{aligned}
\mathcal{I}_v^{(i)} &:= [a, \infty) \times [b, \infty), \\
\mathcal{Z}_v'^{(i)} &:= (-\infty, a + \delta_{i,r-1} \cdot m] \times [a, \infty), \\
\mathcal{Z}_v''^{(i)} &:= (-\infty, b - \delta_{i,r-1} \cdot n] \times [b, \infty),
\end{aligned}$$

and

$$\mathcal{Z}_v^{(i)} := (-\infty, a + \delta_{i,r-1} \cdot m] \times (\infty, b - \delta_{i,r-1} \cdot n].$$

For $u \in \mathcal{I}_v^{(i)}$, $u \neq v$, there is an arrow $f_{v,u}^{(i)} : Z_v^{(i)} \to Z_u^{(i)}$ of degree 0. Next, for $u \in \mathcal{Z}_v'^{(i)}$ there is an arrow $h_{v,u}'^{(i)} : Z_v^{(i)} \to X_u^{(i+1)}$ of degree 1. Similarly, for $u \in \mathcal{Z}_v''^{(i)}$ there is an arrow $h_{v,u}''^{(i)} : Z_v^{(i)} \to Y_u^{(i+1)}$ of degree 1. Finally, for $u \in \mathcal{Z}_v^{(i)}$ there is an arrow $e_{v,u}''^{(i)} : Z_v^{(i)} \to Z_u^{(i+1)}$ of degree 2.

Now we describe the relations. Let $f : X \to Y$ and $g : Y \to Z$ be arrows of degree p and q, respectively. If there is an arrow $h : X \to Z$ of degree $p + q$, then we have the relation $gf = h$, otherwise we have the relation $gf = 0$ (we note that some of the relations are described directly in Section 5).

We also introduce the following notation

$$f'^{(i)}_{v,v} := \mathrm{Id}_{X^{(i)}_v},\ v \in I'_i, \qquad f''^{(i)}_{v,v} := \mathrm{Id}_{Y^{(i)}_v},\ v \in I''_i,$$

and

$$f^{(i)}_{v,v} := \mathrm{Id}_{Z^{(i)}_v},\ v \in I_i,$$

for $i \in [0, r-1]$.

Now we describe the shift Σ. First, for $i \in [0, r-1]$ we put

$$\mathbf{v}'_i := (1 + \delta_{i,r-1} \cdot m, 1 + \delta_{i,0} \cdot m), \qquad \mathbf{v}''_i := (1 - \delta_{i,r-1} \cdot n, 1 - \delta_{i,0} \cdot n),$$

and

$$\mathbf{v}_i := (1 + \delta_{i,r-1} \cdot m, 1 - \delta_{i,r-1} \cdot n).$$

If $i \in [0, r-1]$, then $\Sigma X^{(i)}_v = X^{(i+1)}_{v+\mathbf{v}'_i}$ for each $v \in I'_i$, $\Sigma Y^{(i)}_v = Y^{(i+1)}_{v+\mathbf{v}''_i}$ for each $v \in I''_i$, and $\Sigma Z^{(i)}_v = Z^{(i+1)}_{v+\mathbf{v}_i}$ for each $v \in I''_i$. We leave it to the reader to write the obvious formulas for the action of the shift on the morphisms.

Finally, we describe the action of the Auslander–Reiten translation τ. Namely, if $i \in [0, r-1]$, then $\tau X^{(i)}_v = X^{(i)}_{v-(1,1)}$ for each $v \in I'_i$, $\tau Y^{(i)}_v = Y^{(i)}_{v-(1,1)}$ for each $v \in I''_i$, and $\tau Z^{(i)}_v = Z^{(i)}_{v-(1,1)}$ for each $v \in I''_i$. Observe that it follows by direct calculations that there exists an indecomposable object X in $\mathcal{K}^b(\operatorname{proj}\Lambda)$ such that $\tau X = \Sigma^p X$ for some $p \in \mathbb{Z}$ if and only if either $r = n - 1$ or $r = 1$ and $m = 0$.

4. Calculations

Throughout this section we fix $(r, n, m) \in \Omega$ such that $r < n$ and we put $\Lambda := \Lambda(r, n, m)$. In this section we calculate $\mathfrak{Z}(\Lambda)$ and $\mathfrak{Z}(\Lambda')$.

