# One-radius results for supermedian functions on $\mathbb{R}^d$, $d \le 2$

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**Abstract**

A classical result states that every lower bounded superharmonic function on $\mathbb{R}^2$ is constant. In this paper the following (stronger) one-circle version is proven. If $f\colon \mathbb{R}^2 \to (-\infty,\infty]$ is lower semicontinuous, $\liminf_{|x|\to\infty} f(x)/\ln|x| \ge 0$, and, for every $x \in \mathbb{R}^2$, $1/(2\pi)\int_0^{2\pi} f(x+r(x)e^{it})\,dt \le f(x)$, where $r\colon \mathbb{R}^2 \to (0,\infty)$ is continuous, $\sup_{x\in\mathbb{R}^2}(r(x)-|x|) < \infty$, and $\inf_{x\in\mathbb{R}^2}(r(x)-|x|) = -\infty$, then $f$ is constant.

Moreover, it is shown that, assuming $r \le c|\cdot| + M$ on $\mathbb{R}^d$, $d \le 2$, and taking averages on $\{y \in \mathbb{R}^d\colon |y-x| \le r(x)\}$, such a result of Liouville type holds for supermedian functions if and only if $c \le c_0$, where $c_0 = 1$, if $d = 2$, whereas $2.50 \le c_0 \le 2.51$, if $d = 1$.

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## 1 Introduction and results

It is a well-known fact that every lower bounded superharmonic function on $\mathbb{R}^2$ is constant. We recall that superharmonic functions $u$ on $\mathbb{R}^2$ are lower semicontinuous functions on $\mathbb{R}^2$ such that $u > -\infty$, $u \not\equiv \infty$, and, for every circle $S(x,\rho)$ of center $x \in \mathbb{R}^2$ and radius $\rho > 0$, the average $\sigma_{x,\rho}(u)$ of $u$ on $S(x,\rho)$ is at most $u(x)$. In this note, we shall present the following stronger result (where, as usual, we do not distinguish between $\mathbb{C}$ and $\mathbb{R}^2$).

**Theorem 1.1.** *Let $r$ be a strictly positive real function on $\mathbb{R}^2$ such that*

(i) *$r$ is continuous,*

(ii) $\limsup_{|x|\to\infty}(r(x)-|x|) < \infty$,

(iii) $\liminf_{|x|\to\infty}(r(x)-|x|) = -\infty$.[1]

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[1] Having (i), properties (ii),(iii) are equivalent to $\sup_{x\in\mathbb{R}^2}(r(x)-|x|) < \infty$, $\inf_{x\in\mathbb{R}^2}(r(x)-|x|) = -\infty$, respectively.

*Let $f > -\infty$ be a lower semicontinuous numerical function on $\mathbb{R}^2$ such that*

$$\liminf_{|x|\to\infty} f(x)/\ln|x| \ge 0 \tag{1.1}$$

*and $f$ is $(\sigma, r)$-supermedian, that is,*

$$\sigma_{x,r(x)}(f) := \frac{1}{2\pi}\int_0^{2\pi} f(x + r(x)e^{it})\,dt \;\le\; f(x) \qquad (x \in \mathbb{R}^2). \tag{1.2}$$

*Then $f$ is constant.*

**Remarks 1.2.** 1. Obviously, $r\colon \mathbb{R}^2 \to (0,\infty)$ has the properties (i) – (iii), if there exists $L \in (0,1)$ such that, for all $x, y \in \mathbb{R}^2$,

$$|r(x) - r(y)| \le L\,|x - y|. \tag{1.3}$$

However, assuming only that $|r(x) - r(y)| < |x - y|$ (which implies (i) and (ii)), even the conclusion breaks down. Indeed, if $f := (1 - |x|)^+$ and $r := |x| + 2 + (|x| + 1)^{-1}$, then, for all $x, y \in \mathbb{R}^2$, $\sigma_{x,r(x)}(f) = 0 \le f(x)$ and the inequality $|r(x) - r(y)| < |x - y|$ holds, since $(|x| + 1)^{-1} > (|y| + 1)^{-1}$ if $|x| < |y|$. In fact, none of the properties (i), (ii), (iii) may be dropped (see Section 3). Moreover, since the function $-\ln(|\cdot|^2 + 1)$ is superharmonic, it is clear that (1.1) cannot be replaced by $\liminf_{|x|\to\infty} f(x)/\ln|x| > -\infty$.

