Harmonic measures for a point may form a square

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1 Introduction

Let X be an open set in $\mathbb{R}^d$, $d \geq 2$, such that X^c is non-polar, if $d = 2$, and let $x \in X$. In [2] it is shown that any convex combination of harmonic measures $\mu_x^{U_1}, \dots, \mu_x^{U_k}$, where $U_1, \dots, U_k$ are open neighborhoods of x in X, can be approximated by a sequence $(\mu_x^{W_n})$ of harmonic measures such that each W_n is an open neighborhood of x in $U_1 \cup \dots \cup U_k$. Harmonic measures μ_x^U, $x \in U$, U open in X, are obtained by balayage (with respect to X) of the Dirac measure ε_x at x on $X \setminus U$: $\mu_x^U = \varepsilon_x^{X \setminus U}$. They are special cases of measures μ on X which are representing measures for x with respect to the convex cone $\mathcal{P}(X)$ of all continuous real potentials on X, that is, such that $\mu(p) \leq p(x)$ for all $p \in \mathcal{P}(X)$. The convex set $\mathcal{M}_x(\mathcal{P}(X))$ of these representing measures is compact and metrizable with respect to the topology of weak convergence. By [4], it is known since forty years that the set $(\mathcal{M}_x(\mathcal{P}(X)))_e$ of extreme points of $\mathcal{M}_x(\mathcal{P}(X))$ consists of the balayage measures ε_x^A, where A is a Borel set in X. By elementary convergence properties, the set of harmonic measures μ_x^U, $x \in U$, U open in X, is dense in $(\mathcal{M}_x(\mathcal{P}(X)))_e$. Therefore, by the theorem of Krein-Milman, the approximation result above implied that the set $(\mathcal{M}_x(\mathcal{P}(X)))_e$ of extreme points in $\mathcal{M}_x(\mathcal{P}(X))$ is dense in $\mathcal{M}_x(\mathcal{P}(X))$.

It seemed to be widely believed that $\mathcal{M}_x(\mathcal{P}(X))$ is a simplex, and hence a Poulsen simplex. We shall disprove this belief by exhibiting open neighborhoods U_0, U_1, U_2, U_3 of x in X such that the harmonic measures $\mu_x^{U_j} \in (\mathcal{M}_x(\mathcal{P}(X)))_e$, $0 \leq j \leq 3$, are pairwise different and satisfy

$$\mu_x^{U_0} + \mu_x^{U_2} = \mu_x^{U_1} + \mu_x^{U_3}. \tag{1.1}$$

Assuming without loss of generality that x is the origin, the open sets U_j, $0 \leq j \leq 3$, will be related in a very simple way. For $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, let

$$T(y_1, y_2, y_3, \dots, y_d) := (-y_2, y_1, y_3, \dots, y_d),$$

that is, T turns the (y_1, y_2)-part of y counterclockwise around the origin by $\pi/2$. Then we shall have

$$U_j = T^j(U_0), \qquad 0 \leq j \leq 3.$$

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More precisely, given any open ball U centered at 0 (or any other non-empty T-invariant connected open neighborhood U of 0) and any non-polar Borel set A in $\{y \in U\colon y_1 > 0,\, y_2 = 0\}$ which is finely closed in U, (2.12) holds if we take

$$U_j := U \setminus T^j(A), \qquad 0 \le j \le 3.$$

The measures $\mu_0, \mu_1, \mu_2, \mu_3$ will form a square, in the sense that their densities with respect to a T-invariant measure τ on $\overline{U}$ form a square in $L^2(\overline{U}, \tau)$ (see Corollary 2.5).

In fact, equalities like (1.1) show that none of the compact convex sets $\mathcal{M}_x(S(W))$, $x \in W$, W open in X, is a simplex (see Corollary 3.2). By definition, $\mathcal{M}_x(S(W))$ is the set of all measures μ on X which are representing measures for x with respect to the convex cone $S(W)$ of all $\mathcal{P}(X)$-bounded continuous real functions on X which are superharmonic on W. $\mathcal{M}_x(S(W))$ is a closed face of $\mathcal{M}_x(\mathcal{P}(X))$ and

$$\{\varepsilon_x^{A \cup W^c} \colon A \text{ Borel set in } W\} \subset (\mathcal{M}_x(S(W)))_e$$

(with equality if W is regular; see [1, VII.9.5]). We note that $S(X) = \mathcal{P}(X)$, and hence $\mathcal{M}_x(S(X)) = \mathcal{M}_x(\mathcal{P}(X))$.