First we describe the homomorphism spaces between the indecomposable objects in $\mathcal{K}^b(\operatorname{proj}\Lambda)$ and their shifts.

Lemma 4.1. *Let $i \in [0, r-1]$ and $v \in I_i$. If $p \in \mathbb{N}$, then*

$$\mathrm{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(Z^{(i)}_v, \Sigma^p Z^{(i)}_v) = \begin{cases} \mathbb{F} \cdot f^{(i)}_{v,v} & p = 0, \\ 0 & p > 0. \end{cases}$$

Proof. Put $Z := Z^{(i)}_v$. The claim is obvious if neither $p - 1$ nor p is divisible by r.

First assume that $r > 1$ and $p = qr + 1$ for some $q \in \mathbb{N}$. Then $\Sigma^p Z = Z^{(i+1)}_{v+q(r+m,r-n)+\mathbf{v}_i}$ and one easily verifies that $v+q(r+m, r-n)+\mathbf{v}_i \notin \mathcal{Z}^{(i)}_v$.

Now assume that $p = qr$ for some $q \in \mathbb{N}$. In this case $\Sigma^p Z = Z^{(i)}_{v+q(r+m,r-n)}$. We leave it to the reader to verify that $v + q(r + m, r -$

$n) \in \mathcal{I}_v^{(i)}$ if and only if $q = 0$. Finally, one also shows that $v + q(r + m, r - n) \notin \mathcal{Z}_v^{(0)}$ provided $r = 1$. □

Lemma 4.2. *Let $i \in [0, r-1]$ and $v = (a, b) \in I_i'$. If $p \in \mathbb{N}_+$, then*

$$\operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(X_v^{(i)}, \Sigma^p X_v^{(i)}) = \begin{cases} \mathbb{F} \cdot f'^{(i)}_{v, v + \frac{p}{r}(r+m, r+m)} & r \mid p \text{ and } \frac{p}{r} \leq \frac{b + \delta_{i,0} \cdot m - a}{r+m}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(X_v^{(i)}, X_v^{(i)}) = \begin{cases} \mathbb{F} \cdot f'^{(i)}_{v,v} + \mathbb{F} \cdot e'^{(i)}_{v,v} & r = 1 \text{ and } a \leq b. \\ \mathbb{F} \cdot f'^{(i)}_{v,v} & \text{otherwise.} \end{cases}$$

Proof. The method of the proof is analogous to that of the proof of the previous lemma, hence we leave it to the reader. □

Lemma 4.3. *Let $i \in [0, r-1]$ and $v = (a, b) \in I_i''$. If $p \in \mathbb{N}$, then*

$$\operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(Y_v^{(i)}, \Sigma^p Y_v^{(i)}) = \begin{cases} \mathbb{F} \cdot f''^{(i)}_{v,v} & p = 0, \\ \mathbb{F} \cdot e''^{(i)}_{v, v + \frac{p-1}{r}(r-n, r-n) + \mathbf{v}_i''} & r \mid p - 1 \text{ and } \\ & \quad \frac{1}{n-r} \leq \frac{p-1}{r} \leq \frac{b + 1 - a - \delta_{i,0} \cdot n}{n - r}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Similar as above. □

Assume that $r = n - 1$ and fix $q \in \mathbb{N}$. Observe that in this case $\tau Y_v^{(i)} = \Sigma^r Y_v^{(i)}$ for each $i \in [0, r-1]$ and $v \in I_i''$. Moreover, for each $i \in [0, r-1]$ and $v = (a, b) \in I_i''$ such that $b - a = q + \delta_{i,0} \cdot n$, there exists unique $p \in \mathbb{Z}$ such that $Y_v^{(i)} = \Sigma^p Y_{(0, n+q)}^{(0)}$. We put $\varepsilon(i, v) := (-1)^{n \cdot p}$ in the above situation. We define the natural transformation $\eta'^{(q)} : \operatorname{Id} \to \Sigma^n$ by setting $\eta'^{(q)}_X := \varepsilon(i, v) \cdot e''^{(i)}_{v, v + \mathbf{v}_i'' - (1,1)}$ if $X = Y_v^{(i)}$ for $i \in [0, r-1]$ and $v = (a, b) \in I_i''$ such that $b - a = q + \delta_{i,0} \cdot n$, and $\eta'^{(q)}_X := 0$ if X is an indecomposable object of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ not isomorphic to $Y_v^{(i)}$ for some $i \in [0, r-1]$ and $v = (a, b) \in I_i''$ such that $b - a = q + \delta_{i,0} \cdot n$. One easily verifies that $\eta'^{(q)} \in \mathfrak{Z}_n(\Lambda)$.