2. In Theorem 1.1, we may just as well assume that $f$ does not attain the value $\infty$ (it suffices to consider the functions $f_n := \min\{f, n\}$, $n \in \mathbb{N}$, which are $(\sigma, r)$-supermedian provided $f$ is $(\sigma, r)$-supermedian; if these functions are constant, then $f := \lim_{n\to\infty} f_n$ is constant).

Let us assume, for a moment, that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and $(\sigma, r)$-*median*, that is, such that

$$\sigma_{x,r(x)}(f) = f(x) \qquad (x \in \mathbb{R}^2).$$

P.C. Fenton [2] showed that $f$ has to be constant provided $f$ is lower bounded, $r$ is continuous and, for some $x_0 \in \mathbb{R}^2$, the set $\{x \in \mathbb{R}^2\colon r(x) > |x - x_0|\}$ is bounded, a requirement which may be replaced by the weaker property (ii) (see Remark 2.1). If $f$ is bounded, then (ii) alone (without any further assumption on $r$) is sufficient to conclude that $f$ is constant ([7, Theorem 1.1], cf. also [5]). On the other hand, there exist $r\colon \mathbb{R}^2 \to (0,\infty)$ and a continuous $(\sigma, r)$-median function $f$ on $\mathbb{R}^2$ such that $r \le 4(|\cdot| + 1)$ and $\min f(\mathbb{R}^2) = 0$, $\max f(\mathbb{R}^2) = 1$ (see [5, Proposition 6.1] or [7, Section 5]).

An essential step for the strong version [7, Theorem 1.1] of Liouville's theorem consists in proving that, assuming (ii), every lower semicontinuous $(\sigma, r)$-supermedian function $f \ge 0$ on $\mathbb{R}^2$ attains a minimum. It immediately implies that constant functions are the only lower semicontinuous, lower bounded functions $f$ on $\mathbb{R}^2$ which are $(\lambda, r)$-*supermedian*, that is, which have the property that, for every $x \in \mathbb{R}^2$, the average $\lambda_{x,r(x)}(f)$ of $f$ on the (closed) disk $B(x, r(x))$ is at most $f(x)$ (see [7, Corollary 6.1]). We recall that $(\lambda, r)$-supermedian functions are $(\sigma, \tilde r)$-supermedian for some function $0 < \tilde r \le r$ (cf. [7, Section 6]). The following result shows that the existence of a minimum fails, if (ii) is replaced by an inequality $r \le c|\cdot| + M$, where $c > 1$.

**Proposition 1.3.** *Let $c > 1$, $M > 0$, and $r(x) := \max\{c|x|, M\}$, $x \in \mathbb{R}^2$. Then the functions $r^{-\alpha}$ on $\mathbb{R}^2$ are $(\sigma, r)$-supermedian (and $(\lambda, r)$-supermedian) provided $\alpha > 0$ is sufficiently small.*

So, assuming that $r \le c|\cdot| + M$, where $c, M \in (0,\infty)$, a result of Liouville type for $(\lambda, r)$-supermedian functions on $\mathbb{R}^2$ holds if and only if $c \le 1$.

On the real line, this will turn out to be strikingly different. By the following proposition, such a result of Liouville type on $\mathbb{R}$ holds if and only if $c \le c_0$, where $c_0 \in [2.50, 2.51]$ is the unique solution to the equation

$$(t+1)\ln(t+1) + (t-1)\ln(t-1) = 2t$$

in $(1,\infty)$ (see Section 5).