As long as we have certain symmetries, the same is true for many potential-theoretic settings (Riesz potentials, heat equation, Laplace-Kohn operator on the Heisenberg group, general sub-Laplacians on stratified Lie algebras).

Finally, we shall see that in many parabolic cases (whenever we have a space-time structure), we do not need symmetries to prove that the compact convex sets $\mathcal{M}_x(\mathcal{P}(X))$ and $\mathcal{M}_x(S(W))$, $x \in W$, W open in X, are never simplices.

2 Results based on global symmetries

Let $(X, \mathcal{W})$ be a balayage space. A homeomorphism on X is called an automorphism of $(X, \mathcal{W})$, if $\mathcal{W} \circ T = \mathcal{W}$ (cf. [1, Sect. VI.8]), and hence, for all $x \in X$ and $A \subset X$,

$$(2.1) \qquad T(\varepsilon_x^A) = \varepsilon_{T(x)}^{T(A)}.$$

Throughout this section, let us fix $x_0 \in X$ and suppose that we have automorphisms R, T of $(X, \mathcal{W})$ and a Borel set A in X such that

$$Rx_0 = Tx_0 = x_0, \quad T^4 = I \text{ (identity)}\ , \quad \varepsilon_{x_0}^A \neq 0,$$

the fine closures (or even the closures) of the sets

$$A_j := T^j(A), \qquad 0 \le j \le 3,$$

are pairwise disjoint, and

$$(2.2) \qquad R = I \quad \text{on } A_1 \cup A_3, \qquad R = T^2 \quad \text{on } A_0 \cup A_2.$$

EXAMPLES 2.1. For $x = (x_1, x_2, x_3, \dots, x_d) \in \mathbb{R}^d$, $d \ge 2$, let

$$Rx := (-x_1, x_2, x_3, \dots, x_d) \quad \text{and} \quad Tx := (-x_2, x_1, x_3, \dots, x_d),$$

that is, R is reflection at the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$ and T, as in the Introduction, turns the (x_1, x_2)-part of x counterclockwise by $\pi/2$. Let X be a domain in $\mathbb{R}^d$ such that $x_0 := 0 \in X$ and $R(X) = T(X) = X$, and let

$$A \subset \{x \in X : x_1 > 0,\, x_2 = 0\} =: H_2^+$$

such that 0 is not contained in the closure of A and the $(d-1)$-dimensional Hausdorff measure of A is strictly positive.

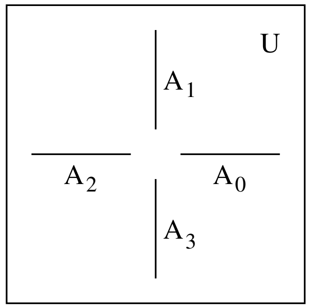

Figure 1. The sets A_0, A_1, A_2, A_3

Then our assumptions are satisfied in each of the following cases:

(i) Classical case (X^c non-polar, if $d = 2$),

(ii) Riesz potentials ($\alpha \in (1, 2)$),

(iii) Heisenberg group (or – more generally – stratified Lie algebras),

(iv) Heat equation on $\mathbb{R}^{d-1} \times \mathbb{R}$, $d \geq 3$ (with $A \subset \{x \in H_2^+ : x_d < 0\}$).

Let us now return to the general case and let U be an open neighborhood of x_0 in X such that

$$R(U) = T(U) = U \qquad \text{and} \qquad \varepsilon_{x_0}^{A \cup U^c}(U) > 0$$

(for example, $U = X$). We define

$$\mu_j := \varepsilon_{x_0}^{A_j \cup A_{j+1} \cup U^c}, \qquad 0 \leq j \leq 3,$$

where we take $A_4 := A_0$. If A is closed, then μ_j is the harmonic measure $\mu_{x_0}^{U_j}$ for the open set

$$U_j := U \setminus (A_j \cup A_{j+1}).$$

By (2.1),

$$\mu_j = T^j \mu_0, \qquad 1 \leq j \leq 3. \tag{2.3}$$

LEMMA 2.2. *The measures $\mu_0, \mu_1, \mu_2, \mu_3$ have pairwise different supports.*