Proposition 4.4. *Let $p \in \mathbb{N}_+$. Then*

$$\mathfrak{Z}_p(\Lambda) = \begin{cases} \prod_{q \in \mathbb{N}} \mathbb{F} \cdot \eta'^{(q)} & (r, p) = (n-1, n), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Fix $\eta \in \mathfrak{Z}_p(\Lambda)$. Lemma 4.1 implies that $\eta_{Z_v^{(i)}} = 0$ for each $v \in I_i$ and $i \in [0, r-1]$.

Now we show that $\eta_X = 0$ if $X = X_v^{(i)}$ for $v = (a,b) \in I_i'$ and $i \in [0, r-1]$. It follows from from Lemma 4.2 that we may assume that $r \mid p$ and $\frac{p}{r} \leq \frac{b+\delta_{i,0}m-a}{r+m}$. In this case $\Sigma^p X = X_u^{(i)}$ and $\eta_X = \lambda \cdot f_{v,u}'^{(i)}$ for some $\lambda \in \mathbb{F}$, where $u := v + \frac{p}{r}(r+m, r+m)$. Let $f := g_{v,v'}'^{(i)}$, where $v' := (a, 0)$. Then $\Sigma^p f = g_{u,u'}'^{(i)}$, where $u' := v' + \frac{p}{r}(r+m, r-n)$. Observe that $\Sigma^p f \circ \eta_X = \lambda \cdot g_{v,u'}'^{(i)}$. On the other hand $\eta_{Z_{v'}^{(i)}} \circ f = 0$, as we have already proved that $\eta_{Z_{v'}^{(i)}} = 0$. Since $\Sigma^p f \circ \eta_X = \eta_{Z_{v'}^{(i)}} \circ f$, it follows that $\lambda = 0$, and hence $\eta_X = 0$.

Now assume that $(r,p) \neq (n-1,n)$ and let $Y := Y_v^{(i)}$ for some $v = (a,b) \in I_i''$ and $i \in [0, r-1]$. We prove by induction on $b - a - \delta_{i,0} \cdot n$ that $\eta_Y = 0$. If $b = a + \delta_{i,0} \cdot n$, then the claim follows from Lemma 4.3 (due to the assumption $(r,p) \neq (n-1,n)$). Now assume that $b > a + \delta_{i,0} \cdot n$. Lemma 4.3 implies that we may assume that r divides $p-1$ and $\frac{1}{n-r} \leq \frac{p-1}{r} \leq \frac{b+1-a-\delta_{i,0}\cdot n}{n-r}$. In this case $\Sigma^p Y = Y_u^{(i+1)}$ and $\eta_Y = \lambda \cdot e_{v,u}''^{(i)}$ for some $\lambda \in \mathbb{F}$, where $u := v + \frac{p-1}{r}(r-n, r-n) + \mathbf{v}_i''$. Let $v' := (a, b-1)$ and $Y' := Y_{v'}^{(i)}$. By the induction hypothesis $\eta_{Y'} = 0$, thus $\eta_Y \circ f_{v',v}''^{(i)} = \Sigma^p f_{v',v}''^{(i)} \circ \eta_{Y'} = 0$. On the other hand, using once more the assumption $(r,p) \neq (n-1,n)$, we get $u \in \mathcal{Y}_{v'}''^{(i)}$, hence $\eta_Y \circ f_{v',v}''^{(i)} = \lambda \cdot e_{v',u}''^{(i)}$ and the claim follows.

