Combinatorial and Geometric Aspects of the Representation Theory of Finite Group Schemes

Rolf Farnsteiner

Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn Str. 4, 24098 Kiel, Germany

E-mail address: rolf@math.uni-kiel.de

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Introduction

These notes grew out of a series of lectures given at East China Normal University (Shanghai) in July of 2006. The primary objective of my talks was two-fold: to provide a first introduction to the theories of Hopf algebras and algebraic groups on the one hand, and to delineate the rôle of certain combinatorial and geometric techniques within their representation theory on the other. Accordingly, our exposition emphasizes the discussion of illustrative examples and plausibility considerations highlighting the salient points of the theory. Proofs have been included primarily when they serve this purpose, and they often only sketch the main ideas rather than embarking on detailed discussions.

These notes are exclusively concerned with the "classical" theory of cocommutative Hopf algebras, leaving the very active field of quantum groups out of the account. Our Hopf algebras will also appear in a geometric guise, as affine algebraic group schemes. Since our point of view veils the historical sources of the subject matter, let me briefly mention that the axioms of our algebras first occurred in the work of the topologist Heinz Hopf [**72**], defining what are now called graded Hopf algebras.

Using broad brushstrokes the first chapter provides a quick tour of Hopf algebras, their associated group schemes, and restricted Lie algebras. Since the selection of the topics is dictated by what is relevant for the later sections, many interesting aspects have been left out of the account. The reader who is interested in a more in depth introduction may consult [**81, 89**] for the abstract theory of Hopf algebras, and [**80, 123**] for group schemes. The techniques involved in the representation theory of infinitesimal groups are often inductive, that is, one reduces problems to the consideration of "small" groups, whose structure is well enough understood to be amenable to the methods from the abstract representation theory of Artin algebras. The reduction process involves subgroups as well as quotients, and §5 provides a brief introduction to the latter. For almost all of our purposes, it suffices to know that the Hopf algebras associated to quotients are in fact quotient algebras. Accordingly, the notion of faithfully flat ring extensions, that is necessary for a more conceptual understanding of quotients, has been omitted.

In Chapter II we introduce two important tools that are defined in the wider context of self-injective algebras. The complexity of a module measures the polynomial rate of growth of its minimal projective resolution. We explore its relationship with the growth of certain Ext-groups and exhibit special features for Hopf algebras. The Heller operator is shown to induce a bijection on the set of isoclasses of non-projective indecomposable modules, leading to a necessary cohomological condition for an algebra to have finite representation type. The study of the algebras belonging to this class was the origin of many interesting developments in modern representation theory, such as the Auslander-Reiten theory of almost split sequences or tilting theory. Representation-finite algebras are one constituent of a trichotomy, whose other parts are the tame and wild algebras, which we briefly discuss in the final section.

In the context of finite group schemes, the complexity can also be interpreted as the dimension of a certain affine variety, the cohomological support variety of a module. These varieties have been available for modules over group algebras of finite groups for quite some time. The extension to arbitrary cocommutative Hopf algebras is based on the fundamental work by Friedlander-Suslin

[**63**], who showed that the cohomology ring of a finite group scheme is finitely generated. We discuss representation theoretic support spaces, which, via the notion of a p-point, provide a non-cohomological interpretations of support varieties. Generalizing features from rank varieties, p-points are flat homomorphisms $k[X]/(X^p) \longrightarrow k\mathcal{G}$, whose associated pull-back functors $\operatorname{mod} k\mathcal{G} \longrightarrow \operatorname{mod} k[X]/(X^p)$ are used to study $\mathcal{G}$-modules.

As a first application, we show that modules over finite group schemes of tame representation type have complexity at most 2. For Frobenius kernels of smooth groups, the combination of rank varieties with basic results on nilpotent orbits yields the determination of the representation-finite and tame blocks. In fact, the latter are Morita equivalent to tame blocks of the first Frobenius kernel of SL(2). In particular, these blocks are special biserial and of domestic representation type. Section 5 is devoted to a summary of recent progress concerning support spaces. The notion of a π-point, set forth by Friedlander-Pevtsova in [**61**], allows the geometric study of group schemes over arbitrary fields and also includes infinite-dimensional modules.

In Chapter IV we introduce another geometric tool, given by varieties of diagonalizable subgroups. These are essential to understanding the structural impact of conditions on rank varieties of restricted Lie algebras. For Lie algebras of smooth groups, much information is derived from the so-called root space decomposition associated to a maximal torus. Since any two such tori are mapped onto each other by an automorphism of the ambient algebra, any maximal torus will do. By contrast, for arbitrary restricted Lie algebras the information encoded in the root space decomposition is highly sensitive to the choice of the torus. Schemes of tori (diagonalizable groups of height ≤ 1) are introduced to study all tori simultaneously and to consider algebraic families of Lie algebras.

Chapter V is concerned with the combinatorial tool given by the representation theory of bound quivers. We quickly review the fundamental results of Gabriel and Nazarova, Donovan-Freislich concerning basic algebras and the representation type of quivers. Representations of quivers correspond to modules over hereditary algebras, a class that substantially differs from algebras of measures. The notion of a trivial extension affords the passage from hereditary algebras to self-injective algebras. The concomitant doubling process involving the corresponding Gabriel quivers turns out to match the formation of the so-called McKay quivers, which we introduce in Section 5.

The final Chapter combines the aforementioned techniques to study those finite group schemes, whose principal blocks have finite or tame representation type. We begin by reviewing the classical case concerning semi-simple Hopf algebras, which is covered by Nagata's Theorem on linearly reductive group schemes. In Section 2 we collect basic results on tensor products and explain the connection between the Ext-quivers and the McKay quivers of certain algebras. The latter arise in conjunction with the classification of the linearly reductive finite subgroup schemes of SL(2), the so-called binary polyhedral group schemes.

These notes are preliminary in the sense that they do not address the recent developments on π-points and Auslander-Reiten theory. Nevertheless, I feel that the current exposition may provide the interested reader with the background necessary to follow these emerging directions.

Finally, it is a pleasure to thank the Mathematics Department of East China Normal University in general and Professor Bin Shu in particular for their hospitality during my stay in Shanghai.

CHAPTER I

Finite Group Schemes

1. Basic Properties of Hopf Algebras

Throughout these notes k will denote a field. The standard example of a Hopf algebra is the group algebra kG of some abstract group G. Recall that the G-modules correspond to the kG-modules. Accordingly, the module categories of group algebras enjoy special features that those of ordinary algebras do not have: If M and N are G-modules, then the vector spaces $M \otimes_k N$ and $\mathrm{Hom}_k(M, N)$ carry the structure of a G-module by setting

$$g\boldsymbol{.}(m \otimes n) := g.m \otimes g.n \quad \text{and} \quad (g\boldsymbol{.}\varphi)(m) := g\varphi(g^{-1}m)$$

for every $g \in G$, $m \in M$, $n \in N$, $\varphi \in \mathrm{Hom}_k(M, N)$, respectively.

Since these structures extend to kG, we should try to understand them without reference to G. Let's first look at tensor products: The spaces M and N are kG-modules, so $M \otimes_k N$ carries the structure of a $kG \otimes_k kG$-module by defining

$$(a \otimes b)\boldsymbol{.}(m \otimes n) := a.m \otimes b.n \quad \forall\ a, b \in kG,\ m \in M,\ n \in N.$$

Note that the map

$$G \times G \longrightarrow kG \otimes_k kG \quad ; \quad (g, h) \mapsto g \otimes h$$

induces an isomorphism $k(G \times G) \cong kG \otimes_k kG$. Hence the diagonal map $g \mapsto (g, g)$ gives rise to an algebra homomorphism $\Delta : kG \longrightarrow kG \otimes_k kG$. By definition, the kG-module structure on $M \otimes_k N$ is the pull-back of the $kG \otimes_k kG$-structure along Δ.

In order to understand the kG-structure of $\mathrm{Hom}_k(M, N)$, we observe that the map $g \mapsto g^{-1}$ induces an isomorphism $\eta : kG \longrightarrow kG^{\mathrm{op}}$ from kG to its *opposite algebra* kG^{op}. The space $\mathrm{Hom}_k(M, N)$ obtains the structure of a $kG \otimes_k kG^{\mathrm{op}}$-module via

$$((a \otimes b)\boldsymbol{.}\varphi)(m) := a\varphi(bm) \quad \forall\ a, b \in kG,\ m \in M,\ \varphi \in \mathrm{Hom}_k(M, N).$$

Our G-structure corresponds to the pull-back of this structure along the algebra homomorphism $(\mathrm{id}_{kG} \otimes \eta) \circ \Delta$.

The maps Δ and η have various properties that will be listed in the definition below. One obvious relation is $(\Delta \otimes \mathrm{id}_{kG}) \circ \Delta = (\mathrm{id}_{kG} \otimes \Delta) \circ \Delta$, ensuring that the natural k-linear identification $(X \otimes_k Y) \otimes_k Z \cong X \otimes_k (Y \otimes_k Z)$ is an isomorphism of kG-modules.

Unless mentioned otherwise, a *k-algebra* Λ is meant to be associative with an identity element that acts on all (left) modules via the identity operator. We will occasionally write $m : \Lambda \otimes_k \Lambda \longrightarrow \Lambda$ for the multiplication map. Given a k-algebra Λ, a Λ-module M, and k-linear maps $\varphi : V \longrightarrow \Lambda$, $\psi : W \longrightarrow M$ originating in some k-spaces V, W, we denote by $\varphi \hat{\otimes} \psi : V \otimes_k W \longrightarrow M$ the linear map given by $(\varphi \hat{\otimes} \psi)(v \otimes w) := \varphi(v)\psi(w)\ \ \forall\ v \in V,\ w \in W$.

Definition. Let H be a k-algebra, $\Delta : H \longrightarrow H \otimes_k H$, $\varepsilon : H \longrightarrow k$, and $\eta : H \longrightarrow H$ be k-linear maps. We say that (H, Δ, ε) is a *bialgebra* if

(1) Δ and ε are homomorphisms of k-algebras,
(2) $(\Delta \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \Delta) \circ \Delta$ (co-associativity),
(3) $(\mathrm{id}_H \hat{\otimes} \varepsilon) \circ \Delta = \mathrm{id}_H = (\varepsilon \hat{\otimes} \mathrm{id}_H) \circ \Delta$ (counit).

If, in addition, we have

(4) $(\eta \hat{\otimes} \mathrm{id}_H) \circ \Delta = \varepsilon\, 1 = (\mathrm{id}_H \hat{\otimes} \eta) \circ \Delta$,

then $(H, \Delta, \varepsilon, \eta)$ is referred to as a *Hopf algebra*.

Remarks. (i) If $H = kG$ is the group algebra of a group G, then ε is the unique homomorphism such that $\varepsilon(g) = 1 \ \ \forall\, g \in G$.

(ii) If one writes down the axioms for a k-algebra H as commutative diagrams involving the multiplication m and the canonical map $k \longrightarrow H$, then (2) and (3) follow by dualizing these diagrams, that is, by making all arrows point in the opposite direction.

(iii) When dealing with the *comultiplication* Δ it is convenient to use the so-called Heyneman-Sweedler notation

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

For instance, (3) and (4) read

$$\sum_{(h)} h_{(1)}\varepsilon(h_{(2)}) = h = \sum_{(h)} \varepsilon(h_{(1)})h_{(2)}$$

and

$$\sum_{(h)} \eta(h_{(1)})h_{(2)} = \varepsilon(h) = \sum_{(h)} h_{(1)}\eta(h_{(2)}),$$

respectively. The main advantage lies in the treatment of iterated coproducts:

$$((\Delta \otimes \mathrm{id}_H) \circ \Delta)(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = ((\mathrm{id}_H \otimes \Delta) \circ \Delta)(h).$$

The identity $((\varepsilon \hat{\otimes} \mathrm{id}_H) \otimes \mathrm{id}_H) \circ (\Delta \otimes \mathrm{id}_H) \circ \Delta = \Delta$ then simply reads

$$\sum_{(h)} \varepsilon(h_{(1)})h_{(2)} \otimes h_{(3)} = \sum_{(h)} h_{(1)} \otimes h_{(2)}$$

for every $h \in H$.

(iv) The maps ε and η are called the *counit* and the *antipode* of the Hopf algebra H, respectively. The antipode is a homomorphism $H \longrightarrow H^{\mathrm{op}}$. It is bijective whenever H is finite-dimensional (cf. [**102**]).

(v) The ideal $H^\dagger := \ker \varepsilon$ is customarily referred to as the *augmentation ideal* of the Hopf algebra H.

An algebra homomorphism $f : H \longrightarrow H'$ between two bialgebras is a *bialgebra homomorphism* if $(f \otimes f) \circ \Delta = \Delta' \circ f$ and $\varepsilon' \circ f = \varepsilon$. If H and H' are Hopf algebras with antipodes η and η', respectively, then a *Hopf algebra homomorphism* additionally satisfies $\eta' \circ f = f \circ \eta$.

An element $g \neq 0$ of a Hopf algebra H is called *group-like* if $\Delta(g) = g \otimes g$. The set $G(H)$ of group-like elements of H is a subgroup of the group of units of H, the inverse of $g \in G(H)$ being given by $\eta(g)$. Moreover, $G(H) \subseteq H$ is a linearly independent subset, so that $G(H)$ is finite whenever H has finite dimension.

We say that $x \in H$ is *primitive*, provided $\Delta(x) = x \otimes 1 + 1 \otimes x$. The set $\mathrm{Lie}(H)$ of primitive elements is a subspace, that is closed under the Lie bracket $[x, y] := xy - yx$ of H. In other words, $\mathrm{Lie}(H)$ is a Lie subalgebra of the commutator algebra H^- of the associative algebra H.

A subalgebra $H' \subseteq H$ of a Hopf algebra H is referred to as a *Hopf subalgebra* provided $\Delta(H') \subseteq H' \otimes_k H'$ and $\eta(H') \subseteq H'$. An ideal I of a bialgebra H is called a *bi-ideal* if $\Delta(I) \subseteq H \otimes_k I + I \otimes_k H$ and $\varepsilon(I) = (0)$. In that case, the factor space H/I canonically obtains the structure of a bialgebra.

If, in addition, H is a Hopf algebra and $\eta(I) \subseteq I$, then I is called a *Hopf ideal* and H/I inherits the Hopf algebra structure from H.

EXAMPLES. (1) Let T be an indeterminate over k. We can endow the polynomial ring $k[T]$ and its localization $k[T]_T$ at T with the following Hopf algebra structures:

(a) $\Delta(T) = T \otimes 1 + 1 \otimes T$, $\varepsilon(T) = 0$, $\eta(T) = -T$.

(b) $\Delta(T) = T \otimes_k T$, $\varepsilon(T) = 1$, $\eta(T) = T^{-1}$.

Note that the Hopf algebra $k[T]_T$ is the group algebra $k\mathbb{Z}$ of the group $\mathbb{Z}$ of integers.

(2) Let $n \in \mathbb{N}$, and consider the bialgebra $k[\mathrm{Mat}_n] := k[X_{ij}, 1 \leq i, j \leq n]$, whose comultiplication and counit are given by

$$\Delta(X_{ij}) := \sum_{\ell=1}^{n} X_{i\ell} \otimes X_{\ell j} \quad ; \quad \varepsilon(X_{ij}) = \delta_{ij},$$

respectively. Note that the polynomial $\det((X_{ij}))$ belongs to $G(k[\mathrm{Mat}_n])$.

(3) Given $q \in k \setminus \{0\}$, we consider the k-algebra $k_q[k^2] := k\langle x, y\rangle/(yx - qxy)$. Then $k_q[k^2]$ has a bialgebra structure given by

$$\Delta(x) = x \otimes 1 + y \otimes x, \quad \Delta(y) = y \otimes y, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 1.$$

This bialgebra is often referred to as *Manin's quantum plane.*

Let H be a bialgebra, Λ be any k-algebra. Given linear maps $\varphi, \psi : H \longrightarrow \Lambda$, we define the *convolution* $\varphi * \psi : H \longrightarrow \Lambda$ of φ and ψ via

$$(\varphi * \psi)(h) := \sum_{(h)} \varphi(h_{(1)})\psi(h_{(2)}) \qquad \forall\ h \in H.$$

LEMMA 1.1. *Let H be a Hopf algebra. Then the following statements hold:*

(1) *The dual space H^* is a k-algebra with multiplication given by convolution.*

(2) *If H is finite-dimensional, then H^* is a Hopf algebra with comultiplication*

$$\Delta^*(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)} :\Leftrightarrow f(ab) = \sum_{(f)} f_{(1)}(a) f_{(2)}(b) \quad \forall\ a, b \in H,$$

and counit and antipode defined via $\varepsilon^*(f) = f(1)$ and $\eta^*(f) = f \circ \eta$, respectively.

PROOF. We illustrate the existence of an identity element: Note that

$$(\varepsilon * f)(h) = \sum_{(h)} \varepsilon(h_{(1)}) f(h_{(2)}) = \sum_{(h)} f(\varepsilon(h_{(1)})h_{(2)}) = f(h) \quad \forall\ h \in H.$$

Consequently, ε is the identity element of H^*. □

Suppose that $\dim_k H < \infty$. The group-like elements of H^* are the *characters* of H, and the group $X(H) := G(H^*)$ of k-algebra homomorphisms $H \longrightarrow k$ is referred to as the *character group* of H. The space $\mathrm{Lie}(H^*)$ consists of the *derivations* $H \longrightarrow k$, that is, of all linear maps ψ such that $\psi(ab) = \psi(a)\varepsilon(b) + \varepsilon(a)\psi(b) \ \ \forall\ a, b \in H$.

EXAMPLE. Consider the Hopf algebra $k[T]$ with

$$\Delta(T) = T \otimes 1 + 1 \otimes T, \quad \varepsilon(T) = 0, \quad \eta(T) = -T.$$

Suppose that $\mathrm{char}(k) = p > 0$. Given $r \in \mathbb{N}$, the binomial formula yields $\Delta(T^{p^r}) = T^{p^r} \otimes 1 + 1 \otimes T^{p^r}$. Accordingly, (T^{p^r}) is a Hopf ideal, and $k[\mathbb{G}_{a(r)}] := k[T]/(T^{p^r})$ has the structure of a Hopf algebra.

We put $t := T + (T^{p^r})$, so that $k[\mathbb{G}_{a(r)}]$ has basis $\{t^i \ ; \ 0 \leq i \leq p^r - 1\}$. Let $\{\delta_i \ ; \ 0 \leq i \leq p^r - 1\}$ be the dual basis within $k\mathbb{G}_{a(r)} := k[\mathbb{G}_{a(r)}]^*$. Then we have

$$(\delta_i * \delta_j)(t^m) = \begin{cases} \binom{i+j}{i} & i + j = m \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $\delta_i * \delta_j = \binom{i+j}{i}\delta_{i+j}$, where the right-hand term is understood to be zero whenever $i + j \notin \{0, \dots, p^r - 1\}$. This readily yields $\delta_i^p = 0$ for $i \geq 1$, and it follows that the map $X_i \mapsto \delta_{p^i}$ induces an isomorphism

$$k[X_0, \dots, X_{r-1}]/(X_0^p, \dots, X_{r-1}^p) \longrightarrow k\mathbb{G}_{a(r)}$$

of k-algebras. Thus, as an algebra, $k\mathbb{G}_{a(r)}$ is isomorphic to the group algebra of the p-elementary abelian group $(\mathbb{Z}/(p))^r$. However, the Hopf algebras $k\mathbb{G}_{a(r)}$ and $k(\mathbb{Z}/(p))^r$ are not isomorphic: The dual of $k\mathbb{G}_{a(r)}$ is local (its group scheme is *infinitesimal*), while the dual of $k(\mathbb{Z}/(p))^r$ is separable (its group scheme is *étale*).

Let M be a k-vector space. The linear map

$$\tau_M : M \otimes_k M \longrightarrow M \otimes_k M \quad ; \quad m \otimes n \mapsto n \otimes m$$

is called the *flip*.

Definition. A bialgebra H is called *cocommutative* if $\tau_H \circ \Delta = \Delta$.

Note that a bialgebra is cocommutative exactly when its dual algebra is commutative.

Examples. (1) The Hopf algebras kG, $k[T]$, $k[T]_T$, and $k\mathbb{G}_{a(r)}$ are cocommutative.
(2) The bialgebras $k_q[k^2]$ and $k[\mathrm{Mat}_n]$ are not cocommutative.

Lemma 1.2. *Let H be a Hopf algebra. Then the following statements hold:*
(1) *If $H' \subseteq H$ is a finite-dimensional subalgebra such that $\Delta(H') \subseteq H' \otimes_k H'$, then H' is a Hopf subalgebra of H.*
(2) *If H is finite-dimensional and $I \subsetneq H$ is an ideal such that $\Delta(I) \subseteq I \otimes_k H + H \otimes_k I$, then I is a Hopf ideal of H.*

Proof. (1) Let $f \in \mathrm{End}_k(H')$ be a k-linear map. By assumption, we have $(f * \mathrm{id}_{H'})(h) = \sum_{(h)} f(h_{(1)})h_{(2)} \in H'$ for every $h \in H'$. Hence

$$\psi : \mathrm{End}_k(H') \longrightarrow \mathrm{End}_k(H') \quad ; \quad f \mapsto f * \mathrm{id}_{H'}$$

is a well-defined linear map. Given $f \in \ker \psi$ and $h \in H'$, we obtain, observing $\Delta(H') \subseteq H' \otimes_k H'$,

$$f(h) = \sum_{(h)} f(h_{(1)})\varepsilon(h_{(2)}) = \sum_{(h)} f(h_{(1)})h_{(2)}\eta(h_{(3)}) = \sum_{(h)} \psi(f)(h_{(1)})\eta(h_{(2)}) = 0,$$

so that ψ is injective. Since H' is finite-dimensional, there exists an element $g \in \mathrm{End}_k(H')$ such that

$$g * \mathrm{id}_{H'} = \varepsilon|_{H'}1.$$

A computation similar to the one above then implies $g(h) = \eta(h)$ for every $h \in H'$, so that $\eta(H') \subseteq H'$. Consequently, H' is a Hopf subalgebra.

(2) We consider $U := \{f \in \mathrm{End}_k(H) \ ; \ f(I) \subseteq I\}$ as well as

$$\psi : \mathrm{End}_k(H) \longrightarrow \mathrm{End}_k(H) \quad ; \quad f \mapsto f * \mathrm{id}_H \, .$$

Since I is an ideal satisfying $\Delta(I) \subseteq I \otimes_k H + H \otimes_k I$, we have $\psi(U) \subseteq U$. As ψ is injective and U has finite dimension, there exists $g \in U$ such that $g * \mathrm{id}_H = \varepsilon$. Consequently, $\eta = g \in U$, so that $\eta(I) \subseteq I$. As a result,

$$\varepsilon \cdot 1 = \eta * \mathrm{id}_H = \psi(\eta) \in U,$$

implying $\varepsilon(x)1 \in I$ for every $x \in I$. Thus, $I \neq H$ is contained in $\ker \varepsilon$. □

We continue by recording an important homological property of Hopf algebras. Recall that a finite-dimensional k-algebra Λ is referred to as a *Frobenius algebra* if it admits a nondegenerate bilinear form $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k$ such that

$$(ab, c) = (a, bc) \qquad \forall\ a, b, c \in \Lambda.$$

Forms with the latter property are referred to as *associative.* Associative forms correspond to linear maps $\lambda : H \longrightarrow k$, the correspondence being given by

$$(a, b)_\lambda := \lambda(ab) \qquad \forall\ a, b \in \Lambda.$$

The associative form $(\ ,\ )$ is usually not symmetric. Its departure from symmetry is measured by the *Nakayama automorphism* $\mu : \Lambda \longrightarrow \Lambda$ that is given by

$$(b, a) = (\mu(a), b) \qquad \forall\ a, b \in \Lambda.$$

Our next results show that Hopf algebras are Frobenius algebras affording a Nakayama automorphism of finite order.

Given a Hopf algebra H, we let $\int_H^\ell := \{x \in H\ ;\ hx = \varepsilon(h)x \ \ \forall\ h \in H\}$ be the space of *left integrals* of H. Analogously, $\int_H^r := \{x \in H\ ;\ xh = \varepsilon(h)x \ \ \forall\ h \in H\}$ is the space of *right integrals* of H.

Theorem 1.3 ([**118, 85**]). *Let H be a finite-dimensional Hopf algebra.*

(1) *We have* $\dim_k \int_H^\ell = 1$.

(2) *If $\lambda \in \int_{H^*}^\ell \setminus \{0\}$, then $(\ ,\ )_\lambda$ is a nondegenerate, associative form.* □

Owing to (1), there exists an algebra homomorphism $\zeta_\ell : H \longrightarrow k$ such that $xh = \zeta(h)x$ for every $h \in H$ and $x \in \int_H^\ell$. The function ζ_ℓ is called the *left modular function* of H. The unique algebra homomorphism $\zeta_r : H \longrightarrow k$ with $hx = \zeta(h)x$ for every $h \in H$ and $x \in \int_H^r$ is referred to as the *right modular function* of H. Note that

$$\zeta_\ell = \zeta_r \circ \eta^{-1}.$$

Given an algebra homomorphism (a character) $\gamma : H \longrightarrow k$, the map $\mathrm{id}_H * \gamma$ is readily seen to be an automorphism of the algebra H. In fact,

$$X(H) \longrightarrow \mathrm{Aut}_k(H) \ \ ; \ \ \gamma \mapsto \mathrm{id}_H * \gamma$$

is group homomorphism from the character group $X(H)$ to the automorphism group $\mathrm{Aut}_k(H)$ of the algebra H. In particular, the automorphisms $\mathrm{id}_H * \gamma$ have finite order.

Proposition 1.4 ([**92, 56**]). *Let H be a finite-dimensional Hopf algebra with antipode η and left modular function ζ_ℓ. Then $\eta^{-2} \circ (\mathrm{id}_H * \zeta_\ell)$ is a Nakayama automorphism of H.*

PROOF. Let $\pi \in \int^{\ell}_{H^*}$ be a non-zero left integral of H^*, so that

$$(x,y)_\pi = \pi(xy)$$

endows H with the structure of a Frobenius algebra. Directly from the defining property of π, we obtain

$$(*) \qquad \pi(h)1 = \sum_{(h)} \pi(h_{(2)})h_{(1)} \qquad \forall\ h \in H.$$

Since the map $H \longrightarrow H^*$; $h \mapsto \pi.h$ is an isomorphism of right H-modules, there exists a unique element $u_\pi \in H$ such that

$$\pi.u_\pi = \varepsilon.$$

In other words, we have

$$\pi(u_\pi h) = \varepsilon(h) \qquad \forall\ h \in H.$$

In view of $\pi((u_\pi h - \varepsilon(h)u_\pi)x) = 0$ for all $h, x \in H$, we see that $u := u_\pi$ is a non-zero right integral of H. Given $x \in H$, we now obtain, observing $(*)$,

$$\begin{aligned}\sum_{(u)} \eta(u_{(1)})\pi(u_{(2)}x) &= \sum_{(u),(x)} \eta(u_{(1)})u_{(2)}x_{(1)}\pi(u_{(3)}x_{(2)}) = \sum_{(u),(x)} \varepsilon(u_{(1)})x_{(1)}\pi(u_{(2)}x_{(2)}) \\ &= \sum_{(x)} x_{(1)}\pi(ux_{(2)}) = \sum_{(x)} x_{(1)}\varepsilon(x_{(2)}) = x.\end{aligned}$$

Thus, letting μ be the Nakayama automorphism relative to $(\ ,\ )_\pi$, we have

$$\mu^{-1}(x) = \sum_{(u)} \eta(u_{(1)})\pi(u_{(2)}\mu^{-1}(x)) = \sum_{(u)} \pi(xu_{(2)})\eta(u_{(1)}).$$

Using $(*)$ again, we compute $\eta^{-2} \circ \mu^{-1}$:

$$\begin{aligned}(\eta^{-2} \circ \mu^{-1})(x) &= \sum_{(u)} \pi(xu_{(2)})\eta^{-1}(u_{(1)}) = \sum_{(u),(x)} \pi(x_{(2)}u_{(3)})x_{(1)}u_{(2)}\eta^{-1}(u_{(1)}) \\ &= \sum_{(u),(x)} \pi(x_{(2)}u_{(2)})x_{(1)}\varepsilon(u_{(1)}) = \sum_{(x)} \pi(x_{(2)}u)x_{(1)} = \sum_{(x)} \zeta_r(x_{(2)})\pi(u)x_{(1)} \\ &= \sum_{(x)} \zeta_r(x_{(2)})x_{(1)} = (\mathrm{id}_H * \zeta_r)(x).\end{aligned}$$

It follows that

$$\mu = (\mathrm{id}_H * \zeta_r)^{-1} \circ \eta^{-2} = (\mathrm{id}_H * (\zeta_r \circ \eta)) \circ \eta^{-2} = \eta^{-2} \circ (\mathrm{id}_H * (\zeta_r \circ \eta^{-1})) = \eta^{-2} \circ (\mathrm{id}_H * \zeta_\ell),$$

as desired. □

If H is cocommutative, then $\eta^2 = \mathrm{id}_H$, so that $\mathrm{id}_H * \zeta$ is a Nakayama automorphism of finite order. Since the antipode has finite order, this result also holds in the general case [**102**]. Integrals and modular functions are usually hard to compute. If $H = kG$ is the group algebra of a finite group G, then the element $\sum_{g \in G} g$ is an integral, and ε is the modular function. In particular, group algebras are *symmetric* algebras (i.e., they afford a symmetric, nondegenerate associative form). By contrast, the Hopf algebras associated to arbitrary finite group schemes are usually not symmetric.

In our prefatory remarks we emphasized the fact that tensor products of modules over a Hopf algebra H are also H-modules. Given H-modules M and N, we define H-module structures on

$M \otimes_k N$ and $\mathrm{Hom}_k(M,N)$ in analogy with the special case kG:

$$h.(m \otimes n) := \sum_{(h)} h_{(1)}m \otimes h_{(2)}n \qquad \forall\ m \in M,\ n \in N,\ h \in H$$

and

$$(h.\varphi)(m) := \sum_{(h)} h_{(1)}\varphi(\eta(h_{(2)})m) \qquad \forall\ m \in M,\ \varphi \in \mathrm{Hom}_k(M,N),\ h \in H.$$

Let me indicate the utility of these concepts by giving a result that illustrates the way in which tensor products can be exploited.

LEMMA 1.5. *Let H be a finite-dimensional Hopf algebra, P be a projective H-module, M be an arbitrary H-module. Then $P \otimes_k M$ is projective and injective.*

PROOF. Recall that the functors $\mathrm{Hom}_k(P \otimes_k M, -)$ and $\mathrm{Hom}_k(P, \mathrm{Hom}_k(M,-))$ are naturally equivalent. Direct computation shows that this equivalence induces an equivalence

$$\mathrm{Hom}_H(P \otimes_k M, -) \cong \mathrm{Hom}_H(P, \mathrm{Hom}_k(M,-)).$$

Consequently, the left-hand functor is, as the composite of two exact functors, exact. Thus, $P \otimes_k M$ is projective, and since H is a Frobenius algebra, $P \otimes_k M$ is also injective. □

Using (1.5) one obtains identities for Ext-groups such as

$$\mathrm{Ext}^n_H(M,N) \cong \mathrm{Ext}^n_H(k, \mathrm{Hom}_k(M,N)) \qquad \forall\ n \geq 0,$$

where the right-hand groups coincide with the Hochschild cohomology groups of the augmented algebra (H, ε) with coefficients in the H-module $\mathrm{Hom}_k(M,N)$.

We conclude our survey of basic properties by quoting a Hopf algebra freeness theorem that was first verified in [**92**] for cocommutative Hopf algebras.

THEOREM 1.6 ([**91**]). *Let K be a Hopf subalgebra of the finite-dimensional Hopf algebra H. Then H is a free left and right K-module.* □

2. Group Schemes

In this section we introduce the geometric interpretation of the theory of cocommutative Hopf algebras. For a more thorough discussion we refer to [**80**] and [**123**]. Throughout, M_k and Ens will denote the categories of commutative k-algebras and sets, respectively. A functor $\mathcal{X} : M_k \longrightarrow \mathrm{Ens}$ is called a *k-functor.* The k-functors we will primarily be interested in are the so-called *affine schemes*: given a commutative k-algebra A, we consider the k-functor

$$\mathrm{Spec}_k(A) : M_k \longrightarrow \mathrm{Ens} \ \ ; \quad R \mapsto \mathrm{Alg}_k(A,R),$$

where $\mathrm{Alg}_k(A,R)$ is the set of k-algebra homomorphisms from A to R. An affine scheme is called *algebraic* if A is a finitely generated k-algebra. Accordingly, a k-functor $\mathcal{X}$ is affine algebraic if and only if there exist polynomials $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$ such that

$$\mathcal{X}(R) = \{(x_1, \ldots, x_n) \in R^n \ ; \ f_i(x_1, \ldots, x_n) = 0 \ , \ 1 \leq i \leq m\},$$

for every commutative k-algebra R. Thus, the *k-rational points* $\mathcal{X}(k)$ of the affine algebraic schemes are the affines varieties from classical algebraic geometry. Note, however, that the *coordinate ring* $k[\mathcal{X}]$ of an affine scheme $\mathcal{X} = \mathrm{Spec}_k(k[\mathcal{X}])$ may have nilpotent elements.

Group functors take values in the category Gr of groups. In order to see the connection with Hopf algebras, we require the following basic result:

THEOREM 2.1 (Yoneda's Lemma). *Let A, B be two commutative k-algebras. The assignment $\Phi \mapsto \Phi_A(\mathrm{id}_A)$ is a bijection between the set of natural transformations $\mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ and $\mathrm{Alg}_k(B, A)$. In other words, for every natural transformation $\Phi : \mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ there exists a unique homomorphism $\varphi : B \longrightarrow A$ of k-algebras such that $\Phi_R(\lambda) = \lambda \circ \varphi$ for every $\lambda \in \mathrm{Alg}_k(A, R)$ and $R \in M_k$.*

PROOF. Each natural transformation $\Phi : \mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ is determined by $\Phi_A(\mathrm{id}_A) \in \mathrm{Spec}_k(B)(A)$: If $\lambda : A \longrightarrow R$ is a homomorphism of commutative k-algebras, then

$$\Phi_R(\lambda) = \Phi_R(\lambda \circ \mathrm{id}_A) = \Phi_R(\mathrm{Spec}_k(A)(\lambda)(\mathrm{id}_A)) = \mathrm{Spec}_k(B)(\lambda)(\Phi_A(\mathrm{id}_A)) = \lambda \circ \Phi_A(\mathrm{id}_A).$$

Since the above equation defines a natural transformation, our result follows. □

Let $\Phi : \mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ be a natural transformation. The corresponding homomorphism $B \longrightarrow A$ is often denoted Φ^* and referred to as the *comorphism* associated to Φ. In view of (2.1) Φ^* is uniquely determined by

$$\Phi_R(\lambda) = \lambda \circ \Phi^* \qquad \forall\ \lambda \in \mathrm{Spec}_k(A)(R),\ R \in M_k.$$

Let A and B be commutative k-algebras. Then there is a natural equivalence

$$\Pi : \mathrm{Spec}_k(A) \times \mathrm{Spec}_k(B) \longrightarrow \mathrm{Spec}_k(A \otimes_k B)$$

sending a pair (x, y) of algebra homomorphisms with values in R to the unique map

$$x \hat{\otimes} y : A \otimes_k B \longrightarrow R \ \ ; \ \ a \otimes b \mapsto x(a)y(b).$$

DEFINITION. A k-functor $\mathcal{G} : M_k \longrightarrow \mathrm{Gr}$ from M_k to the category Gr of groups is called a *k-group functor*. We say that $\mathcal{G}$ is an *affine group scheme* if the k-functor $\mathcal{G}$ is affine. If the representing algebra $k[\mathcal{G}]$ is finitely generated, then the affine group scheme is called *algebraic*. In that case $\mathcal{G}$ is often referred to as an *(affine) algebraic k-group*.

Directly from the definition we obtain that
(a) the multiplication $(m_R : \mathcal{G}(R) \times \mathcal{G}(R) \longrightarrow \mathcal{G}(R))_{R \in M_k}$, and
(b) the inverse map $(\iota_R : \mathcal{G}(R) \longrightarrow \mathcal{G}(R))_{R \in M_k}$
are natural transformations.

EXAMPLES. (1) Consider the k-group functor $\mathrm{GL}_n : M_k \longrightarrow \mathrm{Gr}$

$$\mathrm{GL}_n(R) := \{(x_{ij}) \in \mathrm{Mat}_n(R) \ ; \ \det((x_{ij})) \text{ is invertible}\}, \qquad R \in M_k.$$

Observe that $\mathrm{GL}_n = \mathrm{Spec}_k(k[\mathrm{Mat}_n]_{\det(X_{ij})})$ is an affine algebraic group, the so-called *general linear group*.

(2) The *additive group* $\mathbb{G}_a$ is defined by means of $\mathbb{G}_a : M_k \longrightarrow \mathrm{Gr}$; $\mathbb{G}_a(R) := (R, +)$ for every $R \in M_k$.

(3) Consider $\mu : M_k \longrightarrow \mathrm{Gr}$; $\mu(R) := (R^\times, \cdot)$ for every $R \in M_k$. The k-group functor $\mu = \mathrm{GL}_1$ is represented by the Hopf algebra $k[T]_T$.

(4) Suppose that $\mathrm{char}(k) = p > 0$. Given $r \in \mathbb{N}$, we consider the group $\mathbb{G}_{a(r)}$ that is given by

$$\mathbb{G}_{a(r)}(R) := \{x \in \mathbb{G}_a(R) \ ; \ x^{p^r} = 0\}.$$

Note that $\mathbb{G}_a$ and $\mathbb{G}_{a(r)}$ are represented by the commutative Hopf algebras $k[T]$ and $k[T]/(T^{p^r})$, respectively. For $\mathbb{G}_a$ the points of R are identified with the values of $x(T)$, where $x \in \mathrm{Spec}_k(k[T])(R)$. The operation of $\mathbb{G}_a$ is induced by the comultiplication of $k[T]$:

$$x(T) + y(T) = (x\hat{\otimes}y)(T \otimes 1 + 1 \otimes T) = ((x\hat{\otimes}y) \circ \Delta)(T).$$

Inverses are given by the antipode:

$$-x(T) = x(-T) = (x \circ \eta)(T).$$

For every $R \in M_k$ the group $\mathbb{G}_{a(r)}(R) = \mathrm{Spec}_k(k[T]/(T^{p^r}))(R)$ is contained in $\mathbb{G}_a(R)$, and the inclusion is induced by the surjective map $k[T] \longrightarrow k[T]/(T^{p^r})$. This is an example of a closed subgroup of an algebraic group.

The group GL_n is an example of a reduced group scheme. Recall the bialgebra $k[\mathrm{Mat}_n]$. This algebra represents the monoid functor Mat_n that associates to every commutative k-algebra R the multiplicative monoid $\mathrm{Mat}_n(R)$ of $(n \times n)$-matrices with coefficients in R. Since this algebra is *reduced* (i.e., zero is the only nilpotent element), its localization $k[\mathrm{Mat}_n)]_{\det(X_{ij})}$ has the same property. The algebraic groups represented by reduced Hopf algebras, are called *reduced group schemes*. They are determined by their k-rational points in case k is algebraically closed. By contrast, $\mathbb{G}_{a(r)}(k) = \{0\}$, so the k-rational points don't provide any information in this case.

Suppose that $(A, \Delta, \varepsilon, \eta)$ is a commutative Hopf algebra. Given $R \in M_k$, we define a multiplication on $\mathrm{Spec}_k(A)(R)$ via convolution:

$$(x * y)(a) = \sum_{(a)} x(a_{(1)})y(a_{(2)}) \tag{1}$$

for $x, y \in \mathrm{Spec}_k(A)$ and $a \in A$. (Observe that we need the commutativity of R to ensure that $x * y$ is indeed a homomorphism of k-algebras.) Then $(\mathrm{Spec}_k(A)(R), *)$ is a group with identity element ε and inverse $x^{-1} = x \circ \eta$. These operations endow $\mathrm{Spec}_k(A)$ with the structure of an affine group scheme.

The following result shows that all affine group schemes arise in this fashion.