Proof. We may assume without loss of generality that $A \subset U$ and A is finely closed in U. Then the measures μ_j are supported by $A_j \cup A_{j+1} \cup U^c$ and, by [1, VI.9.4],

$$\varepsilon_{x_0}^{A_0 \cup U^c} = \mu_0|_{A_0 \cup U^c} + (\mu_0|_{A_1})^{A_0 \cup U^c}.$$

By assumption $\varepsilon_{x_0}^{A_0 \cup U^c}(U) > 0$. Hence $\mu_0(A_0) > 0$ or $\mu_0(A_1) > 0$. By (2.3), for every $1 \leq j \leq 3$,

$$\mu_j(A_j) = \mu_0(A_0) \qquad \text{and} \qquad \mu_j(A_{j+1}) = \mu_0(A_1).$$

A consideration of the possible cases for the values of $\mu_0(A_k)$, $k \in \{0, 1\}$, immediately shows that the measures $\mu_0, \mu_1, \mu_2, \mu_3$ have pairwise different supports. $\square$

To see how the relation between R and T can be exploited let us first consider the simple case, where $U = X$.

THEOREM 2.3. *Assume that $U = X$. Then $\mu_0+\mu_2 = \mu_1+\mu_3$, where the measures $\mu_0, \mu_1, \mu_2, \mu_3 \in \mathcal{M}_x(\mathcal{P}(X))$ are pairwise different. In particular, $\mathcal{M}_x(\mathcal{P}(X))$ is not a simplex.*

Moreover, let ν_0 denote the restriction of μ_0 on the fine closure of $A_0 = A$. Then $\mu_0 = \nu_0 + T\nu_0$, $\|\mu_2 - \mu_1\| = \|\mu_1 - \mu_0\|$ (where $\|\cdot\|$ denotes total variation), and the signed measures $\mu_2 - \mu_1$ and $\mu_1 - \mu_0$ have disjoint supports, $\mu_2 - \mu_1 \perp \mu_1 - \mu_0$.

Proof. We may suppose that A is finely closed. Let

$$\nu_j := 1_{A_j}\mu_j, \qquad 0 \le j \le 3.$$

Then, by (2.3), for every $1 \le j \le 3$,

$$\nu_j = T^j\nu_0 \quad \text{and} \quad \|\nu_j\| = \|\nu_0\|. \tag{2.4}$$

By (2.1) and (2.2),

$$R\mu_0 = \varepsilon_{x_0}^{R(A_0\cup A_1)} = \varepsilon_{x_0}^{A_2\cup A_1} = \mu_1.$$

Defining $\nu_0' := 1_{A_1}\mu_0$ we have $\mu_0 = \nu_0 + \nu_0'$. Hence, by (2.2),

$$\mu_1 = R\mu_0 = R\nu_0 + R\nu_0' = T^2\nu_0 + \nu_0',$$

where $T^2\nu_0$ is supported by A_2 and ν_0' is supported by A_1. So $\nu_0' = \nu_1$, that is, $\mu_0 = \nu_0 + \nu_1$. By (2.3) and (2.4), we see that

$$\mu_j = \nu_j + \nu_{j+1}, \qquad 0 \le j \le 3,$$

where $\nu_4 := \nu_0$. Thus

$$\mu_0 + \mu_2 = \nu_0 + \nu_1 + \nu_2 + \nu_3 = \mu_1 + \mu_3.$$

By Lemma 2.2, the measures $\mu_0, \mu_1, \mu_2, \mu_3$ are pairwise different. Moreover,

$$\mu_1 - \mu_0 = \nu_2 - \nu_0 \quad \text{and} \quad \mu_2 - \mu_1 = \nu_3 - \nu_1.$$

So the signed measures $\mu_1 - \mu_0$ and $\mu_2 - \mu_1$ are orthogonal and, by (2.4),

$$\|\mu_1 - \mu_0\| = 2\|\nu_0\| = \|\mu_2 - \mu_1\|.$$

□

The following result is more subtle (note that the identity $\mu_0 = \nu_0 + T\nu_0$ in the proof of Theorem 2.3 is the special case, where $U = X$). Its immediate consequence is stronger and more useful than Theorem 2.3 (see Corollaries 2.5 and 3.2).

PROPOSITION 2.4. *There exists a measure σ on $A_0 \cup U^c$ such that*

$$\mu_0 = \sigma + T\sigma. \tag{2.5}$$

Proof. Again, we may suppose that $A \subset U$ and A is finely closed in U.