Finally, assume that $(r,p) = (n-1,n)$. In this case for each $i \in [0, r-1]$ and $v \in I_i''$ there exists $\lambda_{i,v} \in \mathbb{F}$ such that $\eta_{Y_v^{(i)}} = \lambda_{i,v} \cdot e_{v,v+\mathbf{v}_i''-(1,1)}''^{(i)}$. Since η commutes up to sign with Σ, $\lambda_{i,v} = \varepsilon(i,v) \cdot \lambda_{0,(0,n+b-a-\delta_{i,0}\cdot n)}$ for each $i \in [0, r-1]$ and $v = (a,b) \in I_i''$, hence the claim follows. □

Again assume that $r = n-1$ and fix $q \in \mathbb{N}$. Similarly as before we define the natural transformation $\eta''^{(q)} : \mathrm{Id} \to \Sigma^n$ by $\eta_X''^{(q)} := e_{v,v+\mathbf{v}_i''-(1,1)}''^{(i)}$ if $X = Y_v^{(i)}$ for $i \in [0, r-1]$ and $v = (a,b) \in I_i''$ such that $b - a = q + \delta_{i,0} \cdot n$, and $\eta_X'^{(q)} := 0$ if X is an indecomposable object of $\mathcal{K}^b(\operatorname{proj} \Lambda)$ not isomorphic to $Y_v^{(i)}$ for some $i \in [0, r-1]$ and $v = (a,b) \in I_i''$ such that $b - a = q + \delta_{i,0} \cdot n$. One easily verifies that $\eta''^{(q)} \in \mathfrak{Z}_n'(\Lambda)$.

We have the following variant of the above proposition, which is proved analogously.

Proposition 4.4'. *Let $p \in \mathbb{N}_+$. Then*

$$\mathfrak{Z}_p(\Lambda) = \begin{cases} \prod_{q \in \mathbb{N}} \mathbb{F} \cdot \eta''^{(q)} & (r,p) = (n-1,n), \\ 0 & \textit{otherwise}. \end{cases}$$ □

Our next aim is to calculate $\mathfrak{Z}_0(\Lambda) = \mathfrak{Z}_0'(\Lambda)$. Let Id denote the natural transformation $\mathrm{Id} \to \mathrm{Id}$ which associates to an object X of $\mathcal{K}^b(\operatorname{proj} \Lambda)$ the identity map. Obviously $\mathrm{Id} \in \mathfrak{Z}_0(\Lambda)$.

Assume that $r = 1$ and $m = 0$, and fix $q \in \mathbb{N}$. Observe that in this case $\tau X_v^{(0)} = \Sigma^{-1} X_v^{(0)}$ for each $v \in I_0'$. We define the natural transformation $\eta^{(q)} : \mathrm{Id} \to \mathrm{Id}$ by setting $\eta_X^{(q)} := e'^{(0)}_{v,v}$ if $X = X_v^{(0)}$ for $v = (a, b) \in I_0'$ such that $b - a = q$, and $\eta_X^{(q)} := 0$ if X is an indecomposable object of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ not isomorphic to $X_v^{(0)}$ for some $v = (a, b) \in I_0'$ such that $b - a = q$. One easily verifies that $\eta^{(q)} \in \mathfrak{Z}_0(\Lambda)$.

Proposition 4.5. *We have*

$$\mathfrak{Z}_0(\Lambda) = \begin{cases} \mathbb{F} \cdot \mathrm{Id} \oplus \prod_{q \in \mathbb{N}} \mathbb{F} \cdot \eta^{(q)} & r = 1 \text{ and } m = 0, \\ \mathbb{F} \cdot \mathrm{Id} & \text{otherwise.} \end{cases}$$

Proof. Fix $\eta \in \mathfrak{Z}_0(\Lambda)$. Let $\lambda \in \mathbb{F}$ be such that $\eta_{Z^{(0)}_{(0,0)}} = \lambda \cdot f^{(0)}_{(0,0),(0,0)}$.

First we show by induction on i that for each $i \in [0, r - 1]$ there exists $v \in I_i$ such that $\eta_{Z_v^{(i)}} = \lambda \cdot f^{(i)}_{v,v}$. This claim is obvious for $i = 0$, thus assume that $i > 0$. Fix $u \in I_{i-1}$ such that $\eta_{Z_u^{(i-1)}} = \lambda \cdot f^{(i-1)}_{u,u}$. Put $v := u + \mathbf{v}_{i-1} - (1, 1)$. Lemma 4.1 implies that $\eta_{Z_v^{(i)}} = \mu \cdot f^{(i)}_{v,v}$ for some $\mu \in \mathbb{F}$. Moreover, $\mu \cdot e^{(i-1)}_{u,v} = \eta_{Z_v^{(i)}} \circ e^{(i-1)}_{u,v} = e^{(i-1)}_{u,v} \circ \eta_{Z_u^{(i-1)}} = \lambda \cdot e^{(i-1)}_{u,v}$, hence $\mu = \lambda$.