PROPOSITION 2.2. *Let A be a commutative k-algebra such that $\mathrm{Spec}_k(A)$ is a group scheme. Then A has the structure of a Hopf algebra such that the group structure on $\mathrm{Spec}_k(A)$ is given by (1).*

PROOF. Let m be the multiplication on $\mathcal{G} := \mathrm{Spec}_k(A)$. Then $m : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ is a natural transformation. We compose this transformation with the natural equivalence $\Pi^{-1} : \mathrm{Spec}_k(A \otimes_k A) \longrightarrow \mathcal{G} \times \mathcal{G}$. By Yoneda's Lemma, there exists an algebra homomorphism $\Delta : A \longrightarrow A \otimes_k A$ such that

$$(m \circ \Pi^{-1})(x) = x \circ \Delta \qquad \forall\, x \in \mathrm{Spec}_k(A \otimes_k A)(R),\ R \in M_k.$$

Let $g, h \in \mathcal{G}(R)$. Since $\Pi^{-1}(g\hat{\otimes}h) = (g, h)$, we have

$$(g \cdot h)(a) = ((g\hat{\otimes}h) \circ \Delta)(a) = \sum_{(a)} g(a_{(1)})h(a_{(2)}) = (g * h)(a)$$

for every $a \in A$. Yoneda's Lemma also provides an algebra homomorphism $\eta : A \longrightarrow A$ such that $g^{-1} = g \circ \eta$ for every element $g \in \mathcal{G}(R)$. The Hopf algebra axioms are now readily seen to correspond to the group axioms. □

REMARK. Since $(g^{-1})^{-1} = g$ for every $g \in \mathcal{G}(R)$, we conclude that the antipode η of a commutative Hopf algebra H satisfies $\eta^2 = \mathrm{id}_H$. In view of (1.1) the antipode of a finite-dimensional cocommutative Hopf algebra satisfies the same identity. By the same token, an algebra homomorphism $\varphi : H \longrightarrow H'$ between two finite-dimensional cocommutative Hopf algebras satisfying $\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$ is a homomorphism of Hopf algebras.

A natural transformation $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ between two k-group functors is a *homomorphism of k-group functors*. Yoneda's Lemma then shows that the homomorphisms between affine group schemes correspond to the Hopf algebra homomorphisms $k[\mathcal{H}] \longrightarrow k[\mathcal{G}]$. More precisely, we have:

PROPOSITION 2.3. *The categories of affine group schemes and commutative Hopf algebras are anti-equivalent.*

PROOF. We only have to understand how to retrieve A from $\mathrm{Spec}_k(A)$. For any scheme $\mathcal{X}$, we define $k[\mathcal{X}]$ to be the set of natural transformations $\mathcal{X} \longrightarrow \mathrm{Spec}_k(k[T])$. This set naturally has the structure of a k-algebra. In case $\mathcal{X} = \mathrm{Spec}_k(A)$, Yoneda's Lemma provides an identification $k[\mathcal{X}] \cong \mathrm{Spec}_k(k[T])(A) \cong A$. □

DEFINITION. Let $\mathcal{X}$ be a k-functor. Then the k-algebra $k[\mathcal{X}]$ of all natural transformations $\mathcal{X} \longrightarrow \mathrm{Spec}_k(k[T])$ is called the *function algebra* of $\mathcal{X}$. If $\mathcal{X}$ is an affine scheme, then $k[\mathcal{X}]$ is the coordinate ring of $\mathcal{X}$.

Given a homomorphism $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ of affine group schemes, we consider the functor $\ker \varphi$ that is defined by $\ker \varphi\,(R) := \ker \varphi_R$ for every $R \in M_k$. Let $\varphi^* : k[\mathcal{H}] \longrightarrow k[\mathcal{G}]$ be the Hopf algebra homomorphism corresponding to φ, and let $I := k[\mathcal{G}]\varphi^*(k[\mathcal{H}]^\dagger)$ be the ideal generated by the image of the augmentation ideal $k[\mathcal{H}]^\dagger$ of $k[\mathcal{H}]$. Then $x \in \ker \varphi\,(R)$ if and only if $x(I) = (0)$. In other words, the functor $\ker \varphi$ is represented by the Hopf algebra $k[\mathcal{G}]/I$.

Suppose that k is algebraically closed, and let A be a finitely generated, commutative k-algebra. Every finite-dimensional semi-simple subalgebra $S \subseteq A$ gives rise to a finite subset of mutually orthogonal idempotents of A. Since A is noetherian, it follows that A possesses a unique maximal finite-dimensional semi-simple subalgebra $\pi_0(A)$. Moreover, if A is a Hopf algebra, then $\pi_0(A)$ is a Hopf subalgebra of A. (For arbitrary fields, one considers separable subalgebras instead of semi-simple ones.)

Now let $\mathcal{G} = \mathrm{Spec}_k(A)$ be an affine algebraic k-group. We put $\pi_0(\mathcal{G}) := \mathrm{Spec}_k(\pi_0(A))$, and consider the homomorphism $\pi : \mathcal{G} \longrightarrow \pi_0(\mathcal{G})$ that is given by restriction. The subgroup $\mathcal{G}^0 := \ker \pi$ is called the *connected component* of $\mathcal{G}$.

DEFINITION. An affine group scheme $\mathcal{G} := \mathrm{Spec}_k(k[\mathcal{G}])$ is *connected* if $k[\mathcal{G}]$ possesses exactly one idempotent.

Note that $\mathcal{G}^0$ is a connected, affine algebraic group scheme. An arbitrary affine algebraic group $\mathcal{G} = \mathrm{Spec}_k(A)$ is connected if and only if the prime ideal spectrum of A is connected.

3. Algebras of Measures

In this section we interpret cocommutative Hopf algebras as the algebras of measures on the finite algebraic k-groups. *For simplicity we assume throughout that the ground field k is algebraically closed.*

Definition. An affine group scheme $\mathcal{G}$ is said to be *finite* if its coordinate ring $k[\mathcal{G}]$ is finite-dimensional. Given such a scheme $\mathcal{G}$, we call $k\mathcal{G} := k[\mathcal{G}]^*$ the *algebra of measures* on $\mathcal{G}$. The number $\mathrm{ord}(\mathcal{G}) := \dim_k k[\mathcal{G}]$ is referred to as the *order* of the finite algebraic group $\mathcal{G}$.

By definition, $k\mathcal{G}$ is a finite-dimensional cocommutative Hopf algebra. Our previous results now show that the category of finite-dimensional cocommutative Hopf algebras is equivalent to the category of finite group schemes. Indeed, if H is such a Hopf algebra, then $\mathcal{G}_H := \mathrm{Spec}_k(H^*)$ is a finite group scheme such that $H \cong k\mathcal{G}_H$.

Algebras of measures can be viewed as "group algebras" of finite group schemes: Given a k-vector space V, we consider the k-functor $V_a : M_k \longrightarrow \mathrm{Ens}$; $V_a(R) := V \otimes_k R$. In particular, we can consider $V := k\mathcal{G}$. Note that $k\mathcal{G} \otimes_k R$ has the structure of a Hopf algebra over R with comultiplication

$$\Delta_R : k\mathcal{G} \otimes_k R \longrightarrow k\mathcal{G} \otimes_k k\mathcal{G} \otimes_k R \cong (k\mathcal{G} \otimes_k R) \otimes_R (k\mathcal{G} \otimes_k R) \;\; ; \;\; h \otimes x \mapsto \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes x.$$

There is an embedding $\iota_{\mathcal{G}} : \mathcal{G} \hookrightarrow k\mathcal{G}_a$, which interprets an element $g : k[\mathcal{G}] \longrightarrow R$ of $\mathcal{G}(R)$ as a homomorphism $k[\mathcal{G}] \otimes_k R \longrightarrow R$ of R-algebras. Since $\mathrm{Hom}_R(k[\mathcal{G}] \otimes_k R, R) \cong \mathrm{Hom}_k(k[\mathcal{G}], R) \cong k\mathcal{G} \otimes_k R$, this amounts to identifying $\mathcal{G}(R)$ with the group $G(k\mathcal{G} \otimes_k R)$ of group-like elements of $k\mathcal{G} \otimes_k R$. Given any morphism $f : \mathcal{G} \longrightarrow V_a$, there exists a unique k-linear map $\hat{f} : k\mathcal{G} \longrightarrow V$ such that

$$f_R = (\hat{f} \otimes \mathrm{id}_R) \circ (\iota_{\mathcal{G}})_R \qquad \forall\; R \in M_k.$$

The interested reader may consult [**122**] for more details.

Definition. Let $\mathcal{G}$ be a k-group, V be a k-vector space. We say that V is a $\mathcal{G}$-*module* if there exists a natural transformation $\mathcal{G} \times V_a \longrightarrow V_a$ such that, for every $R \in M_k$, the map

$$\mathcal{G}(R) \times (V \otimes_k R) \longrightarrow V \otimes_k R$$

is an action of the group $\mathcal{G}(R)$ on $V \otimes_k R$ by R-linear transformations.

Suppose that $\mathcal{G}$ is a finite algebraic group. The aforementioned universal property of $k\mathcal{G}$ entails that the notions "$\mathcal{G}$-module" and "$k\mathcal{G}$-module" coincide: every $\mathcal{G}$-module possesses a unique $k\mathcal{G}$-module structure and vice versa. We shall therefore use the two notions interchangeably.

Examples. (1) Let $\mathrm{char}(k) = p > 0$ and consider the group $\mathbb{G}_{a(r)}$. We have observed before that $k\mathbb{G}_{a(r)} \cong k[X_0, \ldots, X_{r-1}]/(X_0^p, \ldots, X_{r-1}^p)$ as an algebra. Recall that the generator X_i corresponds to the functional δ_{p^i}, where $\{\delta_0, \ldots, \delta_{p^r-1}\}$ is the basis dual to the canonical basis $\{1, t, \ldots, t^{p^r-1}\}$ of the coordinate ring $k[T]/(T^{p^r})$ of $\mathbb{G}_{a(r)}$. Direct computation shows that

$$\Delta(\delta_i) = \sum_{j=0}^{i} \delta_j \otimes \delta_{i-j} \;\; ; \;\; \varepsilon(\delta_i) = \delta_{i,0} \;\; ; \;\; \eta(\delta_i) = (-1)^i \delta_i.$$

Thus, $\mathrm{Lie}(k\mathbb{G}_{a(r)}) = k\delta_1$, while $G(k\mathbb{G}_{a(r)}) = \{\delta_0\}$. In particular, the algebra $k\mathbb{G}_{a(r)}$ is generated by $\mathrm{Lie}(k\mathbb{G}_{a(r)})$ if and only if $r = 1$.

(2) Given $r > 0$, we consider the finite group scheme $\mu_{(r)} : M_k \longrightarrow \mathrm{Gr}$,

$$\mu_{(r)}(R) := \{x \in R \ ; \ x^r = 1\}.$$

Note that $\mu_{(r)} = \mathrm{Spec}_k(k[T]/(T^r - 1))$. If $t := T + (T^r - 1)$, then $\Delta(t) = t \otimes t$, $\varepsilon(t) = 1$ and $\eta(t) = t^{r-1}$. Given $g, h \in k\mu_{(r)}$, we have

$$(g * h)(t^i) = g(t^i)h(t^i),$$

whence $k\mu_{(r)} \cong k^r$ is semi-simple. If $\{\delta_0, \dots, \delta_{r-1}\}$ is the basis dual to $\{1, t, \dots, t^{r-1}\}$, then the δ_i are the primitive idempotents of $k\mu_{(r)}$. Note that

$$\Delta(\delta_i) = \sum_{j=0}^{r-1} \delta_j \otimes \delta_{i-j} \quad ; \quad \varepsilon(\delta_i) = \delta_{i,0} \quad ; \quad \eta(\delta_i) = \delta_{-i},$$

where the subscripts are considered elements of $\mathbb{Z}/(r)$. Moreover, $\mathrm{Lie}(k\mu_{(r)}) = (0)$ for $p \nmid r$, and $\dim_k \mathrm{Lie}(k\mu_{(r)}) = 1$, otherwise.

So far, the characteristic of the underlying base field k has not played a major rôle. The following fundamental result shows how the classical characteristic zero theory differs from the modular theory. It says that algebraic groups in characteristic zero are reduced and hence are completely determined by their k-rational points. We require the following result from commutative algebra.

THEOREM 3.1 (Krull's Intersection Theorem,[**35**]). *Let R be a commutative noetherian ring, $\mathrm{Max}(R)$ be the set of maximal ideals of R. Then*

$$\bigcap_{\mathfrak{M} \in \mathrm{Max}(R)} \bigcap_{n \in \mathbb{N}} \mathfrak{M}^n = (0).$$

Let A be a commutative k-algebra. A k-linear map, $d : A \longrightarrow M$ with values in an A-module M is called a *derivation* of A into M, provided

$$d(xy) = x.d(y) + y.d(x) \qquad \forall\, x, y \in A.$$

Given a derivation $d : A \longrightarrow M$, the map

$$\mathrm{id}_A \otimes d : A \otimes_k A \longrightarrow A \otimes_k M \quad ; \quad a \otimes b \mapsto a \otimes d(b)$$

is a derivation of the k-algebra $A \otimes_k A$ with values in the $(A \otimes_k A)$-module $A \otimes_k M$, whose structure is defined by

$$(a \otimes b).(a' \otimes m) := aa' \otimes b.m.$$

Let H be a commutative Hopf algebra with augmentation ideal $I := \ker \varepsilon$. We consider the map

$$\pi_H : H \longrightarrow I/I^2 \quad ; \quad h \mapsto h - \varepsilon(h)1 + I^2,$$

which is readily seen to be a derivation. Hence $\mathrm{id}_H \otimes \pi_H$ is a derivation of $H \otimes_k H$ with values in $H \otimes_k (I/I^2)$. Consequently,

$$d_H := (\mathrm{id}_H \otimes \pi_H) \circ \Delta$$

is a derivation of H into $H \otimes_k (I/I^2)$, where the latter space is the tensor product of the H-modules H and I/I^2.

THEOREM 3.2 (Cartier). *Suppose that* $\mathrm{char}(k) = 0$. *Then every finitely generated commutative Hopf algebra H over k is reduced.*

PROOF. Since the k-algebra H is finitely generated, the vector space I/I^2 is finite-dimensional. Hence there are elements $x_1, \ldots, x_r \in I$ such that their residue classes form a basis of I/I^2. Using multi-index notation, we write

$$x^m := x_1^{m_1} \cdots x_r^{m_r} \quad \text{and} \quad |m| := \sum_{i=1}^{r} m_i$$

for $m \in \mathbb{N}_0^r$. The following claim says that the graded algebra, defined by the powers of the augmentation ideal I, is a polynomial ring in r variables.

(i) *Let $n \geq 1$. The residue classes $\{x^m + I^{n+1} \ ; \ |m| = n\}$ form a basis of I^n/I^{n+1}.*

Given $i \in \{1, \ldots, r\}$, we consider the k-linear map

$$f_i : I/I^2 \longrightarrow k \quad ; \quad \bar{x}_j \mapsto \delta_{ij}$$

as well as

$$\tilde{f}_i : H \otimes_k I/I^2 \longrightarrow H \quad ; \quad a \otimes v \mapsto f_i(v)a.$$

For $a, b \in H$ and $v \in I/I^2$, we obtain

$$a.(b \otimes v) = \sum_{(a)} a_{(1)}b \otimes a_{(2)}v = \sum_{(a)} a_{(1)}b \otimes \varepsilon(a_{(2)})v = (\sum_{(a)} a_{(1)}\varepsilon(a_{(2)}))b \otimes v = ab \otimes v,$$

so that the map $\tilde{f}_i$ is H-linear.

Consequently, the map

$$d_i : H \longrightarrow H \quad ; \quad h \mapsto \tilde{f}_i \circ d_H(h)$$

is a derivation of H with $d_i(h) = \sum_{(h)} f_i(\pi_H(h_{(2)}))h_{(1)}$ for all $h \in H$. We thus obtain

$$(\varepsilon \circ d_i)(h) = \sum_{(h)} \varepsilon(h_{(1)})f_i(\pi_H(h_{(2)})) = f_i(\sum_{(h)} \varepsilon(h_{(1)})\pi_H(h_{(2)})) = (f_i \circ \pi_H)(h),$$

whence

$$(*) \qquad d_i(x_j) \equiv \delta_{ij} \mod I$$

for $1 \leq i \leq r$. Let $\underline{1}_i \in \mathbb{N}_0^r$ be the element with coordinates δ_{ij}. Since d_i is a derivation, $(*)$ implies

$$d_i(x^m) \equiv m_i x^{m-\underline{1}_i} \mod I^{|m|}.$$

In particular, $d_i(I^n) \subseteq I^{n-1}$ for all $n \geq 1$. Thus, if

$$\sum_{|m|=n} \alpha_m x^m \equiv 0 \mod I^{n+1},$$

then, applying d_i, we obtain

$$0 = \sum_{|m|=n} m_i \alpha_m x^{m-\underline{1}_i} \mod I^n,$$

so that induction implies $m_i\alpha_m = 0$. Since $\mathrm{char}(k) = 0$, we conclude that $\alpha_m = 0$. ◇

(ii) *If $h^2 = 0$, then $h \in \bigcap_{n\in\mathbb{N}} I^n$.*

We have $h \in I$, and if $h \notin \bigcap_{n\in\mathbb{N}} I^n$, then there exists $n \in \mathbb{N}$ with $h \in I^n \setminus I^{n+1}$. We write

$$h = \sum_{|m|=n} \alpha_m x^m + z,$$

with $\alpha_m \in k$ and $z \in I^{n+1}$. Our assumption in conjunction with (i) implies

$$\sum_{m+m'=t} \alpha_m \alpha_{m'} = 0$$

for all $t \in \mathbb{N}_0^r$ with $|t| = 2n$. Upon ordering the elements of $\mathbb{N}_0^r$ lexicographically we obtain $0 = \alpha_{\tilde{m}}$, where $\tilde{m} = \max_{|m|=n}\{\alpha_m \neq 0\}$, a contradiction. $\diamond$

If $\mathfrak{M} \unlhd H$ is a maximal ideal of H, then Hilbert's Nullstellensatz provides an algebra homomorphism $\lambda : H \longrightarrow k$ such that $\mathfrak{M} = \ker \lambda$. Direct computation shows that λ induces an automorphism

$$\psi_\lambda : H \longrightarrow H \quad ; \quad h \mapsto \sum_{(h)} \lambda(h_{(1)})h_{(2)}$$

of H, whose inverse is $\psi_{\lambda\circ\eta}$. Since

$$(\varepsilon \circ \psi_\lambda)(h) = \sum_{(h)} \lambda(h_{(1)})\varepsilon(h_{(2)}) = \lambda(\sum_{(h)} h_{(1)}\varepsilon(h_{(2)})) = \lambda(h)$$

for every $h \in H$, it follows that $\Psi_\lambda(\mathfrak{M}) = I$.

Let $h \in H$ be nilpotent. Without loss of generality, we may assume that $h^2 = 0$. Given a maximal ideal $\mathfrak{M} = \ker \lambda$, (ii) implies that $\psi_\lambda(h) \in \bigcap_{n\in\mathbb{N}} I^n$, whence $h \in \bigcap_{n\in\mathbb{N}} \mathfrak{M}^n$. Krull's Intersection Theorem now yields $h = 0$, as desired. $\square$

Corollary 3.3. *If H is a finite-dimensional, cocommutative Hopf algebra such that H^* is reduced, then there exists a finite group G such that $H \cong kG$. In particular, all finite-dimensional cocommutative Hopf algebras of characteristic zero are semi-simple.*

Proof. By assumption, H^* is a finite-dimensional reduced algebra and thus is a product of copies of k. We let $G := X(H^*)$ be the character group of H^*, endowed with the convolution product. Then $\mathrm{ord}(G) = \dim_k H$, and $G \subseteq H$ is linearly independent. Consequently, the canonical map $kG \longrightarrow H$ is an isomorphism of Hopf algebras.

If $\mathrm{char}(k) = 0$, then Cartier's Theorem ensures that H^* is reduced, so that there exists a finite group G with $H \cong kG$. Owing to Maschke's Theorem, the latter algebra is semi-simple. $\square$

Since we will be mainly interested in questions related to the representation type of an algebra, (3.3) shows that we will ultimately be studying Hopf algebras that are defined over fields of positive characteristic.

Let H be a finite-dimensional Hopf algebra with counit ε. As an algebra, H decomposes into its blocks

$$H = \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n.$$

Here $\mathcal{B}_0$ is the block to which the *trivial H-module k*, with H acting via ε, belongs. In other words, $\mathcal{B}_0$ is determined by the property $\varepsilon(\mathcal{B}_0) \neq (0)$. This block is usually referred to as the *principal block* $\mathcal{B}_0(H)$ of H. It is not true in general that the principal block of a Hopf algebra is a Hopf subalgebra. In fact, if H is commutative, then $\mathcal{B}_0(H)$ is a Hopf subalgebra if and only if $\mathcal{B}_0(H) = H$ ($\mathcal{B}_0(H)$ is local with unique non-zero idempotent e_0. Thus, if $\mathcal{B}_0(H)$ is a Hopf subalgebra, then $\Delta(e_0) = e_0 \otimes e_0$, and e_0 is invertible, whence $e_0 = 1$).

Definitions. Let $\mathcal{G}$ be an affine group scheme. A subfunctor $\mathcal{H} \subseteq \mathcal{G}$ is called a (*closed*) *subgroup* if there exists a Hopf ideal $I \subseteq k[\mathcal{G}]$ such that $\mathcal{H}(R) = \mathcal{V}(I)(R) := \{g \in \mathcal{G}(R) \; ; \; g(I) = (0)\}$ for every commutative k-algebra R.

A homomorphism $\mathcal{G} \longrightarrow \mathcal{G}'$ of affine group schemes is called a *closed embedding* if the associated comorphism $k[\mathcal{G}'] \longrightarrow k[\mathcal{G}]$ of k-algebras is surjective.

Let $\mathcal{N} \subseteq \mathcal{G}$ be a subgroup. We say that $\mathcal{N}$ is a *normal* subgroup of $\mathcal{G}$ if $\mathcal{N}(R)$ is normal in $\mathcal{G}(R)$ for every $R \in M_k$.

Note that the comorphism $c^* : k[\mathcal{G}] \longrightarrow k[\mathcal{G}] \otimes_k k[\mathcal{G}]$ of the conjugation $c : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$; $(g, h) \mapsto ghg^{-1}$ is given by $c^*(a) = \sum_{(a)} a_{(1)}\eta(a_{(3)}) \otimes a_{(2)}$. Hence $\mathcal{N} = \mathcal{V}(I)$ is normal if and only if $c^*(I) \subseteq k[\mathcal{G}] \otimes_k I$.

Suppose that $\operatorname{char}(k) = p > 0$, and consider the group $\mu_{(r)}$, where $r = p^s\ell$ with p not dividing ℓ. The map $x \mapsto (x^{p^s}, x^\ell)$ is an isomorphism $\mu_{(r)} \cong \mu_{(\ell)} \times \mu_{(p^s)}$. Note that the first factor is represented by $k[T]/(T^\ell - 1) \cong k^\ell$, while the second has local coordinate ring $k[T]/(T^{p^s} - 1)$. We now prove that finite algebraic groups always afford such a decomposition.

Theorem 3.4. *Let $\mathcal{G} = \operatorname{Spec}_k(A)$ be a finite algebraic group. Then $\mathcal{G}$ is a semidirect product*

$$\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}},$$

where $\mathcal{G}^0$ is normal and $\mathcal{G}_{\mathrm{red}} \cong \pi_0(\mathcal{G})$. As a k-functor, the connected component $\mathcal{G}^0$ is represented by the principal block $\mathcal{B}_0(A)$ of A.

Proof. We decompose $A = \bigoplus_{i=0}^n \mathcal{B}_i$ into its blocks, and denote the (central) primitive idempotents by $\{e_0, \ldots, e_n\}$ with $\mathcal{B}_0 := \mathcal{B}_0(A) = Ae_0$. Since each block $\mathcal{B}_i = Ae_i$ is local and of the form $\mathcal{B}_i = ke_i \oplus \operatorname{Rad}(\mathcal{B}_i)$, the subalgebra $S := \sum_{i=0}^n ke_i$ is the largest semi-simple subalgebra of A, and we have $A = S \oplus \operatorname{Rad}(A)$.

As $\operatorname{Rad}(A)$ is the set of nilpotent elements of A, it is a Hopf ideal. Consider the closed subgroup $\mathcal{G}_{\mathrm{red}} := \mathcal{V}(\operatorname{Rad}(A))$ of $\mathcal{G}$.

Note that $\{e_i \otimes e_j \ ; \ 0 \leq i, j \leq n\}$ is the set of orthogonal primitive idempotents of $A \otimes_k A$. Since $\Delta(e_i)$ is an idempotent of $A \otimes_k A$, we have $\Delta(e_i) \in S \otimes_k S$. Consequently, S is a Hopf subalgebra of A, and the composition $\lambda \circ \iota$ of the canonial projection $A \stackrel{\lambda}{\longrightarrow} A/\operatorname{Rad}(A)$ with the inclusion $\iota : S \hookrightarrow A$ is an isomorphism $S \cong A/\operatorname{Rad}(A)$ of Hopf algebras. Accordingly, the corresponding isomorphism $\operatorname{Spec}_k(\lambda \circ \iota) : \mathcal{G}_{\mathrm{red}} \longrightarrow \pi_0(\mathcal{G})$ factors as

$$\mathcal{G}_{\mathrm{red}} \stackrel{\operatorname{Spec}_k(\lambda)}{\longrightarrow} \mathcal{G} \stackrel{\pi}{\longrightarrow} \pi_0(\mathcal{G}),$$

so that $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$. By definition, the closed subgroup $\mathcal{G}^0 = \ker \pi$ is normal in $\mathcal{G}$. Since $\mathcal{G}^0 = \mathcal{V}(AS^\dagger)$ and $AS^\dagger = \bigoplus_{i=1}^n \mathcal{B}_i$, we see that the representing algebra $A/AS^\dagger$ is isomorphic to $\mathcal{B}_0(A)$. □

The foregoing result can also be interpreted at the level of Hopf algebras. Recall from (3.3) that, given a reduced finite algebraic group $\mathcal{G}$, we have $k\mathcal{G} \cong kG$, where $G = \operatorname{Alg}_k(k[\mathcal{G}], k) = \mathcal{G}(k)$ is the finite group of k-rational points of $\mathcal{G}$. Now let $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$ be an arbitrary finite algebraic group. Since the connected component $\mathcal{G}^0$ is represented by a local algebra, we have

$$\mathcal{G}(k) = \mathcal{G}^0(k) \cdot \mathcal{G}_{\mathrm{red}}(k) = \mathcal{G}_{\mathrm{red}}(k).$$

It follows that

$$k\mathcal{G} \cong k\mathcal{G}^0 \# k\mathcal{G}_{\mathrm{red}} \cong k\mathcal{G}^0 \# k\mathcal{G}(k) \cong (k\mathcal{G}^0)\mathcal{G}(k)$$

is the *smash product* of $k\mathcal{G}^0$ with the group algebra of the group of k-rational points of $\mathcal{G}$. The right-hand term interprets the smash product as a *skew group algebra*.

Definition. A finite group scheme $\mathcal{G}$ is *infinitesimal* if its coordinate ring $k[\mathcal{G}]$ is local. In that case, $k\mathcal{G}$ is also called the *distribution algebra* of $\mathcal{G}$ and one also writes $k\mathcal{G} = \operatorname{Dist}(\mathcal{G})$ (see also the comments below).

Note that an affine algebraic k-group $\mathcal{G}$ is infinitesimal if and only if $\mathcal{G}(k) = \{1\}$. By Cartier's Theorem, any infinitesimal group of characteristic zero is trivial.

When studying the representations of a cocommutative Hopf algebra one thus has to understand the following disciplines:

(a) The modular representation theory of finite groups. By now, this field is rather well-understood.

(b) The representation theory of the infinitesimal group $\mathcal{G}^0$.

(c) The fusion of (a) and (b).

It has turned out that the methods figuring prominently in (a), such as the Mackey decomposition theorem and the defect theory of blocks, usually break down for infinitesimal group schemes. This has ultimately led to the approach via the geometric methods to be outlined in Chapters III and IV.

For the remainder of this section we assume that $\mathrm{char}(k) = p > 0$.

Examples. (1) The groups $\mathbb{G}_{a(r)}$ and $\mu_{(p^r)}$ are infinitesimal for every $r > 0$.

(2) Given $r \in \mathbb{N}$, we consider the closed subgroup of GL(2) given by

$$\mathrm{GL}(2)_r(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)(R) \ ; \ a^{p^r} = 1 = d^{p^r} \ , \ b^{p^r} = 0 = c^{p^r}\}$$

for every commutative k-algebra R. The group $\mathrm{GL}(2)_r$ is readily seen to be the kernel of the homomorphism

$$F^r : \mathrm{GL}(2) \longrightarrow \mathrm{GL}(2) \ ; \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^{p^r} & b^{p^r} \\ c^{p^r} & d^{p^r} \end{pmatrix}.$$

Accordingly,

$$k[\mathrm{GL}(2)_r] \cong k[\mathrm{GL}(2)]/(\{X_{11}^{p^r} - 1, X_{22}^{p^r} - 1, X_{12}^{p^r}, X_{21}^{p^r}\}),$$

so that $\mathrm{GL}(2)_r$ is finite. Since $\mathrm{GL}(2)_r(k) = \{1\}$, the group scheme $\mathrm{GL}(2)_r$ is infinitesimal.

(3) Let $\mathcal{L} : M_k \longrightarrow \mathrm{Gr}$ be the group scheme, given by

$$\mathcal{L}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ a^p = 1 = d^p \ , \ b^{p^2} = b^p \ , \ c^p = 0\}$$

for every $R \in M_k$. Then $\mathcal{L}$ is a closed subgroup of SL(2) such that $\mathcal{L}(k) = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; \ b \in \mathbb{F}_p\} \cong \mathbb{Z}/(p)$. Moreover, using Proposition 3.5 below, we see that $\mathcal{L}^0 = \mathrm{SL}(2)_1 := \ker F|_{\mathrm{SL}(2)}$. Consequently,

$$\mathcal{L} = \mathrm{SL}(2)_1 \rtimes \mathcal{U},$$

where, for every commutative k-algebra R, the R-points of $\mathcal{U}$ are the group

$$\mathcal{U}(R) = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \ ; \ b^p = b\}.$$

The second example above shows how one can generate infinitesimal groups. The relevant notion in this context is that of the Frobenius homomorphism. If V is a k-vector space, we denote by $V^{(r)}$ the k-vector space with underlying abelian group V and action given by

$$\alpha \cdot v := \alpha^{p^{-r}} v \qquad \forall \ \alpha \in k, \, v \in V.$$

Given an affine k-group $\mathcal{G}$, we let $\mathcal{G}^{(r)} := \mathrm{Spec}_k(k[\mathcal{G}]^{(r)})$ be the affine group scheme defined by the twisted coordinate ring of $\mathcal{G}$.

Definition. Let $\mathcal{G}$ be an affine algebraic group scheme over the algebraically closed field k of characteristic $p > 0$. The homomorphism $F : \mathcal{G} \longrightarrow \mathcal{G}^{(1)}$ satisfying

$$F_R : \mathcal{G}(R) \longrightarrow \mathcal{G}^{(1)}(R) \ \ ; \ \ F_R(\lambda)(x) = \lambda(x)^p \quad \forall\ \lambda \in \mathcal{G}(R),\, x \in k[\mathcal{G}],\, R \in M_k$$

is called the *Frobenius homomorphism* of $\mathcal{G}$. The kernel $\mathcal{G}_r$ of its iterate $F^r : \mathcal{G} \longrightarrow \mathcal{G}^{(r)}$ is referred to as the *r-th Frobenius kernel* of $\mathcal{G}$.

Remarks. (1) Note that $F_R(\lambda)$ is indeed a k-linear map:

$$F_R(\lambda)(\alpha \cdot x) = \lambda(\alpha^{\frac{1}{p}} x)^p = \alpha F_R(\lambda)(x)$$

for every $x \in k[\mathcal{G}]^{(1)}$ and $\alpha \in k$.

(2) We have $F^r = \mathrm{Spec}_k(\varphi_r)$, where

$$\varphi_r : k[\mathcal{G}]^{(r)} \longrightarrow k[\mathcal{G}] \ \ ; \ \ x \mapsto x^{p^r}$$

is the comorphism of F^r. It follows that $k[\mathcal{G}_r] = k[\mathcal{G}]/k[\mathcal{G}]\varphi_r(k[\mathcal{G}]^\dagger) = k[\mathcal{G}]/k[\mathcal{G}]\{x^{p^r} \,;\, x \in k[\mathcal{G}]^\dagger\}$. Consequently, $\mathcal{G}_r$ is an infinitesimal k-group whenever $\mathcal{G}$ is algebraic.

(3) Certain problems on representations of connected algebraic groups can already be decided on sufficiently large Frobenius kernels. For instance, two finite-dimensional $\mathcal{G}$-modules are isomorphic if and only if their restrictions to a suitable Frobenius kernel $\mathcal{G}_r$ enjoy this property (see [**80**, (I.9.8)]).

Examples. (1) The group $\mathbb{G}_{a(r)}$ is the r-th Frobenius kernel of $\mathbb{G}_a$.

(2) For $r \geq 1$, let

$$\mathcal{A}_{[r]}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)(R) \ ; \ a^{p^r} = 1 = d^{p^r} \ ,\ b^{p^2} = 0 = c^p\} \quad , \quad R \in M_k.$$

Then $\mathcal{A}_{[r]} \subseteq \mathrm{GL}(2)_r$ is an infinitesimal subgroup of $\mathrm{GL}(2)$ containing $\mathrm{GL}(2)_1$, which is not a Frobenius kernel of $\mathrm{GL}(2)$.

Proposition 3.5. *Let $\mathcal{G}$ be a finite algebraic k-group. Then $\mathcal{G}$ is infinitesimal if and only if $\mathcal{G} = \mathcal{G}_r$ for some $r \geq 0$.*

Proof. Suppose that $\mathcal{G}$ is infinitesimal, and put $I := k[\mathcal{G}]^\dagger$. By assumption, the ideal I is nilpotent, so there exists an integer $r \geq 0$ such that $x^{p^r} = 0$ for every $x \in I$. By our remark above, this implies that $\mathcal{G}_r = \mathcal{V}(k[\mathcal{G}]\{x^{p^r} \,;\, x \in I\}) = \mathcal{V}((0)) = \mathcal{G}$. □

Definition. Let $\mathcal{G}$ be an infinitesimal k-group. Then $\mathrm{ht}(\mathcal{G}) := \min\{r \in \mathbb{N}_0 \ ; \ \mathcal{G} = \mathcal{G}_r\}$ is called the *height* of $\mathcal{G}$.

The distribution algebra of an infinitesimal group scheme is a special case of a more general construction that applies to arbitrary affine group schemes. We briefly indicate the definition; a thorough account can be found in Jantzen's book [**80**, I.§7]. Let $\mathcal{G} = \mathrm{Spec}_k(A)$ be an affine group scheme. Then

$$\mathrm{Dist}(\mathcal{G}) := \{h \in A^* \ ; \ h((A^\dagger)^r) = (0) \text{ for some } r \in \mathbb{N}\}$$

is a subalgebra of A^*. If $\mathcal{G}$ is algebraic, then the definition of the comultiplication for finite-dimensional A still works, and $\mathrm{Dist}(\mathcal{G})$ has the structure of a Hopf algebra. This Hopf algebra is called the *distribution algebra* of $\mathcal{G}$. If $\mathcal{G}$ is a finite algebraic group, then $\mathrm{Dist}(\mathcal{G}) \subseteq k\mathcal{G}$ with equality holding if and only if $\mathcal{G}$ is infinitesimal.

DEFINITION. Let $\mathcal{G}$ be an affine algebraic k-group. Then $\mathrm{Lie}(\mathcal{G}) := \mathrm{Lie}(\mathrm{Dist}(\mathcal{G}))$ is called the *Lie algebra of* $\mathcal{G}$.

Let $x \in \mathrm{Lie}(\mathcal{G})$. Then we have $x(ab) = x(a)\varepsilon(b) + \varepsilon(a)x(b)$ for $a, b \in k[\mathcal{G}]$, so that $x((k[\mathcal{G}]^\dagger)^2) = (0)$. Consequently, the natural map $\mathrm{Dist}(\mathcal{G}_r) \hookrightarrow \mathrm{Dist}(\mathcal{G})$ induces an isomorphism $\mathrm{Lie}(\mathcal{G}_r) \cong \mathrm{Lie}(\mathcal{G})$.

Let $\mathcal{G}$ be an affine algebraic k-group. Since $e_k = \mathcal{V}(k[\mathcal{G}]^\dagger)$ is a normal subgroup, the comorphism $c^* : k[\mathcal{G}] \longrightarrow k[\mathcal{G}] \otimes_k k[\mathcal{G}]$ of the conjugation action satisfies $c^*(k[\mathcal{G}]^\dagger) \subseteq k[\mathcal{G}] \otimes_k k[\mathcal{G}]^\dagger$. Hence we have an action, defined by

$$k[\mathcal{G}]^* \otimes_k \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{Dist}(\mathcal{G}) \;\; ; \;\; (\varphi \cdot \psi)(a) := ((\varphi \hat{\otimes} \psi) \circ c^*)(a) = \sum_{(a)} \varphi(a_{(1)}\eta(a_{(3)}))\psi(a_{(2)})$$

for every $a \in k[\mathcal{G}]$. Direct computation shows that $\varphi \cdot \psi = \sum_{(\varphi)} \varphi_{(1)} * \psi * \eta^*(\varphi_{(2)})$ for $\varphi, \psi \in \mathrm{Dist}(\mathcal{G})$. Thus, our action specializes to the (left) adjoint representation of the Hopf algebra $\mathrm{Dist}(\mathcal{G})$. One verifies that

$$\varphi \cdot \psi \in \mathrm{Lie}(\mathcal{G}) \qquad \forall \; \varphi \in k[\mathcal{G}]^*, \; \psi \in \mathrm{Lie}(\mathcal{G}).$$

In particular, the group $\mathcal{G}(k)$ acts on $\mathrm{Lie}(\mathcal{G})$ via the *adjoint representation*:

$$g \cdot \psi := g * \psi * g^{-1} \qquad \forall \; g \in \mathcal{G}(k), \; \psi \in \mathrm{Lie}(\mathcal{G}).$$

Finally, we have $\varphi \cdot \psi = \varphi * \psi - \psi * \varphi \quad \forall \; \varphi, \psi \in \mathrm{Lie}(\mathcal{G})$.

4. Restricted Lie Algebras

Given a Hopf algebra H, we have defined the associated Lie algebra $\mathrm{Lie}(H)$ via

$$\mathrm{Lie}(H) := \{x \in H \; ; \; \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

The subspace $\mathrm{Lie}(H)$ is closed under the commutator product $[x, y] := xy - yx$ of H, that is, $\mathrm{Lie}(H)$ is a Lie subalgebra of the Lie algebra $(H, [\,,\,])$. If $\mathrm{char}(k) = p > 0$ we also have $x^p \in \mathrm{Lie}(H)$ for every $x \in \mathrm{Lie}(H)$. Lie algebras with the latter property are examples of restricted Lie algebras. The abstract notion of a restricted Lie algebra arose first in work by N. Jacobson [**75, 76**] concerning a Galois theory for purely inseparable field extensions of exponent 1.

Throughout this section we assume that k is a field of characteristic $\mathrm{char}(k) = p > 0$. Given a Lie algebra $\mathfrak{g}$, the left multiplication effected by the element $x \in \mathfrak{g}$ is customarily denoted $\mathrm{ad}\, x : \mathfrak{g} \longrightarrow \mathfrak{g} \; ; \; y \mapsto [x, y]$. If $(\mathfrak{g}, [\,,\,])$ is a Lie algebra over k, and R is a commutative k-algebra, then $\mathfrak{g} \otimes_k R$ obtains the structure of a Lie algebra over R via $[x \otimes r, y \otimes s] := [x, y] \otimes rs$ for all $x, y \in \mathfrak{g}, \; r, s \in R$.