Let us fix $q_0 \in \mathcal{P}(X)$, $q > 0$, and define $q := q_0 + q_0 \circ T + q_0 \circ T^2 + q_0 \circ T^3$. Then $q \in \mathcal{P}(X)$, $q > 0$, and $q \circ T = q$. Let

$$\rho_0 := (1/2)\varepsilon_{x_0} \quad \text{and} \quad B := A_0 \cup A_1 \cup U^c$$

so that

$$\mu_0 = (\rho_0 + T\rho_0)^B \quad \text{and} \quad \rho_0(q) = q(x_0) < \infty. \tag{2.6}$$

We claim that there exist measures $\sigma_0, \sigma_1, \sigma_2, \dots$ on $A_0 \cup U^c$ and measures $\rho_1, \rho_2, \dots$ on A_2 such that, for every $n \in \{0\} \cup \mathbb{N}$,

$$(\rho_n + T\rho_n)^B = \sigma_n + T\sigma_n + (\rho_{n+1} + T\rho_{n+1})^B \quad \text{and} \quad \rho_{n+1}(q) \le \rho_n(q)/2. \tag{2.7}$$

Then, by (2.6) and (2.7), the proof will be finished taking $\sigma := \sum_{n=0}^{\infty} \sigma_n$.

To prove (2.7) we suppose that $n \in \{0\} \cup \mathbb{N}$ and that we have a measure ρ_n on $\{x_0\} \cup A_2$. Let

$$A_2' := A_0 \cup A_1 \cup A_3 \cup U^c, \qquad A_3' := A_0 \cup A_1 \cup A_2 \cup U^c,$$

$$\tau_n := \rho_n^{A_2'}|_{A_0 \cup U^c}, \quad \rho_{n+1} := (T\rho_n)^{A_3'}|_{A_2}, \quad \sigma_{n+1} = \tau_n + T^2\rho_{n+1}.$$

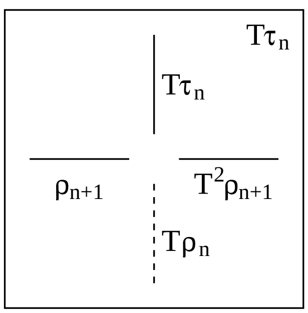

Figure 2. Decomposition of $(T\rho_n)^{A_3'}$

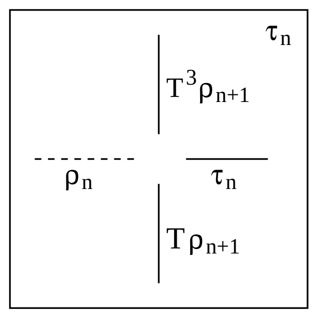

Figure 3. Decomposition of $\rho_n^{A_2'}$

We first observe that

$$(T\rho_n)^{A_3'} = \rho_{n+1} + T\tau_n + T^2\rho_{n+1} \quad \text{and} \quad \rho_n^{A_2'} = T^3\rho_{n+1} + \tau_n + T\rho_{n+1} \tag{2.8}$$

(see Figures 2 and 3). Indeed, by (2.1),

$$T(\rho_n^{A_2'}) = (T\rho_n)^{A_3'}, \tag{2.9}$$

and hence $T\tau_n$ is the part of $(T\rho_n)^{A_3'}$ on $T(A_0 \cup U^c) = A_1 \cup U^c$. Moreover, by (2.1) (with R in place of T),

$$(R(T\rho_n))^{R(A_3')} = R((T\rho_n)^{A_3'}). \tag{2.10}$$

The measure $T\rho_n$ is supported by $\{x_0\} \cup A_3$, and $R = I$ on $\{x_0\} \cup A_1 \cup A_3$, $R = T^2$ on $A_0 \cup A_2$. Therefore $R(T\rho_n) = T\rho_n$ and $R(A_3') = A_3'$. So (2.10) implies that $(T\rho_n)^{A_3'} = R((T\rho_n)^{A_3'})$ and hence

$$(T\rho_n)^{A_3'}|_{A_0} = R((T\rho_n)^{A_3'}|_{A_2}) = R\rho_{n+1} = T^2\rho_{n+1}.$$

So the first identity in (2.8) holds, and the second follows immediately, by (2.9).