**Proposition 1.4.** 1. *Let $r\colon \mathbb{R} \to (0,\infty)$ and $M > 0$ such that*

$$r(x) \le c_0|x| + M \qquad \textit{for all } x \in \mathbb{R} \setminus [-M, M]. \tag{1.4}$$

*Then every lower semicontinuous $(\lambda, r)$-supermedian function $f > -\infty$ on $\mathbb{R}$ which is lower bounded (or satisfies $\liminf_{|x|\to\infty} f(x)/\ln|x| \ge 0$) is constant.*

2. *If, however, $c > c_0$, $M > 0$, and $r := \max(c|\cdot|, M)$, then the function $r^{-\alpha}$ is $(\lambda, r)$-supermedian provided $\alpha > 0$ is sufficiently small.*

**Remark 1.5.** As in the 2-dimensional case, the condition $\liminf_{|x|\to\infty} f(x)/\ln|x| \ge 0$ cannot be replaced by $\liminf_{|x|\to\infty} f(x)/\ln|x| > -\infty$. Indeed, let $r(x) := c_0(|x|+1)$ and $f(x) := -\ln(|x|+1)$, $x \in \mathbb{R}$. We recall that $\int_0^t f(s)\,ds = t - (t+1)\ln(t+1)$, $t \ge 0$. If $x \ge 0$, then

$$\begin{aligned} a &:= (c_0+1)(x+1) = x + r(x) + 1, \\ b &:= (c_0-1)(x+1) < r(x) - x + 1 = |x - r(x)| + 1, \end{aligned}$$

and hence

$$\lambda_{x,r(x)}(f) < 1 - \frac{1}{2c_0(x+1)}(a\ln a + b\ln b) = f(x).$$

By symmetry, $\lambda_{x,r(x)}(f) < f(x)$ holds as well if $x < 0$.

Finally, let us recall that every bounded harmonic function on $\mathbb{R}^d$, $d \ge 1$, is constant. Hence the results of Liouville type, which we discussed until now, are special cases of one-radius results for harmonic functions on open sets $U$. The trivial requirement that $r$ be at most the distance to $U^c$, if $U \ne \mathbb{R}^d$, implies the existence of a real $M > 0$ such that

$$r \le |\cdot| + M \tag{1.5}$$

(which may justify considering (1.5) as a natural assumption on $r$ in the case $U = \mathbb{R}^d$).

One radius-results for harmonic functions have a long history (see the survey papers [11, 3] and the references therein). If $U$ is an arbitrary open set in $\mathbb{R}^d$, then every continuous $(\lambda, r)$-median functions $f\colon U \to \mathbb{R}$ admitting a (sub)harmonic minorant and a (super)harmonic majorant is harmonic (provided (1.5) holds, if $U = \mathbb{R}^d$).

Let us now return to continuous bounded $(\sigma, r)$-median functions. If $d = 1$, the corresponding result fails almost trivially both for $\mathbb{R}$ and $(-1,1)$ (in the first case consider $f(x) := \sin x$ and $r(x) := 2\pi$, for the interval see e.g. [1, Section IV.3], cf. also [9]). We already mentioned the positive result for $U = \mathbb{R}^2$. On the unit disk, however, there exists a continuous function $0 \le f \le 1$ having the one-circle property which is not harmonic (see [8] and [6]). The corresponding *general* problems for $\mathbb{R}^d$, $d \ge 3$, are unsolved, both for the open unit ball and the entire space (if, however, $r$ is Lipschitz with constant $L \in (0,1)$, the answer is positive [10, Theorem 2]).

## 2 Proof of Theorem 1.1

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be lower semicontinuous such that (1.1) holds and $f$ is $(\sigma, r)$-supermedian, where $r\colon \mathbb{R}^2 \to (0,\infty)$ satisfies (i), (ii), and (iii). For all $x \in \mathbb{R}^2$ and $\rho > 0$, let

$$B(x,\rho) := \{y \in \mathbb{R}^2 \colon |y-x| \le \rho\} \quad \text{and} \quad S(x,\rho) := \{y \in \mathbb{R}^2 \colon |y-x| = \rho\}.$$

By (ii), there is a real $M > 0$ such that

$$r(x) \le |x| + M \qquad \text{for all } x \in B(0,M)^c.$$

If $f$ is lower bounded, then, by [5, Proposition 2.1] (see also [7]),

$$f(x_0) \le f \quad \text{for some } x_0 \in B(0, M+2). \tag{2.1}$$

In fact, a short look at the proof for (2.1) reveals that it is valid as well under our weaker assumption (1.1). So we may suppose without loss of generality that $f(0) = 0$ and $f \ge 0$ (we can replace $f$ by the function $x \mapsto f(x_0+x) - f(x_0)$). Since $f$ is lower semicontinuous, we know, by (1.2), that

$$f = 0 \text{ on } S(x, r(x)), \quad \text{whenever } x \in \mathbb{R}^2 \text{ such that } f(x) = 0. \tag{2.2}$$