DEFINITION. A *restricted Lie algebra* $(\mathfrak{g}, [p])$ is a pair consisting of a Lie algebra $\mathfrak{g}$ and a map $[p] : \mathfrak{g} \longrightarrow \mathfrak{g} \,; x \mapsto x^{[p]}$ such that

(1) $\mathrm{ad}\, x^{[p]} = (\mathrm{ad}\, x)^p \quad \forall \; x \in \mathfrak{g}$,

(2) $(\alpha x)^{[p]} = \alpha^p x^{[p]} \quad \forall \; \alpha \in k, \; x \in \mathfrak{g}$,

(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, where the s_i are given by the identity $(\mathrm{ad}\,(x \otimes T + y \otimes 1))^{p-1}(x \otimes 1) = \sum_{i=1}^{p-1} i\, s_i(x, y) \otimes T^{i-1}$ in $\mathfrak{g} \otimes_k k[T]$.

A map $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying (1)-(3) is called a *p-map*.

If $(\mathfrak{g}, [p])$ is a restricted Lie algebra, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (an ideal $\mathfrak{m} \subseteq \mathfrak{g}$) is called a *p-subalgebra* (*a p-ideal*) if $x^{[p]} \in \mathfrak{h} \;\; \forall\, x \in \mathfrak{h}$ ($x^{[p]} \in \mathfrak{m} \;\; \forall\, x \in \mathfrak{m}$). The notions of homomorphisms and factor algebras of restricted Lie algebras are defined in the canonical fashion.

EXAMPLES. (1) Given an associative k-algebra Λ, its *commutator algebra* $(\Lambda^-, [\,,\,])$ with product $[x, y] = xy - yx$ is restricted with respect to the ordinary p-th power operator $x \mapsto x^p$.

(2) Let $\Lambda := \mathrm{Mat}_n(k)$ be the algebra of $(n \times n)$-matrices over k. The corresponding restricted Lie algebra, which we denote by $\mathfrak{gl}(n)$, is called the *general linear Lie algebra.* Note that the *special linear Lie algebra*

$$\mathfrak{sl}(n) := \{x \in \mathfrak{gl}(n) \;;\; \mathrm{tr}(x) = 0\}$$

is a p-ideal of $\mathfrak{gl}(n)$.

(3) For $n = 2$, we shall refer to the basis $\{e, h, f\}$ defined via

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;;\; h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;;\; f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

as the *standard basis* of $\mathfrak{sl}(2)$.

Suppose that $\mathfrak{g}$ is an *abelian* Lie algebra, i.e., $[x, y] = 0$ for $x,\, y \in \mathfrak{g}$. Then the p-maps on $\mathfrak{g}$ are just the p-semilinear maps. These are determined by their values on a basis. To see how this generalizes to arbitrary restricted Lie algebras, we quote several basic results, whose proofs can be found in [**77, 115**].

THEOREM 4.1. *Let $\mathfrak{g}$ be a Lie algebra with basis $(e_i)_{i \in I}$. Suppose there exist $x_i \in \mathfrak{g}$ $(i \in I)$ such that*

$$(\mathrm{ad}\, e_i)^p = \mathrm{ad}\, x_i \qquad \forall\; i \in I.$$

Then there exists a unique p-map $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $e_i^{[p]} = x_i$ for every $i \in I$. □

The foregoing result enables us to construct simple examples.

EXAMPLES. (1) We consider the (2n+1)-dimensional *Heisenberg algebra* $\mathfrak{h}_n$ with basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$. The Lie product is given by

$$[x_i, y_j] = \delta_{ij} z \;\;;\;\; [x_i, x_j] = 0 = [y_i, y_j] \;\;;\;\; [z, \mathfrak{h}_n] = (0).$$

We endow $\mathfrak{h}_n$ with the following p-maps:

(a) $x_i^{[p]} = 0 = y_i^{[p]} \;\;;\;\; z^{[p]} = 0.$

(b) $x_i^{[p]} = 0 = y_i^{[p]} \;\;;\;\; z^{[p]} = z.$

(2) Let $\mathfrak{g} = kt \oplus kx$, $[t, x] = x$. Then $\mathfrak{g}$ possesses exactly one p-map, namely the one satisfying

$$t^{[p]} = t \;\;;\;\; x^{[p]} = 0.$$

(3) We endow the vector space $\mathfrak{sl}(2) \oplus k$ with the structure of a Lie algebra by setting

$$[(x, \alpha), (y, \beta)] = [(x, y), 0] \qquad \forall\; x, y \in \mathfrak{sl}(2),\; \alpha, \beta \in k.$$

In other words, $\mathfrak{sl}(2) \oplus k$ is the direct sum of $\mathfrak{sl}(2)$ and the one-dimensional Lie algebra. This Lie algebra supports essentially three p-structures, defining four-dimensional restricted Lie algebras $\mathfrak{sl}(2)_0$, $\mathfrak{sl}(2)_s$, and $\mathfrak{sl}(2)_n$. In the following formulae we consider the basis e, h, f, v_0 of $\mathfrak{sl}(2) \oplus k$, where $v_0 := 1$:

(a) $\mathfrak{sl}(2)_0$: $e^{[p]} = 0 \;;\; h^{[p]} = h \;;\; f^{[p]} = 0 \;;\; v_0^{[p]} = 0.$

(b) $\mathfrak{sl}(2)_s$: $e^{[p]} = 0$; $h^{[p]} = h + v_0$; $f^{[p]} = 0$; $v_0^{[p]} = 0$.

(c) $\mathfrak{sl}(2)_n$: $e^{[p]} = 0$; $h^{[p]} = h$; $f^{[p]} = v_0$; $v_0^{[p]} = 0$.

We shall see later that the representation theories of these three restricted Lie algebras are rather different.

(4) Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, R be a commutative k-algebra. The extended Lie algebra $\mathfrak{g} \otimes_k R$ has the structure of a restricted R-Lie algebra via

$$(x \otimes r)^{[p]} := x^{[p]} \otimes r^p \quad \forall\, x \in \mathfrak{g},\ r \in R.$$

The following subsidiary result explains how many different p-maps can be defined on a given Lie algebra. We denote by

$$C(\mathfrak{g}) := \{x \in \mathfrak{g}\ ;\ [x, y] = 0 \quad \forall\, y \in \mathfrak{g}\}$$

the *center* of the Lie algebra $\mathfrak{g}$.

LEMMA 4.2. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, $\{p\} : \mathfrak{g} \longrightarrow \mathfrak{g}$ be a map. Then the following statements are equivalent:*

(1) *$\{p\}$ is a p-map.*

(2) *There exists a p-semilinear map $f : \mathfrak{g} \longrightarrow C(\mathfrak{g})$ such that $\{p\} = [p] + f$.* □

Let $\mathfrak{g}$ be a Lie algebra with *universal enveloping algebra* $(U(\mathfrak{g}), \iota)$. By definition, $U(\mathfrak{g})$ is an associative k-algebra and $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})^-$ is a homomorphism of Lie algebras satisfying the following universal property: for any associative k-algebra Λ and any homomorphism $f : L \longrightarrow \Lambda^-$ of Lie algebras there exists a unique homomorphism $\varphi : U(\mathfrak{g}) \longrightarrow \Lambda$ of associative algebras such that $\varphi \circ \iota = f$.

It will be convenient to employ multi-index notation. Let Λ be a k-algebra, $a := (a_1, \ldots, a_\ell) \in \Lambda^\ell$, and $i = (i_1, \ldots, i_\ell) \in \mathbb{N}_0^n$. Then we put

$$a^i := \prod_{j=1}^{\ell} a_j^{i_j}.$$

Given ℓ-tuples $r = (r_1, \ldots, r_\ell)$; $s = (s_1, \ldots, s_\ell) \in \mathbb{N}_0^\ell$, we define

$$r \leq s :\Leftrightarrow r_i \leq s_i \quad 1 \leq i \leq \ell.$$

We also put $\tau := (p - 1, \ldots, p - 1)$.

The following result is usually referred to as the PBW-Theorem:

THEOREM 4.3 (Poincaré-Birkhoff-Witt). *Let $\mathfrak{g}$ be a Lie algebra with basis $\{x_1, \ldots, x_\ell\}$. Then $\{\iota(x)^n\ ;\ n \in \mathbb{N}_0^\ell\}$ is a basis of $U(\mathfrak{g})$ over k.* □

In particular, $\iota : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is an embedding, and we will henceforth consider $\mathfrak{g}$ a subalgebra of $U(\mathfrak{g})^-$. Since $x \mapsto x \otimes 1 + 1 \otimes x$ is a homomorphism $\mathfrak{g} \longrightarrow (U(\mathfrak{g}) \otimes_k U(\mathfrak{g}))^-$ of Lie algebras, there is a unique extension $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes_k U(\mathfrak{g})$ to a homomorphism of associative k-algebras. By the same token, there exist unique homomorphisms $\eta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\mathrm{op}}$ and $\varepsilon : U(\mathfrak{g}) \longrightarrow k$ such that $\eta(x) = -x$ and $\varepsilon(x) = 0$ for every $x \in \mathfrak{g}$. Consequently, $U(\mathfrak{g})$ is a Hopf algebra that is generated by $\mathfrak{g}$.

Many features from the theory of finite-dimensional Hopf algebras lose their validity for $U(\mathfrak{g})$. For instance, the global dimension of $U(\mathfrak{g})$ coincides with $\dim_k \mathfrak{g}$. By contrast, Frobenius algebras never have finite non-zero global dimension. Moreover, $U(\mathfrak{g})$ is free of zero divisors. In view of

(1.3), this shows that $k \cdot 1$ is the only finite-dimensional Hopf subalgebra of $U(\mathfrak{g})$. Thus, while $\mathrm{Dist}(\mathcal{G}) \cong U(\mathrm{Lie}(\mathcal{G}))$ in case $\mathrm{char}(k) = 0$ (cf. [**31**, II.§6]), it will follow from (4.6) below that these algebras are not isomorphic over fields of positive characteristic.

Definition. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with universal enveloping algebra $U(\mathfrak{g})$. Let $I \subseteq U(\mathfrak{g})$ be the two-sided ideal generated by $\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\}$. Then

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/I$$

is called the *restricted enveloping algebra* of $\mathfrak{g}$.

If $(\mathfrak{g}, [p])$ is a restricted Lie algebra with basis $\{x_1, \ldots, x_\ell\}$, we define $z_i := x_i^p - x_i^{[p]} \in U(\mathfrak{g})$. Then one can modify the PBW-Theorem to show that $\{x^i z^j \ ; \ 0 \leq i \leq \tau, \ j \in \mathbb{N}_0^\ell\}$ is a basis of $U(\mathfrak{g})$ over k (cf. [**75**]). Denoting by $\iota : \mathfrak{g} \longrightarrow U_0(\mathfrak{g})$ the natural map, we thus obtain:

Corollary 4.4. *The set $\{\iota(x)^r \ ; \ 0 \leq r \leq \tau\}$ is a basis of $U_0(\mathfrak{g})$ over k. In particular, ι is injective, and $\dim_k U_0(\mathfrak{g}) = p^{\dim_k \mathfrak{g}}$.* □

Accordingly, we will henceforth consider $\mathfrak{g}$ a subalgebra of $U_0(\mathfrak{g})^-$. Since the ideal I is generated by primitive elements, I is a Hopf ideal, and $U_0(\mathfrak{g})$ inherits the Hopf algebra structure from $U(\mathfrak{g})$. As $U_0(\mathfrak{g})$ is generated by primitive elements, it is a cocommutative Hopf algebra. Note that $U_0(\mathfrak{g})$ has the following universal property: for any k-algebra Λ and any homomorphism $f : \mathfrak{g} \longrightarrow \Lambda^-$ of restricted Lie algebras, there exists a unique homomorphism $\varphi : U_0(\mathfrak{g}) \longrightarrow \Lambda$ of associative k-algebras such that $\varphi|_{\mathfrak{g}} = f$.

Example. For restricted enveloping algebras, modular functions and Nakayama automorphisms can be written down explicitly. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with basis $\{e_1, \ldots, e_n\}$, and corresponding basis $\{e^r \ ; \ 0 \leq r \leq \tau\}$ of $U_0(\mathfrak{g})$. We consider the linear form

$$\psi : U_0(\mathfrak{g}) \longrightarrow k \ \ ; \ \ \sum_{0 \leq r \leq \tau} \alpha_r e^r \mapsto \alpha_\tau .$$

Since each e_i is primitive, we have $\Delta(e^r) = \sum_{0 \leq s \leq r} \binom{r}{s} e^s \otimes e^{r-s}$, where $\binom{r}{s} := \prod_{i=1}^n \binom{r_i}{s_i}$. For $\varphi \in U_0(\mathfrak{g})^*$ we thus obtain

$$(\varphi * \psi)(e^r) = \sum_{0 \leq s \leq r} \binom{r}{s} \varphi(e^s)\psi(e^{r-s}) = \delta_{r,\tau}\varphi(1) = \varepsilon(\varphi)\psi(e^r).$$

Hence ψ is a left integral of the commutative Hopf algebra $U_0(\mathfrak{g})^*$. It was shown in [**45**] that $\psi(e^r x) = \psi((x + \mathrm{tr}(\mathrm{ad}\, x)1)e^r)$ for every $x \in \mathfrak{g}$. Owing to (1.4), the unique automorphism $\mu : U_0(\mathfrak{g}) \longrightarrow U_0(\mathfrak{g})$ satisfying $\mu(x) = x + \mathrm{tr}(\mathrm{ad}\, x)1$ for every $x \in \mathfrak{g}$ is a Nakayama automorphism of $U_0(\mathfrak{g})$. By the same token, the map $x \mapsto \mathrm{tr}(\mathrm{ad}\, x)1$ gives rise to the left modular function of $U_0(\mathfrak{g})$.

In contrast to group algebras of finite groups, restricted enveloping algebras are usually not symmetric. In fact, they are symmetric precisely when $\mathrm{tr}(\mathrm{ad}\, x) = 0$ for every $x \in \mathfrak{g}$, which was first observed by Schue [**110**].

Let $\mathcal{G}$ be a reduced affine algebraic k-group. Then $\mathcal{G}$ acts on $k\mathcal{G}_r$ via the adjoint representation Ad, and a combination of [**80**, (I.9.7)] and [**80**, (I.8.8)] shows that the character $g \mapsto \det(\mathrm{Ad}(g))$ defines a modular function of $k\mathcal{G}_r$. As noted in [**80**, (I.9.7)], these results are also valid for arbitrary infinitesimal groups of height 1. Our next result implies that this also yields the above formula for the Nakayama automorphism of $U_0(\mathfrak{g})$.

We will return to the structure of restricted Lie algebras in Chapter IV when we study schemes of tori. Presently, we are interested in the interpretation of Lie algebras as infinitesimal groups of height ≤ 1. We record the following basic result from ring theory:

Proposition 4.5. *Let $\Lambda' \subseteq \Lambda$ be a subring of the artinian ring Λ such that $\Lambda = \Lambda' + \mathrm{Rad}(\Lambda)$. Then we have $\Lambda = \Lambda'$.* □

Theorem 4.6. *Let $\mathcal{G}$ be an infinitesimal k-group.*
(1) *There is an embedding $U_0(\mathrm{Lie}(\mathcal{G})) \hookrightarrow k\mathcal{G}$ of Hopf algebras.*
(2) *The group $\mathcal{G}$ has height ≤ 1 if and only if $U_0(\mathrm{Lie}(\mathcal{G})) \cong k\mathcal{G}$.*

Proof. Let $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$. The universal property of $U_0(\mathfrak{g})$ guarantees the existence of an algebra homomorphism $\psi : U_0(\mathfrak{g}) \longrightarrow k\mathcal{G}$ such that $\psi|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$. Thus, $((\psi \otimes \psi) \circ \Delta)|_{\mathfrak{g}} = (\Delta \circ \psi)|_{\mathfrak{g}}$, $\varepsilon_{\mathcal{G}} \circ \psi = \varepsilon_{\mathfrak{g}}$, and $(\eta_{\mathcal{G}} \circ \psi)|_{\mathfrak{g}} = (\psi \circ \eta_{\mathfrak{g}})|_{\mathfrak{g}}$, so that ψ is in fact a homomorphism of Hopf algebras.

Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$ over k. By (4.4), the set $\{x^r \ ; \ 0 \leq r \leq \tau\}$ is a basis of $U_0(\mathfrak{g})$ over k. Let $\{\delta_r \ ; \ 0 \leq r \leq \tau\}$ be the dual basis of the commutative Hopf algebra $k[\mathfrak{g}] := U_0(\mathfrak{g})^*$. Direct computation shows that $\delta_r * \delta_s = \binom{r+s}{r}\delta_{r+s}$. If ϵ_i denotes the n-tuple with i-th entry 1 and all other entries zero, then the map $X_i \mapsto \delta_{\epsilon_i}$ induces an isomorphism $k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p) \cong k[\mathfrak{g}]$ of k-algebras. Moreover, $k[\mathfrak{g}]^\dagger = (\delta_{\epsilon_1}, \ldots, \delta_{\epsilon_n})$.

Recall that $\mathfrak{g}$ is the space of derivations $k[\mathcal{G}] \longrightarrow k$, which is isomorphic to $k[\mathcal{G}]^\dagger/(k[\mathcal{G}]^\dagger)^2$. Application of this argument to $k[\mathfrak{g}]$ shows that $\dim_k \mathrm{Lie}(U_0(\mathfrak{g})) = n$, so that $\mathfrak{g} = \mathrm{Lie}(U_0(\mathfrak{g}))$.

(1) Consider the transpose map $\psi^t : k[\mathcal{G}] \longrightarrow k[\mathfrak{g}]$. Since $\psi|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$, we see that ψ^t induces an isomorphism $k[\mathcal{G}]^\dagger/(k[\mathcal{G}]^\dagger)^2 \cong k[\mathfrak{g}]^\dagger/(k[\mathfrak{g}]^\dagger)^2$. Consequently,

$$k[\mathfrak{g}] = \psi^t(k[\mathcal{G}]) + (k[\mathfrak{g}]^\dagger)^2 = \psi^t(k[\mathcal{G}]) + \mathrm{Rad}(k[\mathfrak{g}])^2,$$

so that (4.5) implies that ψ^t is surjective. Consequently, ψ is injective.

(2) Suppose that $\mathcal{G}$ is infinitesimal of height ≤ 1. Then $x^p = 0$ for every $x \in k[\mathcal{G}]^\dagger$. Since $\dim_k \mathfrak{g} = n$, the local algebra $k[\mathcal{G}]$ is generated by n elements of $k[\mathcal{G}]^\dagger$. Thus, the resulting surjective homomorphism $k[X_1, \ldots, X_n] \longrightarrow k[\mathcal{G}]$ factors through the truncated polynomial ring $k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$. Hence $\dim_k \mathcal{G} = \dim_k k[\mathcal{G}] \leq p^n = \dim_k U_0(\mathfrak{g})$, and the injection ψ is surjective. □

Remark. Thanks to (4.6), the functors $\mathcal{G} \mapsto \mathrm{Lie}(\mathcal{G})$ and $\mathfrak{g} \mapsto \mathrm{Spec}_k(U_0(\mathfrak{g})^*)$ induce equivalences between the categories of infinitesimal groups of height ≤ 1 and restricted Lie algebras, respectively.

Proposition 4.7. *Let $\mathcal{G}$ be an infinitesimal k-group. Then there exists $n \in \mathbb{N}_0$ with $\dim_k k\mathcal{G} = p^n$.*

Proof. Let A be a local, commutative Hopf algebra. We show that A has dimension a power of p. We proceed inductively, and consider the Hopf subalgebra $B := \{a^p \ ; \ a \in A\}$. If $B = k1$, then $\mathcal{G} := \mathrm{Spec}_k(A)$ is an infinitesimal group of height ≤ 1 with function algebra A. Hence our assertion follows from a consecutive application of (4.4) and (4.6).

Alternatively, (1.6) implies that A is free over B. Moreover, $\mathrm{rk}_B(A) = \dim_k A/AB^\dagger$. Since $A/AB^\dagger$ is also a local Hopf algebra, the inductive hypothesis ensures that $\dim_k B$ and $\mathrm{rk}_B(A)$ are p-powers. Hence $\dim_k A = \mathrm{rk}_B(A) \dim_k B$ is also a p-power. □

5. Quotients

Let $\mathcal{G}$ be an affine algebraic k-group, $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup. We would like to define a quotient $\mathcal{G}/\mathcal{N}$. The naive approach, setting $\mathcal{G}/\mathcal{N}(R) := \mathcal{G}(R)/\mathcal{N}(R)$ does not work, as the resulting k-functor is not necessarily representable (see the example below). What we need is a *categorical quotient*, i.e., a map $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ with kernel $\mathcal{N}$ that satisfies the following universal property:

- If $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ is a homomorphism of affine group schemes such that $\mathcal{N} \subseteq \ker \varphi$, then there exists a unique homomorphism $\psi : \mathcal{G}/\mathcal{N} \longrightarrow \mathcal{H}$ such that $\psi \circ \pi = \varphi$.

Let us see what this condition amounts to: Let $\pi^* : k[\mathcal{G}/\mathcal{N}] \longrightarrow k[\mathcal{G}]$ be the associated Hopf algebra homomorphism. Setting $I := \ker \pi^*$, we consider the subgroup $\mathcal{H} := \mathcal{V}(I) \subseteq \mathcal{G}/\mathcal{N}$ as well as the canonical embedding $\iota : \mathcal{H} \longrightarrow \mathcal{G}/\mathcal{N}$. By construction, there is a homomorphism $\gamma^* : k[\mathcal{H}] \longrightarrow k[\mathcal{G}]$ of Hopf algebras such that $\gamma^* \circ \iota^* = \pi^*$. By the universal property, the corresponding homomorphism $\gamma : \mathcal{G} \longrightarrow \mathcal{H}$ induces a unique map $\omega : \mathcal{G}/\mathcal{N} \longrightarrow \mathcal{H}$ with $\omega \circ \pi = \gamma$. Accordingly, we have $\pi = \iota \circ \gamma = (\iota \circ \omega) \circ \pi$, and unicity implies $\iota \circ \omega = \mathrm{id}_{\mathcal{G}/\mathcal{N}}$. Thus, $\omega^* \circ \iota^* = \mathrm{id}_{k[\mathcal{G}/\mathcal{N}]}$, so that $I = \ker \iota^* = (0)$. We therefore make the following definition:

Definition. A homomorphism $\pi : \mathcal{G} \longrightarrow \mathcal{H}$ between two affine k-groups is a *quotient map* if the corresponding Hopf algebra map $\pi^* : k[\mathcal{H}] \longrightarrow k[\mathcal{G}]$ is injective.

It is by no means clear that for any normal subgroup $\mathcal{N} \trianglelefteq \mathcal{G}$ a quotient map with kernel $\mathcal{N}$ exists. It turns out that the algebra $k[\mathcal{G}]^{\mathcal{N}}$ of invariants gives rise to the quotient group. Thus, if $\mathcal{G}$ is algebraic, then the question as to whether $\mathcal{G}/\mathcal{N}$ inherits this property is related to Hilbert's fourteenth problem. Of course, for finite algebraic groups we don't need to worry about such issues.

Theorem 5.1. *Let $\mathcal{G}$ be an affine k-group, $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup. Then there exists a quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{H}$ with kernel $\mathcal{N}$. If $\mathcal{G}$ is algebraic, so is $\mathcal{H}$.* □

Remarks. (1) By the universal property, the pair $(\mathcal{H}, \pi)$ is unique up to isomorphism. One thus writes $\mathcal{G}/\mathcal{N} := \mathcal{H}$ and calls $\mathcal{G}/\mathcal{N}$ the *factor group of* $\mathcal{G}$ *by* $\mathcal{N}$.

(2) Note that the quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ is usually not surjective at each point. Suppose it is, then there exists $\lambda \in \mathcal{G}(k[\mathcal{G}/\mathcal{N}])$ such that $\pi(\lambda) = \mathrm{id}_{k[\mathcal{G}/\mathcal{N}]}$. Accordingly, the comorphism $\pi^* : k[\mathcal{G}/\mathcal{N}] \longrightarrow k[\mathcal{G}]$ is split injective.

Example. Consider the group $\mathbb{G}_{a(2)}$ with coordinate ring $k[\mathbb{G}_{a(2)}] = k[T]/(T^{p^2})$. We write $t := T + (T^{p^2})$ and observe that the primitive element t^p generates a Hopf ideal $I \trianglelefteq k[\mathbb{G}_{a(2)}]$. Thus, the corresponding normal subgroup $\mathcal{N} := \mathcal{V}(I)$ is isomorphic to $\mathbb{G}_{a(1)}$. Consider the group scheme

$$\mathcal{X} : M_k \longrightarrow \mathrm{Gr} \ \ ; \quad R \mapsto \mathcal{G}(R)/\mathcal{N}(R).$$

Then $\pi := (\pi_R : \mathcal{G}(R) \longrightarrow \mathcal{X}(R))_{R \in M_k}$ is a homomorphism of k-group functors. If $\mathcal{X} = \mathrm{Spec}_k(k[\mathcal{X}])$ is representable, then $\pi^* : k[\mathcal{X}] \longrightarrow k[\mathcal{G}]$ is split injective, and there exists an ideal $J \trianglelefteq k[\mathcal{G}]$ such that

$$k[\mathcal{G}] = k[\mathcal{X}] \oplus J.$$

Since $k[\mathcal{G}]$ is free over $k[\mathcal{X}]$, we have $\dim_k k[\mathcal{X}] = p$. Consequently, $x^p = 0$ for every $x \in k[\mathcal{X}]^\dagger$, so that $k[\mathcal{X}]^\dagger \subseteq \bigoplus_{i=p}^{p^2-1} kt^i$. The condition $\Delta(k[\mathcal{X}]) \subseteq k[\mathcal{X}] \otimes_k k[\mathcal{X}]$ then implies $k[\mathcal{X}] = k[t^p]$. Thus, $x^p \in k[\mathcal{X}]$ for every $x \in k[\mathcal{G}]$, and the p-th power map is trivial on J. Since $k[\mathcal{G}]^\dagger = k[\mathcal{X}]^\dagger \oplus J$, we obtain $x^p = 0$ for every $x \in k[\mathcal{G}]^\dagger$, a contradiction.

For finite algebraic groups, we could have defined quotients via the following argument: If $\mathcal{N} \trianglelefteq \mathcal{G}$ is a normal subgroup, then, as we shall see below, $k\mathcal{G}k\mathcal{N}^\dagger$ is a Hopf ideal of $k\mathcal{G}$. In view of the equivalence between the categories of finite group schemes and finite-dimensional cocommutative Hopf algebras, there exists a finite group scheme, $\mathcal{G}/\mathcal{N}$ say, such that

$$k(\mathcal{G}/\mathcal{N}) \cong k\mathcal{G}/(k\mathcal{G}k\mathcal{N}^\dagger).$$

We now show that this brute force definition agrees with the previous one. In this context, the following notion is convenient:

Let $\mathcal{G}$ be a finite algebraic k-group. If η denotes the antipode of $k\mathcal{G}$, then the action given by

$$h.x := \sum_{(h)} h_{(1)} x\, \eta(h_{(2)}) \qquad \forall\ h, x \in k\mathcal{G}$$

is called the *(left) adjoint representation of* $k\mathcal{G}$.

Proposition 5.2. *Let $\mathcal{G}$ be a finite algebraic k-group, $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup. Then $k\mathcal{G}k\mathcal{N}^\dagger$ is a Hopf ideal, and $k(\mathcal{G}/\mathcal{N}) \cong k\mathcal{G}/(k\mathcal{G}k\mathcal{N}^\dagger)$.*

Proof. Consider the natural map $\mathcal{G} \times \mathcal{N} \longrightarrow \mathcal{N}\ ;\ (g, n) \mapsto gng^{-1}$. By the universal property, there exists a unique linear map $k(\mathcal{G} \times \mathcal{N}) \longrightarrow k\mathcal{N}$ extending the above action. Direct computation, using $k(\mathcal{G} \times \mathcal{N}) \cong k\mathcal{G} \otimes_k k\mathcal{N}$, shows that this map is given by the adjoint representation. Accordingly,

$$h.x \in k\mathcal{N} \quad \forall\ h \in k\mathcal{G},\ x \in k\mathcal{N}.$$

Given $x \in k\mathcal{N}^\dagger$ and $h \in k\mathcal{G}$, we therefore have

$$xh = \sum_{(h)} \varepsilon(h_{(1)}) x h_{(2)} = \sum_{(h)} h_{(1)} \eta(h_{(2)}) x h_{(3)} = \sum_{(h)} h_{(1)} (\eta(h_{(2)}).x) \in k\mathcal{G}k\mathcal{N}^\dagger.$$

Thus, $k\mathcal{G}k\mathcal{N}^\dagger$ is a two-sided ideal, and it readily follows that it is also a Hopf ideal.

Let $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ and $\iota : \mathcal{N} \hookrightarrow \mathcal{G}$ be the quotient map and the canonical embedding, respectively. Since $\pi^* : k[\mathcal{G}/\mathcal{N}] \longrightarrow k[\mathcal{G}]$ is injective, the induced map $\hat{\pi} : k\mathcal{G} \longrightarrow k(\mathcal{G}/\mathcal{N})$ is surjective. By the same token, $\hat{\iota} : k\mathcal{N} \longrightarrow k\mathcal{G}$ is injective. Since $\pi \circ \iota \equiv 1$, we have $\hat{\pi} \circ \hat{\iota} = \varepsilon$. Consequently, $k\mathcal{G}k\mathcal{N}^\dagger \subseteq \ker \hat{\pi}$.

Consider the cocommutative Hopf algebra $H := k\mathcal{G}/(k\mathcal{G}k\mathcal{N}^\dagger)$ as well as $\mathcal{H} := \operatorname{Spec}_k(H^*)$. There results a factorization $\hat{\pi} = \hat{\zeta} \circ \hat{\gamma}$, with a surjective homomorphism $\hat{\gamma} : k\mathcal{G} \longrightarrow H$ of Hopf algebras. Note that $\hat{\gamma}$ corresponds to a homomorphism $\gamma : \mathcal{G} \longrightarrow \mathcal{H}$. Since $\hat{\gamma} \circ \hat{\iota} = \varepsilon$, we have $\gamma \circ \iota \equiv 1$, whence $\mathcal{N} \subseteq \ker \gamma$. The universal property now provides a homomorphism $\omega : \mathcal{G}/\mathcal{N} \longrightarrow \mathcal{H}$ such that $\omega \circ \pi = \gamma$. Consequently,

$$(\hat{\omega} \circ \hat{\zeta}) \circ \hat{\gamma} = \hat{\omega} \circ \hat{\pi} = \hat{\gamma},$$

so that the surjectivity of $\hat{\gamma}$ yields $\hat{\omega} \circ \hat{\zeta} = \mathrm{id}_H$. As a result, $\hat{\zeta}$ is injective, so that $\ker \hat{\pi} = \ker(\hat{\zeta} \circ \hat{\gamma}) = \ker \hat{\gamma} = k\mathcal{G}k\mathcal{N}^\dagger$. ∎

CHAPTER II

Complexity and Representation Type

Let Λ be a finite-dimensional k-algebra. By the theorem of Krull-Remak-Schmidt, each finite dimensional Λ-module M decomposes in an essentially unique fashion into a direct sum

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

of indecomposables. Thus, when studying the category $\operatorname{mod} \Lambda$ of finite-dimensional Λ-modules, attention first focusses on the classification of indecomposable Λ-modules. The classical theory of semi-simple algebras is concerned with the case, where all indecomposables are simple. Wedderburn's Theorem then shows that the block decomposition of Λ is particularly nice: There exist division algebras $\Delta_1, \dots, \Delta_r$ over k with

$$\Lambda \cong \operatorname{Mat}_{n_1}(\Delta_1) \oplus \cdots \oplus \operatorname{Mat}_{n_r}(\Delta_r).$$

Of course, determining the simple Λ-modules may still be rather difficult.

By definition, semi-simple algebras have (up to isomorphism) only finitely many indecomposables. In the 1950's, people working on representations of finite groups started to look at algebras with the latter property. Later, the notion of tameness was developed and it turned out that for algebras that are neither representation-finite nor tame, the abovementioned classification of indecomposable modules is rather hopeless.

Aside from providing the fundamental notions, this chapter introduces two important tools: The Heller operator and the complexity of a module. Our basic references are [**8, 4, 12, 13**].

1. The Heller Operator

Let Λ be a finite-dimensional self-injective algebra, $M \in \operatorname{mod} \Lambda$. Given a minimal projective resolution $(P_i, \partial_i)_{i \geq 0}$ of M, the syzygies $\Omega^n_\Lambda(M) := \ker \partial_{n-1}$ $(n \geq 1)$ are uniquely determined up to isomorphism. Hence $\Omega_\Lambda := \Omega^1_\Lambda$ is a well-defined operator on the isoclasses of Λ-modules. This operator is customarily referred to as the *Heller operator* or *loop space operator*. Directly from the definition we obtain $\Omega^m_\Lambda \circ \Omega^n_\Lambda = \Omega^{m+n}_\Lambda$ for $m, n \geq 0$. Note that Ω_Λ induces a functor on the *stable module category* $\underline{\operatorname{mod}}\Lambda$. This category has the same objects as $\operatorname{mod} \Lambda$ and morphisms

$$\underline{\operatorname{Hom}}_\Lambda(M, N) := \operatorname{Hom}_\Lambda(M, N)/P(M, N),$$

where $P(M, N)$ is the subspace of those homomorphisms that factor through a projective module.

Dually, we can construct for $n \geq 1$ operators Ω^{-n}_Λ by setting $\Omega^{-n}_\Lambda(M) :=$ coker ∂^{n-1}, where $(E_i, \partial^i)_{i \geq 0}$ is a minimal injective resolution of M.

Given $M \in \operatorname{mod} \Lambda$, the theorem of Krull-Remak-Schmidt yields a decomposition

$$M = M_{\text{pf}} \oplus (\text{proj.}),$$

in which the first summand is the sum of all non-projective indecomposable constituents of M. In the following, we shall consider the full subcategory $(\operatorname{mod} \Lambda)_{\text{pf}}$ of $\operatorname{mod} \Lambda$ consisting of those finite-dimensional Λ-modules M for which $M = M_{\text{pf}}$. Since Λ is self-injective, each object of $(\operatorname{mod} \Lambda)_{\text{pf}}$ is characterized by the property that it admits no non-zero projective submodules.

Since $\Omega^n_\Lambda(M)$ is defined via a minimal projective resolution of M, Schanuel's Lemma implies that, given an arbitrary projective resolution $(P_i, \partial_i)_{i\geq 0}$ of M, the module $\Omega^n_\Lambda(M)$ is a direct summand of $\ker \partial_{n-1}$. Moreover, $\Omega^n_\Lambda(M) \in (\mathrm{mod}\,\Lambda)_{\mathrm{pf}}$ for all $n \in \mathbb{Z} \setminus \{0\}$, so that $\Omega^n_\Lambda(M) \cong (\ker \partial_{n-1})_{\mathrm{pf}}$. The objects of $(\mathrm{mod}\,\Lambda)_{\mathrm{pf}}$ are referred to as *projective-free* modules.

LEMMA 1.1. *Let M, $N \in (\mathrm{mod}\,\Lambda)_{\mathrm{pf}}$. Then the following statements hold:*
(1) $\Omega^{-1}_\Lambda(\Omega_\Lambda(M)) \cong M \cong \Omega_\Lambda(\Omega^{-1}_\Lambda(M))$
(2) $\Omega_\Lambda(M \oplus N) \cong \Omega_\Lambda(M) \oplus \Omega_\Lambda(N)$.
(3) $\Omega^{-1}_\Lambda(M \oplus N) \cong \Omega^{-1}_\Lambda(M) \oplus \Omega^{-1}_\Lambda(N)$.
(4) *M is indecomposable if and only if $\Omega_\Lambda(M)$ is indecomposable.*
(5) *M is indecomposable if and only if $\Omega^{-1}_\Lambda(M)$ is indecomposable.*

PROOF. (1) We show that the projective cover (P_M, ε_M) of M is an injective hull of $\Omega_\Lambda(M)$. We denote this injective hull by I. Since Λ is self-injective, P_M is an injective Λ-module. Hence there exists a Λ-linear map $f : I \longrightarrow P_M$ such that $f|_{\Omega_\Lambda(M)}$ is the canonical inclusion $\Omega_\Lambda(M) \hookrightarrow P_M$. Thus, f is injective, and there exists a Λ-linear map $\sigma : P_M \longrightarrow I$ such that $\sigma \circ f = \mathrm{id}_I$. We thus obtain a decomposition $P_M = (\ker \sigma) \oplus (\mathrm{im}\, f)$. Since $\ker \varepsilon_M = \Omega_\Lambda(M) \subseteq \mathrm{im}\, f$, the map $\varepsilon|_{\ker \sigma} : \ker \sigma \longrightarrow M$ is injective and M contains the injective module $\ker \sigma$. Hence $\ker \sigma$ is a direct summand of M and our assumption $M \in (\mathrm{mod}\,\Lambda)_{\mathrm{pf}}$ yields $\ker \sigma = (0)$. Consequently, $P_M \cong I$ is the injective hull of $\Omega_\Lambda(M)$, so that $\Omega^{-1}_\Lambda(\Omega_\Lambda(M) \cong M$. The other assertion is proved analogously.

(4) Suppose M to be indecomposable, and write $\Omega_\Lambda(M) = X \oplus Y$. Since $\Omega^{-1}_\Lambda(\Omega_\Lambda(M)) \cong M$, (3) yields
$$M \cong \Omega^{-1}_\Lambda(X) \oplus \Omega^{-1}_\Lambda(Y),$$
so that we may assume without loss of generality that $\Omega^{-1}_\Lambda(X) = (0)$. As a result, $X = X_{\mathrm{pf}} \cong \Omega_\Lambda(\Omega^{-1}_\Lambda(X)) = (0)$, proving that $\Omega_\Lambda(M)$ is indecomposable. □

REMARK. The condition of self-injectivity is essential for the validity of (4). Consider for instance the three-dimensional local algebra $\Lambda := k[X, Y]/(X^2, XY, Y^2)$. If we define the Heller operator as above, then $\Omega_\Lambda(k) \cong \mathrm{Rad}(\Lambda)$ is isomorphic to the semi-simple Λ-module $k \oplus k$.

In the sequel, we shall write $M \cong N \oplus (\mathrm{proj.})$ to indicate that M is isomorphic to the direct sum of N and an unspecified projective module.

LEMMA 1.2. *Let Λ be a finite-dimensional Hopf algebra, M, N be finite-dimensional Λ-modules. Then we have*
$$\Omega^r_\Lambda(M) \otimes_k \Omega^s_\Lambda(N) \cong \Omega^{r+s}_\Lambda(M \otimes_k N) \oplus (\mathrm{proj.})$$
for every $r, s \geq 0$.

PROOF. We first consider the case, where $s = 0$. Let $(P_n, \partial_n)_{n\geq 0}$ be a minimal projective resolution of M. Thanks to (I.1.5), $(P_n \otimes_k N, \partial_n \otimes \mathrm{id}_N)_{n\geq 0}$ is a projective resolution of $M \otimes_k N$, so that our observations above imply
$$\Omega^r_\Lambda(M) \otimes_k N \cong (\ker \partial_{r-1}) \otimes_k N \cong \ker(\partial_{r-1} \otimes \mathrm{id}_N) \cong \Omega^r_\Lambda(M \otimes_k N) \oplus (\mathrm{proj.}).$$
Applying the analogous result for $r = 0$, we obtain
$$\begin{aligned}\Omega^r_\Lambda(M) \otimes_k \Omega^s_\Lambda(N) &\cong \Omega^r_\Lambda(M \otimes_k \Omega^s_\Lambda(N)) \oplus (\mathrm{proj.}) \cong \Omega^r_\Lambda(\Omega^s_\Lambda(M \otimes_k N) \oplus (\mathrm{proj.})) \oplus (\mathrm{proj.})\\ &\cong \Omega^{r+s}_\Lambda(M \otimes_k N) \oplus (\mathrm{proj.}),\end{aligned}$$
as desired. □

2. Complexity

The notion of the complexity of a module, first introduced by Alperin for group algebras of finite groups [**1**], has proven to be an effective tool in representation theory. Like the Heller operator, it makes methods from homological algebra amenable to applications.