Next we show that $\eta_{Z_v^{(i)}} = \lambda \cdot f^{(i)}_{v,v}$ for each $i \in [0, r - 1]$ and $v \in I_i$. We proceed in a similar way as above in the following steps. First, we fix $u \in I_i$ such that $\eta_{Z_u^{(i)}} = \lambda \cdot f^{(i)}_{u,u}$. Next we show the claim for all $v \in \mathcal{I}_u^{(i)}$ (using $f^{(i)}_{u,v}$) and finally we prove it for arbitrary $v \in I_i$ (using $f^{(i)}_{v,v'}$ for some $v' \in \mathcal{I}_v^{(i)} \cap \mathcal{I}_u^{(i)}$).

Further we apply the same method and Lemma 4.3 in order to show that $\eta_{Y_v^{(i)}} = \lambda \cdot f''^{(i)}_{v,v}$ for each $i \in [0, r - 1]$ and $v \in I_i''$. In this case we use $g''^{(i)}_{v,u}$ for some $u \in \mathcal{Y}_v^{(i)}$. Moreover, we use Lemma 4.2 in order to prove similarly that $\eta_{X_v^{(i)}} = \lambda \cdot f'^{(i)}_{v,v}$ for each $i \in [0, r - 1]$ and $v \in I_i'$ provided $r > 1$.

Finally assume that $r = 1$. The analogous arguments to those presented above and Lemma 4.2 imply that for each $v \in I_0'$ there exists $\lambda_v \in \mathbb{F}$ such that $\eta_{X_v^{(0)}} = \lambda \cdot f'^{(0)}_{v,v} + \lambda_v \cdot e'^{(0)}_{v,v}$. Observe that for each $v = (a, b) \in I_0''$ there exists unique $p \in \mathbb{Z}$ such that $X_v^{(i)} = \Sigma^p X^{(0)}_{(0,b-a)}$. Consequently, it follows that $\lambda_v = \lambda_{(0,b-a)}$ for each $v = (a, b) \in I_0'$, since η and Σ. Finally, we prove by induction on $a \in \mathbb{N}$ that $\lambda_{(0,a)} = 0$ if $m > 0$, proceeding similarly as we did in the proof of Proposition 4.4. □

Finally, we describe the multiplication in $\mathfrak{Z}(\Lambda)$ and $\mathfrak{Z}'(\Lambda)$.

Proposition 4.6. *Let* $q_1, q_2 \in \mathbb{N}$.

(1) *If* $r = n - 1$, *then* $\eta'^{(q_1)} \cdot \eta'^{(q_2)} = 0 = \eta''^{(q_1)} \cdot \eta''^{(q_2)}$.
(2) *If* $r = 1$ *and* $m = 0$, *then* $\eta^{(q_1)} \cdot \eta^{(q_2)} = 0$.

(3) *If* $r = 1$, $n = 2$, *and* $m = 0$, *then* $\eta^{(q_1)} \cdot \eta'^{(q_2)} = 0 = \eta^{(q_1)} \cdot \eta''^{(q_2)}$.

Proof. Direct calculations. □

5. Infinite global dimension

Throughout this section we fix $(n, m) \in \mathbb{N}^2$ such that $n > 0$ and we put $\Lambda := \Lambda(n, n, m)$. In this case $\operatorname{gl.dim} \Lambda = \infty$.

First we describe the full subcategory of $\mathcal{K}^b(\operatorname{proj} \Lambda)$ formed by the indecomposable objects. Thoroughly speaking it is the full subcategory of the category described in Section 3 given by the X-vertices. More precisely, we have the following quiver with relations whose path category is equivalent to the full subcategory of $\mathcal{K}^b(\operatorname{proj} \Lambda)$ formed by the indecomposable objects. The vertices of this quiver are $X_v^{(i)}$ for $i \in [0, n-1]$ and $v \in I_i'$, where I_i' is defined as in Section 3, i.e.

$$I_i' := \{(a, b) \in \mathbb{Z}^2 \mid a \leq b + \delta_{i,0} \cdot m\}.$$

Next, for $i \in [0, n-1]$ and $v = (a, b) \in I_i'$ we define $\mathcal{I}_v'^{(i)}$ and $\mathcal{X}_v'^{(i)}$ by

$$\mathcal{I}_v'^{(i)} := [a, b + \delta_{i,0} \cdot m] \times [b, \infty)$$

and

$$\mathcal{X}_v'^{(i)} := (-\infty, a + \delta_{i,n-1} \cdot m] \times [a, b + \delta_{i,0} \cdot m].$$