The set A'_2 is the disjoint union of B and A_3, the set A'_3 is the disjoint union of B and A_2. Thus, by [1, VI.9.4] and (2.8) (see also Figures 2 and 3),

$$\rho_n^B = \rho_n^{A'_2}|_B + (\rho_n^{A'_2}|_{A_3})^B = T^3\rho_{n+1} + \tau_n + (T\rho_{n+1})^B,$$

$$(T\rho_n)^B = (T\rho_n)^{A'_3}|_B + ((T\rho_n)^{A'_3}|_{A_2})^B = T\tau_n + T^2\rho_{n+1} + \rho_{n+1}^B.$$

By definition, $\sigma_{n+1} = \tau_n + T^2\rho_{n+1}$. Hence we see that

$$(\rho_n + T\rho_n)^B = \sigma_{n+1} + T\sigma_{n+1} + (\rho_{n+1} + T\rho_{n+1})^B. \tag{2.11}$$

Finally, since $q \circ T = q$, we obtain that

$$2\rho_{n+1}(q) = (\rho_{n+1} + T^2\rho_{n+1})(q) \le (T\rho_n)^{A'_3}(q) \le (T\rho_n)(q) = \rho_n(q).$$

□

Let us fix a T-invariant measure τ on X such that the measure μ_0 is absolutely continuous with respect to τ and

$$\varphi_0 := \frac{d\mu_0}{d\tau} \in L^2(X,\tau)$$

(we shall not distinguish between functions in $\mathcal{L}^2(X,\tau)$ and equivalence classes in $L^2(X,\tau)$). Of course, a possible choice would be $\tau := \mu_0 + \mu_1 + \mu_2 + \mu_3$ and then $0 \le \varphi_0 \le 1$. In (i), (iii), and (iv) of the Examples 2.1, τ could be $(d-1)$-dimensional Hausdorff measure provided U has a smooth boundary.

Defining $\varphi_j := \varphi_0 \circ T^{-j}$ it is easily verified that, for every $1 \le j \le 3$,

$$\|\varphi_j\|_2 = \|\varphi_0\|_2 \quad \text{and} \quad \mu_j = T^j(\varphi_0\tau) = \varphi_j\tau.$$

On $L^2(X,\tau)$ we have the inner product

$$\langle\varphi,\psi\rangle := \int \varphi\psi\, d\tau,$$

the norm $\|\varphi\|_2 := \langle\varphi,\varphi\rangle^{1/2}$, and $\varphi \perp \psi$ if and only if $\langle\varphi,\psi\rangle = 0$.

COROLLARY 2.5. $\mu_0 + \mu_2 = \mu_1 + \mu_3$. *The measures* $\mu_0, \mu_1, \mu_2, \mu_3$ *are pairwise different, and the densities* $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ *form a square in* $L^2(X,\tau)$.

Proof. By Lemma 2.2, the measures $\mu_0, \mu_1, \mu_2, \mu_3$ are pairwise different. By Theorem 2.4, there exists a measure σ on $A_0 \cup U^c$ such that $\mu_0 = \sigma + T\sigma$. By (2.3),

$$\mu_0 + \mu_2 = \sigma + T\sigma + T^2\sigma + T^3\sigma = \mu_1 + \mu_3,$$

and hence

$$\varphi_0 + \varphi_2 = \varphi_3 + \varphi_1. \tag{2.12}$$

Of course, σ is absolutely continuous with respect to τ and

$$s := \frac{d\sigma}{d\tau} \le \varphi_0.$$

Defining $s_j := s \circ T^{-j}$ we obtain that, for every $0 \le j \le 3$,

$$T^j \sigma = s_j \tau \quad \text{and} \quad \varphi_j = s_j + s_{j+1}$$

(where, of course, $s_4 := s_0$). Moreover,

$$\psi_1 := \varphi_1 - \varphi_0 = s_2 - s_0 \quad \text{and} \quad \psi_2 := \varphi_2 - \varphi_1 = s_3 - s_1,$$

where, by the T-invariance of τ, $\langle s_2, s_0 \rangle = \langle s_3, s_1 \rangle$ and, for every $0 \le j \le 3$,

$$\|s_j\|_2 = \|s_0\|_2, \qquad \langle s_{j+1}, s_j \rangle = \langle s_1, s_0 \rangle.$$

This immediately implies that $\|\psi_1\|_2 = \|\psi_2\|_2$ and

$$\langle \psi_1, \psi_2 \rangle = \langle s_2, s_3 \rangle - \langle s_2, s_1 \rangle - \langle s_4, s_3 \rangle + \langle s_0, s_1 \rangle = 0.$$

Together with (2.12) this proves that $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ form a square in $L^2(X, \tau)$. □

Corollary 2.5 has the following immediate consequence.