Let $(a_i)_{i\geq 0}$ be a sequence of natural numbers. We call

$$\operatorname{gr}((a_i)_{i\geq 0}) := \min\{s \in \mathbb{N} \cup \{\infty\} \ ; \ \exists\ \lambda > 0 \text{ such that } a_n \leq \lambda n^{s-1} \quad \forall\ n \geq 1\}$$

the *polynomial rate of growth* of the sequence $(a_i)_{i\geq 0}$. If $\mathcal{V} := (V_i)_{i\geq 0}$ is a sequence of finite-dimensional k-vector spaces, then we write $\operatorname{gr}(\mathcal{V}) := \operatorname{gr}((\dim_k V_i)_{i\geq 0})$.

Throughout this section, Λ is assumed to be a finite-dimensional k-algebra.

Definition. Let M be a finite-dimensional Λ-module, $\mathcal{P} := (P_i, \partial_i)_{i\geq 0}$ be a minimal projective resolution of M. Then $\operatorname{cx}_\Lambda(M) := \operatorname{gr}(\mathcal{P})$ is called the *complexity* of M.

Remarks. (1) Since any two minimal projective resolutions are isomorphic, the complexity of a module is well-defined.

(2) The modules of complexity zero are precisely the modules of finite projective dimension. Thus, if Λ is self-injective, then a Λ-module M has complexity zero if and only if it is projective.

By our last observation, semi-simple algebras are characterized by the property that they are self-injective with all their modules having complexity zero. In the following chapters we want to provide similar characterizations for algebras of finite- and tame representation types. We begin with an interpretation of the complexity in terms of extension groups. Let $\mathcal{S}$ be a complete set of representatives for the isomorphism classes of the simple Λ-modules. The projective cover of the simple Λ-module S will be denoted $P(S)$.

The following result, due to Alperin-Evens [**3**], relates the complexity of a module to the growth of certain Ext-groups.

Proposition 2.1. *Let M be a finite-dimensional Λ-module. Then*

$$\operatorname{cx}_\Lambda(M) = \max_{S\in\mathcal{S}} \operatorname{gr}((\operatorname{Ext}^n_\Lambda(M,S))_{n\geq 0}).$$

Proof. Given a minimal projective resolution $(P_n)_{n\geq 0}$ of M, we decompose each P_n into its indecomposable constituents and write $P_n \cong \bigoplus_{T\in\mathcal{S}} \ell_{n,T} P(T)$. Basic properties of Ext yield

$$\dim_k \operatorname{Ext}^n_\Lambda(M,S) = \sum_{T\in\mathcal{S}} \ell_{n,T} \dim_k \operatorname{Hom}_\Lambda(P(T),S) = \ell_{n,S} \dim_k \operatorname{Hom}_\Lambda(S,S)$$

for every $S \in \mathcal{S}$. Consequently,

$$\operatorname{cx}_\Lambda(M) = \max_{S\in\mathcal{S}} \operatorname{gr}((\ell_{n,S})_{n\geq 0}) = \max_{S\in\mathcal{S}} \operatorname{gr}((\operatorname{Ext}^n_\Lambda(M,S))_{n\geq 0}),$$

as desired. □

In certain cases, the rate of growth of a sequence $\mathcal{V} = (V_i)_{i\geq 0}$ of finite-dimensional k-vector spaces can also be characterized via power series. The *Poincaré series* $P_\mathcal{V}(t) \in k[[t]]$ is defined via

$$P_\mathcal{V}(t) = \sum_{i\geq 0} (\dim_k V_i)\, t^i.$$

If $\bigoplus_{i\geq 0} V_i$ is a finitely generated module, over a finitely generated, commutative, graded k-algebra $A = \bigoplus_{i\geq 0} A_i$, then the Hilbert-Serre Theorem provides a polynomial $f(t) \in k[t]$ such that

$$P_{\mathcal{V}}(t) = \frac{f(t)}{\prod_{j=1}^{s}(1-t^{k_j})},$$

where $k_1, \dots, k_s$ are the degrees homogeneous generators of A as an A_0-algebra. It then follows that $\mathrm{gr}(\mathcal{V})$ is the order of the pole of $P_{\mathcal{V}}(t)$ at $t = 1$, cf. [**13**, (5.3)] for more details.

We continue by collecting a few basic properties of the complexity of modules.

PROPOSITION 2.2. *Let $\Gamma \subseteq \Lambda$ be a subalgebra, $M \in \mathrm{mod}\,\Lambda$, $N \in \mathrm{mod}\,\Gamma$.*
(1) *If $S_1, \dots, S_n$ are the composition factors of M, then $\mathrm{cx}_\Lambda(M) \leq \max_{1\leq i\leq n} \mathrm{cx}_\Lambda(S_i)$.*
(2) *If Λ is a projective left Γ-module, then $\mathrm{cx}_\Gamma(M) \leq \mathrm{cx}_\Lambda(M)$.*
(3) *If Λ is a projective right Γ-module, then $\mathrm{cx}_\Lambda(\Lambda \otimes_\Gamma N) \leq \mathrm{cx}_\Gamma(N)$.*
(4) *If Λ is a Hopf algebra, then $\mathrm{cx}_\Lambda(M) \leq \mathrm{cx}_\Lambda(k)$.*

PROOF. (2),(3) Let $\mathcal{P} := (P_i)_{i\geq 0}$ be a minimal projective resolution of the Λ-module M. By assumption, the functor $\mathrm{Hom}_\Gamma(P_i, -) \cong \mathrm{Hom}_\Lambda(P_i, \mathrm{Hom}_\Gamma(\Lambda, -))$ is exact. Hence each P_i is a projective Γ-module, and $\mathrm{cx}_\Gamma(M) \leq \mathrm{gr}(\mathcal{P}) = \mathrm{cx}_\Lambda(M)$.

If $\mathcal{Q} := (Q_i)_{i\geq 0}$ is a minimal projective resolution of N, then the adjoint isomorphism

$$\mathrm{Hom}_\Lambda(\Lambda \otimes_\Gamma Q_i, -) \cong \mathrm{Hom}_\Gamma(Q_i, -)$$

yields the exactness of the left-hand functor, so that $\Lambda\otimes_\Gamma \mathcal{Q} := (\Lambda\otimes_\Gamma Q_i)_{i\geq 0}$ is a projective resolution of the induced module $\Lambda \otimes_\Gamma N$. Consequently, $\mathrm{cx}_\Lambda(\Lambda \otimes_\Gamma N) \leq \mathrm{gr}(\Lambda \otimes_\Gamma \mathcal{Q}) \leq \mathrm{gr}((\Lambda \otimes_k Q_i)_{i\geq 0}) = \mathrm{cx}_\Gamma(N)$.

(4) Let $\mathcal{P} := (P_i)_{i\geq 0}$ be a minimal projective resolution of the trivial Λ-module k. Thanks to (I.1.5), the complex $\mathcal{P} \otimes_k M := (P_i \otimes_k M)_{i\geq 0}$ is a projective resolution of M. Hence $\mathrm{cx}_\Lambda(M) \leq \mathrm{gr}(\mathcal{P} \otimes_k M) = \mathrm{cx}_\Lambda(k)$. □

EXAMPLE. Suppose that $\mathrm{char}(k) = p > 0$, and consider the Hopf algebra

$$k\mathbb{G}_{a(2)} \cong k[X, Y]/(X^p, Y^p) \cong k[X]/(X^p) \otimes_k k[Y]/(Y^p).$$

Let $\mathcal{P} = (P_i)_{i\geq 0}$ be a minimal projective resolution of the trivial $k[X]/(X^p)$-module k, i.e., $P_i = k[X]/(X^p)$ for every $i \geq 0$. Setting $Q_i := \sum_{j=0}^{i} P_j \otimes_k P_{i-j}$, we obtain a minimal projective resolution $\mathcal{Q} := (Q_i)_{i\geq 0}$ of the $k\mathbb{G}_{a(2)}$-module $k \otimes_k k \cong k$. Since $\dim_k Q_i = (i+1)p^2$, we have $\mathrm{cx}_{k\mathbb{G}_{a(2)}}(k) = 2$. One can iterate this process to see that $\mathrm{cx}_{k\mathbb{G}_{a(r)}}(k) = r$.

In view of Proposition 2.1, we also have

$$\mathrm{cx}_{k\mathbb{G}_{a(r)}}(k) = \mathrm{gr}(\mathrm{H}^*(\mathbb{G}_{a(r)}, k)).$$

The structure of the cohomology ring $\mathrm{H}^*(\mathbb{G}_{a(r)}, k)$ is well-understood: For $p > 2$ there exists an isomorphism

$$\mathrm{H}^*(\mathbb{G}_{a(r)}, k) \cong k[X_1, \dots X_r, Y_1, \dots, Y_r]/(Y_1^2, \dots, Y_r^2) \cong \bigotimes_{i=1}^{r} k[X_i] \otimes_k \bigotimes_{j=1}^{r} k[Y_j]/(Y_j^2),$$

of graded vector spaces, where $\deg(X_i) = 2$ and $\deg(Y_i) = 1$, see [**22**, (7.6)]. As a result, the Poincaré series of $\mathrm{H}^*(\mathbb{G}_{a(r)}, k)$ has the form

$$P(t) = \prod_{i=1}^{r} P_{K[X_i]}(t) \prod_{j=1}^{r} P_{K[Y_j]/(Y_j^2)}(t) = \prod_{i=1}^{r} \frac{1}{1-t^2} \prod_{j=1}^{r}(1+t) = \frac{1}{(1-t)^r},$$

so that $\mathrm{gr}(\mathrm{H}^*(\mathbb{G}_{a(r)}, k)) = r$.

As noted earlier, the algebra $k\mathbb{G}_{a(r)}$ is isomorphic to the group algebra $k(\mathbb{Z}/(p))^r)$ of the p-elementary abelian group of rank r. Such groups play an important rôle in the representation theory of finite groups.

THEOREM 2.3 ([**3**]). *Suppose that* $\mathrm{char}(k) = p > 0$. *Let G be a finite group, M be a finite-dimensional G-module. Then*

$$\mathrm{cx}_{kG}(M) = \max_{E\in\mathfrak{E}} \mathrm{cx}_{kE}(M),$$

where $\mathfrak{E}$ is the set of p-elementary abelian subgroups of G. □

DEFINITION. A finite-dimensional non-projective indecomposable Λ-module M is said to be *periodic* if there exists $n > 0$ such that $\Omega^n_\Lambda(M) \cong M$.

EXAMPLES. (1) Suppose that $\mathrm{char}(k) = p > 0$. Let $\Lambda = k\mathbb{G}_{a(1)} = k[X]/(X^p)$ and write $x := X + (X^p)$. Since

$$(0) \longrightarrow kx^{p-1} \longrightarrow k[X]/(X^p) \xrightarrow{x\cdot} k[X]/(X^p) \xrightarrow{\varepsilon} k \longrightarrow (0)$$

defines the initial terms of a minimal projective resolution of the trivial module k, we have $\Omega^2_{k\mathbb{G}_{a(1)}}(k) \cong k$.

(2) Let $\mathfrak{b} := kt \oplus kx$ be the two-dimensional non-abelian restricted Lie algebra with product and p-map given by

$$[t, x] = x \ \text{ and } \ t^{[p]} = t \ \ ; \ \ x^{[p]} = 0,$$

respectively. The simple $U_0(\mathfrak{b})$-modules are of the form k_λ, where the character $\lambda : U_0(\mathfrak{b}) \longrightarrow k$ is defined via

$$\lambda(x) = 0 \ \ ; \ \ \lambda(t) \in \mathbb{F}_p.$$

We therefore identify the character group of $U_0(\mathfrak{g})$ with $\mathbb{Z}/(p)$, and write the convolution as addition. Let $\underline{1} : U_0(\mathfrak{b}) \longrightarrow k$ be the character defined by $\underline{1}(x) = 0$; $\underline{1}(t) = 1$. Since $P(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(kt)} k_\lambda$ is indecomposable projective, with $\mathrm{Rad}(P(\lambda))/\mathrm{Rad}^2(P(\lambda)) \cong k_{\lambda+\underline{1}}$ and $\mathrm{Soc}(P(\lambda)) \cong k_{\lambda-\underline{1}}$, there results an exact sequence

$$(0) \longrightarrow k_\lambda \longrightarrow P(\lambda + \underline{1}) \longrightarrow P(\lambda) \longrightarrow k_\lambda \longrightarrow (0),$$

whence $\Omega^2_{U_0(\mathfrak{b})}(k_\lambda) \cong k_\lambda$.

(3) Assume that $p \geq 3$, and consider the restricted Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$. The p-subalgebra $\mathfrak{b} := kh \oplus ke$ is isomorphic to the algebra of (2). Given $\lambda : U_0(\mathfrak{b}) \longrightarrow k$, we let

$$Z(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_\lambda$$

be the baby Verma module with highest weight λ. Since $U_0(\mathfrak{g})$ is a projective right $U_0(\mathfrak{b})$-module, the exact sequence of (2) yields a short exact sequence

$$(0) \longrightarrow Z(\lambda) \longrightarrow U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} P(\lambda + \underline{1}) \longrightarrow U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} P(\lambda) \longrightarrow Z(\lambda) \longrightarrow (0),$$

whose two middle terms are projective. Consequently,

$$\Omega^2_{U_0(\mathfrak{g})}(Z(\lambda)) \oplus (\mathrm{proj.}) \cong Z(\lambda).$$

As $Z(\lambda)$ is indecomposable, it is thus either periodic or projective. It turns out that projectivity occurs precisely when $\lambda(t) = p - 1$. The corresponding Verma module is the *Steinberg module* St.

REMARK. Note that every periodic Λ-module M has complexity $\mathrm{cx}_\Lambda(M) \leq 1$. Indeed, periodicity implies the existence of a minimal projective resolution $(P_i, \partial_i)_{i\geq 0}$ satisfying $P_{i+r} \cong P_i$ for some $r > 0$. Hence the dimensions of the P_i are bounded, and $\mathrm{cx}_\Lambda(M) \leq 1$. The converse is not true in general, as the example of the four-dimensional truncated quantum plane $\Lambda_q := k\langle x, y\rangle/(\{yx - qxy, x^2, y^2\})$ shows: The algebra Λ_q is a Frobenius algebra and the unique automorphism $\mu : \Lambda_q \longrightarrow \Lambda_q$ with $x \mapsto qx$ and $y \mapsto q^{-1}y$ is a Nakayma automorphism. Given $t \in k \setminus \{0\}$, we consider the two-dimensional indecomposable Λ_q-module $M_t := \Lambda_q(x + ty)$. Direct computations shows $\Omega_{\Lambda_q}(M_t) \cong M_{-q^{-1}t}$. Consequently, $\mathrm{cx}_{\Lambda_q}(M_t) = 1$, while M_t is not periodic whenever q is not a root of unity.

3. Representation Type

In this section we subdivide the category of finite-dimensional associative algebras according to their representation type, a notion that provides a measure of how complicated the module category $\mathrm{mod}\,\Lambda$ of a k-algebra Λ is. For self-injective algebras the representation type turns out to be related to upper bounds on the complexity of modules.

DEFINITION. An algebra Λ is *representation-finite* or of *finite representation type* if it admits only finitely many isoclasses of finite-dimensional indecomposable modules.

From linear algebra we know that the indecomposable modules of the truncated polynomial ring $k[X]/(X^n)$ are cyclic, so that such an algebra has finite representation type. In fact, the truncated polynomial rings in one variable belong to a particularly tractable class of representation-finite algebras, the so-called Nakayama algebras.

Given a finite-dimensional Λ-module M, the *Loewy length* $\ell\ell(M)$ is defined via

$$\ell\ell(M) := \min\{i \geq 0 \ ; \ \mathrm{Rad}(\Lambda)^i M = (0)\}.$$

We say that M is *uniserial* if it possesses exactly one composition series. Equivalently, the *Loewy series* $(\mathrm{Rad}(\Lambda)^i M)_{i\geq 0}$ is a composition series of M.

DEFINITION. An algebra Λ is called a *Nakayama algebra* if all of its projective indecomposable modules and all of its injective indecomposable modules are uniseral.

The usual duality between injective left Λ-modules and projective right Λ-modules implies that Λ is a Nakayama algebra if and only if the projective indecomposable modules for Λ and its opposite algebra Λ^{op} are uniserial. Consequently, factor algebras of Nakayama algebras by powers of their radicals are also Nakayama algebras.

By work of Kupisch [**84**], Nakayama algebras are well-understood. We will return to the description of their Morita equivalence classes once we have discussed path algebras of quivers. For the moment we record the following basic result.

PROPOSITION 3.1. *Let Λ be a Nakayama algebra. Then every indecomposable Λ-module is uniserial, and Λ has finite representation type.*

Proof. Let J be the radical of Λ. We prove the first assertion by induction on the Loewy length $\ell\ell(\Lambda)$ of Λ, the case $\ell\ell(\Lambda) = 1$ being trivial. Assuming $\ell := \ell\ell(\Lambda) \geq 2$, we consider an indecomposable Λ-module M. If $J^{\ell-1}M = (0)$, then M is an indecomposable module for the Nakayama algebra $\Lambda/J^{\ell-1}$, and the inductive hypothesis yields the assertion. Alternatively, there exists a simple left ideal $S \subseteq J^{\ell-1}$ with $SM \neq (0)$. We can therefore find $m \in M \setminus \{0\}$ such that

$$\psi_m : S \longrightarrow M \quad ; \quad s \mapsto s.m$$

is injective. Hence there is a map $\hat{\psi}_m : M \longrightarrow E(S)$ to the injective envelope $E(S)$ of S, whose composite with ψ_m is the canonical inclusion $S \hookrightarrow E(S)$. As $E(S)$ is uniserial, we can find $i \geq 0$ with $\hat{\psi}_m(M) = J^iE(S)$. Consequently, $J^{\ell-i}M \subseteq \ker \hat{\psi}_m$, while $J^{\ell-1}M \not\subseteq \ker \hat{\psi}_m$. As a result, $i = 0$, so that $\hat{\psi}_m$ is surjective and $J^{\ell-1}E(S) \neq (0)$. Since the uniserial projective cover $\pi : P \longrightarrow E(S)$ of $E(S)$ satisfies $\ell(P) = \ell\ell(P) \leq \ell = \ell\ell(E(S)) = \ell(E(S))$, we have $P \cong E(S)$. As M is indecomposable, it now follows that $\hat{\psi}_m$ is an isomorphism. Thus, M is uniserial.

As an upshot of the above, every indecomposable Λ-module M has a simple top and is thus of the form

$$M \cong P(S)/J^iP(S) \quad ; \quad 0 \leq i \leq \ell\ell(\Lambda),$$

for some simple Λ-module S. Consequently, Λ has finite representation type. □

Remark. Let Λ be a local algebra of finite representation type. Then $\dim_k \operatorname{Rad}(\Lambda)/\operatorname{Rad}^2(\Lambda) \leq 1$, so that Λ is a truncated polynomial ring $k[X]/(X^n)$. In particular, every commutative algebra of finite representation type is a direct sum of truncated polynomial rings: $\Lambda \cong \bigoplus_{i=1}^m k[X]/(X^{n_i})$.

Here is a simple necessary condition for a self-injective algebra to have finite representation type.

Theorem 3.2 (Heller [**68**]). *Let Λ be self-injective. If Λ has finite representation type, then every indecomposable non-projective Λ-module is periodic.*

Proof. Let $\mathfrak{X}$ be the set of isomorphism types of the non-projective indecomposable Λ-modules. Owing to (1.1), the map $\Omega_\Lambda|_{\mathfrak{X}}$ is bijective. Since $\mathfrak{X}$ is finite, there exists $n \geq 1$ such that $\Omega^n_\Lambda|_{\mathfrak{X}} = \mathrm{id}_{\mathfrak{X}}$. This implies our result. □

The converse of Heller's Theorem does not hold in general. For instance, the group algebra of the quaternion group

$$Q := \langle x, y \quad ; \quad x^2 = y^2 \ , \ yxy^{-1} = x^{-1} \rangle$$

of order 8 over a field of characteristic 2 is known to possess only periodic modules: One verifies $\Omega^4_{kQ}(k) \cong k$ (see [**25**, (XII.7)]) and applies (1.2) to obtain $\Omega^4_{kQ}(M) \cong M$ for every non-projective indecomposable kQ-module. By the above remark, the non-commutative local algebra kQ does not have finite representation type (in fact, it is tame). In Chapter VI we shall see that distribution algebras of infinitesimal groups afford the converse of Heller's Theorem.

Among the algebras of infinite representation type one distinguishes between those, whose indecomposable modules can be parametrized by one-dimensional varieties and those requiring more parameters.

Definition. A k-algebra Λ is said to be *tame* if it is not representation-finite, and if for every $d > 0$ there exist $(\Lambda, k[X])$-bimodules $M_1, \ldots, M_{n(d)}$ that are finitely generated and free over $k[X]$, so that all but finitely many isoclasses of the d-dimensional indecomposable Λ-modules are of the form $[M_i \otimes_{k[X]} k_\lambda]$ for some $i \in \{1, \ldots, n(d)\}$ and $\lambda \in \mathrm{Alg}_k(k[X], k)$.

REMARK. The tame local algebras were classified by Ringel [**106**]. The list of the tame symmetric local algebras, which plays a prominent rôle in the classification of the tame blocks of group algebras of finite groups, can be found in [**36**, Chap.III].

An easily described class of tame algebras is given by the so-called biserial algebras. If M is a non-simple, indecomposable Λ-module, then $\mathrm{Soc}(M) \subseteq \mathrm{Rad}(M)$, and we let

$$\mathrm{Ht}(M) := \mathrm{Rad}(M)/\,\mathrm{Soc}(M)$$

be the *heart* of M.

DEFINITION. A self-injective algebra Λ is *biserial* if for every non-simple projective indecomposable Λ-module P the heart $\mathrm{Ht}(P)$ is a direct sum of at most two uniserial modules.

We will frequently encounter biserial algebras when studying infinitesimal groups of tame representation type. In a more classical context, any Brauer tree algebra (that is, any representation-finite block of a group algebra kG) is biserial (cf. [**2**] for more details). For the moment we record the following:

THEOREM 3.3 ([**28**]). *A biserial algebra is tame or representation-finite.* □

The notion of a wild algebra, introduced by Drozd [**33**], captures a phenomenon first observed by Corner and Brenner: Given any k-algebra Λ, there exists a full exact embedding

$$\mathcal{F}_\Lambda : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,k\langle x, y\rangle$$

from the module category of Λ to the module category of the free algebra $k\langle x, y\rangle$ with two generators.

DEFINITION. A k-algebra Λ is referred to as *wild* if there exists a $(\Lambda, k\langle x, y\rangle)$-bimodule M, that is finitely generated and free over $k\langle x, y\rangle$, such that the functor

$$\mathrm{mod}\,k\langle x, y\rangle \longrightarrow \mathrm{mod}\,\Lambda \quad ; \quad X \mapsto M \otimes_{k\langle x,y\rangle} X$$

preserves indecomposables and reflects isomorphisms.

Thus, if Λ is wild and Γ is any k-algebra, then the functor

$$\mathrm{mod}\,\Gamma \longrightarrow \mathrm{mod}\,\Lambda \quad ; \quad V \mapsto M \otimes_{k\langle x,y\rangle} \mathcal{F}_\Gamma(V)$$

induces an injection between the isomorphism classes of indecomposable modules for Γ and Λ, respectively. Accordingly, a complete classification of the indecomposable modules of a wild algebra is a rather hopeless endeavour.

THEOREM 3.4 (Drozd (cf. [**33, 27**])). *Suppose that k is algebraically closed. Then Λ is either representation-finite, tame or wild.* □

Since periodic modules have complexity 1, Theorem 3.2 implies that

$$\mathrm{cx}_\Lambda(M) \leq 1 \quad \forall\, M \in \operatorname{mod}\Lambda,$$

whenever Λ has finite representation type. There is a similar criterion for certain algebras of tame representation type. This result requires geometric techniques that will be discussed in the following chapter. Its proof is considerably harder as it employs deep results by Crawley-Boevey [**27**] concerning the structure of the Auslander-Reiten quiver of tame algebras. Crawley-Boevey's Theorem implies the following result, which will be sufficient for our purposes.

THEOREM 3.5. *Let k be algebraically closed and suppose that Λ is self-injective and tame. Then, for any $d > 0$, all but finitely many isoclasses of the d-dimensional indecomposable modules have complexity* 1. □

Note that for algebras of finite global dimension the statement of (3.5) provides no information.

CHAPTER III

Support Varieties and Support Spaces

In this chapter we outline the geometric approach to the representation theory of finite algebraic groups. As before, we will be working over a field k of characteristic $p > 0$.

Given a module M of a commutative ring R, we recall that the *support*

$$\mathrm{Supp}(M) := \{\mathfrak{M} \in \mathrm{Max}(R) \ ; \ M_{\mathfrak{M}} \neq (0)\}$$

of M is the set of those maximal ideals $\mathfrak{M} \trianglelefteq R$ for which the localization of M at $\mathfrak{M}$ is not trivial. If R is noetherian and M is finitely generated, then

$$\mathrm{Supp}(M) = Z(\mathrm{ann}_R(M)) := \{\mathfrak{M} \in \mathrm{Max}(R) \ ; \ \mathrm{ann}_R(M) \subseteq \mathfrak{M}\}$$

is the zero locus of the annihilator of M. Thus, if R is an affine k-algebra, then $\mathrm{Supp}(M)$ is an affine variety. In our context, the even cohomology ring $\mathrm{H}^\bullet(\mathcal{G}, k)$ of a finite group scheme $\mathcal{G}$ assumes the rôle of R, with M being given by the cohomology space $\mathrm{H}^*(\mathcal{G}, \mathrm{Hom}_k(N, N))$ associated to a $\mathcal{G}$-module N.

1. The Friedlander-Suslin Theorem

Let $\mathcal{G}$ be a finite algebraic k-group. If M is a $\mathcal{G}$-module, we denote by

$$\mathrm{H}^n(\mathcal{G}, M) := \mathrm{Ext}^n_{\mathcal{G}}(k, M) \qquad (n \geq 0)$$

the *n-th cohomology group of* $\mathcal{G}$ *with coefficients in* M. Note that these groups are just the Hochschild cohomology groups of the augmented algebra $(k\mathcal{G}, \varepsilon)$.

For infinitesimal groups, these cohomology groups were first studied by Hochschild [**70**] in the context of restricted Lie algebras. He related them to the Chevalley-Eilenberg cohomology of the underlying Lie algebra and provided interpretations for H^1 and H^2. Further early results can be found in [**93**].

Given three $\mathcal{G}$-modules X, Y, Z, we recall the *Yoneda product*

$$\mathrm{Ext}^m_{\mathcal{G}}(Y, Z) \times \mathrm{Ext}^n_{\mathcal{G}}(X, Y) \longrightarrow \mathrm{Ext}^{m+n}_{\mathcal{G}}(X, Z).$$

This product endows $\mathrm{Ext}^*_{\mathcal{G}}(X, X) := \bigoplus_{n \geq 0} \mathrm{Ext}^n_{\mathcal{G}}(X, X)$ with the structure of a $\mathbb{Z}$-graded k-algebra. Moreover, the spaces $\mathrm{Ext}^*_{\mathcal{G}}(Y, X)$ and $\mathrm{Ext}^*_{\mathcal{G}}(X, Y)$ are graded left and right $\mathrm{Ext}^*_{\mathcal{G}}(X, X)$-modules, respectively. In particular, $\mathrm{H}^*(\mathcal{G}, M)$ is a graded right module over the *cohomology ring* $\mathrm{H}^*(\mathcal{G}, k)$. This ring is known to be *graded commutative*, i.e., we have

$$yx = (-1)^{\deg(x)\deg(y)} xy$$

for any two homogeneous elements $x, y \in \mathrm{H}^*(\mathcal{G}, k)$. Consequently, the subring

$$\mathrm{H}^\bullet(\mathcal{G}, k) := \begin{cases} \bigoplus_{i \geq 0} \mathrm{H}^{2i}(\mathcal{G}, k) & \text{if } p > 2 \\ \bigoplus_{i \geq 0} \mathrm{H}^{i}(\mathcal{G}, k) & \text{if } p = 2 \end{cases}$$

is a commutative, $\mathbb{Z}$-graded k-algebra (see [**22**, §6] for details).

The following result by Friedlander and Suslin, which generalizes earlier work by Venkov [**121**] and Evens [**37**] for finite groups, and Friedlander-Parshall [**59**] for infinitesimal groups of height ≤ 1, is fundamental for everything that follows.

THEOREM 1.1 (Friedlander-Suslin [**63**]). *Let $\mathcal{G}$ be a finite algebraic k-group, M be a finite-dimensional $\mathcal{G}$-module. Then the following statements hold:*

(1) $\mathrm{H}^\bullet(\mathcal{G}, k)$ *is a finitely generated k-algebra.*

(2) $\mathrm{H}^*(\mathcal{G}, M)$ *is a finitely generated $\mathrm{H}^\bullet(\mathcal{G}, k)$-module.* □

In some cases, the cohomology ring $\mathrm{H}^\bullet(\mathcal{G}, k)$ can be computed explicitly. If $\mathcal{G}$ is smooth, semi-simple and simply connected and p exceeds the Coxeter number of $\mathcal{G}$, then $\mathrm{H}^\bullet(\mathcal{G}_1, k) \cong k[\mathcal{N}]$, where $\mathcal{N} := \{x \in \mathrm{Lie}(\mathcal{G}) \ ; \ x^{[p]^n} = 0 \text{ for some } n \in \mathbb{N}\}$ is the *nilpotent cone* of $\mathrm{Lie}(\mathcal{G})$ (see [**58**]).

Let M be a finite-dimensional $\mathcal{G}$-module, $(P_i, \partial_i)_{i \geq 0}$ be a projective resolution of the trivial $\mathcal{G}$-module k. Since $(P_i \otimes_k M, \partial_i \otimes \mathrm{id}_M)_{i \geq 0}$ is a projective resolution of M, we obtain a homomorphism

$$\Phi_M : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{Ext}^*_{\mathcal{G}}(M, M) \ \ ; \ \ [f] \mapsto [f \hat{\otimes} \mathrm{id}_M]$$

of graded k-algebras. In view of the standard isomorphism $\mathrm{Ext}^*_{\mathcal{G}}(M, M) \cong \mathrm{H}^*(\mathcal{G}, \mathrm{Hom}_k(M, M))$, Theorem 1.1 implies that Φ_M endows the Yoneda algebra with the structure of a finitely generated $\mathrm{H}^\bullet(\mathcal{G}, k)$-module. We define the *cohomological support variety* of M via

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker \Phi_M) \subseteq \mathrm{Max}(\mathrm{H}^\bullet(\mathcal{G}, k)).$$

Since $\ker \Phi_M$ is a homogeneous ideal, the affine variety $\mathcal{V}_{\mathcal{G}}(M)$ is conical.

Our first result concerns the interpretation of the dimension of a support variety. Let

$$A = \bigoplus_{n \geq 0} A_n$$

be a finitely generated, commutative, graded k-algebra. We want to find the growth $\mathrm{gr}(A)$ of the sequence $(A_n)_{n \geq 0}$. By the Noether Normalization Lemma, there exists a graded subalgebra

$$R = \bigoplus_{n \geq 0} R_n$$

of A such that

(a) A is a finitely generated R-module, and

(b) $R \cong k[X_1, \ldots, X_\ell]$, where $\deg(X_i) = d$ for some $d \geq 1$.

Owing to (a), we have $\mathrm{gr}(R) = \mathrm{gr}(A)$, while property (b) implies $\mathrm{gr}(R) = \ell$. The number ℓ is the *Krull dimension* of R, which, by the Cohen-Seidenberg Theorems coincides with the Krull dimension $\dim A$ of A. We therefore obtain

$$\mathrm{gr}(A) = \dim A.$$

In the sequel, we write $\mathrm{cx}_{\mathcal{G}}(M) := \mathrm{cx}_{k\mathcal{G}}(M)$ for any $\mathcal{G}$-module M.

LEMMA 1.2. *Let $\mathcal{G}' \subseteq \mathcal{G}$ be a subgroup of the finite algebraic group $\mathcal{G}$, $M \in \mathrm{mod}\, \mathcal{G}$ be a $\mathcal{G}$-module.*

(1) $\dim \mathcal{V}_{\mathcal{G}}(M) = \mathrm{cx}_{\mathcal{G}}(M)$.

(2) $\dim \mathcal{V}_{\mathcal{G}'}(M) \leq \dim \mathcal{V}_{\mathcal{G}}(M)$.

PROOF. (1) Let $\mathcal{S}$ be a complete set of representatives for the isoclasses of the simple $\mathcal{G}$-modules. Since the algebra $\mathrm{H}^\bullet(\mathcal{G},k)$ is finitely generated, we have

$$\begin{aligned}\dim \mathcal{V}_{\mathcal{G}}(M) &= \dim \mathrm{H}^\bullet(\mathcal{G},k)/\ker\Phi_M = \mathrm{gr}(\mathrm{H}^\bullet(\mathcal{G},k)/\ker\Phi_M)\\ &\leq \mathrm{gr}(\mathrm{Ext}^*_{\mathcal{G}}(M,M)) \leq \max_{S\in\mathcal{S}} \mathrm{gr}(\mathrm{Ext}^*_{\mathcal{G}}(M,S)),\end{aligned}$$

where the last inequality follows from the long exact sequence in cohomology. In view of (II.2.1), the latter number coincides with $\mathrm{cx}_{\mathcal{G}}(M)$.

To verify the reverse inequality, we let $S \in \mathcal{S}$ be a simple $\mathcal{G}$-module. Thanks to (1.1), the space $\mathrm{Ext}^*_{\mathcal{G}}(M,S) \cong \mathrm{H}^*(\mathcal{G},\mathrm{Hom}_k(M,S))$ is a finitely generated $\mathrm{H}^\bullet(\mathcal{G},k)$-module. Since the action is induced by the scalar multiplication $\mathrm{Hom}_k(M,S)\times k \longrightarrow \mathrm{Hom}_k(M,S)$, it factors through the action $\mathrm{Hom}_k(M,S)\times\mathrm{Hom}_k(M,M) \longrightarrow \mathrm{Hom}_k(M,S)$. As a result, $\ker\Phi_M$ annihilates $\mathrm{Ext}^*_{\mathcal{G}}(M,S)$. Consequently,

$$\mathrm{gr}(\mathrm{Ext}^*_{\mathcal{G}}(M,S)) \leq \mathrm{gr}(\mathrm{H}^\bullet(\mathcal{G},k)/\ker\Phi_M) = \dim\mathcal{V}_{\mathcal{G}}(M),$$

so that another application of (II.2.1) yields $\mathrm{cx}_{\mathcal{G}}(M) \leq \dim\mathcal{V}_{\mathcal{G}}(M)$, as desired.

(2). This follows directly from (1), (I.1.6) and (II.2.2). □

As an immediate consequence, we record the following basic criteria for blocks of finite representation type.

COROLLARY 1.3. *Let $\mathcal{B} \subseteq k\mathcal{G}$ be a representation-finite block. Then we have* $\dim\mathcal{V}_{\mathcal{G}}(M) \leq 1$ *for every $M \in \mathrm{mod}\,\mathcal{B}$.* □

EXAMPLE. Suppose that $p \geq 3$. As noted before, the Künneth formula furnishes an isomorphism

$$\mathrm{H}^*(\mathbb{G}_{a(r)},k) \cong k[X_1,\ldots,X_r]\otimes_k \Lambda(Y_1,\ldots,Y_r),$$

where the X_i and Y_i have degrees 2 and 1, respectively (cf. [**22**, (7.6)]). Consequently, $k[X_1,\ldots,X_r]$ is a Noether normalization of $\mathrm{H}^\bullet(\mathbb{G}_{a(r)},k)$ and $\dim\mathcal{V}_{\mathbb{G}_{a(r)}}(k) = r$. Note that this agrees with our earlier observations.

2. The Space of p-Points

We have just seen that the dimension $\dim\mathcal{V}_{\mathcal{G}}(M)$ of the support variety of a $\mathcal{G}$-module M is an important invariant. Unfortunately, support varieties are usually hard to compute. It is therefore desirable to find representation-theoretic realizations of these varieties that do not resort to the computation of cohomology groups. The so-called *rank varieties*, that were first defined in the context of finite groups [**20, 21**], and later generalized successively to restricted Lie algebras [**57, 59**] and infinitesimal groups [**116, 117**], turn out to have all the requisite properties. In recent work [**60, 61**], Friedlander and Pevtsova have unified and extended the foregoing approaches by considering p-points and π-points of finite group schemes. Since their work is very much motivated by the historical origins pertaining to finite groups, we briefly review the by now classical definition of a rank variety.

Throughout this section, k is assumed to be a field of characteristic $\mathrm{char}(k) = p > 0$. Early work by Quillen [**100, 101**] showed that the support variety of the trivial module of a finite group G may be described as the union of the corresponding supports for the p-elementary abelian subgroups of G. In particular, the dimension of $\mathcal{V}_G(k)$ coincides with the maximal rank of all p-elementary abelian subgroups of G, the so-called *p-rank $r_p(G)$ of G*:

THEOREM 2.1 (Quillen's Dimension Theorem,[**100**]).

$$\dim \mathcal{V}_G(k) = r_p(G).$$

Elementary abelian subgroup also arise in a different context, namely in connection with the detection of projectivity.

If M is a projective G-module and $G' \subseteq G$ is a subgroup, then the restriction $M|_{G'}$ is also projective. The question is, on which class of subgroups can projectivity effectively be detected. It is fairly easy to see that a module is projective, if its restriction $M|_P$ to a Sylow-p-subgroup enjoys the same property. The following refinement of this result is much harder to prove, see [**3**]:

THEOREM 2.2 (Chouinard,[**29**]). *A G-module M is projective if and only if its restriction $M|_E$ to every p-elementary abelian subgroup $E \subseteq G$ is projective.* □

Let $E \cong \mathbb{Z}/(p)^r$ be p-elementary abelian with generators $g_1, \dots, g_r$. Setting $x_i := g_i - 1$, the map $X_i \longrightarrow x_i$ induces an isomorphism

$$k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p) \stackrel{\sim}{\longrightarrow} kE.$$

For $\lambda = (\lambda_1, \dots, \lambda_r) \in k^r$, we put $u_\lambda := \sum_{i=1}^r \lambda_i x_i$. The following result, which requires the ground field to be algebraically closed, provides a criterion for the projectivity of an E-module:

LEMMA 2.3 (Dade's Lemma,[**30**]). *Suppose that k is algebraically closed. The E-module M is projective if and only if $M|_{k[u_\lambda]}$ is projective for every $\lambda \in k^r$.* □

EXAMPLE. Suppose that k is not algebraically closed and let $K : k$ be a finite field extension, $\alpha \in K \setminus k$. Consider the group $E := \mathbb{Z}/(p) \times \mathbb{Z}/(p)$. Then $kE \cong k[X, Y]/(X^p, Y^p)$, with canonical generators x, y. We consider the k-space $M := K^p$ and define a kE-action on M via the $(p \times p)$-matrices

$$x_M := \sum_{i=1}^{p-1} E_{i,i+1} \quad \text{and} \quad y_M := \alpha x_M.$$

If $(\lambda_1, \lambda_1) \in k^2 \setminus \{0\}$, then $\mathrm{rk}(\lambda_1 x_M + \lambda_2 y_M) = \frac{(p-1)\dim_k M}{p}$, so that $M|_{k[\lambda_1 x + \lambda_2 y]}$ is projective. On the other hand, comparing Loewy lengths, we get $\ell\ell(M) = p$, while $\ell\ell(kE) = 2p-1$, implying that M is not a projective kE-module.

In view of the above, we assume from now on that k is algebraically closed, and proceed as follows: Given a p-elementary abelian group E, we consider a subspace $V \subseteq kE$ with $\dim_k V = \mathrm{rk}(E)$, whose nonzero elements $u \in V \setminus \{0\}$ have the following property:

$u^p = 0$, and $k[u]$ is a local algebra of dimension p such that $kE|_{k[u]}$ is free.

For an E-module M, one defines

$$\widehat{\mathcal{V}}_E(M) := \{u \in V \ ; \ M|_{k[u]} \text{ is not free}\} \cup \{0\}.$$

This is a closed subset of V and hence an affine variety. Avrunin and Scott [**9**] showed that $\widehat{\mathcal{V}}_E(M) \cong \mathcal{V}_E(M)$, and they thus obtained an intrinsic characterization of the cohomological support variety.