Let us use the technique developed in [2]. We define an increasing sequence $(\alpha_n)$ of continuous real functions on the unit circle $S := S(0,1)$ by $\alpha_0 := r(0)$ and

$$\alpha_n(u) := \alpha_{n-1}(u) + r(\alpha_{n-1}(u)u) \qquad (n \in \mathbb{N}).$$

By induction, we conclude from (2.2) that, for all $u \in S$ and $n \in \mathbb{N}$,

$$f(\alpha_n(u)u) = 0. \tag{2.3}$$

Since $r$ is continuous and strictly positive, we obtain immediately that $\lim \alpha_n = \infty$. So there exists $n \in \mathbb{N}$ such that

$$\alpha_n > M \quad \text{on } S.$$

For the moment, let us fix $u \in S$. We claim that

$$f(\alpha u) = 0 \quad \text{for every } \alpha \ge \alpha_n(u). \tag{2.4}$$

Indeed, suppose that (2.4) does not hold. Then there exists a maximal real $a$ such that $a \ge \alpha_n(u)$ and $f(\alpha u) = 0$ for every $\alpha \in [\alpha_n(u), a]$. We may join the points $y_0 := au$ and $y_1 := -\alpha_n(-u)u$ continuously by an arc $\gamma : [0,1] \to \mathbb{R}^2$ contained in the set

$$\{\alpha u \colon a \ge \alpha \ge \alpha_n(u)\} \cup \{\alpha_n(v)v \colon v \in S\}. \tag{2.5}$$

In particular, $f \circ \gamma = 0$ by (2.3). Fix $0 < \beta \le r(y_0)$ and let $z := y_0 + \beta u$ so that $|z - y_0| \le r(y_0)$. Clearly, $|z - y_1| \ge r(y_1)$, since the origin is contained in the line segment from $y_1$ to $y_0$, $r(y_1) \le |y_1| + M$, and $a + \beta \ge \alpha_n(u) > M$. By continuity of $r$, there exists $s \in [0,1]$ such that $|z - \gamma(s)| = r(\gamma(s))$. Since $f(\gamma(s)) = 0$, we conclude by (2.2) that $f(z) = 0$. Thus $a \ge a + r(y_0)$, a contradiction proving (2.4).

Fixing $R > 0$ such that $\alpha_n \le R$ on $S$, we therefore know that $f = 0$ on $\mathbb{R}^2 \setminus B(0,R)$. Since $\liminf_{|x|\to\infty}(r(x) - |x|) = -\infty$, there exists $x \in \mathbb{R}^2$ such that $|x| - r(x) > R$, and hence

$$f = 0 \quad \text{on } B(x, r(x)). \tag{2.6}$$

Suppose that there is a point $y \in \mathbb{R}^2$ such that $f(y) > 0$. We define

$$t := \sup\{s \in [0,1] \colon f(sx + (1-s)y) > 0\} \quad \text{and} \quad z := tx + (1-t)y.$$

Since $r(z) > 0$, there exists a point $\tilde{y} \in [y,z]$ such that $|\tilde{y} - z| < r(z)$ and $f(\tilde{y}) > 0$. By (2.6), $|\tilde{y} - x| > r(x)$. By continuity of $r$, we conclude that there exists $\tilde{z} \in (z,x)$ such that $|\tilde{y} - \tilde{z}| = r(\tilde{z})$. So $f(\tilde{z}) > 0$, by (2.2). However, by definition of $z$, $f = 0$ on $(z,x)$. Thus there is no point $y \in \mathbb{R}^2$ such that $f(y) > 0$, that is, $f$ is identically zero, and the proof of Theorem 1.2 is finished.

**Remark 2.1.** If $f$ is even continuous and $(\sigma, r)$-median, then (iii) is not needed to conclude that $f$ is constant.