COROLLARY 2.6. *$\mathcal{M}_{x_0}(S(U))$ is not a simplex.*

3 Result based on local symmetries

In the situation of a harmonic space (which excludes Riesz potentials) we may localize our assumptions.

THEOREM 3.1. *Assume that $(\tilde{X}, \tilde{\mathcal{W}})$ is a harmonic space (so that swept measures $\varepsilon_x^{B \cup U^c}$, $x \in U$, U open in $\tilde{X}$, $B \subset U$, are supported by $\overline{U}$) . Let W be an open neighborhood of $x_0 \in \tilde{X}$ and let $S(W)$ denote the set of all $\mathcal{P}(\tilde{X})$-bounded continuous functions on the closure of W which are ($\tilde{\mathcal{W}}$-)superharmonic on W.[1]*

Moreover, let us suppose that there exist open neighborhoods U and X of $x_0 \in \tilde{X}$ such that $x_0 \in U$, $\overline{U} \subset W \cap X$, and the assumptions of Section 2 are satisfied with respect to the restriction $(X, \mathcal{W})$ of $(\tilde{X}, \tilde{\mathcal{W}})$ on X.

Then $\mathcal{M}_{x_0}(S(W))$ is not a simplex.

Proof. It suffices to observe that balayage of ε_{x_0} on $A_j \cup U^c$ with respect to $(X, \mathcal{W})$ coincides with the balayage of ε_{x_0} on $A_j \cup U^c$ with respect to $(\tilde{X}, \tilde{\mathcal{W}})$. □

The following result for the classical case holds as well for the heat equation and for the harmonic structure given by the Laplace-Kohn operator on the Heisenberg group (or – more generally – by a sub-Laplacian on a stratified Lie group).

COROLLARY 3.2. *Let X be any non-empty open set in $\mathbb{R}^d$, $d \ge 2$, equipped with the classical harmonic structure (X^c non-polar, if $d = 2$), and let $x \in X$.*

Then none of the compact convex sets $\mathcal{M}_x(\mathcal{P}(X))$ and $\mathcal{M}_x(S(W))$, $x \in W$, W open in X, is a simplex.

[1]Let us note again that $S(\tilde{X}) = \mathcal{P}(\tilde{X})$.

4 Parabolic cases

In many parabolic cases, including the heat equation, we may prove that we do not get simplexes in a much simpler way, without using any symmetries.

Let us suppose that $(X, \mathcal{W})$ is a balayage space such that points are polar and $1 \in \mathcal{W}$ (for simplicity). Moreover, we assume the existence of a sequence (H_n) of pairwise disjoint transit sets in X such that $\bigcup_{n=1}^{\infty} H_n$ is finely dense in X. We recall from [3] that a subset H of X is called a *transit set* if it is closed and $\hat{R}_1^H = 0$ on H. If H is a transit set, then, for all (finite) measures ν on H, $A \subset H$, and $B \subset X$,

$$\nu^{A \cup B} = \nu^B, \tag{4.1}$$

since, for every $p \in \mathcal{P}(X)$, $\hat{R}_p^{A \cup B} \le \hat{R}_p^H + \hat{R}_p^B = \hat{R}_p^B$ on H.

EXAMPLES 4.1. 1. Let $X = \mathbb{R}^d \times \mathbb{R}$, $d \ge 1$, and let $(X, \mathcal{W})$ be the harmonic space given by the heat equation. (Unlike in the previous sections, here *one* space variable is sufficient.) Then every hyperplane $H_c := \{x \in X : x_{d+1} = c\}$, $c \in \mathbb{R}$, is a transit set, and the union of the hyperplanes H_c, c rational, is finely dense in X.