Let $\mathfrak{A}_p := k[T]/(T^p)$ be the p-dimensional truncated polynomial ring, with generator $t := T + (T^p)$. Given a finite group scheme $\mathcal{G}$, every element $u \in k\mathcal{G}$ satisfying $u^p = 0$ defines an algebra homomorphism

$$\alpha_u : \mathfrak{A}_p \longrightarrow k\mathcal{G} \ \ ; \ \ t \mapsto u.$$

The freeness requirement occurring for finite groups amounts to saying that α_u is left flat. In general, any homomorphism $\varphi : A \longrightarrow B$ of associative k-algebras, induces a pull-back functor $\varphi^* : \operatorname{mod} B \longrightarrow \operatorname{mod} A$, where $\varphi^*(M)$ is the A-module with underlying k-space M and action

$$a.m := \varphi(a)m \qquad \forall\, a \in A,\ m \in M.$$

We say that φ is *left flat* if $\varphi^*(B)$ is a flat A-module. In case A and B are finite-dimensional, this amounts to saying that $\varphi^*(B)$ is a projective A-module. If the algebra A is local, then $\varphi^*(B)$ is free, and φ is necessarily injective. Thus, if $\alpha_u : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is left flat, then $k[u] \subseteq k\mathcal{G}$ is a k-algebra of dimension p.

Elementary abelian p-groups are abelian p-groups. In the context of finite group schemes, the latter are the reduced abelian unipotent groups. A finite algebraic k-group $\mathcal{U}$ is referred to as *unipotent* if the trivial module k is the only simple $\mathcal{U}$-module. For example, the Frobenius kernels $\mathbb{G}_{a(r)}$ of the additive group $\mathbb{G}_a$ are unipotent.

Definition ([**60**]). Let $\mathcal{G}$ be a finite group scheme. A homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is a *p-point* if

(P1) α is left flat, and

(P2) there exists an abelian, unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ such that $\operatorname{im} \alpha \subseteq k\mathcal{U}$.

Two p-points α and β are *equivalent* ($\alpha \sim \beta$) if, for every $M \in \operatorname{mod} \mathcal{G}$, we have

$$\alpha^*(M) \text{ is projective } \Leftrightarrow \beta^*(M) \text{ is projective.}$$

The set of equivalence classes of p-points will be denoted $P(\mathcal{G})$.

Remarks. (1) Condition (P2) looks somewhat contrived and mysterious, but it turns out to be of vital importance. In general, p-points are hard to come by and the equivalence relation $\sim$ is difficult to compute. However, using (P2) many problems can be reduced to the consideration of abelian unipotent group schemes. For such a group $\mathcal{U}$, general theory (cf. [**123**]) provides an isomorphism

$$k\mathcal{U} \cong k[X_1, \ldots, X_n]/(X_1^{p^{r_1}}, \ldots, X_n^{p^{r_n}}),$$

with the latter algebra being amenable to computations.

(2) By considering algebra homomorphisms rather than homomorphism of Hopf algebras we disregard the coalgebra structure. It turns out, however, that at the level of abelian unipotent groups every equivalence class $x \in P(\mathcal{U})$ can be represented by a homomorphism of Hopf algebras (see [**43**] for more details).

The following important projectivity criterion, which provides some control over the equivalence relation $\sim$, underscores the importance of abelian unipotent subgroups.

Proposition 2.4 ([**60**]). *Let $n \geq 1$ and consider the algebra $A := k[X,Y,Z]/(X^p, Y^p, Z^n)$, whose canonical generators are denoted x, y, z. Let $M \in \operatorname{mod} A$ be an A-module. Then M is projective as a $k[x]$-module if and only if M is projective as a $k[x+yz]$-module.*

PROOF. Observe that the subalgebras $k[x]$ and $k[x+yz]$ are both isomorphic to $\mathfrak{A}_p$. By decomposing an $\mathfrak{A}_p$-module N into its cyclic direct summands, we see that N is projective if and only if the endomorphism t_N associated to its canonical generator $t \in \mathfrak{A}_p$ satisfies the equivalent conditions $\ker t_N = \operatorname{im} t_N^{p-1}$ and $\ker t_N^{p-1} = \operatorname{im} t_N$.

Suppose that $M|_{k[x]}$ is projective and consider the A-module $N := \ker(x+yz)_M / \operatorname{im}(x+yz)_M^{p-1}$. A rather involved computation shows that the operator z_N is injective. As z is nilpotent, we obtain $N = (0)$, so that $M|_{k[x+yz]}$ is projective. □

We illustrate the relevance of condition (P2) by considering the case of infinitesimal groups $\mathcal{G}$ of height 1. Thanks to (I.4.6), $k\mathcal{G} \cong U_0(\mathfrak{g})$ is the restricted enveloping algebra of the restricted Lie algebra $\mathfrak{g} := \operatorname{Lie}(\mathcal{G})$.

Let $\alpha : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{g})$ be a p-point. Then α factors through the restricted enveloping algebra $U_0(\mathfrak{u})$ of a suitable abelian, p-unipotent p-subalgebra $\mathfrak{u} \subseteq \mathfrak{g}$. In this case, the generators $x_1, \ldots, x_n$ of the algebra $U_0(\mathfrak{u})$ are primitive elements and they satisfy $x_i^{p^{r_i}} = 0$. Repeated application of (2.4) shows that α is equivalent to the p-point $\alpha_0 : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{u})$ given by

$$\alpha_0(t) = \sum_{i=1}^{n} \zeta_i x_i^{p^{r_i - 1}},$$

where $(\zeta_1, \ldots, \zeta_n) \in k^n \setminus \{0\}$. Thus, we may assume that $u := \alpha(t)$ belongs to $\mathfrak{u} \subseteq \mathfrak{g}$ and satisfies the identity $u^{[p]} = 0$. Consequently, the element u belongs to the *nullcone*

$$\widehat{\mathcal{V}}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$$

of $\mathfrak{g}$. In view of (I.4.4), given $v, w \in \widehat{\mathcal{V}}_{\mathfrak{g}} \setminus \{0\}$, the $U_0(\mathfrak{g})$-module $M_v := U_0(\mathfrak{g}) \otimes_{k[v]} k$ is $k[w]$-projective if and only if $w \notin kv$. Thus, the line $k\alpha_0(t)$ only depends on the equivalence class $[\alpha]$, and we obtain a bijection

$$P(\mathcal{G}) \xrightarrow{\sim} \operatorname{Proj}(\widehat{\mathcal{V}}_{\mathfrak{g}}) \ \ ; \ \ [\alpha] \mapsto k\alpha_0(t).$$

We will refine this result at the end of this section.

Returning to the general situation, we observe that every algebra homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ induces a homomorphism

$$\alpha^\bullet : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{H}^\bullet(\mathfrak{A}_p, k)$$

of commutative, graded k-algebras. The even cohomology ring of $\mathfrak{A}_p$ is known to be isomorphic to the polynomial ring $k[X]$, where X has degree 2.

Let $\mathcal{V}$ be a conical affine variety, whose coordinate ring is a finitely generated, commutative, graded k-algebra

$$R = \bigoplus_{n \geq 0} R_n \ \ ; \ \ R_0 = k.$$

Recall that

$$\operatorname{Proj}(\mathcal{V}) \cong \{\ker \varphi \ ; \ \varphi \in \operatorname{Hom}_{\mathrm{gr}}(R, k[X]), \ \operatorname{im} \varphi \neq k\}$$

is computable via non-trivial graded homomorphisms with values in $k[X]$. Thus, the hope is that $\alpha \mapsto \ker \alpha^\bullet$ provides a connection between $P(\mathcal{G})$ and $\operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(k))$.

The following Proposition elicits the cohomological aspects of the equivalence relation on p-points. The key notion is that of the Carlson module associated to a cohomology class $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$. By general theory, we have an isomorphism $\mathrm{H}^n(\mathcal{G}, k) \cong \operatorname{Hom}_{\mathcal{G}}(\Omega^n_{\mathcal{G}}(k), k)$, allowing us to view the representing cocycle as a map $\hat{\zeta} : \Omega^n_{\mathcal{G}}(k) \longrightarrow k$. Then

$$L_\zeta := \ker \hat{\zeta}$$

is called the *Carlson module* of ζ.

PROPOSITION 2.5 ([**60**]). *Let $\mathcal{G}$ be a finite group scheme over k.*
(1) *If $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is flat, then*

$$\ker \alpha^\bullet \cap (\mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}) = \{\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\} \ ; \ \alpha^*(L_\zeta) \text{ is not projective}\}$$

for every $n \geq 0$.
(2) *If $\alpha, \beta : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ are flat maps such that $\ker \alpha^\bullet \not\subseteq \ker \beta^\bullet$, then there exist $n \geq 1$ and $\zeta \in \mathrm{H}^{2n}_{\mathcal{G}}(k), k) \setminus \{0\}$ such that $\alpha^*(L_\zeta)$ is not projective and $\beta^*(L_\zeta)$ is projective.*

PROOF. (1) As α is flat, the corresponding exact functor

$$\alpha^* : \mathrm{mod}\, \mathcal{G} \longrightarrow \mathrm{mod}\, \mathfrak{A}_p$$

takes projectives to projectives. Consequently, α^* sends a minimal projective resolution of $M \in \mathrm{mod}\, \mathcal{G}$ to a (not necessarily minimal) projective resolution of $\alpha^*(M)$, so that

$$\alpha^*(\Omega^m_{\mathcal{G}}(M)) \cong \Omega^m_{\mathfrak{A}_p}(\alpha^*(M)) \oplus (\text{proj.}) \quad \forall\ m \in \mathbb{N}.$$

Let $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}$ be an arbitrary element. Upon application of α^* to the exact sequence $(0) \longrightarrow L_\zeta \longrightarrow \Omega^{2n}_{\mathcal{G}}(k) \xrightarrow{\hat{\zeta}} k \longrightarrow (0)$ we obtain, observing $\Omega^{2n}_{\mathfrak{A}_p}(k) \cong k$, an exact sequence

$$(*) \qquad (0) \longrightarrow \alpha^*(L_\zeta) \longrightarrow k \oplus (\text{proj.}) \xrightarrow{\alpha^*(\hat{\zeta})} k \longrightarrow (0).$$

The definition of $\alpha^\bullet$ yields

$$\alpha^\bullet(\zeta) = 0 \Leftrightarrow k \subseteq \ker \alpha^*(\hat{\zeta}).$$

If $\alpha^*(L_\zeta)$ is projective, then it is injective and $(*)$ splits. Consequently, k is not a direct summand of $\alpha^*(L_\zeta)$, so that $k \not\subseteq \ker \alpha^*(\hat{\zeta})$, whence $\zeta \notin \ker \alpha^\bullet$. Conversely, the assumption $\zeta \notin \ker \alpha^\bullet$ yields the splitting of the sequence $(*)$, whence

$$k \oplus (\text{proj.}) \cong \alpha^*(L_\zeta) \oplus k.$$

By the Theorem of Krull-Remak-Schmidt, the $\mathfrak{A}_p$-module $\alpha^*(L_\zeta)$ is projective.
(2) Since $\alpha^\bullet$ and $\beta^\bullet$ are homomorphisms of degree 0, our assumption provides an element $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k)$ with $\zeta \in \ker \alpha^\bullet \setminus \ker \beta^\bullet$. Thus, $\zeta \neq 0$, and (1) shows that L_ζ has the requisite properties. □

In view of (2.5), equivalent p-points yield the same kernels in cohomology. This is the first step towards proving the existence of a map

$$\Psi_{\mathcal{G}} : P(\mathcal{G}) \longrightarrow \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(k)) \ \ ; \ \ [\alpha] \mapsto \ker \alpha^\bullet.$$

The missing point is to show that $\mathrm{im}\, \alpha^\bullet \neq k$ for all p-points α. This is actually a rather deep result.

LEMMA 2.6 ([**60**]). *Let $\mathcal{G}$ be a finite k-group scheme. If $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is a flat homomorphism, then $\mathrm{im}\, \alpha^\bullet \neq k$.*

PROOF. Since α is flat, $\alpha^*(k\mathcal{G})$ is a projective $\mathfrak{A}_p$-module, so that the Eckmann-Shapiro Lemma furnishes an isomorphism

$$\mathrm{H}^\bullet(\mathfrak{A}_p, k) \cong \mathrm{H}^\bullet(\mathcal{G}, \mathrm{Hom}_{\mathfrak{A}_p}(k\mathcal{G}, k))$$

of $\mathrm{H}^\bullet(\mathcal{G}, k)$-modules, with $\mathrm{H}^\bullet(\mathcal{G}, k)$ operating on $\mathrm{H}^\bullet(\mathfrak{A}_p, k)$ via $\alpha^\bullet$. Thanks to the Friedlander-Suslin Theorem, the space $\mathrm{H}^\bullet(\mathfrak{A}_p, k) \cong k[X]$ is a finitely generated $\mathrm{H}^\bullet(\mathcal{G}, k)$-module, so that $\mathrm{im}\, \alpha^\bullet \neq k$. □

Following Friedlander-Pevtsova [**60**], we associate to every $\mathcal{G}$-module M the set

$$P(\mathcal{G})_M := \{[\alpha] \in P(\mathcal{G}) \ ; \ \alpha^*(M) \text{ is not projective}\}.$$

If these sets were a good model for support varieties, then they should interact nicely with tensor products. The subtlety of the following result resides in p-points not being Hof algebra homomorphisms, so that α^* does not necessarily commute with tensor products. Nevertheless, one can show:

THEOREM 2.7 ([**60**]). *Let $\mathcal{G}$ be a finite group scheme. Then we have*

$$P(\mathcal{G})_{M\otimes_k N} = P(\mathcal{G})_M \cap P(\mathcal{G})_N$$

for all $M, N \in \operatorname{mod} k\mathcal{G}$.

PROOF. Suppose first that $\mathcal{G} = \mathcal{U}$ is abelian and unipotent. Recall that there exist $r_i \geq 1$ such that

$$k\mathcal{U} \cong k[X_1, \ldots, X_n]/(X_1^{p^{r_1}}, \ldots, X_n^{p^{r_n}}).$$

We denote the canonical generators by the corresponding lower case letters. Thus, by defining a comultiplication $\Delta_1 : k\mathcal{U} \longrightarrow k\mathcal{U} \otimes_k k\mathcal{U}$ via

$$\Delta_1(x_i) = x_i \otimes 1 + 1 \otimes x_i \ \ ; \ \ 1 \leq i \leq n$$

we see that $k\mathcal{U}$ can be considered as the restricted enveloping algebra $U_0(\mathfrak{u})$ of the abelian restricted Lie algebra $\mathfrak{u} := \bigoplus_{i=1}^n \bigoplus_{j=0}^{r_i-1} kx_i^{p^j}$. Let $\Delta_0 : k\mathcal{U} \longrightarrow k\mathcal{U} \otimes_k k\mathcal{U}$ be the originally given comultiplication. Since tensor products of projective modules over Hopf algebras are projective, both comultiplications are flat. The well-known identity $\Delta_0(u) \equiv u \otimes 1 + 1 \otimes u \mod(\operatorname{Rad}(k\mathcal{U}) \otimes_k \operatorname{Rad}(k\mathcal{U}))$, in conjunction with $k\mathcal{U}$ being generated by Δ_1-primitive elements, yields

$$\Delta_0(u) \equiv \Delta_1(u) \qquad \mod(\operatorname{Rad}(k\mathcal{U}) \otimes_k \operatorname{Rad}(k\mathcal{U})) \qquad \forall\ u \in k\mathcal{U}.$$

We let $\Delta_{i,*} : P(\mathcal{U}) \longrightarrow P(k\mathcal{U} \otimes_k k\mathcal{U})$ be the canonical map induced by Δ_i, sending p-points of $\mathcal{U}$ to flat maps $\mathfrak{A}_p \longrightarrow k\mathcal{U} \otimes_k k\mathcal{U}$. Then Proposition 2.4 yields

$$P(\mathcal{U})_{M\otimes N} = \Delta_{0,*}^{-1}(P(k\mathcal{U} \otimes_k k\mathcal{U})_{M\otimes_k N}) = \Delta_{1,*}^{-1}(P(k\mathcal{U} \otimes_k k\mathcal{U})_{M\otimes_k N}),$$

so that we may consider the tensor product relative to the Hopf structure given by Δ_1. In other words, $k\mathcal{U}$ is a restricted enveloping algebra.

Let $[\alpha]$ be an element of $P(\mathcal{U})$. By our observations concerning p-points of restricted enveloping algebras, we may assume that $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{U}$ is a homomorphism of Hopf algebras, where $\mathfrak{A}_p = U_0(kt)$ is the restricted enveloping algebra of the one-dimensional nil-cyclic Lie algebra kt. Thus, the functor α^* commutes with tensor products, and our result follows from the representation theory of $U_0(kt)$.

In the general case, a p-point $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ factors through an abelian, unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$. Hence there exists a p-point $\beta : \mathfrak{A}_p \longrightarrow k\mathcal{U}$ such that $\iota_{\mathcal{U}} \circ \beta = \alpha$. Here $\iota_{\mathcal{U}}$ denotes the canonical inclusion, whose associated pull-back functor commutes with tensor products. $\square$

REMARK. As is shown in [**43**, §1], Theorem 2.7 loses its validity, if one considers arbitrary flat maps $\mathfrak{A}_p \longrightarrow k\mathcal{G}$ instead of p-points.

The conceptual importance of (2.7) manifests itself in the following result that justifies the notion of a support space:

THEOREM 2.8 ([**60**]). *Let $\mathcal{G}$ be a finite group scheme. Then*

$$\{P(\mathcal{G})_M \ ; \ M \in \operatorname{mod} k\mathcal{G}\}$$

is the set of closed subsets of a Noetherian topology on $P(\mathcal{G})$. □

In particular, we can speak of the Krull dimension of the closed subspace $P(\mathcal{G})_M$ of $P(\mathcal{G})$.

THEOREM 2.9 ([**60**]). *The map*

$$\Psi_{\mathcal{G}} : \begin{cases} P(\mathcal{G}) & \longrightarrow & \operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(k)) \\ \alpha & \mapsto & \ker \alpha^{\bullet} \end{cases}$$

is a homeomorphism such that

$$\Psi_{\mathcal{G}}^{-1}(\operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(M))) = P(\mathcal{G})_M$$

for every $M \in \operatorname{mod} \mathcal{G}$. Moreover, $\Psi_{\mathcal{G}}$ is natural relative to flat homomorphisms $\mathcal{G} \longrightarrow \mathcal{H}$. □

REMARK. By definition, the Carlson module L_ζ belongs to the principal block $\mathcal{B}_0(\mathcal{G})$ of $k\mathcal{G}$. Theorem 2.9 and Proposition 2.5 thus state that the equivalence relation $\sim$ is completely determined on $\operatorname{mod} \mathcal{B}_0(\mathcal{G})$.

COROLLARY 2.10. *We have $\mathcal{V}_{\mathcal{G}}(L_\zeta) = Z(\zeta)$ for any $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}$.*

PROOF. Let $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ be a p-point of $\mathcal{G}$, and consider the induced map $\alpha^{\bullet} : \mathrm{H}^{\bullet}(\mathcal{G}, k) \longrightarrow \mathrm{H}^{\bullet}(\mathfrak{A}_p, k)$. In view of (2.5(1)), we have $\zeta \in \ker \alpha^{\bullet}$ if and only if the pull-back $\alpha^*(L_\zeta)$ is not projective. Thanks to (2.9), the latter condition is equivalent to $\ker \alpha^{\bullet} \supseteq \ker \Phi_{L_\zeta}$, so that

$$(\zeta) \subseteq \ker \alpha^{\bullet} \Leftrightarrow \ker \Phi_{L_\zeta} \subseteq \ker \alpha^{\bullet}.$$

By virtue of (2.9), this implies $\operatorname{Proj}(Z(\zeta)) = \operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(L_\zeta))$, whence $Z(\zeta) = \mathcal{V}_{\mathcal{G}}(L_\zeta)$. □

Given a restricted Lie algebra $(\mathfrak{g}, [p])$, we recall the bijection

$$\operatorname{Proj}(\widehat{\mathcal{V}}_{\mathfrak{g}}) \longrightarrow P(\mathfrak{g}) \ ; \ ku \mapsto [\alpha_u].$$

If M is a $U_0(\mathfrak{g})$-module, then the pre-image of $P(\mathfrak{g})_M$ under this map is the projective space of the *rank variety*

$$\widehat{\mathcal{V}}_{\mathfrak{g}}(M) := \{u \in \widehat{\mathcal{V}}_{\mathfrak{g}} \ ; \ M|_{U_0(ku)} \text{ is not projective}\} \cup \{0\}$$

of M, which is a closed, conical subset of the nullcone $\widehat{\mathcal{V}}_{\mathfrak{g}}$. The above map is actually a homeomorphism; see also the work by Jantzen [**79**] and Friedlander-Parshall [**57**] concerning the relationship between the nullcone $\widehat{\mathcal{V}}_{\mathfrak{g}}$ and the cohomological support variety $\mathcal{V}_{\mathfrak{g}}(k)$. We record the following consequence:

COROLLARY 2.11. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then we have*

$$\dim \widehat{\mathcal{V}}_{\mathfrak{g}}(M) = \mathrm{cx}_{U_0(\mathfrak{g})}(M)$$

for every $M \in \operatorname{mod} U_0(\mathfrak{g})$. □

3. Tameness and Complexity

In this section we shall employ support varieties to verify a useful necessary condition for the tameness of blocks of cocommutative Hopf algebras. This result was first announced by Rickard [**103**] in 1990, yet his proof is not correct. We therefore follow the approach of [**42**], which is valid for blocks of Hopf algebras associated to finite group schemes and related classes of algebras. Given such a group scheme $\mathcal{G}$ and a $\mathcal{G}$-module M, we recall that $\mathrm{cx}_{\mathcal{G}}(M) := \mathrm{cx}_{k\mathcal{G}}(M)$ denotes the complexity of M.

Throughout, our base field k is assumed to be algebraically closed.

THEOREM 3.1 ([**42**]). *Let $\mathcal{G}$ be a finite group scheme, $\mathcal{B} \subseteq k\mathcal{G}$ be a tame block. Then $\mathrm{cx}_{\mathcal{G}}(M) \leq 2$ for every $M \in \mathrm{mod}\,\mathcal{B}$.*

PROOF. We outline the proof, since it rests on the geometric notions we have studied in the foregoing sections. Let $\{S_1, \ldots, S_n\}$ be a complete set of representatives for the simple $\mathcal{B}$-modules. We set

$$\mathcal{V}_{\mathcal{B}} := \mathcal{V}_{\mathcal{G}}(\bigoplus_{i=1}^{n} S_i).$$

Our Theorem is a consequence of the following assertion, valid for an arbitrary block $\mathcal{B} \subseteq k\mathcal{G}$:

($*$) *If $m := \dim \mathcal{V}_{\mathcal{B}} \geq 3$, then $\mathcal{B}$ is not tame.*

The idea is to produce a family of indecomposable $\mathcal{B}$-modules that violates Theorem II.3.5. Using the Noether Normalization Lemma, one first proves the existence of $d > 0$ and a familiy $(\zeta_\alpha)_{\alpha \in k}$ with $\zeta_\alpha \in \mathrm{H}^{2d}(\mathcal{G}, k) \setminus \{0\}$ for all $\alpha \in k$, such that

(a) $\dim(Z(\zeta_\alpha) \cap \mathcal{V}_{\mathcal{B}}) = m - 1$ for $\alpha \in k$, and
(b) $\dim(Z(\zeta_\alpha) \cap Z(\zeta_\beta) \cap \mathcal{V}_{\mathcal{B}}) = m - 2$ for $\alpha \neq \beta \in k$.

Now consider the $\mathcal{B}$-module $M_{\mathcal{B}} := \bigoplus_{i=1}^{n} S_i$ as well as the Carlson modules $(L_{\zeta_\alpha})_{\alpha \in k}$. These give rise to exact sequences

$$(0) \longrightarrow L_{\zeta_\alpha} \otimes_k M_{\mathcal{B}} \longrightarrow \Omega^{2d}_{\mathcal{G}}(k) \otimes_k M_{\mathcal{B}} \stackrel{\hat{\zeta}_\alpha \otimes \mathrm{id}_{M_{\mathcal{B}}}}{\longrightarrow} M_{\mathcal{B}} \longrightarrow (0).$$

Owing to (2.7), (2.9) and (2.10) we have

$$\mathcal{V}_{\mathcal{G}}(L_{\zeta_\alpha} \otimes_k M_{\mathcal{B}}) = \mathcal{V}_{\mathcal{G}}(L_{\zeta_\alpha}) \cap \mathcal{V}_{\mathcal{G}}(M_{\mathcal{B}}) = Z(\zeta_\alpha) \cap \mathcal{V}_{\mathcal{B}}.$$

By considering indecomposable summands of $L_{\zeta_\alpha} \otimes_k M_{\mathcal{B}}$ belonging to $\mathcal{B}$, we finally obtain a family $(N_\alpha)_{\alpha \in k}$ of indecomposable $\mathcal{B}$-modules with

(i) $\dim_k N_\alpha \leq (\dim_k M_{\mathcal{B}})(\dim_k \Omega^{2d}_{\mathcal{G}}(k))$, and
(ii) $\dim \mathcal{V}_{\mathcal{G}}(N_\alpha) = \dim \mathcal{V}_{\mathcal{B}} - 1$, and
(iii) $\dim(\mathcal{V}_{\mathcal{G}}(N_\alpha) \cap \mathcal{V}_{\mathcal{G}}(N_\beta)) = \dim \mathcal{V}_{\mathcal{B}} - 2$ for all $\alpha \neq \beta$.

In view of (iii), these modules are pairwise non-isomorphic, and (1.2) yields $\mathrm{cx}_{\mathcal{G}}(N_\alpha) = \dim \mathcal{V}_{\mathcal{G}}(N_\alpha) \geq 2$. Owing to (i), Theorem II.3.5 shows that $\mathcal{B}$ is not tame.

Now let $\mathcal{B}$ be a tame block. Thanks to ($*$), we obtain $\mathrm{cx}_{\mathcal{G}}(S_i) \leq \mathrm{cx}_{\mathcal{G}}(M_{\mathcal{B}}) \leq 2$, so that an application (II.2.2) completes the proof. □

By (II.2.2(4)) we have $\mathrm{cx}_{\mathbb{G}_{a(2)}}(M) \leq \mathrm{cx}_{\mathbb{G}_{a(2)}}(k) = 2$ for every $M \in \mathrm{mod}\, k\mathbb{G}_{a(2)}$. However, $k\mathbb{G}_{a(2)} \cong k[X,Y]/(X^p, Y^p)$ is tame only if $p = 2$. Consequently, the converse of (3.1) does not hold in general.

Remark. By combining (I.1.6), (II.2.2) and (3.1), we see that Hopf subalgebras of tame or finite representation type “come close to” being tame or representation-finite. In fact, by special properties of extensions of group algebras, the representation type is inherited by subgroups of finite groups (cf. [**15**, Prop.2]). As we shall see later, such a result does not hold in the context of infinitesimal group schemes.

4. Blocks of Frobenius Kernels of Smooth Groups

Throughout this section, we fix a reduced (=smooth) affine group scheme $\mathcal{G}$ and assume that $\mathcal{G}$ is defined over the algebraically closed field k of characteristic $\mathrm{char}(k) = p \geq 3$. We will illustrate how rank varieties and nilpotent orbits can be employed to classify the representation-finite and tame blocks of the Frobenius kernels of such groups. Let us begin by looking at an important example.

Example. Consider the first Frobenius kernel $\mathrm{SL}(2)_1$ of $\mathrm{SL}(2)$. According to (I.4.6), the algebra $k\,\mathrm{SL}(2)_1$ is isomorphic to $U_0(\mathfrak{sl}(2))$, the restricted enveloping algebra of the Lie algebra of (2×2)-matrices of trace zero.

The block structure of $U_0(\mathfrak{sl}(2))$ was determined by Pollack [**98**]. The algebra $U_0(\mathfrak{sl}(2))$ possesses exactly $\frac{p+1}{2}$ blocks. There is one simple block, corresponding to the p-dimensional Steinberg module $\mathrm{St} = L(p-1)$, and $\ell := \frac{p-1}{2}$ blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\ell-1}$, with $\mathcal{B}_i$ having two simple modules $L(i)$ and $L(p{-}2{-}i)$ of dimensions $i{+}1$ and $p{-}1{-}i$, respectively. Pollack also proved that $U_0(\mathfrak{sl}(2))$ has infinite representation type, which of course also follows from (1.3) and the fact that $\widehat{\mathcal{V}}_{\mathfrak{sl}(2)}$ has dimension 2. The tameness of these blocks was verified independently by Drozd [**34**], Fischer [**55**], and Rudakov [**109**]. The indecomposable $U_0(\mathfrak{sl}(2))$-modules were determined by Premet [**99**].

We shall return to these blocks once we have discussed path algebras. For the time being, we just record

$$\dim_k \mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(L(i), L(j)) = \begin{cases} 2 & i+j = p-2 \\ 0 & \text{otherwise.} \end{cases}$$

For every $r \in \mathbb{N}$, we define

$$\ell_r(\mathcal{G}) := r \dim \mathcal{G}/B(\mathcal{G}),$$

where $B(\mathcal{G})$ is a Borel subgroup of $\mathcal{G}$. Since all Borel subgroups of $\mathcal{G}$ are conjugate, the number $\ell_r(\mathcal{G})$ does not depend on the choice of $B(\mathcal{G})$.

Given $n \geq 0$, we denote by

$$\mathfrak{n}_n := \bigoplus_{n=0}^{n-1} kx^{[p]^i} \quad ; \quad x^{[p]^n} = 0 \neq x^{[p]^{n-1}}$$

the n-dimensional *nil-cyclic* Lie algebra. The smooth group giving rise to this Lie algebra is the group $\mathcal{W}_n$ of *Witt vectors* of length n (cf. [**31**, V,§5]). Thus, we have

$$\mathrm{Lie}(\mathcal{W}_n) = \mathfrak{n}_n \qquad \forall\ n \geq 0.$$

The following result shows in particular that the representation-finite blocks of $k\mathcal{G}_r$ are Nakayama algebras.

THEOREM 4.1 ([**40**]). *Let $\mathcal{G}$ be a smooth group with unipotent radical $\mathcal{U}$ of dimension $n := \dim \mathcal{U}$. Let $\mathcal{B} \subseteq k\mathcal{G}_r$ be a block.*

(1) *Suppose that $\mathcal{B}$ is representation-finite.*
  (a) *If $r \geq 2$, then $\mathcal{B}$ is simple, and $\mathcal{G}$ is reductive (i.e., $\mathcal{U} = e_k$),*
  (b) *If $r = 1$, then there exists $s \in \{0,1\}$ such that $\mathcal{B} \cong \mathrm{Mat}_{p^{\ell_1(\mathcal{G})}}(k((\mathcal{W}_n)_1 \rtimes \mu_{(p^s)}))$.*

(2) *If $\mathcal{B}$ is tame, then $\mathcal{G}$ is reductive and $\mathcal{B} \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G})-1}}(\mathcal{C})$ for a (tame) block $\mathcal{C} \subseteq k\,\mathrm{SL}(2)_1$.*

PROOF. By way of illustration, we shall only consider (1) for $r = 1$, and in the case where the group $\mathcal{G}$ is reductive. Then $n = 0$ and we have to show that the block $\mathcal{B}$ is simple. Since the Frobenius kernels of $\mathcal{G}$ and its connected component $\mathcal{G}^0$ coincide, we may also assume that $\mathcal{G}$ is connected.

The group $\mathcal{G}(k)$ acts on $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ via the adjoint representation, which we may also consider as an action

$$\mathrm{Ad} : \mathcal{G}(k) \longrightarrow \mathrm{Aut}_k(U_0(\mathfrak{g})).$$

Let S be a simple module belonging to the block $\mathcal{B} \subseteq U_0(\mathfrak{g})$. For $g \in \mathcal{G}(k)$ we set

$$S^{(g)} := ((\mathrm{Ad}\, g)^{-1})^*(S).$$

Direct computation shows that

$$\widehat{\mathcal{V}}_{\mathfrak{g}}(S^{(g)}) = (\mathrm{Ad}\, g)(\widehat{\mathcal{V}}_{\mathfrak{g}}(S)) \qquad \forall\ g \in \mathcal{G}(k).$$

Since S is simple and $\mathcal{G}(k)$ is connected, we have $S \cong S^{(g)} \quad \forall\ g \in \mathcal{G}(k)$, so that the variety $\widehat{\mathcal{V}}_{\mathfrak{g}}(S) \subseteq \widehat{\mathcal{V}}_{\mathfrak{g}} \subseteq \mathfrak{g}$ is invariant under the adjoint action.

We first show that $\dim \widehat{\mathcal{V}}_{\mathfrak{g}}(S) \neq 1$. Otherwise, each irreducible component of the conical variety $\widehat{\mathcal{V}}_{\mathfrak{g}}(S)$ is a $\mathcal{G}(k)$-stable line, and $\mathfrak{g}$ contains a line $kx \neq (0)$ with $x^{[p]} = 0$ that is invariant under the adjoint representation. This contradicts the fact that the group $\mathcal{G}$ is reductive (cf. [**73**]).

Since $\mathcal{B}$ has finite representation type, the rank variety $\widehat{\mathcal{V}}_{\mathfrak{g}}(S)$ has dimension ≤ 1 (cf. (1.3)). By what we have just seen, this implies $\dim \hat{\mathcal{V}}_{\mathfrak{g}}(S) = 0$. Consequently, S is projective, and the self-injective algebra $\mathcal{B}$ is simple. $\square$

REMARK. The scarcity of tame or representation-finite blocks for Frobenius kernels of smooth reductive groups closely parallels the situation for blocks of groups of Lie type in the defining characteristic. The reader is referred to [**74**, §8] for more details.

5. π-Points and Jordan Type

The concept of a p-point has turned out to be merely the beginning of far-reaching investigations involving π-points and Jordan types. While π-points incorporate the treatment of infinite-dimensional modules for group schemes over arbitrary fields, the notion Jordan type refines the defining property of support spaces by investigating the isomorphism types of the $\mathfrak{A}_p$-modules $\alpha^*(M)$. In this section we sketch a few results of the relevant articles [**61, 62, 24**].

Throughout, k denotes a field of positive characteristic $p > 0$. If $\mathcal{G} := \mathrm{Spec}_k(k[\mathcal{G}])$ is an affine group scheme over k and $R \in M_k$ is a commutative k-algebra, then

$$\mathcal{G}_R : M_R \longrightarrow \mathrm{Gr} \ \ ; \quad S \mapsto \mathcal{G}(S)$$

is a group functor, which is the composite of $\mathcal{G}$ with the forgetful functor $M_R \longrightarrow M_k$ induced by the structure map $k \longrightarrow R$. In fact, being represented by the R-algebra $k[\mathcal{G}] \otimes_k R$, the group functor $\mathcal{G}_R$ is affine. If K is an extension field of k, and $\mathcal{G}$ is a finite k-group, then $\mathcal{G}_K$ is a finite K-group with algebra of measures

$$K\mathcal{G} := k\mathcal{G} \otimes_k K.$$

The isomorphism $\mathfrak{A}_p \otimes_k K \cong K[T]/(T^p)$ can be interpreted as a special case. We shall denote this algebra by $\mathfrak{A}_{p,K}$. The main innovation, expounded in [**61**], is the consideration of p-points over arbitrary field extensions:

Definition. Let $\mathcal{G}$ be a finite group scheme over k. Given a field extension $K:k$, a *π-point*

$$\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$$

is a left flat homomorphism of K-algebras, such that there exists an abelian, unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$ with $\operatorname{im}\alpha_K \subseteq K\mathcal{U}$.

We may now investigate $k\mathcal{G}$-modules M by considering their extensions $M_K := M \otimes_k K$ as $K\mathcal{G}$-modules as well as the structure of their pullbacks $\alpha_K^*(M_K) \in \operatorname{mod}\mathfrak{A}_{p,K}$. In their paper [**61**], Friedlander and Pevtsova define an equivalence relation via

$$\alpha_K \sim \beta_L \;:\Leftrightarrow\; \alpha_K^*(M_K) \text{ is projective if and only if } \beta_L^*(M_L) \text{ is projective} \quad \forall\, M \in \operatorname{mod} k\mathcal{G}$$

for π-points $\alpha_K : \mathfrak{A}_{p.K} \longrightarrow K\mathcal{G}$ and $\beta_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}$. We let $\Pi(\mathcal{G})$ be the set of equivalence classes of π-points.

If $k \subseteq K \subseteq L$ are fields, then $\mathfrak{A}_{p,K} \otimes_K L \cong \mathfrak{A}_{p,L}$, and base change associates to a π-point $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$ a π-point $\alpha_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}$. Given any $k\mathcal{G}$-module M, we have $M_K \otimes_K L \cong M_L$, whence

$$\alpha_L^*(M_L) \cong \alpha_K^*(M_K) \otimes_K L.$$

This implies:

Proposition 5.1. *Let $\mathcal{G}$ be a finite group scheme over k. Two π-points $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$ and $\beta_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}$ are equivalent if and only if $\alpha_F \sim \beta_F$ for some field F containing K and L.* □

Let me comment on the technical aspect of the definition involving the existence of an abelian, unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$. By not insisting on $\mathcal{U}$ to be defined over k, we introduce a degree of freedom that enables us to reduce many questions to the consideration of p-points over algebraically closed fields. Here is the relevant result:

Lemma 5.2. *Let $\mathcal{G}$ be a finite k-group scheme, K be an extension field of k.*
(1) *A π-point $\alpha_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}_K$ of $\mathcal{G}_K$ can be considered as a π-point $\widehat{\alpha}_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}$ of $\mathcal{G}$.*
(2) *The map $\Pi(\mathcal{G}_K) \longrightarrow \Pi(\mathcal{G})$; $[\alpha_L] \mapsto [\widehat{\alpha}_L]$ is well-defined.*

Proof. Since $L\mathcal{G}_K \cong L\mathcal{G}$, it is clear that $\widehat{\alpha}_L$ is a flat homomorphism. By assumption there exists an abelian, unipotent subgroup $\mathcal{U} \subseteq (\mathcal{G}_K)_L$ such that $\operatorname{im}\alpha_L \subseteq L\mathcal{U}$. Consequently, $\operatorname{im}\widehat{\alpha}_L \subseteq L\mathcal{U}$, so that $\widehat{\alpha}_L$ is a π-point for $\mathcal{G}$. □

Given a finite k-group scheme $\mathcal{G}$, the Universal Coefficient Theorem (cf. [**108**, (8.22)]) provides an isomorphism

$$\mathrm{H}^*(\mathcal{G}_K, K) \cong \mathrm{H}^*(\mathcal{G}, k) \otimes_k K$$

for any field $K \supseteq k$. We may thus consider $\mathrm{H}^\bullet(\mathcal{G}, k)$ as a k-subalgebra of the K-algebra $\mathrm{H}^\bullet(\mathcal{G}_K, K)$. Using the arguments of Section 2, we see that every π-point $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$ defines a non-trivial homomorphism

$$\alpha_K^\bullet : \mathrm{H}^\bullet(\mathcal{G}_K, K) \longrightarrow \mathrm{H}^\bullet(\mathfrak{A}_{p,K}, K)$$

of graded k-algebras. In particular, $\ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G},k)$ is a homogeneous prime ideal of $\mathrm{H}^\bullet(\mathcal{G},k)$ which is properly contained in the augmentation ideal $\bigoplus_{n\geq 1} \mathrm{H}^{2n}(\mathcal{G},k)$ of $\mathrm{H}^\bullet(\mathcal{G},k)$.