Indeed, it suffices to observe that (iii) has not been used to obtain that $f = \inf f(\mathbb{R}^2)$ outside a compact set, and hence $\gamma := \sup f(\mathbb{R}^2) < \infty$. Then, just using (i) and (ii), we get as well that $\gamma - f = \inf(\gamma - f)(\mathbb{R}^2)$ outside a compact set, that is, $f = \sup f(\mathbb{R}^2)$ outside a compact set. Thus $\inf f(\mathbb{R}^2) = \sup f(\mathbb{R}^2)$, $f$ is constant.

# 3 Examples

Simple examples show that a continuous bounded $(\sigma, r)$-supermedian function $f$ on $\mathbb{R}^2$ may be non-constant, if any of the properties (i), (ii), or (iii) of $r$ is violated.

1. Let $f := (1 - |x|)^+$, $x \in \mathbb{R}^2$. Taking $r(x) := 3$, if $|x| < 2$, and $r(x) := 1$, if $|x| \ge 2$, we observe that, of course, property (i), that is, the continuity of $r$, cannot be omitted (or replaced by lower semicontinuity). Considering $r := |x| + 2 + (|x| + 1)^{-1}$ we already noted in Remark 1.2.1 that (iii) cannot be dropped (of course, for this purpose, it would be sufficient to take $r(x) := |x| + 2$).

2. Finally, let us prove that the conclusion of Theorem 1.1 fails, if property (ii) is omitted. For $x \in \mathbb{R}^2$, let

$$f(x) := \min\{1, |x_1|^{-1}\} \quad \text{and} \quad r(x) := 6 \max\{1, x_1^2\}.$$

Clearly, $0 \le f \le 1$, $f$ and $r$ are continuous functions, and $r$ satisfies (iii), since $r(0,t) = 6$ for every $t \in \mathbb{R}$. To prove that (1.2) holds, we fix $x \in \mathbb{R}^2$ and define

$$a := \max\{|x_1|, 1\}, \qquad A := \{y \in \mathbb{R}^2 \colon |y_1| \le 2a\}.$$

Then $f(x) = a^{-1}$. We shall see that

$$\sigma_{x,r(x)}(A) \le a^{-1}/2 \tag{3.1}$$

and hence

$$\sigma_{x,r(x)}(f) \le \sigma_{x,r(x)}(A) + \sup f(\mathbb{R}^2 \setminus A) \le a^{-1}/2 + a^{-1}/2 = f(x).$$

To prove (3.1) let $\alpha$ denote the maximal angle between the $x_2$-axis and the lines connecting $x$ with one of the four points $y \in S(x, r(x))$ satisfying $|y_1| = 2a$. Then

$$\sigma_{x,r(x)}(A) \le 4\frac{\alpha}{2\pi} = \frac{2}{\pi}\alpha \le \sin\alpha \le \frac{3a}{r(x)}.$$

If $|x_1| \le 1$, then $a = 1$, $r(x) = 6$, and hence $3a/r(x) = 1/2 = 1/(2a)$. If $|x_1| > 1$, then $a = |x_1|$, $r(x) = 6x_1^2$, and hence $3a/r(x) = 1/(2|x_1|) = 1/(2a)$. Thus (3.1) holds

A closer look would reveal that $f$ is $\tilde{r}$-superharmonic with respect to a continuous function $0 < \tilde{r} \le r$ satisfying $\tilde{r}(x) = \tilde{r}(|x_1|, 0)$ and

$$\lim_{t\to\infty} \frac{\tilde{r}(t,0)}{t\ln t} = \frac{2}{\pi}. \tag{3.2}$$

This follows from the fact that, given $t \ge 1$ and $k \in \mathbb{N}$, the point $x := (t, 0)$ and the set $\tilde{A} := \{y \in \mathbb{R}^2 \colon |y_1| \le kt\}$ satisfy

$$\int_{\mathbb{R}^2\setminus\tilde{A}} f\, d\sigma_{x,\rho} \le \sup f(\mathbb{R}^2 \setminus \tilde{A}) \le \frac{1}{k} f(x) \qquad \text{for every } \rho > 0,$$

and, for large $\rho$,

$$\int_{\tilde{A}} f\, d\sigma_{x,\rho} \sim 4 \cdot \frac{1}{2\pi\rho} \int_1^{kt} \frac{1}{\tau}\, d\tau = \frac{2\ln(kt)}{\pi\rho} \sim \frac{2}{\pi} \cdot \frac{t\ln t}{\rho} f(x)$$

(which, incidentally, shows that the limit behavior in (3.2) is optimal for our function $f$).