2. More generally, let $\mathbb{P} = (P_t)_{t>0}$ be a strong Feller right continuous sub-Markov semigroup on a locally compact space X' with countable base and let $\mathcal{W}$ be the set of all excessive functions with respect to $\mathbb{P} \otimes \mathbb{T}$ on $X := X' \times \mathbb{R}$, where $\mathbb{T} = (T_t)_{t>0}$ denotes the semigroup of uniform translation to the left, that is, $T_t(s, \cdot) = \varepsilon_{s-t}$, $s \in \mathbb{R}$. Then $(X, \mathcal{W})$ satisfies our assumptions (taking $H_c := X' \times \{c\}$) provided there exist strictly positive, continuous real functions $u, v \in \mathcal{W}$ such that u/v vanishes at infinity. See [3, Section 7] and [1, V.5.6] for details.

We recall that the previous example is the special case, where $X' = \mathbb{R}^d$ and $\mathbb{P}$ is the Brownian semigroup on $\mathbb{R}^d$.

THEOREM 4.2. *For all $x \in X$ and open neighborhoods U of x, there exist pairwise disjoint compact sets A_0, A_1, A_2, A_3 in U such that (taking $A_4 := A_0$) the harmonic measures $\mu_j := \varepsilon_x^{U \setminus (A_j \cup A_{j+1})}$, $0 \le j \le 3$, are pairwise different and $\mu_0 + \mu_2 = \mu_1 + \mu_3$.*

In particular, there is no $x \in X$ such that $\mathcal{M}_x(\mathcal{P}(X))$ or any $\mathcal{M}_x(S(W))$, $x \in W$, W open in X, is a simplex.

Proof. We fix an open set U in X and $x \in U$. There exists $n_0 \in \mathbb{N}$ such that $x \notin H_n$, $n \ge n_0$. Since $\rho_n := \varepsilon_x^{H_{n_0} \cup H_{n_0+1} \cup \cdots \cup H_n \cup U^c} \le \sum_{k=n_0}^n \varepsilon_x^{H_k \cup U^c}$ and $\lim_{n \to \infty} \rho_n = \varepsilon_x$, there is a transit set H (one of the sets H_n, $n \ge n_0$) such that

$$\sigma := \varepsilon_x^{H \cup U^c}|_{H \cap U} \ne 0.$$

Since points of X are polar and σ does not not charge polar sets, there are pairwise disjoint compact sets A_0, A_1, A_2, A_3 in H such that $\sigma(A_j) > 0$, $j \in \{0, 1, 2, 3\}$. Let

$$\nu := \varepsilon_x^{H \cup U^c}|_{U^c} \quad \text{and} \quad \sigma_j := 1_{A_j} \sigma, \qquad 0 \le j \le 3,$$

so that $\sigma = \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3$. By [1, VI.9.4] and (4.1), for every compact set A in $H \cap U$,

$$\varepsilon_x^{A \cup U^c} = \varepsilon_x^{H \cup U^c}|_{A \cup U^c} + (\varepsilon_x^{H \cup U^c}|_{(H \cap U) \setminus A)})^{A \cup U^c} = \varepsilon_x^{H \cup U^c}|_{A \cup U^c} + (\sigma - 1_A \sigma)^{U^c}.$$

So, for every $0 \le j \le 3$,

$$\mu_j = \varepsilon_x^{A_j \cup A_{j+1} \cup U^c} = \sigma_j + \sigma_{j+1} + \nu + (\sigma - (\sigma_j + \sigma_{j+1}))^{U^c},$$

whence $1_{H\cap U}\mu_j = \sigma_j + \sigma_{j+1}$. Thus the measures $\mu_0, \mu_1, \mu_2, \mu_3$ are pairwise different and

$$\mu_0 + \mu_2 = \sigma + 2\nu + \sigma^{U^c} = \mu_1 + \mu_3.$$

□

References

[1] J. Bliedtner and W. Hansen. *Potential Theory – An Analytic and Probabilistic Approach to Balayage*. Universitext. Springer, Berlin-Heidelberg-New York-Tokyo, 1986.

[2] W. Hansen and I. Netuka. Convexity properties of harmonic measures. *Adv. Math.*, 218(4):1181–1223, 2008.

[3] W. Hansen and I. Netuka. Density of extremal measures in parabolic potential theory. *Math. Ann.*, 2009 (DOI 10.1007/s00208-009-0371-3).

[4] G. Mokobodzki. Éléments extrémaux pour le balayage. In *Séminaire de Théorie du Potentiel, dirigé par M. Brelot, G. Choquet et J. Deny (1969/70), Exp. 5*, page 14. Secrétariat Math., Paris, 1971.

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