Then for each $i \in [0, n-1]$, $v \in I_i'^{(i)}$, and $u \in \mathcal{I}_v'^{(i)}$, $u \neq v$, we have an arrow $f_{v,u}'^{(i)} : X_v^{(i)} \to X_u^{(i)}$, and for each $i \in [0, n-1]$, $v \in I_i'^{(i)}$, and $u \in \mathcal{X}_v'^{(i)}$, we have an arrow $e_{v,u}'^{(i)} : X_v^{(i)} \to X_u^{(i+1)}$. Moreover, we put $f_{v,v}'^{(i)} := \operatorname{Id}_{X_v^{(i)}}$ for each $i \in [0, n-1]$ and $v \in I_i'$. Finally, we have the following relations:

$$f_{u,w}'^{(i)} \circ f_{v,u}'^{(i)} = \begin{cases} f_{v,w}'^{(i)} & w \in \mathcal{I}_v'^{(i)}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \in [0, n-1]$, $v \in I_i'$, $u \in \mathcal{I}_v'^{(i)}$, and $w \in \mathcal{I}_u'^{(i)}$,

$$e_{u,w}'^{(i)} \circ f_{v,u}'^{(i)} = \begin{cases} e_{v,w}'^{(i)} & w \in \mathcal{X}_v'^{(i)}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \in [0, n-1]$, $v \in I_i'$, $u \in \mathcal{I}_v'^{(i)}$, and $w \in \mathcal{X}_u'^{(i)}$,

$$f_{u,w}'^{(i+1)} \circ e_{v,u}'^{(i)} = \begin{cases} e_{v,w}'^{(i)} & w \in \mathcal{X}_v'^{(i)}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \in [0, n-1]$, $v \in I_i'$, $u \in \mathcal{X}_v'^{(i)}$, and $w \in \mathcal{I}_u'^{(i+1)}$, and

$$e_{u,w}'^{(i+1)} \circ e_{v,u}'^{(i)} = 0$$

for $i \in [0, n-1]$, $v \in I_i'$, $u \in \mathcal{X}_v'^{(i)}$, and $w \in \mathcal{X}_u'^{(i+1)}$.

The descriptions of Σ and τ are also analogous to these given in Section 3. Namely, if $i \in [0, n-1]$, then $\Sigma X_v^{(i)} = X_{v+\mathbf{v}_i'}^{(i+1)}$ and $\tau X_v^{(i)} = X_{v-(1,1)}^{(i)}$ for each $v \in I_i'$, where $\mathbf{v}_i' := (1 + \delta_{i,n-1} \cdot m, 1 + \delta_{i,0} \cdot m)$. Consequently, there exists an indecomposable object X in $\mathcal{K}^b(\operatorname{proj}\Lambda)$ such that $\tau X = \Sigma^p X$ for some $p \in \mathbb{Z}$ if and only if $n = 1$ and $m = 0$.

First we observe that Lemma 4.2 is valid in this case without any changes. Namely, we have the following.

Lemma 5.1. *Let $i \in [0, n-1]$ and $v = (a, b) \in I_i'$. If $p \in \mathbb{N}_+$, then*

$$\operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(X_v^{(i)}, \Sigma^p X_v^{(i)}) = \begin{cases} \mathbb{F} \cdot f'^{(i)}_{v, v+\frac{p}{n}(n+m, n+m)} & n \mid p \text{ and } \frac{p}{n} \leq \frac{b+\delta_{i,0} m - a}{n+m}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(X_v^{(i)}, X_v^{(i)}) = \begin{cases} \mathbb{F} \cdot f'^{(i)}_{v,v} + \mathbb{F} \cdot e'^{(i)}_{v,v} & n = 1 \text{ and } a \leq b. \\ \mathbb{F} \cdot f'^{(i)}_{v,v} & \text{otherwise.} \end{cases}$$