# 4 Proof of Proposition 1.3

Let $c > 1$, $M > 0$, and $r := \max\{c|\cdot|, M\}$ so that, for every $\alpha > 0$,

$$r^{-\alpha}(x) = \min\{|cx|^{-\alpha}, M^{-\alpha}\}, \qquad x \in \mathbb{R}^2.$$

We define

$$I(\alpha) := \frac{1}{2\pi} \int_0^{2\pi} |1 + ce^{it}|^{-\alpha}\, dt, \qquad \alpha \ge 0.$$

Then $I(0) = 1$ and

$$I'(0) = -\frac{1}{2\pi} \int_0^{2\pi} \ln|1 + ce^{it}|\, dt = -\ln c < 0.$$

So there exists $\alpha_0 > 0$ such that $I < 1$ on $(0, \alpha_0]$. Let us fix $\alpha \in (0, \alpha_0]$ and $x \in \mathbb{R}^2$. If $c|x| > M$, then $r(x) = c|x|$ and hence

$$\sigma_{x,r(x)}(r^{-\alpha}) \le \sigma_{x,r(x)}(c^{-\alpha}|\cdot|^{-\alpha}) = \frac{c^{-\alpha}}{2\pi} \int_0^{2\pi} |x + c|x|e^{it}|^{-\alpha}\, dt = |cx|^{-\alpha} I(\alpha) < r^{-\alpha}(x).$$

If $c|x| \le M$, then $r(x) = M$, and hence $\sigma_{x,r(x)}(r^{-\alpha}) \le M^{-\alpha} = r^{-\alpha}(x)$. Thus $r^{-\alpha}$ is $(\sigma, r)$-supermedian.

Since $\lambda_{x,\rho} = 2\rho^{-2} \int_0^\rho \sigma_{x,s}\, s ds$ (and $\int_0^{2\pi} \ln|1 + ce^{is}|\, ds = 2\pi \ln 1 = 0$, if $s \in (0, 1)$), we obtain similarly that $r^{-\alpha}$ is $(\lambda, r)$-supermedian provided $\alpha > 0$ is sufficiently small.

# 5 Proof of Proposition 1.4

Let us define

$$\psi(t) := (t+1)\ln(t+1) + (t-1)\ln(t-1) - 2t, \qquad 1 < t < \infty.$$

Then $\psi$ is continuous, $\lim_{t\to 1}\psi(t) = 2\ln 2 - 2 < 0$, $\lim_{t\to\infty}\psi(t) = \infty$. Moreover,

$$\psi'(t) = \ln(t+1) + \ln(t-1) = \ln(t^2-1),$$

hence $\psi$ is strictly decreasing on $(0, \sqrt{2}\,)$ and strictly increasing on $(\sqrt{2}, \infty)$. So there exists $c_0 \in (1,\infty)$ such that $\psi(c_0) = 0$,

$$\psi < 0 \text{ on } (1, c_0), \quad \text{ and } \quad \psi > 0 \text{ on } (c_0, \infty).$$

In fact, $2,50 < c_0 < 2,51$ (since $\psi(2.50) < 0$ and $\psi(2.51) > 0$).