Let $K:k$ be a field extension. Given an element $\zeta \in \mathrm{H}^{2n}(\mathcal{G},k)\setminus\{0\}$, we denote by $L_{\zeta,K}$ the kernel of the map $\hat{\zeta}_K := \hat{\zeta}\hat{\otimes}\mathrm{id}_K : \Omega^{2n}_{\mathcal{G}}(k)\otimes_k K \longrightarrow K$. The corresponding cohomology class ζ_K belongs to $\mathrm{H}^\bullet(\mathcal{G},k) \subseteq \mathrm{H}^\bullet(\mathcal{G}_K,K)$. The following subsidiary result indicates the significance of the ideals $\ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G},k)$:

Proposition 5.3 ([**61**]). *Let $\mathcal{G}$ be a finite group scheme over k, $\alpha_K : \mathfrak{A}_{p.K} \longrightarrow K\mathcal{G}$ be a π-point. Given an element $\zeta \in \mathrm{H}^{2n}(\mathcal{G},k)\setminus\{0\}$, the following statements are equivalent:*
(1) $\zeta_K \in \ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G},k)$.
(2) $\alpha_K^*(L_{\zeta,K})$ *is not projective.*

Proof. By applying α_K^* to the map $\hat{\zeta}_K$, we obtain a short exact sequence

$$(0) \longrightarrow \alpha_K^*(L_{\zeta,K}) \longrightarrow \alpha_K^*(\Omega^{2n}_{\mathcal{G}}(k)\otimes_k K) \stackrel{\alpha_K^*(\hat{\zeta}_K)}{\longrightarrow} K \longrightarrow (0).$$

Since α_K^* sends projectives to projectives, the isomorphisms $\Omega^{2n}_{\mathcal{G}}(k)\otimes_k K \cong \Omega^{2n}_{\mathcal{G}_K}(K)\oplus(\text{proj.})$ and $\Omega^{2n}_{\mathfrak{A}_{p,K}}(K)\cong K$ give rise to a short exact sequence

$$(0) \longrightarrow \alpha_K^*(L_{\zeta,K}) \longrightarrow K \oplus (\text{proj.}) \stackrel{(f,g)}{\longrightarrow} K \longrightarrow (0),$$

where the cohomology class of f corresponds to $\alpha_K^\bullet(\zeta_K)$. We may now argue as in the proof of (2.5). □

In view of (5.1), the equivalence relation of π-points interacts nicely with field extensions. As this also holds for the associated prime ideals, the results of Section 2 are amenable to applications. Using (2.9), one obtains for π-points α_k, β_L:

$$\alpha_K \sim \beta_L \;\Leftrightarrow\; \ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G},k) = \ker \beta_L^\bullet \cap \mathrm{H}^\bullet(\mathcal{G},k).$$

Consequently, $[\alpha_K] \mapsto \ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G},k)$ is a well-defined, injective map. Friedlander and Pevtsova also show that one obtains all relevant homogeneous prime ideals of $\mathrm{H}^\bullet(\mathcal{G},k)$ in this fashion.

Let $R = \bigoplus_{n\geq 0} R_n$ be a commutative, graded k-algebra such that $R_0 = k$. Recall that

$$\mathrm{Proj}(R) := \{\mathfrak{p} \trianglelefteq R\ ;\ \mathfrak{p}\text{ homogeneous prime ideal, } \mathfrak{p} \subsetneq \bigoplus_{n\geq 1} R_n\}$$

is the projective space of R. If k is algebraically closed and $R = k[\mathcal{V}]$ is the coordinate ring of a conical variety $\mathcal{V}$, then $\mathrm{Proj}(\mathcal{V})$ is given by the maximal elements of $\mathrm{Proj}(k[\mathcal{V}])$.

Lemma 5.4. *Suppose $R = \bigoplus_{n\geq 0} R_n$ be a finitely generated k-algebra with $R_0 = k$. Given $\mathfrak{p} \in \mathrm{Proj}(R)$, let K be an algebraically closed extension field of the field of fractions $Q(R/\mathfrak{p})$. Then there exists a maximal element $\mathfrak{P} \in \mathrm{Proj}(R\otimes_k K)$ such that $\mathfrak{P}\cap R = \mathfrak{p}$.*

Proof. We first consider the case where $\mathfrak{p} = (0)$, so that R is an integral domain and $K \supseteq Q(R)$ contains its field of fractions. The canonical inclusion $R \hookrightarrow K$ induces an injective homomorphism

$$\gamma : R \longrightarrow K[T] \;\;;\;\; \sum_{n\geq 0} r_n \mapsto \sum_{n\geq 0} r_n T^n$$

of graded k-algebras. We consider the extended graded K-homomorphism

$$\hat{\gamma} : R\otimes_k K \longrightarrow K[T] \;\;;\;\; x\otimes\alpha \mapsto \alpha\gamma(x).$$

Since K is algebraically closed, the observations of Section 2 show that the ideal $\mathfrak{P} := \ker \hat{\gamma}$ is a maximal element of $\operatorname{Proj}(R \otimes_k K)$. By construction, we have $\mathfrak{P} \cap R = (0)$.

In the general case, we consider the finitely generated, graded integral domain $R' := R/\mathfrak{p}$. By the above, there exists a maximal element $\mathfrak{P}' \in \operatorname{Proj}(R' \otimes_k K)$ with $\mathfrak{P}' \cap R' = (0)$. We let $\pi : R \otimes_k K \longrightarrow R' \otimes_k K$ be the canonical graded homomorphism with $\ker \pi = \mathfrak{p} \otimes_k K$ and consider the maximal element $\mathfrak{P} := \pi^{-1}(\mathfrak{P}') \in \operatorname{Proj}(R \otimes_k K)$. Directly from the definitions we obtain

$$\mathfrak{P} \cap R = \pi^{-1}(\mathfrak{P}') \cap \pi^{-1}(R') \cap R = \pi^{-1}(\mathfrak{P}' \cap R') \cap R = \ker \pi \cap R = \mathfrak{p},$$

as desired. □

THEOREM 5.5 ([**61**]). *Let $\mathcal{G}$ be a finite group scheme over k. Then*

$$\Psi_{\mathcal{G}} : \Pi(\mathcal{G}) \longrightarrow \operatorname{Proj}(\mathrm{H}^\bullet(\mathcal{G}, k)) \quad ; \quad [\alpha_K] \mapsto \ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G}, k)$$

is a bijective map.

PROOF. It remains to verify the surjectivity of the map $\Psi_{\mathcal{G}}$. Given $\mathfrak{p} \in \operatorname{Proj}(\mathrm{H}^\bullet(\mathcal{G}, k))$, we let K be the algebraic closure of the field of fractions of $R/\mathfrak{p}$. We have a commutative diagram

$$\begin{array}{ccc} \Pi(\mathcal{G}_K) & \xrightarrow{\Psi_{\mathcal{G}_K}} & \operatorname{Proj}(\mathrm{H}^\bullet(\mathcal{G}_K, K)) \\ \Big\downarrow{\scriptstyle \mathrm{res}_1} & & \Big\downarrow{\scriptstyle \mathrm{res}_2} \\ \Pi(\mathcal{G}) & \xrightarrow{\Psi_{\mathcal{G}}} & \operatorname{Proj}(\mathrm{H}^\bullet(\mathcal{G}, k)), \end{array}$$

where res_2 is induced by base change and res_1 is defined in Lemma 5.2. Lemma 5.4 provides a maximal element $\mathfrak{P} \in \operatorname{Proj}(\mathrm{H}^\bullet(\mathcal{G}_K, K))$ with $\mathfrak{p} = \mathrm{res}_2(\mathfrak{P})$. Thanks to (2.9), we can find $x \in \Pi(\mathcal{G}_K)$ with $\Psi_{\mathcal{G}_K}(x) = \mathfrak{P}$. Consequently, $\mathrm{res}_1(x) \in \Pi(\mathcal{G})$ is the desired pre-image of $\mathfrak{p}$ under $\Psi_{\mathcal{G}}$. □

Let M be an arbitrary $\mathcal{G}$-module (not necessarily finite-dimensional). In analogy with Section 2 one defines the Π-*support of M* via

$$\Pi(\mathcal{G})_M := \{[\alpha_K] \in \Pi(\mathcal{G}) \ ; \ \alpha_K^*(M_K) \text{ is not projective}\}.$$

The sets $\Pi(\mathcal{G})_M$ enjoy the usual properties with respect to tensor products and short exact sequences. We summarize the relevant results of [**61**] in the following:

THEOREM 5.6. *Let $\mathcal{G}$ be a finite group scheme over a field k.*
(1) *The set $\{\Pi(\mathcal{G})_M \ ; \ \dim_k M < \infty\}$ is the set of closed sets of a Noetherian topology on $\Pi(\mathcal{G})$.*
(2) *For every subset $\mathcal{X} \subseteq \Pi(\mathcal{G})$ there exists a $\mathcal{G}$-module M such that $\mathcal{X} = \Pi(\mathcal{G})_M$.*
(3) *The map $\Psi_{\mathcal{G}} : \Pi(\mathcal{G}) \longrightarrow \operatorname{Proj}(\mathrm{H}^\bullet(\mathcal{G}, k))$ is a homeomorphism such that*

$$\Psi_{\mathcal{G}}^{-1}(\operatorname{Proj}(\mathrm{H}^\bullet(\mathcal{G}, k)/\ker \Phi_M) = \Pi(\mathcal{G})_M$$

for every finite-dimensional $\mathcal{G}$-module M.
(4) *If M is a finite-dimensional $\mathcal{G}$-module, then* $\dim \Pi(\mathcal{G})_M = \mathrm{cx}_{\mathcal{G}}(M) - 1$. □

Instead of asking whether the pull-back $\alpha_K^*(M_K)$ of a finite-dimensional $\mathcal{G}_K$-module is projective, one can equally well investigate the isomorphism type of the $\mathfrak{A}_{p,K}$-module $\alpha_K^*(M_K)$. This point of view is adopted in the articles [**24, 62**], which study generic properties as well as modules of constant Jordan type.

Let $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$ be a π-point. If $M \in \operatorname{mod} \mathcal{G}$, then $\alpha_K^*(M_K) \in \operatorname{mod} \mathfrak{A}_{p,K}$ uniquely decomposes as

$$\alpha_K^*(M_K) \cong \bigoplus_{i=1}^{p} \alpha_{K,i}(M)[i],$$

where $[i]$ represents the (up to isomorphism) unique indecomposable $\mathfrak{A}_{p,K}$-module of dimension i. We will interpret the right-hand side as a base change of an $\mathfrak{A}_{p,k}$-module N, that is,

$$\alpha_K^*(M_K) \cong N_K,$$

with $N = \bigoplus_{i=1}^{p} \alpha_{K,i}(M)[i] \in \operatorname{mod} \mathfrak{A}_{p,k}$. The isomorphism class of N is the *Jordan type of M with respect to α_K*, denoted $\operatorname{Jt}(M, \alpha_K)$.

Let α_K and β_L be π-points, $M \in \operatorname{mod} \mathcal{G}$ be a $\mathcal{G}$-module. In [**62**], the authors introduce a relation by setting

$$\alpha_K \trianglelefteq_M \beta_L \ :\Leftrightarrow \dim_K \operatorname{im} t^m_{\alpha_K^*(M_K)} \leq \dim_K \operatorname{im} t^m_{\beta_L^*(M_K)} \quad \forall\ m \in \{1, \ldots, p\}.$$

This condition is readily seen to be equivalent to

$$\alpha_K \trianglelefteq_M \beta_L \ :\Leftrightarrow \ \sum_{i=j}^{p} (i-j)\alpha_{K,i}(M_K) \leq \sum_{i=j}^{p} (i-j)\beta_{L,i}(M_L) \quad 1 \leq j \leq p.$$

In view of [**26**, (6.2.2)], the latter relation corresponds to the usual dominance ordering on the partitions of $\dim_k M$, associated to $\alpha_K^*(M_K)$ and $\beta_L^*(M_L)$, respectively. More precisely,

$$\alpha_K \sim_M \beta_L :\Leftrightarrow \operatorname{Jt}(M, \alpha_K) = \operatorname{Jt}(M, \beta_L)$$

defines an equivalence relation on the set $\operatorname{Pt}(\mathcal{G})$ of π-points and $\trianglelefteq_M$ is a partial ordering on the set of its equivalence classes.

This point of view provides new invariants of $\mathcal{G}$-modules and leads to the definition of classes of $\mathcal{G}$-modules that are only beginning to be understood. In [**62**] the authors show:

THEOREM 5.7. *Let $\mathcal{G}$ be a finite k-group scheme and M be a finite-dimensional $\mathcal{G}$-module. Then*

$$\tilde{\Pi}(\mathcal{G})_M := \{x \in \Pi(\mathcal{G}) \ ; \exists\ \alpha_K \in x \ \textit{such that} \ \operatorname{Jt}(M, \alpha_K) \ \textit{is not of maximal for} \trianglelefteq_M\}$$

is a closed subspace of $\Pi(\mathcal{G})$. □

The non-maximal support space $\tilde{\Pi}(\mathcal{G})_M$ coincides with $\Pi(\mathcal{G})_M$ if and only if $\Pi(\mathcal{G})_M \neq \Pi(\mathcal{G})$.

Recall that a finite-dimensional $\mathcal{G}$-module M is projective if and only if $\Pi(\mathcal{G})_M = \emptyset$. If $\tilde{\Pi}(\mathcal{G})_M = \emptyset$, then M is of *constant Jordan type*, that is, $\operatorname{Jt}(M, \alpha_K) = \operatorname{Jt}(M, \beta_L)$ for any two π-points α_K and β_L of $\mathcal{G}$. This interesting class of modules was investigated in recent papers by Carlson-Friedlander-Pevtsova [**24**] and Carlson-Friedlander [**23**]. The full subcategory of $\operatorname{mod} \mathcal{G}$, whose objects are of constant Jordan type, is closed under direct sums and direct summands.

CHAPTER IV

Varieties of Tori

As we have seen in Sections III.2 and III.3, the determination of the representation type of finite algebraic groups leads to conditions on rank varieties. This leaves us with the problem of interpreting the ramifications of these conditions for the structure of the underlying groups. For Frobenius kernels of smooth groups, our knowledge of nilpotent orbits suffices to reduce the problem to the study of SL(2), a group whose representation theory is well enough understood to provide us with complete answers. The failure of the classical Lie-Kolchin Theorem for arbitrary infinitesimal groups already indicates the problems one encounters when leaving the classical environment of Frobenius kernels associated to smooth group schemes. Schemes of tori and their associated algebraic families of Lie algebras help us deal with this problem for arbitrary infinitesimal groups.

The material of this chapter is taken from [**48, 49**].

1. Algebraic Families of Vector Spaces

Throughout, we let A be a finitely generated integral domain over the algebraically closed field k with associated scheme $\mathcal{X} := \mathrm{Spec}_k(A)$. Given a finite-dimensional k-vector space V, we consider the free A-module $V \otimes_k A$. For an A-submodule $P \subseteq V \otimes_k A$, and $x \in \mathcal{X}(k)$, we denote by $P(x) := (\mathrm{id}_V \hat{\otimes} x)(P) \subseteq V$ the subspace of V obtained by specialization along x. If P is a direct summand of the A-module $V \otimes_k A$, then $P(x) \cong P \otimes_A k_x$, where k_x denotes the one-dimensional A-module afforded by x. We will be studying the algebraic family $(P(x))_{x\in\mathcal{X}(k)}$ of subspaces of V.

Let X be a variety, $x \in X$ be a point. Then

$$\dim_x X := \max\{\dim Y \ ; \ Y \subseteq X \text{ is an irreducible component of } X \text{ containing } x\}$$

is called the *local dimension* of X at x. A function $f : X \longrightarrow \mathbb{N}_0$, defined on a variety X, is called *upper semicontinuous* if for every $n \in \mathbb{N}_0$ the subset $f^{-1}(\{m \in \mathbb{N}_0 \ ; \ m \geq n\}) \subseteq X$ is closed. Given a morphism $\varphi : X \longrightarrow Y$ of varieties, we denote by $\varepsilon_\varphi(x) := \dim_x \varphi^{-1}(\varphi(x))$ the local dimension of $\varphi^{-1}(\varphi(x))$ at the point $x \in X$. The following basic result is referred to as semicontinuity of fiber dimension, see [**35**, §14].

Proposition 1.1. *Let $\varphi : X \longrightarrow Y$ be a morphism of varieties. Then the map*

$$X \longrightarrow \mathbb{N}_0 \ \ ; \ \ x \mapsto \varepsilon_\varphi(x)$$

is upper semicontinuous. □

Proposition 1.2. *Let $Y \subseteq V$ be a conical closed subset, and suppose that $P \subseteq V \otimes_k A$ is an A-direct summand. Then the following statements hold:*

(1) *The function $\delta_{P\cap Y} : \mathcal{X}(k) \longrightarrow \mathbb{N}_0$; $x \mapsto \dim P(x) \cap Y$ is upper semicontinuous.*

(2) *If $d := \min\{\dim P(x) \cap Y \ ; \ x \in \mathcal{X}(k)\}$, then $U := \delta^{-1}_{P\cap Y}(d)$ is a dense, open subset of $\mathcal{X}(k)$.*

PROOF. (1) Consider the variety $\mathcal{X}(k) \times Y$ as well as the subset

$$Z := \{(x, y) \in \mathcal{X}(k) \times Y \ ; \ \exists\, p \in P \text{ such that } (\mathrm{id}_V \hat{\otimes} x)(p) = y\}.$$

By assumption, there exists a submodule $Q \subseteq V \otimes_k A$ such that $V \otimes_k A = P \oplus Q$. Let $\{v_1, \ldots, v_n\}$ be a basis of V. We thus have projection maps

$$\mathrm{pr}_j : V \longrightarrow k \ \ ; \ \ \sum_{i=1}^{n} \alpha_i v_i \mapsto \alpha_j \qquad 1 \leq j \leq n$$

as well as

$$\mathrm{pr}_Q : V \otimes_k A \longrightarrow Q \ \ ; \ \ p + q \mapsto q.$$

Let $(a_{ij}) \in \mathrm{Mat}_n(A)$ be the $(n \times n)$-matrix, given by

$$\mathrm{pr}_Q(v_j \otimes 1) = \sum_{i=1}^{n} v_i \otimes a_{ij} \qquad 1 \leq i \leq n.$$

Let $(x, y) \in \mathcal{X}(k) \times Y$. Direct computation now shows that

$$(x, y) \in Z \ \text{ if and only if } \ \sum_{j=1}^{n} x(a_{ij})\mathrm{pr}_j(y) = 0 \qquad 1 \leq i \leq m.$$

Since $\mathrm{pr}_j|_Y \in k[Y]$ for $1 \leq j \leq n$ and $\tilde{a} : \mathcal{X}(k) \longrightarrow k \,; x \mapsto x(a)$ belongs to $k[\mathcal{X}(k)]$ for every $a \in A$, it follows that

$$Z = \{(x, y) \in \mathcal{X}(k) \times Y \ ; \ [\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \mathrm{pr}_j](x, y) = 0 \quad 1 \leq i \leq n\}$$

is closed.

We consider the restriction $\pi : Z \longrightarrow \mathcal{X}(k)$ of the projection onto the first factor. Given $x \in \mathcal{X}(k)$, the canonical map

$$\iota_x : Y \longrightarrow \mathcal{X}(k) \times Y \ \ ; \ \ y \mapsto (x, y)$$

induces an isomorphism $\pi^{-1}(x) \cong P(x) \cap Y$, so that $\varepsilon_\pi(x, y) = \dim_y P(x) \cap Y$ for every point $(x, y) \in Z$. Since Y is a conical, closed subset of V, the neutral element 0 belongs to Y and $\dim_0 Y = \dim Y$. This also applies to the closed, conical subset $P(x) \cap Y$. Consider the morphism $\gamma : \mathcal{X}(k) \longrightarrow Z \ ; \ x \mapsto (x, 0)$. The above observations yield

$$\varepsilon_\pi \circ \gamma = \delta_{P \cap Y},$$

and Proposition 1.1 ensures that this map is upper semicontinuous.

(2) By definition of d, the set $U = \mathcal{X}(k) \setminus \{x \in \mathcal{X}(k) \ ; \delta_{P \cap Y}(x) \geq d + 1\}$ is a non-empty open subset of $\mathcal{X}(k)$. Since A is an integral domain, the variety $\mathcal{X}(k)$ is irreducible, so that U lies dense in $\mathcal{X}(k)$. □

COROLLARY 1.3. *Suppose that $P \subseteq V \otimes_k A$ is an A-direct summand.*
(1) *The function $\delta_P : \mathcal{X}(k) \longrightarrow \mathbb{N}_0 \ ; \ x \mapsto \dim_k P(x)$ is constant.*
(2) *If $P \neq (0)$, then $P(x) \neq (0)$ for every $x \in \mathcal{X}(k)$.*

PROOF. (1) By assumption, there exists an A-submodule $Q \subseteq V \otimes_k A$ such that $V \otimes_k A = P \oplus Q$. Consequently, $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$. Let $d_P := \min\{\dim_k P(x) \ ; \ x \in \mathcal{X}(k)\}$ and $d_Q := \min\{\dim_k Q(x) \ ; \ x \in \mathcal{X}(k)\}$. Since $\mathcal{X}(k)$ is irreducible, (1.2) yields $d_P + d_Q = \dim_k V$. Consequently, $d_P \leq \dim_k P(x) = \dim_k V - \dim_k Q(x) \leq \dim_k V - d_Q = d_P$ for an arbitrary $x \in \mathcal{X}(k)$.

(2) Suppose there is $x_0 \in \mathcal{X}(k)$ such that $P(x_0) = (0)$. According to (1) we then have $P(x) = (0)$ for every $x \in \mathcal{X}(k)$. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Given $p \in P$, we write $p = \sum_{i=1}^n v_i \otimes a_i$. Since

$$0 = (\mathrm{id}_V \hat{\otimes} x)(p) = \sum_{i=1}^n x(a_i)v_i \qquad \forall\ x \in \mathcal{X}(k)$$

we see that the zero locus $Z(I)$ of the ideal $I := (\{a_1, \ldots, a_n\}) \subseteq A$ is all of $\mathcal{X}(k)$. Hilbert's Nullstellensatz now yields $I = (0)$, whence $p = 0$. Consequently, $P = (0)$, a contradiction. □

Given an A-direct summand $P \subseteq V \otimes_k A$, and a subset $U \subseteq \mathcal{X}(k)$, we define

$$I_U(P) := \bigcap_{x \in U} P(x).$$

In the sequel, we are going to provide conditions ensuring that $I_U(P)$ is a non-zero subspace of V which is stable under certain group actions.

Definition. Let V be a finite-dimensional vector space, $\mathcal{F}$ be a finite set of closed subsets of V. A family $(W_i)_{i \in I}$ of subspaces of V is said to be *$\mathcal{F}$-regular*, provided every subspace W_i contains an element of $\mathcal{F}$.

Lemma 1.4. *Let $\mathcal{F}$ be a finite set of closed subsets of the finite-dimensional k-vector space V. If $P \subseteq V \otimes_k A$ is an A-submodule such that*

(a) *P is an A-direct summand of $V \otimes_k A$, and*

(b) *there exists a dense subset $U \subseteq \mathcal{X}(k)$ such that $(P(x))_{x \in U}$ is $\mathcal{F}$-regular,*

then the subspace $I_{\mathcal{X}(k)}(P) \subseteq V$ is $\mathcal{F}$-regular.

Proof. We write $\mathcal{F} = \{Y_1, \ldots, Y_n\}$. In view of (a), there exists an A-submodule $Q \subseteq V \otimes_k A$ with $V \otimes_k A = P \oplus Q$, so that $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$. Given $y \in V$, we write $y \otimes 1 = p + q$ with $p \in P$ and $q \in Q$. Then $y \in P(x)$ if and only if $(\mathrm{id}_V \hat{\otimes} x)(q) = 0$. By choosing a basis $\{v_1, \ldots, v_m\}$ of V and writing $q = \sum_{i=1}^m v_i \otimes a_i$, we see that the latter condition is equivalent to $x(a_i) = 0$ for $1 \leq i \leq m$. Consequently, the set $Z[y] := \{x \in \mathcal{X}(k)\ ;\ y \in P(x)\}$ is closed, and $Z_j := \{x \in \mathcal{X}(k)\ ;\ Y_j \subseteq P(x)\} = \bigcap_{y \in Y_j} Z[y]$ has the same property. Condition (b) implies $\mathcal{X}(k) = \bar{U} \subseteq \bigcup_{j=1}^n Z_j$. Since A is an integral domain, the variety $\mathcal{X}(k)$ is irreducible, and there exists $j_0 \in \{1, \ldots, n\}$ such that $Z_{j_0} = \mathcal{X}(k)$. It follows that $Y_{j_0} \subseteq I_{\mathcal{X}(k)}(P)$. □

Let $\mathcal{G}$ be an affine algebraic k-group, V be a $\mathcal{G}$-module. We assume that $\mathcal{G}$ acts on A via algebra homomorphisms. Consider the $\mathcal{G}$-module $V \otimes_k A$ with the diagonal operation

$$g{\boldsymbol .}[(v \otimes_k r) \otimes_R (a \otimes_k s)] := g(v \otimes_k r) \otimes_R g(a \otimes_k s)$$

for $g \in \mathcal{G}(R)$, $r, s \in R$, $v \in V$, $a \in A$.

Lemma 1.5. *Let $P \subseteq V \otimes_k A$ be a $\mathcal{G}$-stable A-submodule that is also an A-direct summand of $V \otimes_k A$. If $U \subseteq \mathcal{X}(k)$ is dense, then $I_U(P) \subseteq V$ is a $\mathcal{G}$-submodule.*

Proof. One verifies the identity

$$V \cap P = I_U(P)$$

and observes that the left-hand side is a $\mathcal{G}$-submodule of V. □

2. Schemes of Tori

Recall that k is an algebraically closed field of characteristic $p > 0$. Throughout this section we will only consider finite-dimensional restricted Lie algebras. Such a Lie algebra $(\mathfrak{t}, [p])$ is called a *torus* if and only if $\widehat{\mathcal{V}}_{\mathfrak{t}} = \{0\}$. Tori are necessarily abelian and linearly reductive: Every $U_0(\mathfrak{t})$-module is completely reducible.

If $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ is the Lie algebra of a smooth group, then much of the structure of $\mathfrak{g}$ can be detected via the so-called *root space decomposition*. One picks a maximal torus $\mathfrak{t} \subseteq \mathfrak{g}$ and decomposes $\mathfrak{g}$ into its eigenspaces relative to $\mathfrak{t}$:

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{t}) \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad ; \quad R \subseteq \mathfrak{g}^* \setminus \{0\}.$$

Here $C_{\mathfrak{g}}(\mathfrak{t}) := \{x \in \mathfrak{g} \ ; \ [t, x] = 0 \ \ \forall\, t \in \mathfrak{t}\}$ is the *centralizer of* $\mathfrak{t}$ *in* $\mathfrak{g}$, and $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \ ; \ [t, x] = \alpha(t)x \ \ \forall\, t \in \mathfrak{t}\} \neq (0)$ is the *root space* for the *root* $\alpha \in R$. Since $\mathcal{G}$ is smooth, any two maximal tori are conjugate under the adjoint representation, so it doesn't really matter which maximal torus we take. The following example shows that this is no longer true for arbitrary restricted Lie algebras.

Example. For $p \geq 5$ we consider the *Witt algebra* $W(1) := \mathrm{Der}_k(k[X]/(X^p))$ of the derivations of the truncated polynomial ring $k[X]/(X^p)$. Since the p-th power of a derivation is again a derivation, $(W(1), p)$ is a restricted Lie algebra. Let ∂ be the derivation induced by $\frac{d}{dX}$ and set $x := X + (X^p)$ as well as $e_i := x^{i+1}\partial$ for $-1 \leq i \leq p-2$. Then $\{e_{-1}, \dots, e_{p-2}\}$ is a basis of $W(1)$, and we have

$$[e_i, e_j] = (j - i)e_{i+j} \quad ; \quad e_i^p = \delta_{i,0}e_0,$$

where the product is understood to be zero whenever $i + j$ does not lie within $\{-1, \dots, p-2\}$. It follows that the p-subalgebra $\mathfrak{t} := ke_0$ is self-centralizing and hence is a maximal torus of $W(1)$.

We define another basis via $f_i := (x+1)^{i+1}\partial$ for $-1 \leq i \leq p-2$. Then $f_0^p = f_0$ and $[f_i, f_j] = (j - i)f_{i+j}$. In particular, $\mathfrak{t}' := kf_0$ is another maximal torus of $W(1)$. However, now the subscripts have to be interpreted $\mathrm{mod}(p)$, e.g., $[f_1, f_{p-2}] = -3f_{-1}$. Thus, while the root space decomposition relative to $\mathfrak{t}$ induces a $\mathbb{Z}$-grading, we have a grading with respect to the group $\mathbb{Z}/(p)$ in the latter case. The $\mathbb{Z}$-grading is better to work with because we can for instance read off that $\mathrm{ad}\, e_1$ is a nilpotent transformation.

These observations already indicate that $\mathfrak{t}$ and $\mathfrak{t}'$ are really different. In fact, they cannot be mapped onto each other by any automorphism of $W(1)$. Direct computation shows that $W(1)_{(0)} := \sum_{i=0}^{p-2} k\, e_i$ is the unique p-subalgebra of codimension 1 (here we need $p \geq 5$). Hence it is fixed by any automorphism $\varphi \in \mathrm{Aut}_p(W(1))$, and $\varphi(\mathfrak{t}) \subseteq W(1)_{(0)}$. In particular, $\varphi(\mathfrak{t}) \neq \mathfrak{t}'$.

The foregoing example illustrates our predicament. We have to choose a maximal torus without knowing which choice is good for our purposes. Schemes of tori obviate this difficulty by simultaneously studying all tori of a certain isomorphism type.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k, R be a commutative k-algebra. Recall that $\mathfrak{g} \otimes_k R$ obtains the structure of a restricted Lie algebra over R via

$$[x \otimes r, y \otimes s] := [x, y] \otimes rs \quad ; \quad (x \otimes r)^{[p]} = x^{[p]} \otimes r^p \quad \forall\, x, y \in \mathfrak{g},\ r, s \in R.$$

Now let $(\mathfrak{t}, \mathfrak{g})$ be a pair of restricted Lie algebras over k. We consider the k-functor $\mathcal{T}_{\mathfrak{g}} : M_k \longrightarrow \mathrm{Ens}$ that associates to each commutative k-algebra R the set $\mathcal{T}_{\mathfrak{g}}(R)$ of those homomorphisms $\varphi : \mathfrak{t} \otimes_k R \longrightarrow \mathfrak{g} \otimes_k R$ of restricted Lie algebras over R that are split injective R-linear maps. Observe that the set $\mathcal{T}_{\mathfrak{g}}(k)$ of k-rational points is just the set of embeddings $\mathfrak{t} \hookrightarrow \mathfrak{g}$.

THEOREM 2.1 ([**48**]). *Let $\mathfrak{t}$ be a torus. Given a restricted Lie algebra $\mathfrak{g}$, the following statements hold:*

(1) *$\mathcal{T}_{\mathfrak{g}}$ is a smooth, affine, algebraic scheme.*

(2) *If $\mathcal{X} \subseteq \mathcal{T}_{\mathfrak{g}}$ is an irreducible component, then*

$$\dim \mathcal{X} = \dim_k \mathfrak{g} - \dim_k C_{\mathfrak{g}}(\varphi(\mathfrak{t})) \qquad \forall\ \varphi \in \mathcal{X}(k). \qquad \square$$

One main point of (1) is that the connected components of $\mathcal{T}_{\mathfrak{g}}$ coincide with its irreducible components. Thus, if $\mathfrak{t} \subseteq \mathfrak{g}$ is a torus with the embedding $\mathfrak{t} \hookrightarrow \mathfrak{g}$ corresponding to a rational point $x_0 \in \mathcal{T}_{\mathfrak{g}}(k)$, then there exists exactly one irreducible component $\mathcal{X}_{\mathfrak{t}} \subseteq \mathcal{T}_{\mathfrak{g}}$ such that $x_0 \in \mathcal{X}_{\mathfrak{t}}(k)$.

We illustrate (2.1) by considering Lie algebras of smooth groups. Let $\mathcal{G}$ be an affine algebraic group, $\mathrm{Ad} : \mathcal{G}(k) \longrightarrow \mathrm{Aut}_k(\mathrm{Lie}(\mathcal{G}))$ be its adjoint representation. Then $\mathcal{G}(k)$ operates on the affine variety $\mathcal{T}_{\mathfrak{g}}(k)$ via

$$g.\varphi := \mathrm{Ad}(g) \circ \varphi \qquad \forall\ g \in \mathcal{G}(k),\ \varphi \in \mathcal{T}_{\mathfrak{g}}(k).$$

PROPOSITION 2.2. *Let $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ be the Lie algebra of a smooth, connected, affine algebraic group, $\mathfrak{t}$ be a torus. Then the connected components of $\mathcal{T}_{\mathfrak{g}}(k)$ are the $\mathcal{G}(k)$-orbits of $\mathcal{T}_{\mathfrak{g}}(k)$.* $\square$

REMARKS. (1) Let $\mathfrak{t} \overset{x_0}{\hookrightarrow} \mathfrak{g}$ be the canonical embedding. Under the assumptions of (2.2) the morphism $g \mapsto \mathrm{Ad}(g) \circ x_0$ induces a bijective $\mathcal{G}(k)$-equivariant map $\mathcal{G}(k)/\mathrm{Stab}_{\mathcal{G}(k)}(x_0) \longrightarrow \mathcal{X}_{\mathfrak{t}}(k)$ of homogeneous spaces. Owing to [**48**, (1.4)], its differential $T_{\bar{1}}(\mathcal{G}(k)/\mathrm{Stab}_{\mathcal{G}(k)}(x_0)) \longrightarrow \mathrm{T}_{x_0}(\mathcal{X}_{\mathfrak{t}})$ is given by the bijection

$$\mathfrak{g}/C_{\mathfrak{g}}(\mathfrak{t}) \longrightarrow \mathrm{Der}_k(\mathfrak{t}, \mathfrak{g}) \ \ ; \ \ [x] \mapsto \mathrm{ad}\, x|_{\mathfrak{t}}.$$

Consequently, we have an isomorphism $\mathcal{G}(k)/C_{\mathcal{G}(k)}(\mathfrak{t}) \cong \mathcal{X}_{\mathfrak{t}}(k)$. Here $C_{\mathcal{G}(k)}(\mathfrak{t}) = \{g \in \mathcal{G}(k)\ ;\ \mathrm{Ad}(g)(t) = t \ \ \forall\ t \in \mathfrak{t}\}$ is the *centralizer of* $\mathfrak{t}$ *in* $\mathcal{G}(k)$.

(2) One can use schemes of tori to recover the classical conjugacy theorem for maximal tori of Lie algebras of smooth groups, see [**39**] for more details.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, $\mathfrak{t} \subseteq \mathfrak{g}$ be a torus with embedding $\mathfrak{t} \hookrightarrow \mathfrak{g}$ corresponding to a rational point $x_0 \in \mathcal{T}_{\mathfrak{g}}(k)$. Thanks to (2.1), the irreducible component $\mathcal{X}_{\mathfrak{t}}$ is representable: There exists a finitely generated integral domain A such that $\mathcal{X}_{\mathfrak{t}} \cong \mathrm{Spec}_k(A)$. We consider the restricted Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes_k A$. Under the above identification $\mathrm{id}_A \in \mathrm{Spec}_k(A)(A)$ corresponds to an embedding $j : \mathfrak{t} \longrightarrow \tilde{\mathfrak{g}}$ of restricted k-Lie algebras such that the A-submodule $Aj(\mathfrak{t}) \subseteq \tilde{\mathfrak{g}}$ is a direct summand of $\tilde{\mathfrak{g}}$. Since $\mathcal{X}_{\mathfrak{t}} \cong \mathrm{Spec}_k(A)$, any element $\varphi : \mathfrak{t} \longrightarrow \mathfrak{g} \otimes_k R$ of $\mathcal{X}_{\mathfrak{t}}(R)$ is obtained via specialization: If $\varphi \in \mathcal{X}_{\mathfrak{t}}(R)$ corresponds to $x \in \mathrm{Spec}_k(A)(R)$, then we have

$$\varphi = (\mathrm{id}_{\mathfrak{g}} \otimes x) \circ j.$$

For that reason, we call $j : \mathfrak{t} \hookrightarrow \tilde{\mathfrak{g}}$ the *universal embedding.*

Note that j endows the extended Lie algebra $\tilde{\mathfrak{g}}$ with the structure of an infinite-dimensional $U_0(\mathfrak{t})$-module. Since $U_0(\mathfrak{t})$ is commutative and semi-simple, there results a weight space decomposition

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus \bigoplus_{\alpha \in \Phi} \tilde{\mathfrak{g}}_\alpha$$

of $\tilde{\mathfrak{g}}$ relative to $\mathfrak{t}$. Here $\Phi \subseteq \mathfrak{t}^* \setminus \{0\}$ is the set of weights, and for $\alpha \in \Phi \cup \{0\}$ the weight space $\tilde{\mathfrak{g}}_\alpha = \{v \in \tilde{\mathfrak{g}}\ ;\ [j(t), v] = \alpha(t)v \ \ \forall\ t \in \mathfrak{t}\} \neq (0)$ is an A-direct summand of $\tilde{\mathfrak{g}}$. Given an arbitrary element $x \in \mathcal{X}_{\mathfrak{t}}(k)$, we have

$$\mathfrak{g} = \tilde{\mathfrak{g}}_0(x) \oplus \bigoplus_{\alpha \in \Phi} \tilde{\mathfrak{g}}_\alpha(x),$$

where $\tilde{\mathfrak{g}}_\alpha(x) = \{v \in \mathfrak{g} \ ; \ [(\mathrm{id}_{\mathfrak{g}} \hat{\otimes} x)(j(t)), v] = \alpha(t)v \quad \forall \ t \in \mathfrak{t}\}$ for $\alpha \in \Phi \cup \{0\}$. In other words, if $\varphi := (\mathrm{id}_{\mathfrak{g}} \hat{\otimes} x) \circ j$ is the embedding corresponding to $x \in \mathfrak{X}_{\mathfrak{t}}(k)$, then $\tilde{\mathfrak{g}}_\alpha(x)$ is the root space with root $\alpha \circ \varphi^{-1}$ relative to the torus $\varphi(\mathfrak{t}) \subseteq \mathfrak{g}$. In particular, specialization along x_0 yields the root space decomposition

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{t}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

of $\mathfrak{g}$ relative to $\mathfrak{t}$. Thanks to (1.3), this also shows that Φ is a finite set.

Recall that the rank variety of the trivial module of a restricted Lie algebra $(\mathfrak{g}, [p])$ is given by

$$\widehat{\mathcal{V}}_{\mathfrak{g}} = \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}.$$

We will occasionally refer to $\dim \widehat{\mathcal{V}}_{\mathfrak{g}}$ as the *null-rank* of the restricted Lie algebra $\mathfrak{g}$.

Returning to our general set-up, we let $P \subseteq \tilde{\mathfrak{g}}$ be an A-direct summand of $\tilde{\mathfrak{g}}$. By virtue of (1.2), there exists a non-empty open subset $U_P \subseteq \mathfrak{X}_{\mathfrak{t}}(k)$, and a natural number $c_P(\mathfrak{g}, \mathfrak{t}) \in \mathbb{N}_0$ such that

$$\dim P(x) \cap \widehat{\mathcal{V}}_{\mathfrak{g}} = c_P(\mathfrak{g}, \mathfrak{t}) \qquad \forall \ x \in U_P.$$

The number $c_P(\mathfrak{g}, \mathfrak{t})$ is the *generic null-rank* of the algebraic family $(P(x))_{x \in \mathfrak{X}_{\mathfrak{t}}(k)}$ of subspaces of $\mathfrak{g}$. Given a subset $\Psi \subseteq \mathfrak{t}^*$, the A-submodule $\tilde{\mathfrak{g}}^\Psi := \bigoplus_{\alpha \in \Psi} \tilde{\mathfrak{g}}_\alpha$ is a direct summand of $\tilde{\mathfrak{g}}$, and we write $c_\Psi(\mathfrak{g}, \mathfrak{t}) := c_{\tilde{\mathfrak{g}}^\Psi}(\mathfrak{g}, \mathfrak{t})$.

A nilpotent, self-normalizing p-subalgebra of $\mathfrak{g}$ is called a *Cartan subalgebra*. According to general theory, the centralizer $C_{\mathfrak{g}}(\mathfrak{t})$ of the torus $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{t}$ is a maximal torus (cf. [**115**, Chapter II]).