The description of $\mathfrak{Z}_0(\Lambda) = \mathfrak{Z}_0'(\Lambda)$ does not differ either. Namely, let Id denote the natural transformation $\mathrm{Id} \to \mathrm{Id}$ which associates to an object X in $\mathcal{K}^b(\operatorname{proj}\Lambda)$ the identity map. Moreover, if $n = 1$ and $m = 0$, then for $q \in \mathbb{N}$ we define the natural transformation $\eta^{(q)} : \mathrm{Id} \to \mathrm{Id}$ by setting $\eta_X^{(q)} := e'^{(0)}_{v,v}$ if $X = X_v^{(0)}$ for $v = (a, b) \in I_0'$ such that $b - a = q$, and $\eta_X^{(q)} := 0$ if X is an indecomposable object of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ not isomorphic to $X_v^{(0)}$ for some $v = (a, b) \in I_i'$ such that $b - a = q$.

The proof of the following fact is obtained by adapting the arguments from the proof of Proposition 4.5 to the considered case.

Proposition 5.2. *We have*

$$\mathfrak{Z}_0(\Lambda) = \begin{cases} \mathbb{F} \cdot \mathrm{Id} \oplus \prod_{q \in \mathbb{N}} \mathbb{F} \cdot \eta^{(q)} & n = 1, \\ \mathbb{F} \cdot \mathrm{Id} & \text{otherwise.} \end{cases}$$

The situation differs in positive degrees.

We define $\eta : \mathrm{Id} \to \Sigma^n$ in the following way. We put $\eta_X := f'^{(i)}_{v, v+(n+m, n+m)}$ if $X = X_v^{(i)}$ for $i \in [0, n-1]$ and $v = (a, b) \in I_i'$ such that $n + m \leq b + \delta_{i,0} \cdot m - a$, and $\eta_X := 0$ if X is an indecomposable object of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ not isomorphic to $X_v^{(i)}$ for some $i \in [0, n-1]$ and $v = (a, b) \in I_i'$ such that $n+m \leq b+\delta_{i,0} \cdot m-a$. Observe that $\eta^p \neq 0$ for each $p \in \mathbb{N}_+$. Moreover, $\eta^p \in \mathfrak{Z}_{p\cdot n}'(\Lambda)$ for each $p \in \mathbb{N}_+$. More precisely, $\eta_X^p = f_{v, v+p\cdot(n+m, n+m)}$ if $X = X_v^{(i)}$ for $i \in [0, n-1]$ and $v = (a, b) \in I_i'$ such that $p \cdot (n+m) \leq b+\delta_{i,0} \cdot m-a$, and $\eta_X := 0$ if X is an indecomposable object of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ not isomorphic to $X_v^{(i)}$ for some $i \in [0, n-1]$ and $v = (a, b) \in I_i'$ such that $p \cdot (n + m) \leq b + \delta_{i,0} \cdot m - a$. Finally, if $p \in \mathbb{N}_+$, then $\eta^p \in \mathfrak{Z}_{p \cdot n}(\Lambda)$ if and only if either $2 \mid p \cdot n$ or $\operatorname{char}\mathbb{F} = 2$.

We have the following.

Proposition 5.3. *Let $p \in \mathbb{N}_+$. Then*

$$\mathfrak{Z}'_p(\Lambda) = \begin{cases} \mathbb{F} \cdot \eta^{\frac{p}{n}} & n \mid p, \\ 0 & \textit{otherwise}. \end{cases}$$

Moreover,

$$\mathfrak{Z}_p(\Lambda) = \begin{cases} \mathfrak{Z}'_p(\Lambda) & \textit{either } 2 \mid p \textit{ or } \operatorname{char} \mathbb{F} = 2, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. The claim follows by arguments similar to those used before, hence we omit it (compare also the proof of [8, Lemma 5.3]). □

We finish this section with the following.

Proposition 5.4. *Let $q, q_1, q_2 \in \mathbb{N}$. If $n = 1$, then*

$$\eta \cdot \eta^{(q)} = 0 = \eta^{(q_1)} \cdot \eta^{(q_2)}.$$

Proof. Direct calculations. □

6. Main theorem

Throughout this section we fix $(r, n, m) \in \Omega$ and we put $\Lambda := \Lambda(r, n, m)$. We summarize our findings in the theorems describing $\mathfrak{Z}(\Lambda)$ and $\mathfrak{Z}'(\Lambda)$. First we introduce some additional notation.