1. Let $r\colon \mathbb{R} \to (0,\infty)$ and $M > 0$ such that

$$r(x) \le c_0|x| + M \qquad \text{for all } x \in \mathbb{R}\setminus[-M, M].$$

Let $\varphi := \ln^+(|\cdot| - M)$ (so that $\varphi(x) = 0$, if $-(M+1) \le x \le M+1$). We claim that there exists $\tilde M > 0$ such that, for every $x \in \mathbb{R}\setminus[\tilde M, -\tilde M]$,

$$\lambda_{x,r(x)}(\varphi) \le \varphi(x). \tag{5.1}$$

Since $\lim_{x\to\infty}(x/M)\ln\bigl(1-(M/x)\bigr) = -\ln' 1 = -1$, there exists $\tilde M \ge 1 + c_0 + 2M$ such that

$$M\ln(x-M) + c_0 x \ln\frac{x-M}{x} - 1 > 0 \quad \text{ for every } x > \tilde M. \tag{5.2}$$

For a while, let us fix $x \in \mathbb{R}\setminus[-\tilde M, \tilde M]$. To prove (5.1) we may assume, by symmetry, that $x$ is positive. Let $y \in (0, c_0 x + M)$. If $x - y \ge M + 1$, then

$$\varphi(x-y) + \varphi(x+y) \le 2\varphi(x), \tag{5.3}$$

since $\varphi$ is concave on $(M+1, \infty)$. If $M + 1 > x - y \ge -(M+1)$, then (5.3) holds, since $\varphi(x-y) = 0$ and $x + y - M \le x(1+c_0) \le (x-M)^2$. Therefore (5.1) holds, if $x - r(x) \ge -(M+1)$.

Let us assume next that $x - r(x) < -(M+1)$. Then $t := (r(x) - M)/x \in (1, c_0]$,

$$\int_0^{r(x)\pm x} \varphi(s)\,ds = \int_{M+1}^{(t\pm1)x+M} \varphi(s)\,ds$$
$$= \int_1^{(t\pm1)x} \ln s\,ds = (t\pm1)x\ln[(t\pm1)x] - (t\pm1)x + 1.$$

Since $\psi(t) \le \psi(c_0) = 0$, we hence see that

$$\int_{x-r(x)}^{x+r(x)} \varphi(s)\,ds = \psi(t)x + 2tx\ln x + 2 \le 2tx\ln x + 2,$$

that is, $r(x)\lambda_{x,r(x)}(\varphi) \le tx\ln x + 1$. Thus

$$\begin{aligned} r(x)(\varphi(x) - \lambda_{x,r(x)}(\varphi)) &\ge (tx+M)\ln(x-M) - (tx\ln x + 1) \\ &= M\ln(x-M) + tx\ln\frac{x-M}{x} - 1, \end{aligned}$$

where the right side is positive by (5.2), since $t \le c_0$. This finishes the proof of (5.1).

Now let $f > -\infty$ be a lower semicontinuous $(\lambda, r)$-supermedian function on $\mathbb{R}$ such that $\liminf_{|x|\to\infty} f(x)/\ln|x| \ge 0$. To prove that $f$ is constant, we use ideas from [4] and [7]. There exists $x_0 \in [-\tilde{M}, \tilde{M}]$ such that

$$f(x_0) = \inf f([-\tilde{M}, \tilde{M}]).$$

We intend to show that $f \ge f(x_0)$ on $\mathbb{R}$. Then the lower semicontinuity of $f$ and the inequalities $\lambda_{x,r(x)}(f) \le f(x)$, $x \in \mathbb{R}$, will imply that the set $A := \{x \in \mathbb{R} \colon f(x) = f(x_0)\}$ is both closed and open, hence $A = \mathbb{R}$, $f = f(x_0)$.

Fixing $\varepsilon > 0$, it suffices to prove that

$$\tilde{f} := f + \varepsilon\varphi \ge f(x_0).$$

Obviously, $\tilde{f}$ is lower semicontinuous, lower bounded, and $\lim_{|x|\to\infty} \tilde{f}(x) = \infty$. Therefore $\tilde{f}$ attains a minimum on $\mathbb{R}$, and the non-empty set

$$\tilde{A} := \{x \in \mathbb{R} \colon \tilde{f}(x) = \inf \tilde{f}(\mathbb{R})\}$$

is closed. Let $z \in \tilde{A}$ with minimal absolute value. If $|z| > \tilde{M}$, then $\lambda_{z,r(z)}(\tilde{f}) \le \tilde{f}(z)$, and hence $[z - r(z), z + r(z)] \subset \tilde{A}$. This is impossible, by our choice of $z$. Thus $z \in [-\tilde{M}, \tilde{M}]$, and $\tilde{f} \ge \tilde{f}(z) \ge f(z) \ge f(x_0)$.