We illustrate the utility of the generic null-rank by verifying the following subsidiary result:

Lemma 2.3. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a torus. Then $c_0(\mathfrak{g}, \mathfrak{t}) = 0$ if and only if $C_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$.*

Proof. Let $A := k[\mathfrak{X}_{\mathfrak{t}}]$ be the coordinate ring of the component $\mathfrak{X}_{\mathfrak{t}}$. We consider the A-direct summands $Aj(\mathfrak{t})$ and $\tilde{\mathfrak{g}}_0$ of $\tilde{\mathfrak{g}}$, and observe that for $x \in \mathfrak{X}(k)$ and its associated embedding $\varphi := (\mathrm{id}_{\mathfrak{g}} \hat{\otimes} x) \circ j$, the specializations along x are given by

$$(Aj(\mathfrak{t}))(x) = \varphi(\mathfrak{t}) \ \text{ and } \ \tilde{\mathfrak{g}}_0(x) = C_{\mathfrak{g}}(\varphi(\mathfrak{t})),$$

respectively. By definition, there exists a dense open subset $U \subseteq \mathfrak{X}_{\mathfrak{t}}(k)$ such that

$$\dim \widehat{\mathcal{V}}_{\tilde{\mathfrak{g}}_0(x)} = c_0(\mathfrak{g}, \mathfrak{t}) \quad \forall \ x \in U.$$

Suppose that $c_0(\mathfrak{g}, \mathfrak{t}) = 0$ and let $x \in U$. Then $\widehat{\mathcal{V}}_{\tilde{\mathfrak{g}}_0(x)} = \{0\}$ and $C_{\mathfrak{g}}(\varphi(\mathfrak{t}))$ is a torus. Since $C_{\mathfrak{g}}(\varphi(\mathfrak{t}))$ is self-normalizing, it is a Cartan subalgebra of $\mathfrak{g}$. Thus, $(Aj(\mathfrak{t}))(x) = \varphi(\mathfrak{t})$ is a maximal torus of $\mathfrak{g}$, and we conclude that $(Aj(\mathfrak{t}))(x) = \tilde{\mathfrak{g}}_0(x)$. As $Aj(\mathfrak{t})$ and $\tilde{\mathfrak{g}}_0$ are A-direct summands of $\tilde{\mathfrak{g}}$, (1.3) yields

$$\dim_k C_{\mathfrak{g}}(\mathfrak{t}) = \dim_k \tilde{\mathfrak{g}}_0(x_0) = \dim_k \tilde{\mathfrak{g}}_0(x) = \dim_k (Aj(\mathfrak{t}))(x) = \dim_k (Aj(\mathfrak{t}))(x_0) = \dim_k \mathfrak{t},$$

so that $\mathfrak{t} = C_{\mathfrak{g}}(\mathfrak{t})$.

Conversely, assume that $\mathfrak{t} = C_{\mathfrak{g}}(\mathfrak{t})$. Then we have $\dim_k (Aj(\mathfrak{t}))(x_0) = \dim_k \tilde{\mathfrak{g}}_0(x_0)$. As $Aj(\mathfrak{t})$ and $\tilde{\mathfrak{g}}_0$ are A-direct summands of $\tilde{\mathfrak{g}}$, (1.3) implies $(Aj(\mathfrak{t}))(x) = \tilde{\mathfrak{g}}_0(x)$ for every $x \in \mathfrak{X}_{\mathfrak{t}}(k)$. This shows that $c_0(\mathfrak{g}, \mathfrak{t}) = 0$. □

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. We denote by $\mathcal{AUT}(\mathfrak{g})$ the automorphism scheme of $\mathfrak{g}$. For every commutative k-algebra R, $\mathcal{AUT}(\mathfrak{g})(R)$ is the set of automorphisms of the restricted R-Lie algebra $\mathfrak{g} \otimes_k R$. The connected component of $\mathcal{AUT}(\mathfrak{g})$ will be denoted $\mathcal{G}_{\mathfrak{g}}$. A subspace $\mathfrak{n} \subseteq \mathfrak{g}$ is $\mathcal{G}_{\mathfrak{g}}$-*invariant* if $g(\mathfrak{n} \otimes_k R) = \mathfrak{n} \otimes_k R$ for every $R \in M_k$ and $g \in \mathcal{G}_{\mathfrak{g}}(R)$.

The natural operation of $\mathcal{G}_{\mathfrak{g}}$ on $\mathfrak{g}$ induces an action of $\mathcal{G}_{\mathfrak{g}}$ on $\mathcal{T}_{\mathfrak{g}}$: For $g \in \mathcal{G}_{\mathfrak{g}}(R)$ and $\varphi \in \mathcal{T}_{\mathfrak{g}}(R)$ we have $g.\varphi := g \circ \varphi$.

Lemma 2.4. *The following statements hold:*
(1) *Every irreducible component* $\mathcal{X} \subseteq \mathcal{T}_{\mathfrak{g}}$ *is* $\mathcal{G}_{\mathfrak{g}}$*-invariant.*
(2) *Let* $\Psi \subseteq \mathfrak{t}^*$*. Then* $I_{\mathcal{X}_{\mathfrak{t}}(k)}(\tilde{\mathfrak{g}}^{\Psi})$ *is* $\mathcal{G}_{\mathfrak{g}}$*-invariant.*
(3) *Let* $\Psi \subseteq \mathfrak{t}^*$*. Then* $I_{\mathcal{X}_{\mathfrak{t}}(k)}(\tilde{\mathfrak{g}}^{\Psi})$ *is an ideal of* $\mathfrak{g}$*.* □

We illustrate the use of these techniques by considering representation-finite restricted Lie algebras. Recall that $\mathfrak{g}$ is referred to as *solvable* if its *derived series* $(\mathfrak{g}^{(n)})_{n\geq 0}$, which is inductively defined via $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$, contains the ideal (0)of $\mathfrak{g}$. If this holds for the *descending central series* $(\mathfrak{g}^n)_{n\geq 1}$ with $\mathfrak{g}^1 := \mathfrak{g}$ and $\mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]$, then $\mathfrak{g}$ is called *nilpotent*.

Proposition 2.5. *Let* $(\mathfrak{g}, [p])$ *be a restricted Lie algebra such that the principal block* $\mathcal{B}_0(\mathfrak{g}) \subseteq U_0(\mathfrak{g})$ *has finite representation type. Then* $\mathfrak{g}$ *is solvable.*

Proof. According to (II.3.2) and (III.2.11) the null-rank of $\mathfrak{g}$ is bounded by 1, that is, $\dim \widehat{\mathcal{V}}_{\mathfrak{g}} \leq 1$. If $\mathfrak{g}$ has null-rank 0, then $\mathfrak{g}$ is a torus, and thus is in particular abelian. We therefore assume $\dim \widehat{\mathcal{V}}_{\mathfrak{g}} = 1$ and proceed by induction on $\dim_k \mathfrak{g}$, the case $\dim_k \mathfrak{g} = 1$ being trivial.

Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal torus, $\mathcal{X}_{\mathfrak{t}} \subseteq \mathcal{T}_{\mathfrak{g}}$ be the irreducible component containing the embedding $\mathfrak{t} \overset{x_0}{\hookrightarrow} \mathfrak{g}$. We have the weight space decomposition

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus \bigoplus_{\alpha \in \Phi} \tilde{\mathfrak{g}}_{\alpha}$$

of $\tilde{\mathfrak{g}}$ relative to $\mathfrak{t}$. In our current situation, the generic null-rank $c_0(\mathfrak{g}, \mathfrak{t})$ satisfies

$$0 \leq c_0(\mathfrak{g}, \mathfrak{t}) \leq 1.$$

If $c_0(\mathfrak{g}, \mathfrak{t}) = 1$, then there exists a dense open subset $U \subseteq \mathcal{X}_{\mathfrak{t}}(k)$ such that the family $(\tilde{\mathfrak{g}}_0(x))_{x \in U}$ is regular with respect to the one-dimensional irreducible components of the variety $\widehat{\mathcal{V}}_{\mathfrak{g}}$. A consecutive application of (1.4) and (2.4) now implies that

$$\mathfrak{n} := I_{\mathcal{X}_{\mathfrak{t}}(k)}(\tilde{\mathfrak{g}}_0) \subseteq C_{\mathfrak{g}}(\mathfrak{t})$$

is a non-zero p-ideal of $\mathfrak{g}$ that is contained in the Cartan subalgebra $C_{\mathfrak{g}}(\mathfrak{t})$. Thus, $\mathfrak{n}$ is nilpotent and hence solvable. Moreover, the principal block $\mathcal{B}_0(\mathfrak{g}/\mathfrak{n})$ is representation-finite, so that the inductive hypothesis implies the solvability of $\mathfrak{g}/\mathfrak{n}$. Consequently, the Lie algebra $\mathfrak{g}$ is solvable.

It remains to consider the case $c_0(\mathfrak{g}, \mathfrak{t}) = 0$. Let $\alpha \in \Phi$ and consider $\Psi := \mathbb{F}_p\alpha \cap \Phi$. Then $c_{\Psi}(\mathfrak{g}, \mathfrak{t}) = 1$, and the foregoing arguments in conjunction with (2.3) provide a non-zero p-ideal

$$\mathfrak{n} \subseteq \mathfrak{t} \oplus \bigoplus_{i=1}^{p-1} \mathfrak{g}_{i\alpha} =: \mathfrak{h}$$

of $\mathfrak{g}$. We shall show that the p-subalgebra $\mathfrak{h}$ is solvable. Note that $\ker \alpha$ is contained in the center $C(\mathfrak{h})$ of $\mathfrak{h}$. Direct computation shows that $\mathfrak{h}' := \mathfrak{h}/\ker \alpha$ is a restricted Lie algebra with a one-dimensional maximal torus and a one-dimensional nullcone. Hence we may assume that the torus $\mathfrak{t}$ of $\mathfrak{h}$ has dimension $\dim_k \mathfrak{t} = 1$, so that $\mathfrak{t} = kt_0$. Let $\beta \in \mathbb{F}_p\alpha$ be a root. Given $x \in \mathfrak{g}_{\beta} \setminus \{0\}$, there exists $\omega(x) \in k$ with $x^{[p]} = \omega(x)t_0$. Since $\beta(t_0) \neq 0$, the identities

$$0 = [x^{[p]}, x] = \omega(x)[t_0, x] = \omega(x)\beta(t_0)x$$

imply $\omega(x) = 0$, so that $\mathfrak{g}_{\beta} \subseteq \widehat{\mathcal{V}}_{\mathfrak{g}}$. As a result, $\dim_k \mathfrak{g}_{\beta} = 1$ and $\mathfrak{h}_{(\beta)} := \mathfrak{g}_{-\beta} \oplus \mathfrak{t} \oplus \mathfrak{g}_{\beta}$ is a p-subalgebra of $\mathfrak{h}$. The assumption $[\mathfrak{g}_{\beta}, \mathfrak{g}_{-\beta}] \neq (0)$ implies $\mathfrak{h}_{(\beta)} \cong \mathfrak{sl}(2)$, which contradicts $\dim \widehat{\mathcal{V}}_{\mathfrak{h}} = 1$.

As a result, the derived algebra $[\mathfrak{h}, \mathfrak{h}] = \bigoplus_{i=1}^{p-1} \mathfrak{g}_{i\alpha}$ is a p-subalgebra of $\mathfrak{h}$ that is generated by the nil Lie set $\bigcup_{i=1}^{p-1} \mathfrak{g}_{i\alpha}$. The Engel-Jacobson Theorem now implies that $[\mathfrak{h}, \mathfrak{h}]$ is nilpotent. Consequently, $\mathfrak{h}$ is solvable.

Hence $\mathfrak{n}$ is solvable and we may argue as before. □

Refinements of the above arguments yield stronger results. A non-zero Lie algebra $\mathfrak{g}$ which does not possess any non-zero abelian ideals is called *semi-simple*. Fur future reference, we record the following result:

THEOREM 2.6 ([**48**]). *Let* $(\mathfrak{g}, [p])$ *be a semi-simple restricted Lie algebra of characteristic* $p \geq 3$*. If* $\dim \widehat{\mathcal{V}}_{\mathfrak{g}} = 2$*, then* $\mathfrak{g} \cong \mathfrak{sl}(2)$*.* □

CHAPTER V

Quivers and Path Algebras

In this chapter we turn to the combinatorial aspects of the representation theory of group schemes. The methods originate in the abstract representation theory of algebras, where one is interested in understanding module categories up to Morita equivalence (or other equivalences, such as stable equivalence or derived equivalence). Over algebraically closed fields, this amounts to studying the so-called *basic algebras*, whose simple modules are one-dimensional. Such an algebra is given by a quiver and certain relations, giving rise to a presentation that encapsulates a lot of information concerning the structure of the module category. We shall only give a rather short introduction to this wide subject. A detailed account may be found in [**8, 4**].

A *quiver* $Q = (Q_0, Q_1, s, t)$ is an oriented graph on a set of vertices Q_0 and with a set Q_1 of arrows, whose starting points and terminal points are given by the functions $s, t : Q_1 \longrightarrow Q_0$. The *path algebra* $k[Q]$ of the quiver has an underlying vector space, whose basis is the set of all paths of Q. The product of two paths is given by concatenation if this is possible, and is defined to be zero otherwise. We compose arrows like maps. Thus, if p is path starting in $u \in Q_0$ and terminating $v \in Q_0$ and q is a path starting in v and terminating in w, then qp starts in u and ends in w. We shall denote by $k[Q]^+$ the two-sided ideal of $k[Q]$ that is generated by all arrows. Then $(k[Q]^+)^n$ is the ideal of $k[Q]$ consisting of all linear combinations of all paths of length $\geq n$. For each vertex $v \in Q_0$ there is a path e_v of length zero starting and ending in v. The corresponding element of $k[Q]$ is an idempotent with $e_v p = p$ ($qe_v = q$) for all paths p ending in v (for all paths q starting in v).

Let me illustrate the advantages of path algebras by considering a simple example.

Example. Let $\mathrm{char}(k) = p > 0$, and consider the two-dimensional, non-abelian restricted Lie algebra $\mathfrak{b} := kt \oplus kx$, whose multiplication and p-map are given by

$$[t, x] = x \quad ; \quad t^{[p]} = t \ , \ x^{[p]} = 0.$$

The corresponding presentation of its restricted enveloping algebra is

$$U_0(\mathfrak{b}) \cong k\langle t, x\rangle/(\{tx - xt - x, t^p - t, x^p\}).$$

The path algebra presentation, which yields an isomorphic algebra in this case, is

$$U_0(\mathfrak{b}) \cong k[\tilde{A}_{p-1}]/(k[\tilde{A}_{p-1}]^+)^p \quad ; \quad \tilde{A}_{p-1} \text{ clockwise oriented.}$$

Here $\tilde{A}_{p-1}$ is a circle with p vertices. This more complicated presentation immediately tells us that the algebra $U_0(\mathfrak{b})$ is a Nakayama algebra and hence has finite representation type (cf. (II.3.1)).

1. Gabriel's Theorem

Let k be an algebraically closed field. It turns out that every associative algebra is Morita equivalent to an algebra that possesses a presentation like the one given above. The first question is how to associate a quiver Q_Λ to an algebra Λ.

Definition. A k-algebra Λ is called *basic* if $\Lambda = \bigoplus_{i=1}^n P_i$ with the P_i being indecomposable and pairwise non-isomorphic.

Since k is algebraically closed, an algebra is basic if and only if all of its simple modules are one-dimensional.

Let us look again at the above example. The Hopf algebra $U_0(\mathfrak{b})$ is basic, and its simple modules are of the form k_λ, where the algebra homomorphism $\lambda : U_0(\mathfrak{b}) \longrightarrow k$ is uniquely determined by $\lambda(x) = 0$ and $\lambda(t) \in \mathbb{F}_p$. Hence there are exactly p simple $U_0(\mathfrak{b})$-modules, corresponding to the vertices of $\tilde{A}_{p-1}$. Note that the convolution product on the character group $X(U_0(\mathfrak{b}))$ corresponds to the addition of linear froms on $\mathfrak{b}$. We thus obtain

$$\mathrm{Ext}^1_{U_0(\mathfrak{g})}(k_\lambda, k_\gamma) \cong \mathrm{H}^1(U_0(\mathfrak{b}), k_{\gamma-\lambda}).$$

According to early results by Hochschild [**70**], the restricted cohomology groups on the right-hand side can be computed via Lie cocycles, whence

$$\dim_k \mathrm{Ext}^1_{U_0(\mathfrak{g})}(k_\lambda, k_\gamma) = \delta_{\gamma-\lambda,\alpha},$$

where $\alpha : U_0(\mathfrak{b}) \longrightarrow k$ is given by $\alpha(x) = 0$ and $\alpha(t) = 1$. By identifying the simple module k_λ with the value $\lambda(t) \in \mathbb{F}_p$ and taking $\dim_k \mathrm{Ext}_{U_0(\mathfrak{b})}(k_\lambda, k_\gamma)$ arrows from $\lambda(t)$ to $\gamma(t)$, we obtain the quiver $\tilde{A}_{p-1}$.

Definition. Let Λ be a finite-dimensional associative k-algebra, $\mathcal{S}$ a complete set of representatives of the simple Λ-modules. The quiver Q_Λ with set of vertices $\mathcal{S}$ and $\dim_k \mathrm{Ext}^1_\Lambda(S,T)$ arrows from S to T is called the *Ext-quiver* of Λ.

Suppose that $\Lambda = k[Q]/I$ is a finite-dimensional algebra such that $I \subseteq (k[Q]^+)^2$. Basic homological algebra then shows that $Q = Q_\Lambda$. Gabriel's Theorem asserts that any k-algebra is essentially of this type:

Theorem 1.1 ([**64**]). *Let Λ be a finite-dimensional k-algebra. Then there exist $n \geq 2$ and an ideal $(k[Q_\Lambda]^+)^n \subseteq I \subseteq (k[Q_\Lambda]^+)^2$ such that Λ is Morita equivalent to $k[Q_\Lambda]/I$.*

Proof. To ease notation, we put $k[Q_\Lambda]_{\geq n} := (k[Q_\Lambda]^+)^n$. By general theory, the algebra Λ is Morita equivalent to a basic algebra, so that we may assume that all simple Λ-modules are one-dimensional.

Let $\{\varepsilon_1, \ldots, \varepsilon_r\} \subseteq k[Q_\Lambda]$ be the set of paths of length zero. By assumption, there exist orthogonal primitive idempotents $e_1, \ldots, e_r \in \Lambda$ with $1 = \sum_{i=1}^r e_i$. Let J be the Jacobson radical of Λ. Given $i, j \in \{1, \ldots, r\}$, we pick elements $a_{ij\ell} \in e_i J e_j$ such that the residue classes $\bar{a}_{ij\ell}$ form a basis of $e_i J e_j / e_i J^2 e_j$.

Proceeding in several steps, we define a k-linear map $f : k[Q_\Lambda] \longrightarrow \Lambda$. By general theory, we have

$$\mathrm{Ext}^1_\Lambda(S_i, S_j) \cong e_j J e_i / e_j J^2 e_i.$$

Letting $n_{ij} := \dim_k \mathrm{Ext}^1_\Lambda(S_i, S_j)$, we put

- $f(\varepsilon_i) := e_i$ for $1 \leq i \leq r$,
- $f(\alpha_{ij\ell}) := a_{ji\ell}$ for every arrow $\alpha_{ij\ell} : i \to j$ with $1 \leq \ell \leq n_{ij}$.
- $f(p) := \prod_{j=1}^m f(\alpha_j)$ for every path $p = \alpha_1 \cdots \alpha_m$.

In this fashion, we obtain a linear map $f : k[Q_\Lambda] \longrightarrow \Lambda$.

Let $\alpha : i \to j$ and $\beta : \ell \longrightarrow m$ be arrows such that $\ell \neq j$. Then $\beta\alpha = 0$ and $f(\beta)f(\alpha) \in e_m J e_\ell e_j J e_j = (0)$. This shows that f is a homomorphism of k-algebras.

Since every simple Λ-module is one-dimensional, Wedderburns Theorem provides an isomorphism $\Lambda/J \cong \bigoplus_{i=1}^r ke_i$. Moreover, $J/J^2 \cong \bigoplus_{i,j=1}^r e_iJe_j/e_iJ^2e_j$. As a result, the map $\bar{f} : k[Q_\Lambda]/k[Q_\Lambda]_{\geq 2} \longrightarrow \Lambda/J^2$, induced by f, is an isomorphism of k-vector spaces. Consequently, $f(k[Q_\Lambda]) + J^2 = \Lambda$, so that (I.4.5) ensures the surjectivity of f. By the same token, the ideal $I := \ker f$ is contained in $k[Q_\Lambda]_{\geq 2}$. By definition of f, we have $f(k[Q_\Lambda]_{\geq m}) \subseteq J^m$ for all $m \geq 1$. Thus, if $J^n = (0)$, then $k[Q_\Lambda]_{\geq n} \subseteq I$. □

Remarks. (1) As every simple module of the *bound quiver algebra* $k[Q_\Lambda]/I$ is one-dimensional, we cannot expect to obtain isomorphisms $k\mathcal{G} \cong k[Q_\Lambda]/I$. In fact, this is only the case if the group scheme $\mathcal{G}$ is *trigonalizable*, that is, if $\mathcal{G}$ can be embedded into a group of upper triangular matrices. Such groups are necessarily solvable, and the failure of Lie's Theorem in positive characteristic means that there also are solvable groups that are not trigonalizable.

(2) Passage to Morita equivalent algebras not only trivializes the structure of the simple modules, but it also does not interact well with the comultiplication. Thus, we usually do not have a tensor product of modules available when replacing $k\mathcal{G}$ by its bound quiver algebra.

2. Hereditary Algebras

In this section we briefly review basic results concerning representations of hereditary algebras. Aside from group algebras of finite groups, the study of these algebras has been the other major focal point of much of the early work in representation theory. *Throughout, k is assumed to be an algebraically closed field.*

Definition. A finite-dimensional k-algebra Λ is *hereditary* if submodules of projective Λ-modules are projective.

The following result partly explains why hereditary algebras are of interest. In terms of their presentation by quivers and relations, they are the most tractable algebras.

Theorem 2.1. *Let Λ be a finite-dimensional k-algebra. Then the following statements hold:*
(1) *If Λ is hereditary and $I \subseteq \mathrm{Rad}(\Lambda)^2$ is an ideal such that Λ/I is hereditary, then $I = (0)$.*
(2) *Λ is hereditary if and only if Λ is Morita equivalent to $k[Q_\Lambda]$.*

Proof. (1) Let $I \trianglelefteq \Lambda$ be as stated in (1) and put $J := \mathrm{Rad}(\Lambda)$. Using $I \subseteq J$, we obtain an exact sequence
$$(0) \longrightarrow I/IJ \longrightarrow J/IJ \longrightarrow J/I \longrightarrow (0)$$
of (Λ/I)-modules. As $J/I \subseteq \Lambda/I$ is a projective (Λ/I)-module, the sequence splits. Our condition $I \subseteq J^2$ implies that the submodule $I/IJ \subseteq \mathrm{Rad}_{\Lambda/I}(J/IJ)$ is superfluous, so that $I/IJ = (0)$. Consequently, $I = IJ$, whence $I = (0)$.

(2) Direct computation shows that $k[Q]$ is hereditary for any quiver Q without oriented cycles (that is, for a quiver giving rise to a finite-dimensional path algebra).

Suppose that Λ is hereditary with simple modules $S_1, \ldots, S_n$ and corresponding projective covers $P_1, \ldots, P_n$. In view of
$$\mathrm{Ext}^1_\Lambda(S_i, S_j) \cong \mathrm{Hom}_\Lambda(\mathrm{Rad}(P_i), S_j)$$

the condition $\mathrm{Ext}^1_\Lambda(S_i, S_j) \neq (0)$ implies the existence of a surjective homomorphism $\mathrm{Rad}(P_i) \longrightarrow S_j$. As $\mathrm{Rad}(P_i)$ is projective, this map lifts to a surjection $\mathrm{Rad}(P_i) \longrightarrow P_j$ onto the projective cover of S_j. Consequently, $\ell(P_j) < \ell(P_i)$, so that the quiver Q_Λ affords no oriented cycles. We conclude that $k[Q_\Lambda]$ is finite-dimensional with $k[Q_\Lambda]^+ = \mathrm{Rad}(k[Q_\Lambda])$. Our assertion now follows from (1) and (1.1). □

The representation theory of hereditary algebras is very well understood and turns out to be closely related to Lie theory. Given a quiver Q, we denote by $\bar{Q}$ the *underlying graph*, in which every arrow of Q is replaced by a bond.

A finite-dimensional k-algebra Λ is *connected* provided Λ affords only one block. This condition is equivalent to Q_Λ being connected.

Theorem 2.2 ([**64**]). *A finite-dimensional connected path algebra $k[Q]$ is of finite representation type if and only if the underlying graph $\bar{Q}$ of Q is isomorphic to a Dynkin diagram A_n, D_n, or $E_{6,7,8}$. In that case, the dimension vectors of the indecomposable $k[Q]$-modules correspond to the positive roots of the root system of $\bar{Q}$.* □

Remarks. (1) Recall that the *dimension vector* of a Λ-module M is the coordinate vector of its class $[M]$ in the Grothendieck group $K_0(\Lambda)$ relative to the standard basis given by the simple Λ-modules. Thus, if Λ is hereditary of finite representation type, then the lengths of the indecomposable Λ-modules are known. For instance, if Q_Λ is of type A_n, then there exist $n-j+1$ indecomposable Λ-modules of length j.

(2) Owing to (2.2), the representation type of $k[Q]$ does not depend on the orientation of the quiver Q, even though the algebras involved may be rather different. For instance, the module categories of the algebras

$$\Lambda_1 := k[1 \to 2 \to 3] \quad \text{and} \quad \Lambda_2 := k[1 \leftarrow 2 \to 3]$$

have the same distribution of indecomposable modules, yet they are not Morita equivalent: Λ_1 is a Nakayama algebra, while Λ_2 is not. The relationship of these algebras is investigated in tilting theory, whose starting point is the seminal article [**14**] (see also [**4**, (VII.5)]).

Shortly after Gabriel's determination of the representation-finite hereditary algebras, Donovan-Freislich and Nazarova independently classified the tame hereditary algebras:

Theorem 2.3 ([**32**], [**90**]). *A finite-dimensional connected path algebra $k[Q]$ is tame if and only if the underlying graph $\bar{Q}$ of Q is isomorphic to a Euclidean diagram $\tilde{A}_n$, $\tilde{D}_n$, or $\tilde{E}_{6,7,8}$.* □

The foreging result can be used to obtain information on the Ext-quivers of arbitrary algebras of finite or tame representation type.

Recall that the *separated quiver* Q_s of a quiver Q with vertex set $\{1, \dots, n\}$ has $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ and an arrow $\ell \to m'$ for every arrow $\ell \to m$ of Q.

Corollary 2.4. *Let Λ be a finite-dimensional k-algebra. If Λ is tame, then each connected component of the separated quiver $(Q_\Lambda)_s$ of the Ext-quiver Q_Λ is a Dynkin diagram of type A, D, E or a Euclidean diagram of type $\tilde{A}$, $\tilde{D}$, $\tilde{E}$.*

Proof. Let J be the Jacobson radical of Λ. Then the algebra $\Lambda' := \Lambda/J^2$ is representation-finite or tame, has Jacobson radical $J' = J/J^2$ and Ext-quiver $Q_{\Lambda'} = Q_{\Lambda}$. The triangular matrix algebra

$$\Sigma := \begin{pmatrix} \Lambda'/J' & 0 \\ J' & \Lambda'/J' \end{pmatrix}$$

is hereditary with Ext-quiver $Q_{\Sigma} \cong (Q_{\Lambda'})_s$, and the functor

$$F : \text{mod } \Lambda' \longrightarrow \text{mod } \Sigma \ \ ; \ \ M \mapsto \begin{pmatrix} M/J'M \\ J'M \end{pmatrix}$$

reflects isomorphisms and indecomposability. Moreover, F reaches all but finitely many indecomposable Σ-modules, so that Σ is tame or representation-finite. Our assertion now follows from (2.2) and (2.3). □

3. A Criterion for Wildness

As before, we assume k to be algebraically closed. The results of Section 2 yield simple criteria implying the wildness of an algebra Λ. By way of illustration, we consider the following:

Lemma 3.1. *Let Λ be a k-algebra such that $\dim_k \text{Ext}^1_{\Lambda}(S,T) \geq 3$ for two simple Λ-modules S and T. Then Λ is wild.*

Proof. By assumption, the graph $(\bar{Q}_{\Lambda})_s$ underlying the separated quiver has two vertices that are joined by at least 3 bonds. Thus, the connected component of $(\bar{Q}_{\Lambda})_s$ containing these vertices is neither a Dynkin diagram nor a Euclidean diagram of the types given in (2.4). Consequently, Λ is neither representation-finite nor tame, and (II.3.4) yields the result. □

Unfortunately, there are many algebras, whose wildness cannot be recognized via the foregoing criterion. More sophisticated results can be established by studying Galois coverings of quivers, a subject that lies beyond the scope of these notes. We shall be content with one basic fact, involving the quiver and the relations, which works in situations of interest where (3.1) does not apply. The interested reader may consult [**36**, Chap. I] for more details.

Lemma 3.2. *Let Λ be a finite-dimensional k-algebra such that each vertex of Q_{Λ} is the starting point and end point of at least two arrows. If there exist three arrows α, β, γ such that neither $\beta\alpha$ nor $\gamma\alpha$ are summands of a relation of Λ, then the algebra Λ is wild.* □

4. Trivial Extensions

From a homological perspective, hereditary algebras and algebras of measures are far apart. While the former have global dimension ≤ 1, the global dimension of self-injective algebras, such as $k\mathcal{G}$, is finite if and only if $k\mathcal{G}$ is semi-simple. The concept of a trivial extension allows us to pass from any algebra to a symmetric algebra. Moreover, there is a close connection between the representation theory of an algebra Λ and that of its associated trivial extension.

Definition. Let Λ be a finite-dimensional k-algebra. The algebra $T(\Lambda) := \Lambda \ltimes \Lambda^*$, whose multiplication is given by

$$(a, \varphi).(b, \psi) := (ab, a.\psi + \varphi.b) \quad \forall\ a, b \in \Lambda,\ \varphi,\ \psi \in \Lambda^*$$

is called the *trivial extension* of Λ.

In the above definition, we have endowed the dual space Λ^* with the usual structure of a (Λ, Λ)-bimodule by setting

$$(a.\varphi)(x) = \varphi(xa) \text{ and } (\varphi.a)(x) = \varphi(ax) \qquad \forall\, a, x \in \Lambda,\ \varphi \in \Lambda^*.$$

We record the symmetry of $T(\Lambda)$:

LEMMA 4.1. *The linear map*

$$\pi : T(\Lambda) \longrightarrow k \quad ; \quad (a, \varphi) \mapsto \varphi(1)$$

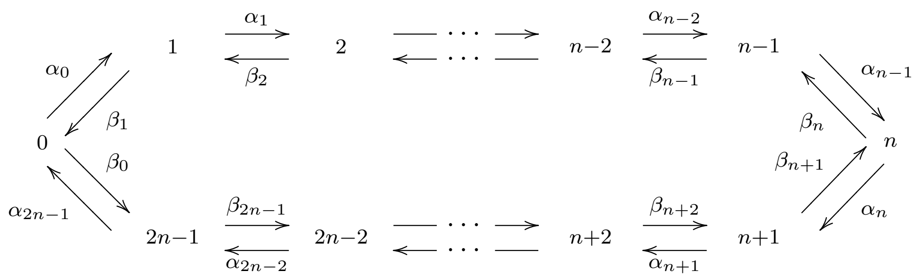
endows $T(\Lambda)$ with the structure of a symmetric algebra. □

Under certain conditions, the quiver and the relations describing $T(\Lambda)$ can be deduced from the corresponding data for Λ. This applies in particular for the so-called *radical square zero hereditary algebras.* By definition, the Ext-quiver of such an algebra is oriented in such a way that there are no paths of length 2. Such an orientation always exists if Q_Λ is a tree, but for a quiver of type $\tilde{A}_n$ this is only possible if n is odd (that is, when the number $n+1$ of vertices is even).

Given a quiver Q, we let $\widehat{Q}$ be the *double of* Q. The quiver $\widehat{Q}$ has the same set of vertices as Q, and for every arrow $\alpha : i \to j$, we add an arrow $\alpha^* : j \to i$. Here is the relevant basic result:

PROPOSITION 4.2. *Let $Q = (Q_0, Q_1, s, t)$ be a connected quiver with $|Q_0| + |Q_1| \geq 4$ and without paths of length ≥ 2. Then the algebra $T(Q) := T(k[Q])$ is isomorphic to $k[\widehat{Q}]/I$, where I is generated by the set $\{\alpha.\alpha^* - \beta.\beta^*\ , (t(\alpha) = t(\beta))\ ;\ \alpha^*.\alpha - \beta^*.\beta\ , (s(\alpha) = s(\beta))\ ;\ \alpha^*.\beta,\ \alpha.\beta^* (\alpha \neq \beta)\}$.* □

EXAMPLE. Let $Q = \tilde{A}_{2n-1}$ be the circle with $2n$ vertices and such that there are no paths of length 2. Then $T(\tilde{A}_{2n-1}) \cong k[\widehat{Q}]/I$, where $\widehat{Q}$ is the double



and $I \subseteq k[\widehat{Q}]$ is the ideal generated by

$$\{\beta_{i+1}\alpha_i - \alpha_{i-1}\beta_i,\ \ \alpha_{i+1}\alpha_i,\ \ \beta_i\beta_{i+1}\ \ ;\ \ i \in \mathbb{Z}/(2n)\}.$$

The algebra $T(\tilde{A}_{2n-1})$ belongs to a particularly tractable class of tame algebras, the so-called *special biserial* algebras that were introduced by Skowroński and Waschbüsch in [**111**]. They are bound quiver algebras $k[Q]/I$, whose presentation satisfies the following conditions:

- Each vertex of Q is the starting point and end point of at most two arrows.
- For any arrow $\alpha \in Q_1$, there exist at most one arrow β and one arrow γ with $\alpha\beta,\ \gamma\alpha \notin I$.

Special biserial algebras are biserial; they appear in the classification of blocks of group algebras with cyclic or dihedral defect groups (cf. [**36, 105**]) as well as in connection with the Gel'fand-Ponomarev classification of the singular Harish-Chandra modules over the Lorentz group [**65**].

The representation theory of special biserial algebras is completely understood. In view of work by Tachikawa [**119**] and Ringel [**107**] (cf. also [**66**]), the algebra $T(Q)$ affords "twice as many" indecomposable modules as $k[Q]$, and thus has the same representation type.

5. McKay Quivers

Up to now, the presence of tensor products in the module category of a finite group scheme has only entered via the definition of support varieties. We shall see in this section how knowledge of tensor products of simple modules helps us understand the block structure of a given Hopf algebra. Our approach is based on McKay's seminal papers [**87, 88**], in which he systematically studied tensor products within the context of complex representations of finite groups. Subsequently, McKay quivers appeared in other areas of representation theory, see for instance [**5, 6**].

Throughout this section, k is an arbitrary field. Given a finite-dimensional Hopf algebra H, we let $K_0(H)$ denote the *Grothendieck group* of $\operatorname{mod} H$. By definition, the group $K_0(H)$ is generated by the set of isomorphism classes of finite-dimensional H-modules with relations $[Y] - [X] - [Z]$, corresponding to short exact sequences $(0) \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow (0)$ of H-modules. Let $\{S_1, \dots, S_n\}$ be a complete set of representatives for the isomorphism classes of the simple H-modules. Recall that

(a) $K_0(H)$ is a free abelian group with basis $\{[S_1], \dots, [S_n]\}$, and

(b) every exact functor $F : \operatorname{mod} H \longrightarrow \operatorname{mod} H$ determines an endomorphism F of $K_0(H)$.

If V is a finite-dimensional H-module, such that

$$[V] = \sum_{j=1}^{n} m_j [S_j],$$

then the coefficient m_j is the Jordan-Hölder multiplicity $[V : S_j]$.

In view of (b), the tensor product endows $K_0(H)$ with the structure of a ring. Given a finite-dimensional H-module V, we consider the integral $(n \times n)$-matrix $(m_{ij})_{1 \le i,j \le n}$ representing the endomorphism $V \otimes_k -$ of $K_0(H)$ relative to the above basis:

$$[V \otimes_k S_j] = \sum_{i=1}^{n} m_{ij} [S_i].$$

We are interested in the quiver associated to this matrix:

Definition. The quiver $\Theta_V(H)$ with set of vertices $\{1, \dots, n\}$ and m_{ij} arrows $i \to j$ is called the *McKay quiver of H relative to V*.

Remark. Our definition differs from McKay's original one in that he considered the opposite quiver. However, the above orientation can also be found in the literature, see [**5, 6, 67**].

Examples. (1) Suppose that every simple H-module is one-dimensional and let $G := X(H)$ be the character group of H. Then the Grothendieck ring $K_0(H)$ is the integral group ring $\mathbb{Z}G$. In that case, the McKay quiver of H relative to the simple H-module k_λ, defined by $\lambda \in G$, is just a union of $\frac{\operatorname{ord}(G)}{\operatorname{ord}(\lambda)}$ oriented cycles $\tilde{A}_{\operatorname{ord}(\lambda)-1}$.

(2) Assuming $p \geq 3$, we consider the restricted enveloping algebra $H = U_0(\mathfrak{sl}(2))$ as well as the standard $\mathfrak{sl}(2)$-module $V = L(1)$. By the modular Clebsch-Gordan formula [**11**], we have

$$[L(1) \otimes_k L(0)] = [L(1)] \ \ ; \ \ [L(1) \otimes_k L(j)] = [L(j-1)] + [L(j+1)] \qquad 1 \leq j \leq p-2$$

as well as

$$[L(1) \otimes_k L(p-1)] = [P(p-2)] = 2[L(p-2)] + 2[L(0)].$$

In particular, the McKay quiver $\Theta_V(U_0(\mathfrak{sl}(2)))$ is connected. The following results show that this is not an accident. We begin with the generalization of Burnside's classical result [**19**] (see also [**16, 114, 104, 94**]):

THEOREM 5.1 (Burnside's Theorem for Hopf algebras). *Let V be a finite-dimensional module for a finite-dimensional Hopf algebra H. If the annihilator $\mathrm{ann}_H(V)$ of V in H does not contain any non-zero Hopf ideals, then every simple H-module S is a submodule of a tensor power $V^{\otimes m}$ for some $m \geq 1$.*

PROOF. Setting $M := \bigoplus_{m\geq 1} V^{\otimes m}$, we obtain

$$M \otimes_k M \cong \bigoplus_{m,n\geq 1} V^{\otimes m} \otimes_k V^{\otimes n} \cong \bigoplus_{m\geq 2} (m-1)V^{\otimes m},$$

so that the inclusion

$$\mathrm{ann}_H(M) = \bigcap_{m\geq 1} \mathrm{ann}_H(V^{\otimes m}) \subseteq \mathrm{ann}_H(M \otimes_k M)$$

holds. Consequently, $\Delta(\mathrm{ann}_H(M)) \subseteq \mathrm{ann}_H(M) \otimes_k H + H \otimes_k \mathrm{ann}_H(M)$. Since H is finite-dimensional, (I.1.2) implies that $\mathrm{ann}_H(M) \subseteq \mathrm{ann}_H(V)$ is a Hopf ideal. In view of our current assumption, the H-module M is faithful.

Let $I \subseteq H$ be a minimal left ideal. Then $I.M \neq (0)$, and there exist $m \in \mathbb{N}$ and $v \in V^{\otimes m}$ such that $I.v \neq (0)$. Since I is a simple H-module, the map $x \mapsto x.v$ thus defines an embedding $I \hookrightarrow V^{\otimes m}$.

As a result, every simple module belonging to $\mathrm{Soc}(H)$ is a submodule of a suitable tensor power of V. By the Theorem of Larson-Sweedler (I.1.3), the algebra H is self-injective. Consequently, every simple H-module occurs in $\mathrm{Soc}(H)$, so that our assertion follows. □

Given any quiver Q and vertices $i, j \in Q_0$, we let $Q(i, j; m)$ be the set of paths of length m starting at the vertex i and terminating at the vertex j.