Let R be a commutative ring graded by $\mathbb{Z}$ and M a graded R-module. We put

$$T(R, M) := \left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \mid r \in R \text{ and } m \in M \right\}.$$

If we endow $T(R, M)$ with the obvious matrix multiplication, then it becomes a graded ring, with the grading inherited from those in R and M. Moreover, for $p \in \mathbb{Z}$ we define the graded R-module $M[p]$ by $M[p]_q := M[p + q]$ for $q \in \mathbb{Z}$.

We view $\mathbb{F}^{\mathbb{N}} := \prod_{q \in \mathbb{N}} \mathbb{F}$ as a graded $\mathbb{F}$-module concentrated in degree 0. Consequently, if $p \in \mathbb{N}$, then the morphism $\mathbb{F}[X^p] \to \mathbb{F}$, $f \mapsto f(0)$, gives $\mathbb{F}^{\mathbb{N}}$ a structure of a graded $\mathbb{F}[X^p]$-module, where in $\mathbb{F}[X^p]$ we have the grading coming from the usual degree of polynomials.

We have the following description of the graded center of Λ.

Theorem 6.1. *Let* $(r,n,m) \in \Omega$ *and* $R := \mathfrak{Z}(\Lambda(r,n,m))$. *Then*

$$R \simeq \begin{cases} T(\mathbb{F}[X], \mathbb{F}^{\mathbb{N}}) & (r,n,m) = (1,1,0) \text{ and } \operatorname{char}\mathbb{F} = 2, \\ T(\mathbb{F}[X^2], \mathbb{F}^{\mathbb{N}}) & (r,n,m) = (1,1,0) \text{ and } \operatorname{char}\mathbb{F} \neq 2, \\ \mathbb{F}[X^n] & r = n,\ (r,m) \neq (1,0), \text{ and } 2 \mid n \cdot \operatorname{char}\mathbb{F}, \\ \mathbb{F}[X^{2n}] & r = n,\ (r,m) \neq (1,0), \text{ and } 2 \nmid n \cdot \operatorname{char}\mathbb{F}, \\ T(\mathbb{F}, \mathbb{F}^{\mathbb{N}} \oplus \mathbb{F}^{\mathbb{N}}[-n]) & (r,n,m) = (1,2,0), \\ T(\mathbb{F}, \mathbb{F}^{\mathbb{N}}[-n]) & (r,m) \neq (1,0) \text{ and } r = n-1, \\ T(\mathbb{F}, \mathbb{F}^{\mathbb{N}}) & (r,m) = (1,0) \text{ and } r \neq n-1, n, \\ \mathbb{F} & \text{otherwise.} \end{cases}$$

In particular, $R_{\mathrm{red}} \neq \mathbb{F}$ *if and only if* $r = n$, *and* $R_{\mathrm{nil}} \neq 0$ *if and only if either* $r = n-1$ *and* $r = 1$ *and* $m = 0$.

Let $(r,n,m) \in \Omega$. Then $\operatorname{gl.dim}\Lambda(r,n,m) = \infty$ if and only if $r = n$. Moreover, there exists an object X in $\mathcal{D}^b(\Lambda(r,n,m))$ such that $\tau X \simeq \Sigma^p X$ for some $p \in \mathbb{Z}$ if and only if either $r = n-1$ and $r = 1$ and $m = 0$. Consequently, the above theorem implies the main theorem of the paper.

The following theorem is the analogue of the above one for "the commutative version" of the graded center.

Theorem 6.2. *Let* $(r,n,m) \in \Omega$ *and* $R := \mathfrak{Z}'(\Lambda(r,n,m))$. *Then*

$$R \simeq \begin{cases} T(\mathbb{F}[X], \mathbb{F}^{\mathbb{N}}) & (r,n,m) = (1,1,0), \\ \mathbb{F}[X^n] & r = n \text{ and } (r,m) \neq (1,0), \\ T(\mathbb{F}, \mathbb{F}^{\mathbb{N}} \oplus \mathbb{F}^{\mathbb{N}}[-n]) & (r,n,m) = (1,2,0), \\ T(\mathbb{F}, \mathbb{F}^{\mathbb{N}}[-n]) & (r,m) \neq (1,0) \text{ and } r = n-1, \\ T(\mathbb{F}, \mathbb{F}^{\mathbb{N}}) & (r,m) = (1,0) \text{ and } r \neq n-1, n, \\ \mathbb{F} & \text{otherwise.} \end{cases}$$

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