2. Finally, let $c > c_0$, $M > 0$, and $r := \max(c|\cdot|, M)$. We have to show that the function $r^{-\alpha}$ is $(\lambda, r)$-supermedian provided $\alpha > 0$ is sufficiently small. To that end we define

$$\Psi(\beta) := (c+1)^\beta + (c-1)^\beta - 2\beta c, \qquad 0 < \beta < \infty.$$

Then $\Psi'(\beta) = (c+1)^\beta \ln(c+1) + (c-1)^\beta \ln(c-1) - 2c$. In particular, $\Psi'(1) = \psi(c) > 0$. So there exists $\alpha \in (0,1)$ such that $\Psi(1-\alpha) < 0$.

Let us now fix $x \in \mathbb{R}$. If $c|x| \le M$, then $r(x) = M$, and hence $\lambda_{x,r(x)}(r^{-\alpha}) \le M^{-\alpha} = r^{-\alpha}(x)$. So let us assume that $c|x| < M$ and hence $r(x) = c|x|$. Then

$$\int_{x-r(x)}^{x+r(x)} |s|^{-\alpha}\,ds = \frac{1}{1-\alpha}|x|^{1-\alpha}\big((c+1)^{1-\alpha} + (c-1)^{1-\alpha}\big) \quad \text{and} \quad 2r(x)|x|^{-\alpha} = 2c|x|^{1-\alpha}.$$

Since $\Psi(1-\alpha) < 0$, we conclude that $\lambda_{x,r(x)}(|\cdot|^{-\alpha}) \le |x|^{-\alpha}$ and hence

$$\lambda_{x,r(x)}(r^{-\alpha}) \le \lambda_{x,r(x)}(c^{-\alpha}|\cdot|^{-\alpha}) \le |cx|^{-\alpha} = r^{-\alpha}(x).$$

Thus $r^{-\alpha}$ is $(\lambda, r)$-supermedian.

**Remark 5.1.** If even $r \le c|\cdot| + M$ for some $c \in (0, c_0)$, then the conclusion in (1) of Proposition 1.4 is still valid, if the condition $\liminf_{|x|\to\infty} f(x)/\ln|x| \ge 0$ is replaced by the weaker assumption $\liminf_{|x|\to\infty} f(x)/\ln|x| > -\infty$. Indeed, by means of the function $\Psi$, we may then prove that, for some $\alpha > 0$, the function $|\cdot|^\alpha$ (which will replace $\varphi$) is $(\lambda, r)$-supermedian.

# References


[1] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. II.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

[2] P. C. Fenton. On sufficient conditions for harmonicity. *Trans. Amer. Math. Soc.*, 253:139–147, 1979.

[3] W. Hansen. Restricted mean value property and harmonic functions. In *Potential theory—ICPT 94 (Kouty, 1994)*, pages 67–90. de Gruyter, Berlin, 1996.

[4] W. Hansen and N. Nadirashvili. Restricted mean value property on $\mathbb{R}^d$, $d \leq 2$. *Expo. Math.*, 13:93–95, 1995.

[5] W. Hansen. A Liouville property for spherical averages in the plane. *Math. Ann.*, 319:539–551, 2001.

[6] W. Hansen. Littlewood's one-circle problem, revisited. *Expo. Math.*, 26(4):365–374, 2008.

[7] W. Hansen. A strong version of Liouville's theorem. *Amer. Math. Monthly*, 115(7):583–595, 2008.

[8] W. Hansen and N. Nadirashvili. Littlewood's one circle problem. *J. London Math. Soc. (2)*, 50(2):349–360, 1994.

[9] W. Hansen and N. Nadirashvili. Harmonic functions and averages on shells. *J. Anal. Math.*, 84:231–241, 2001.

[10] D. Heath. Functions possessing restricted mean value properties. *Proc. Amer. Math. Soc.*, 29:588–595, 1973.

[11] I. Netuka and J. Veselý. Mean value property and harmonic functions. In *Classical and modern potential theory and applications (Chateau de Bonas, 1993)*, volume 430 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 359–398. Kluwer Acad. Publ., Dordrecht, 1994.



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