COROLLARY 5.2. *Let H be a finite-dimensional Hopf algebra, V be a finite-dimensional H-module.*

(1) *We have $[V^{\otimes m} \otimes_k S_j : S_i] = |\Theta_V(H)(i, j; m)|$.*

(2) *If $\mathrm{ann}_H(V)$ contains no non-zero Hopf ideals, then $\Theta_V(H)$ is connected.*

PROOF. (1) Using induction on m, we assume that $m \geq 2$. Note that

$$(*) \qquad \Theta_V(H)(i, j; m) \cong \bigsqcup_{\ell=1}^{n} \Theta_V(H)(\ell, j; m-1) \times \Theta_V(H)(i, \ell; 1).$$

The inductive hypothesis provides a decomposition

$$V^{\otimes(m-1)} \otimes_k S_j \cong \bigoplus_{t=1}^{n} b_t S_t,$$

where $b_t = |\Theta_V(H)(t,j;m-1)|$. Consequently,

$$V^{\otimes m} \otimes_k S_j \cong \bigoplus_{t=1}^{n} b_t(V \otimes_k S_t) \cong \bigoplus_{r=1}^{n}(\sum_{t=1}^{n} m_{rt}b_t)S_r,$$

and $(*)$ implies

$$[V^{\otimes m} \otimes_k S_j : S_i] = \sum_{t=1}^{n} m_{it}b_t = |\Theta_V(H)(i,j;m)|,$$

as desired.

(2) Let $S_1 = k$ be the trivial H-module. Then we have $V^{\otimes m} \otimes_k S_1 \cong V^{\otimes m} \ \ \forall\ m \geq 1$. Given a vertex $i \in \{1, \ldots, n\}$, Burnside's Theorem provides $m \in \mathbb{N}$ with

$$0 \neq [V^{\otimes m} : S_i] = [V^{\otimes m} \otimes_k S_1 : S_i] = |\Theta_V(H)(i,1;m)|.$$

Hence there is a path from i to 1. □

Suppose that $\mathrm{char}(k) \geq 3$. By general theory, the Hopf ideals of $U_0(\mathfrak{sl}(2))$ correspond to the p-ideals of the restricted Lie algebra $\mathfrak{sl}(2)$. As $\mathfrak{sl}(2)$ is simple, every $L(i) \neq L(0)$ satisfies the hypothesis of Corollary 5.2(2). The example of a p-group shows that the converse of (5.2(2)) does not hold in general.

Definition. Let $C = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ be an integral $(n \times n)$-matrix. We say that C is a *generalized Cartan matrix* if

(1) $c_{ii} \leq 2$ for $1 \leq i \leq n$, and
(2) $c_{ij} \leq 0$ for $1 \leq i \neq j \leq n$, and
(3) $c_{ij} = 0$ if and only if $c_{ji} = 0$.

A function $d : \{1, \ldots, n\} \longrightarrow \mathbb{N}\ ;\ i \mapsto d_i$ is an *additive function for C* if

$$\sum_{i=1}^{n} d_i c_{ij} = 0 \quad ; \quad 1 \leq j \leq n.$$

To each generalized Cartan matrix $C \in \mathrm{Mat}_n(\mathbb{Z})$ we associate a valued graph Γ_C, whose vertices are the elements $\{1, \ldots, n\}$. There is a valued edge

$$i \xrightarrow{(|c_{ij}|,|c_{ji}|)} j$$

between i and j whenever $c_{ij} \neq 0$ and $i \neq j$. In addition, there are $2 - c_{ii}$ loops at the vertex i. By property (3), the graph of a generalized Cartan matrix is well-defined.

In the following, we let $I_n \in \mathrm{Mat}_n(\mathbb{Z})$ be the identity matrix.

Lemma 5.3. *Let k be algebraically closed. Suppose that H is a semi-simple, cocommutative Hopf algebra with simple modules $S_1, \ldots, S_n$. If V is a two-dimensional self-dual H-module, and (m_{ij}) is the matrix representing $V \otimes_k -$, then $C := 2I_n - (m_{ij})$ is a symmetric generalized Cartan matrix, and $i \mapsto \dim_k S_i$ defines an additive function for C.*

Proof. The defining conditions (1) and (2) follow immediately. Since H is semi-simple, we have

$$V \otimes_k S_j \cong \bigoplus_{i=1}^{n} m_{ij} S_i,$$

so that commutativity of the tensor product, adjointness, self-duality and Schur's Lemma give

$$m_{ij} = \dim_k \mathrm{Hom}_H(V \otimes_k S_j, S_i) = \dim_k \mathrm{Hom}_H(S_j, V^* \otimes_k S_i) = \dim_k \mathrm{Hom}_H(S_j, V \otimes_k S_i) = m_{ji}.$$

Accordingly, C is symmetric and (3) holds. Since $\dim_k V = 2$, the map $d : \{1, \dots, n\} \longrightarrow \mathbb{N}$; $i \mapsto \dim_k S_i$ is an additive function for C. □

If $H = U_0(\mathfrak{sl}(2))$ and $V = L(1)$, then $C := 2I_p - (m_{ij})$ is not a generalized Cartan matrix, as $c_{p-1,0} = 0$ and $c_{0,p-1} = -2$.

The connected generalized Cartan matrices affording an additive function were classified by Happel, Preiser, and Ringel (see [**67**]). We will only need the following special case of their result:

Theorem 5.4. *Let C be a connected, symmetric generalized Cartan matrix having an additive function. Then the following statements hold:*

(1) *The graph Γ_C belongs to the following list: $\tilde{A}_n$ $(n \geq 1)$, $\tilde{D}_n$ $(n \geq 4)$, $\tilde{E}_{6,7,8}$, $\tilde{L}_n$, $\widetilde{DL}_n$ $(n \geq 2)$.*

(2) *There exists an additive function d_C of C such that any additive function for C is an integral multiple of d_C.* □

Example. If $C = \tilde{A}_n$, then $d_C(i) = 1$ for $1 \leq i \leq n+1$.

CHAPTER VI

Representation-Finite and Tame Group Schemes

This chapter employs the techniques discussed so far to obtain information on those group schemes $\mathcal{G}$, whose Hopf algebras $k\mathcal{G}$ are of finite or tame representation type. Let me begin by recalling what is known for group algebras of finite groups. In view of Maschke's Theorem [**86**], group algebras are semi-simple in case $\mathrm{char}(k) = 0$. We thus assume $\mathrm{char}(k) = p > 0$. To a block $\mathcal{B} \subseteq kG$ one associates a p-group $D_{\mathcal{B}} \subseteq G$ to $\mathcal{B}$, the so-called *defect group*. It turns out that the representation type of $\mathcal{B}$ is reflected by structural properties of $D_{\mathcal{B}}$:

- The block $\mathcal{B}$ is simple if and only if $D_{\mathcal{B}} = \{1\}$ is trivial. This is the generalization of Maschke's classical result.
- Group algebras of finite representation type were first investigated by Higman [**69**], who showed that kG is representation-finite if and only if the Sylow-p-subgroups of G are cyclic. More generally, the block $\mathcal{B} \subseteq kG$ is representation-finite if and only if its defect group $D_{\mathcal{B}}$ is cyclic.
- Thanks to work by Bondarenko and Drozd [**15**], a block $\mathcal{B} \subseteq kG$ is tame if and only if $p = 2$, and if $D_{\mathcal{B}}$ is dihedral, semidihedral, or generalized quaternion.

Defect theory and other special features of group algebras also yield the following facts that proved to be of importance:

- If $H \subseteq G$ is a subgroup and kG is representation-finite (tame), then kH is representation-finite (tame or representation-finite).
- If the principal block $\mathcal{B}_0(G)$ is representation-finite (tame), then any other block of kG is representation-finite (tame or representation-finite).

The latter fact is related to the eponymous property of $\mathcal{B}_0(G)$: The defect group of $\mathcal{B}_0(G)$ is a Sylow-p-subgroup of G, and thus the largest among all defect groups. Consequently, $\mathcal{B}_0(G)$ is commonly thought of as the most complicated block of kG.

Let H be a Hopf algebra. Recall from (I.1.5) that the tensor product $P \otimes_k M$ of a projective H-module P with any H-module M is again projective. Thus, if the principal block $\mathcal{B}_0(H)$ is simple, then k is a projective H-module and hence $M \cong k \otimes_k M$ is projective for every $M \in \mathrm{mod}\, H$. Consequently, H is semi-simple, so that the representation theory of $\mathcal{B}_0(H)$ governs that of H. We shall see later, however, that the paradigm of $\mathcal{B}_0(H)$ being the most complicated block fails for infinitesimal group schemes of tame representation type.

1. Nagata's Theorem

Let H be a finite-dimensional cocommutative Hopf algebra. We know from (I.3.3) that H is semi-simple whenever $\mathrm{char}(k) = 0$. For fields of positive characteristic, Nagata's Theorem determines those group schemes $\mathcal{G}$, whose algebras of measures are semi-simple. Such group schemes are also referred to as *linearly reductive*.

Throughout this section, k denotes an algebraically closed field of characteristic $p > 0$.

DEFINITION. A finite group scheme $\mathcal{G}$ is called *diagonalizable* if its coordinate ring $k[\mathcal{G}]$ is the group algebra of its character group $X(\mathcal{G})$.

If $\mathcal{G}$ is diagonalizable, then we have $\dim_k k\mathcal{G} = \operatorname{ord}(X(\mathcal{G}))$. On the other hand, $\bigoplus_{\lambda\in X(\mathcal{G})} k_\lambda$ is a submodule of $k\mathcal{G}/\operatorname{Rad}(k\mathcal{G})$, so that $\operatorname{Rad}(k\mathcal{G}) = (0)$ and $k\mathcal{G}$ is semi-simple. The following result, which was first established by Hochschild [**71**] for enveloping algebras of restricted Lie algebras, determines all linearly reductive infinitesimal group schemes.

PROPOSITION 1.1. *Let $\mathcal{G}$ be an infinitesimal group. If $k\mathcal{G}$ is semi-simple, then $\mathcal{G}$ is diagonalizable.*

PROOF. We proceed by induction on the order of $\mathcal{G}$, noting that $\mathfrak{g} := \operatorname{Lie}(\mathcal{G}) \neq (0)$ whenever $\mathcal{G} \neq e_k$. By assumption, the trivial $\mathcal{G}$-module k is projective. Since $k\mathcal{G}$ is a free left $k\mathcal{G}_1$-module (cf. (I.1.6)), the trivial $\mathcal{G}_1$-module $k|_{\mathcal{G}_1}$ is also projective, so that $k\mathcal{G}_1$ is semi-simple. According to (I.4.6), this implies the semi-simplicity of the restricted enveloping algebra $U_0(\mathfrak{g})$.

Let $x \in \mathfrak{g}$ with $x^{[p]} = 0$. Owing to (I.4.4) we have

$$U_0(kx) \cong \begin{cases} k[X]/(X^p) & \text{for } x \neq 0 \\ k & \text{otherwise.} \end{cases}$$

By the above arguments, k is a projective $U_0(kx)$-module, so that $x = 0$.

Given an arbitrary element $x \in \mathfrak{g}$, we consider the abelian p-subalgebra $\mathfrak{h} := \sum_{i\geq 0} kx^{[p]^i}$. Thus, the p-map $[p]$ is semilinear on $\mathfrak{h}$, and the above shows that its restriction $[p]|_{\mathfrak{h}}$ is injective and hence bijective. Consequently, there exist $\alpha_1, \dots, \alpha_n \in k$ with $x = \sum_{i=1}^n \alpha_i x^{[p]^i}$. As a result, the left multiplication

$$\operatorname{ad} x : \mathfrak{g} \longrightarrow \mathfrak{g} \ ; \ y \mapsto [x, y]$$

satisfies the polynomial $\sum_{i=1}^n \alpha_i X^{p^i} - X$ and is therefore diagonalizable. Let $\alpha \in k$ be an eigenvalue for $\operatorname{ad} x$. Then there exists $y \in \mathfrak{g}\setminus\{0\}$ such that $[x, y] = \alpha y$. Accordingly, the subspace $V := kx + ky$ is invariant under $\operatorname{ad} y$, and $(\operatorname{ad} y)|_V$ is nilpotent and diagonalizable. Hence $(\operatorname{ad} y)|_V = 0$, so that $\alpha = 0$. Consequently, the Lie algebra $\mathfrak{g}$ is abelian, and, being a commutative semi-simple algebra, $U_0(\mathfrak{g}) \cong k^{\dim_k U_0(\mathfrak{g})}$ is a product of fields. In particular, $U_0(\mathfrak{g})^* \cong kX(U_0(\mathfrak{g}))$ is the group algebra of the character group of $U_0(\mathfrak{g})$, so that $\mathcal{G}_1$ is diagonalizable.

Suppose that $\mathcal{G} \neq \mathcal{G}_1$. By inductive hypothesis, the group $\mathcal{G}_1$ and its factor group $\mathcal{G}/\mathcal{G}_1$ are diagonalizable. By rigidity of tori (cf. [**123**, (7.7)]), the algebra $k\mathcal{G}_1$ is contained in the center of $k\mathcal{G}$.

Let S be a simple $k\mathcal{G}$-module. According to Schur's Lemma, the central subalgebra $k\mathcal{G}_1$ of $k\mathcal{G}$ operates on S via a character. Consequently, $k\mathcal{G}_1$ acts trivially on $\operatorname{End}_k(S)$, and the latter space has the structure of a $\mathcal{G}/\mathcal{G}_1$-module. Since $\mathcal{G}/\mathcal{G}_1$ is linearly reductive, $\operatorname{End}_k(S)$ is a semi-simple $k\mathcal{G}$-module, and there results a decomposition

$$\operatorname{End}_k(S) = \bigoplus_{\lambda\in C} \operatorname{End}_k(S)_\lambda,$$

where $C \subseteq X(\mathcal{G})$, and $\operatorname{End}_k(S)_\lambda = \{\varphi \in \operatorname{End}_k(S) \ ; \ h.\varphi = \lambda(h)\varphi \ \ \forall\, h \in k\mathcal{G}\}$. One readily verifies that $\varphi \circ \psi \in \operatorname{End}_k(S)_{\lambda*\gamma}$ for $\varphi \in \operatorname{End}_k(S)_\lambda$ and $\psi \in \operatorname{End}_k(S)_\gamma$.

Let φ be an element of $\operatorname{End}_k(S)_\lambda$. Since

$$\varphi(hs) = \sum_{(h)} h_{(1)}(\eta(h_{(2)}).\varphi)(s) = (\operatorname{id}_{k\mathcal{G}} * (\lambda \circ \eta))(h)\varphi(s)$$

for $h \in k\mathcal{G}$ and $s \in S$, we see that $\ker \varphi$ is a $\mathcal{G}$-submodule of S. Accordingly, every non-zero element of $\operatorname{End}_k(S)_\lambda$ is invertible.

Direct computation shows that $\mathrm{tr}(h.\varphi) = \varepsilon(h)\,\mathrm{tr}(\varphi)$ for $\varphi \in \mathrm{End}_k(S)$ and $h \in k\mathcal{G}$. This implies that $\mathrm{tr}(\mathrm{End}_k(S)_\lambda) = (0)$ whenever $\lambda \neq \varepsilon$.

Let $\varphi \in \mathrm{End}_k(S)_\lambda \setminus \{0\}$ for some $\lambda \neq \varepsilon$. Owing to (I.1.6) and (I.4.7), the character λ has order p^n for some $n \geq 1$, whence $\varphi^{p^n} \in \mathrm{End}_k(S)_\varepsilon$. By Schur's Lemma, the latter space coincides with $k\,\mathrm{id}_S$, so that there exists $\alpha \in k$ such that $\varphi^{p^n} = \alpha\,\mathrm{id}_S$. Since φ is invertible, we have $\alpha \neq 0$. From the identity

$$\mathrm{tr}(\alpha\,\mathrm{id}_S) = \mathrm{tr}(\varphi^{p^n}) = \mathrm{tr}(\varphi)^{p^n} = 0,$$

we conclude that $\mathrm{tr}(\mathrm{End}_k(S)) = \mathrm{tr}(\mathrm{End}_k(S)_\varepsilon) = (0)$, a contradiction. As a result, $\mathrm{End}_k(S) = \mathrm{End}_k(S)_\varepsilon$ is one-dimensional, so that $\dim_k S = 1$.

In view of the above, the semi-simple $k\mathcal{G}$-module $k[\mathcal{G}] = k\mathcal{G}^*$ decomposes into one-dimensional constituents and we obtain

$$k[\mathcal{G}] = \bigoplus_{\lambda \in D} k[\mathcal{G}]_\lambda,$$

where $D \subseteq X(\mathcal{G})$ and $k[\mathcal{G}]_\lambda = \{x \in k[\mathcal{G}] \ ; \ h.x = \lambda(h)x \quad \forall\ h \in k\mathcal{G}\}$ is the λ-weight space of $k[\mathcal{G}]$. Direct computation shows that $k[\mathcal{G}]_\lambda = k\,(\lambda \circ \eta)$. Hence the Hopf algebra $k[\mathcal{G}]$ is generated by group-like elements, and $\mathcal{G}$ is diagonalizable. □

Theorem 1.2 (Nagata). *Let $\mathcal{G}$ be a finite group scheme. Then the following statements are equivalent:*

(1) *The algebra $k\mathcal{G}$ is semi-simple.*
(2) *The group $\mathcal{G}^0$ is diagonalizable and $p \nmid \mathrm{ord}(\mathcal{G}(k))$.*

Proof. Let $\Lambda := k\mathcal{G}^0$ be the distribution algebra of the connected component $\mathcal{G}^0$ of $\mathcal{G}$. By general theory (cf. (I.3.4)),

$$k\mathcal{G} \cong \Lambda\mathcal{G}(k)$$

is a skew group algebra.

If $k\mathcal{G}$ is semi-simple, then the factor algebra $k\mathcal{G}(k)$ is semi-simple, so that $p \nmid \mathrm{ord}(\mathcal{G}(k))$. Moreover, k is a projective $k\mathcal{G}$-module. Since $k\mathcal{G}$ is a free left Λ-module, it follows that k is also projective over $k\mathcal{G}^0$. Hence $k\mathcal{G}^0$ is semi-simple and (1.1) implies that $\mathcal{G}^0$ is diagonalizable.

Conversely, assuming (2), we note that $P := \Lambda\mathcal{G}(k) \otimes_\Lambda k$ is a projective $\mathcal{G}$-module, which contains the trivial $\Lambda\mathcal{G}(k)$-module $k(\sum_{g \in \mathcal{G}(k)} g \otimes 1)$ as a direct summand. Consequently, k is a projective $\mathcal{G}$-module and $k\mathcal{G}$ is semi-simple (see (I.1.5)). □

2. Tensor Products of Simple and Principal Indecomposable Modules

Throughout, we will be working over an algebraically closed field k. Let $\mathcal{G}$ be a finite algebraic k-group, $\mathcal{N} \trianglelefteq \mathcal{G}$ be a closed, normal subgroup. Under favorable circumstances, which will be seen to hold in all cases of interest, the simple and principal indecomposable $k\mathcal{G}$-modules are given by tensor products of the corresponding modules for $k\mathcal{N}$ and $k(\mathcal{G}/\mathcal{N})$. In particular, the Ext-quiver of $k\mathcal{G}$ will be computable from the structure of tensor products of simple $k(\mathcal{G}/\mathcal{N})$-modules.

In the sequel, we shall often view $(\mathcal{G}/\mathcal{N})$-modules as $\mathcal{G}$-modules via pull-back along the canonical quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$. We begin by studying the underlying set of vertices of the Ext-quiver of $k\mathcal{G}$ in a setting that will turn out to be appropriate for our purposes.

LEMMA 2.1. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the finite algebraic k-group $\mathcal{G}$. If $L_1, \ldots, L_n$ are simple $\mathcal{G}$-modules such that $\{L_n|_{\mathcal{N}}, \ldots, L_n|_{\mathcal{N}}\}$ is a complete set of representatives for the isoclasses of the simple $\mathcal{N}$-modules, then every simple $\mathcal{G}$-module S is of the form*

$$S \cong L_i \otimes_k M$$

for a unique $i \in \{1, \ldots, n\}$ and a unique (up to isomorphism) simple $(\mathcal{G}/\mathcal{N})$-module M.

PROOF. Let $L_1, \ldots, L_n$ and $M_1, \ldots, M_m$ be complete sets of representatives for the isoclasses of the simple $\mathcal{N}$-modules and simple $(\mathcal{G}/\mathcal{N})$-modules, respectively. By our current assumption, each $\mathcal{N}$-module L_i is the restriction of a simple $\mathcal{G}$-module, which we will also denote by L_i. Given a simple $\mathcal{G}$-module S, there exists $i \in \{1, \ldots, n\}$ such that $L_i \hookrightarrow S|_{\mathcal{N}}$. Consequently, the $\mathcal{G}$-linear map

$$\varphi : \mathrm{Hom}_{\mathcal{N}}(L_i, S) \otimes_k L_i \longrightarrow S \ \ ; \ \ f \otimes x \mapsto f(x)$$

is surjective. The image $\mathrm{im}\,\varphi$ is the L_i-isotypic component S_{L_i} of the module $S|_{\mathcal{N}}$. By Schur's Lemma, the $\mathcal{N}$-module S_{L_i} has dimension $\dim_k \mathrm{Hom}_{\mathcal{N}}(L_i, S) \otimes_k L_i$. As a result, φ is also injective, and

$$S \cong \mathrm{Hom}_{\mathcal{N}}(L_i, S) \otimes_k L_i \cong L_i \otimes_k \mathrm{Hom}_{\mathcal{N}}(L_i, S).$$

Note that $\mathrm{Hom}_{\mathcal{N}}(L_i, S)$ has the structure of a $(\mathcal{G}/\mathcal{N})$-module, which is necessarily simple.

Next, we consider the $\mathcal{G}$-module $S := L_i \otimes_k M_r$. By the above, S contains a simple submodule $T \cong L_j \otimes_k M_s$. Upon restriction to $\mathcal{N}$ we obtain

$$(\dim_k M_s) L_j \cong T|_{\mathcal{N}} \hookrightarrow S|_{\mathcal{N}} \cong (\dim_k M_r) L_i,$$

so that $i = j$. Moreover, we have homomorphisms

$$M_s \cong \mathrm{Hom}_{\mathcal{N}}(L_i, L_i \otimes_k M_s) \cong \mathrm{Hom}_{\mathcal{N}}(L_i, T) \hookrightarrow \mathrm{Hom}_{\mathcal{N}}(L_i, S) \cong M_r$$

of $(\mathcal{G}/\mathcal{N})$-modules, whence $r = s$. As a result, the module $S = L_i \otimes_k M_r$ is simple and the pair (i, r) is uniquely determined by S. □

REMARK. If $\mathcal{N} = \mathcal{U}$ is a unipotent normal subgroup of $\mathcal{G}$, then, setting $L_1 = k$, our result specializes to the well-known bijection between the simple $\mathcal{G}/\mathcal{U}$-modules and the simple $\mathcal{G}$-modules.

The technical condition of (2.1) is known to hold for Frobenius kernels of semi-simple, simply connected groups (cf. [**80**, (II.3.15)]). In our projected applications, the normal subgroup $\mathcal{N}$ will be the first Frobenius kernel of the special linear group SL(2). The following subsidiary result shows how this fact can be exploited in the computation of the Ext-quiver of $k\mathcal{G}$.

LEMMA 2.2. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of a finite algebraic k-group $\mathcal{G}$. Suppose that L_1, L_2 and M_1, M_2 are simple $\mathcal{G}$-modules and simple $(\mathcal{G}/\mathcal{N})$-modules, respectively, such that $L_i|_{\mathcal{N}}$ is simple for $i \in \{1, 2\}$.*

(1) *If $\mathrm{Ext}^1_{\mathcal{N}}(S, S) = (0)$ for every simple $\mathcal{N}$-module S, then*

$$\mathrm{Ext}^1_{\mathcal{G}}(L_1 \otimes_k M_1, L_2 \otimes_k M_2) \cong \begin{cases} \mathrm{Ext}^1_{\mathcal{G}/\mathcal{N}}(M_1, M_2) & \text{if } L_1 \cong L_2 \\ \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(M_1, \mathrm{Ext}^1_{\mathcal{N}}(L_1, L_2) \otimes_k M_2) & \text{if } L_1 \not\cong L_2. \end{cases}$$

(2) *If $\mathcal{G}/\mathcal{N}$ is linearly reductive, then*

$$\mathrm{Ext}^1_{\mathcal{G}}(L_1 \otimes_k M_1, L_2 \otimes_k M_2) \cong \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(M_1, \mathrm{Ext}^1_{\mathcal{N}}(L_1, L_2) \otimes_k M_2). \quad \square$$

The following consequence gives a first illustration of the relationship between the Ext-quivers and the McKay quivers of certain groups.

Corollary 2.3. *Let $\mathcal{G}$ be a finite algebraic group, $\mathcal{U} \trianglelefteq \mathcal{G}$ be a unipotent, normal subgroup such that $\mathcal{G}/\mathcal{U}$ is linearly reductive. Then the following statements hold:*

(1) *The* Ext-*quiver $Q_{k\mathcal{G}}$ is isomorphic to the McKay quiver $\Theta_{\mathrm{H}^1(\mathcal{U},k)}(\mathcal{G}/\mathcal{U})$ of $\mathcal{G}/\mathcal{U}$ relative to the $(\mathcal{G}/\mathcal{U})$-module $\mathrm{H}^1(\mathcal{U},k)$.*

(2) *If $\mathrm{H}^1(\mathcal{U},k)$ is a faithful $(\mathcal{G}/\mathcal{U})$-module, then $k\mathcal{G} = \mathcal{B}_0(\mathcal{G})$.*

Proof. Let $S_1, \ldots, S_n$ be a complete set of representatives for the simple $\mathcal{G}$-modules. The unipotent normal subgroup $\mathcal{U}$ acts trivially on the simple modules S_i, so that each S_i is a simple $(\mathcal{G}/\mathcal{U})$-module. Thanks to (2.2(2)), we have

$$\mathrm{Ext}^1_{\mathcal{G}}(S_i, S_j) \cong \mathrm{Hom}_{\mathcal{G}/\mathcal{U}}(S_i, \mathrm{H}^1(\mathcal{U},k) \otimes_k S_j).$$

Thus, if $\mathrm{H}^1(\mathcal{U},k) \otimes_k S_j \cong \bigoplus_{i=1}^n m_{ij} S_i$, then Schur's Lemma yields

$$m_{ij} = \dim_k \mathrm{Hom}_{\mathcal{G}/\mathcal{U}}(S_i, \mathrm{H}^1(\mathcal{U},k) \otimes_k S_j) = \dim_k \mathrm{Ext}^1_{\mathcal{G}}(S_i, S_j),$$

so that (1) follows. Assertion (2) is now a consequence of (V.5.2). □

Our final result of this section provides a criterion for the construction of principal indecomposable modules. For large p, condition (b) of the following Proposition is known to hold in the classical context of Frobenius kernels [**10, 78**].

Proposition 2.4. *Let $\mathcal{G}$ be a finite algebraic group, $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup. Suppose that*

(a) *every simple $\mathcal{N}$-module is the restriction of a $\mathcal{G}$-module, and*

(b) *every principal indecomposable $\mathcal{N}$-module is the restriction of a $\mathcal{G}$-module.*

If $P_1, \ldots, P_n$ are $\mathcal{G}$-modules such that $\{P_1|_{\mathcal{N}}, \ldots, P_n|_{\mathcal{N}}\}$ is a complete set of representatives for the isoclasses of the principal indecomposable $\mathcal{N}$-modules, and $\{Q_1, \ldots, Q_m\}$ is a complete set of representatives for the isoclasses of the principal indecomposable $(\mathcal{G}/\mathcal{N})$-modules, then the $\mathcal{G}$-modules $(P_i \otimes_k Q_r)_{1\le i\le n,\ 1\le r\le m}$ form a complete set of principal indecomposable $\mathcal{G}$-modules. □

Remark. If $\mathcal{G} = \mathcal{U} \rtimes \mathcal{H}$ is a semidirect product of a unipotent group $\mathcal{U}$ and a linearly reductive group $\mathcal{H}$, then $P_1 := k\mathcal{G} \otimes_{k\mathcal{H}} k$ is a projective $k\mathcal{G}$-module such that $P_1|_{\mathcal{U}} \cong k\mathcal{U}$. By (2.4), the principal indecomposable $\mathcal{G}$-modules are of the form $P_1 \otimes_k S \cong k\mathcal{G} \otimes_{k\mathcal{H}} S$, where S is a simple $\mathcal{H}$-module. If $\mathcal{H}$ is diagonalizable, then the simple $\mathcal{H}$-modules are of the form k_λ, with $\lambda \in X(\mathcal{G}) \cong X(\mathcal{H})$ (see also [**47**, (2.4)]).

3. Groups of Finite Representation Type

Let k be an algebraically closed field of characteristic $p > 0$. We now consider those finite algebraic k-groups, whose principal blocks have finite representation type. Recall that Nagata's Theorem gives a complete answer for groups, whose principal blocks are simple.

By Higman's Theorem [**69**], the finite groups with representation-finite principal blocks are precisely those having a cyclic Sylow-p-subgroup. The structure of these groups was investigated by Brauer [**17, 18**]. For instance, the factor group $G/O_{p'}(G)$ of G by its largest normal subgroup of order prime to p, is either trivial, or an automorphism group of a simple group, or its center is cyclic of order p. General theory shows that every simple group possesses some cyclic Sylow subgroup.

Turning to infinitesimal groups, we have already seen in (IV.2.5) that schemes of tori do provide information on the structure of representation-finite Lie algebras.

DEFINITION. An infinitesimal group $\mathcal{G}$ is called *supersolvable* if there exists a chain $e_k = \mathcal{G}_{[0]} \subseteq \mathcal{G}_{[1]} \subseteq \cdots \subseteq \mathcal{G}_{[n]} = \mathcal{G}$ of normal subgroups of $\mathcal{G}$ such that $\mathcal{G}_{[i]}/\mathcal{G}_{[i-1]} \cong \mathbb{G}_{a(1)}, \mu_{(p)}$ for every $i \in \{1, \ldots, n\}$.

Given an infinitesimal group $\mathcal{G}$ with Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$, we recall from (III.1.3), (III.1.2), (I.4.6) and (III.2.11) that $\dim \widehat{\mathcal{V}}_{\mathfrak{g}} \leq 1$ whenever $\mathcal{B}_0(\mathcal{G})$ has finite representation type. The following result (cf. [**47**, (2.1)]]), exploits this property:

PROPOSITION 3.1. *Let $\mathcal{G}$ be an infinitesimal group such that $\dim \widehat{\mathcal{V}}_{\mathfrak{g}} \leq 1$. Then $\mathcal{G}$ is supersolvable.*

PROOF. We shall only sketch a proof of the weaker statement: *If $\mathcal{B}_0(\mathcal{G})$ is representation-finite, then $\mathcal{G}$ is supersolvable.* Proceeding by induction on the order of $\mathcal{G}$, we let $\mathfrak{n} \trianglelefteq \mathfrak{g}$ be a minimal $\mathcal{G}$-invariant p-ideal of $\mathfrak{g}$, relative to the adjoint representation $\mathcal{G} \longrightarrow \mathrm{Aut}_p(\mathfrak{g})$. As in (IV.2.5) one shows that $\mathfrak{g}$ is solvable. Consequently, $\mathfrak{n}$ is abelian, and $\mathfrak{n}^{[p]} \subseteq \mathfrak{n} \cap C(\mathfrak{g})$ is a $\mathcal{G}$-invariant p-ideal of $\mathfrak{g}$. We thus have $\mathfrak{n}^{[p]} = \mathfrak{n}$ or $\mathfrak{n}^{[p]} = (0)$. In the former case, $\mathfrak{n}$ is a torus, so that the connected group $\mathcal{G}$ acts trivially on $\mathfrak{n}$. Alternatively, $\mathfrak{n} \subseteq \widehat{\mathcal{V}}_{\mathfrak{g}}$ has dimension 1. Thus, the normal subgroup $\mathcal{N} \trianglelefteq \mathcal{G}$ corresponding to $\mathfrak{n}$ is isomorphic to $\mu_{(p)}$ or $\mathbb{G}_{a(1)}$. Since the principal block of $\mathcal{G}/\mathcal{N}$ is representation-finite, an application of the inductive hypothesis completes the proof. □

Proposition 3.1 allows us to bring our knowledge of supersolvable groups to bear: The factor group $\mathcal{G}/\mathcal{M}(\mathcal{G})$ of such a group $\mathcal{G}$ by its unique maximal normal diagonalizable subgroup $\mathcal{M}(\mathcal{G})$ is trigonalizable, and thus decomposes into a semidirect product

$$\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes \mathcal{D},$$

with a normal unipotent subgroup $\mathcal{U}$ and a diagonalizable subgroup $\mathcal{D}$. One can show that $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. Moreover, the algebra $\mathcal{B}_0(\mathcal{U} \rtimes \mathcal{D})$ is representation-finite if and only if $k\mathcal{U}$ enjoys this property. However, $k\mathcal{U}$ is local of dimension a p-power. Thus, $k\mathcal{U}$ is of finite representation type if and only if it is isomorphic to a truncated polynomial ring $k[X]/(X^{p^n})$. In particular, $k\mathcal{U}$ is a Nakayama algebra, and the group $\mathcal{U}$ is commutative.

The commutative unipotent groups that give rise to Nakayama algebras are the so-called *V-uniserial* groups. Here the prefix "V" refers to the *Verschiebung* $V_{\mathcal{U}} : \mathcal{U}^{(1)} \longrightarrow \mathcal{U}$, a homomorphism that is the dual of the Frobenius homomorphism of the *Cartier dual* $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$. The V-uniserial groups were classified in [**44**], using Dieudonné modules (see also [**83**]).

In the above, we have sketched the implication (1) ⇒ (2) of the following result:

THEOREM 3.2 ([**47, 50**]). *Let $\mathcal{G}$ be an infinitesimal k-group. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(\mathcal{G})$ has finite representation type.*
(2) *$\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes \mu_{(p^n)}$ is a semidirect product with a V-uniserial normal subgroup $\mathcal{U}$.*
(3) *$k\mathcal{G}$ is a Nakayama algebra.*
(4) *$\dim P(\mathcal{G}_2) \leq 0$.*
(5) *$k\mathcal{G}_2$ is a Nakayama algebra.* □

In particular, finite representation type may be detected on the second Frobenius kernel of an infinitesimal group.

REMARKS. (i) Special cases of (3.2) as well as related results can be found in [**97, 54, 38**].

(ii) The equivalence (1) $\Leftrightarrow$ (4) readily shows that subgroups of representation-finite groups are representation-finite. For finite groups, this fact follows directly from the Mackey Decomposition Theorem.

(iii) Let G be a finite group. The example of the quaternion group shows that the analogue of (1) $\Leftrightarrow$ (4) fails in this context. Moreover, representation-finite group algebras are not necessarily Nakayama algebras. Since the defect group of the principal block of kG is a Sylow-p-subgroup, kG is representation-finite whenever $\mathcal{B}_0(G)$ has this property.

We now turn to the block structure of $k\mathcal{G}$. According to (3.2), each block $\mathcal{B}$ of a representation-finite infinitesimal group $\mathcal{G}$ is a self-injective Nakayama algebra. In general, indecomposable Nakayama algebras are determined by their *Kupisch series*, that is, by the number of simple modules, and the lengths of their projective covers (cf. [**84**], [**4**, (V.3)]). In particular, every self-injective indecomposable Nakayama algebra is Morita equivalent to a bound quiver algebra

$$k[\tilde{A}_{n-1}]/(k[\tilde{A}_{n-1}]^+)^m,$$

whose quiver is a clockwise or counter-clockwise oriented circle with n vertices.

For infinitesimal groups of finite representation type, the parameters n and m of the blocks of $k\mathcal{G}$ were determined in [**46**]. They turn out to be p-powers, whose exponents can be described as orders of certain subgroups of $\mathcal{G}$.

Let me briefly indicate the connection with McKay quivers, as these will appear again for groups of tame representation type. If $\mathcal{B}_0(\mathcal{G})$ is representation-finite, then (3.2) tells us that, for the purpose of describing $\mathcal{B}_0(\mathcal{G})$, we may assume that $\mathcal{G} = \mathcal{U} \rtimes \mu_{(p^n)}$. We may apply (2.3) to see that the Ext-quiver of $k\mathcal{G}$ coincides with the McKay quiver $\Theta_{\mathrm{H}^1(\mathcal{U},k)}(\mu_{(p^n)})$ of $k\mu_{(p^n)}$ relative to the one-dimensional $\mathcal{G}$-module $\mathrm{H}^1(\mathcal{U}, k)$:

$$\mathrm{Ext}^1_{\mathcal{G}}(k_\lambda, k_\gamma) \cong \mathrm{Hom}_{\mu_{(p^n)}}(k_\lambda, \mathrm{H}^1(\mathcal{U}, k) \otimes_k k_\gamma).$$

We have seen in (V.5) that the connected components of such quivers are of the form $\tilde{A}_{q-1}$, with q being a divisor of $p^n = \mathrm{ord}(X(\mu_{(p^n)}))$.

THEOREM 3.3 ([**47**]). *Let $\mathcal{G}$ be a finite group scheme. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ has finite representation type.*

(2) *The algebras $k\mathcal{G}^0$ and $k\mathcal{G}(k)$ have finite representation type, with at least one of them being semi-simple.*

(3) *$k\mathcal{G}$ has finite representation type.* □

Further information concerning blocks of $k\mathcal{G}$ can be found in [**46, 43**].

4. Binary Polyhedral Groups

We now turn to binary polyhedral groups and describe the linearly reductive finite algebraic subgroups of the smooth group scheme SL(2). As before, $T \subseteq \mathrm{SL}(2)$ denotes the standard maximal torus of diagonal matrices. For $m \geq 1$, we let

$$Q_m := \langle x, y \mid x^m = y^2 \,;\, yxy^{-1} = x^{-1} \rangle$$

be the *generalized quaternion group* of order $4m$. In our context, Q_m will occur as the subgroup of $\mathrm{SL}(2)(k)$ generated by the matrices

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } x(\zeta_{2m}) := \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1} \end{pmatrix},$$

where $\zeta_{2m} \in k$ is a primitive $2m$-th root of unity and $(p, 2m) = 1$. We let $\hat{T}$, $\hat{O}$ and $\hat{I}$ be the *binary tetrahedral group*, the *binary octahedral group*, and the *binary icosahedral group* of orders 24, 48 and 120, respectively (cf. [**112**, (4.4)]). Realizations of these groups can be found in [**41**]. When considering binary polyhedral groups we tacitly assume $p \geq 5$ for $\hat{T}$, $\hat{O}$ as well as $p \geq 7$ for $\hat{I}$.

We shall also consider some non-reduced subgroups of SL(2). For $m \in \mathbb{N}$, we let $T_{(m)} \subseteq T$ be the closed subgroup, given by

$$T_{(m)}(R) := \{\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \; ; \; x \in \mu_{(m)}(R)\}$$

for every commutative k-algebra R. Note that $T_{(m)}$ is the unique closed subgroup of T of order m.

Let $h := \begin{pmatrix} \zeta_4 & 1 \\ 1 & \zeta_4 \end{pmatrix} \in \mathrm{GL}(2)(k)$. Then $\mathcal{H}_4 := h^{-1}T_{(4)}h$ is a reduced, closed subgroup of SL(2) such that $\mathcal{H}_4(k) = \langle\omega\rangle \subseteq \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$. Since T is reduced, an application of [**80**, (I.2.6(11))] shows that the latter group coincides with $\mathrm{Nor}_{\mathrm{SL}(2)}(T)(k)$. As $\mathcal{H}_4$ is reduced, this implies $\mathcal{H}_4 \subseteq \mathrm{Nor}_{\mathrm{SL}(2)}(T)$. Thus, $\mathcal{H}_4$ normalizes every closed subgroup of T, and we define

$$N_{(m)} := T_{(m)}\mathcal{H}_4$$

for $m \geq 2$. The construction of the other subgroups of SL(2) requires a few preparatory remarks.

When dealing with reduced algebraic group schemes we shall often make use of the fact that such a group $\mathcal{G}$ is uniquely determined by its group of k-rational points. In fact,

$$\mathcal{G} \mapsto \mathcal{G}(k)$$

defines an equivalence between the category of reduced algebraic group schemes and the category of algebraic groups in the sense of [**113**]. Upon restriction to reduced finite algebraic groups, the above functor provides an equivalence with the category of finite groups (cf. [**122**, (0.16),(0.17)] for more details). Given a finite group G, we let G_k be the reduced finite algebraic group such that $G_k(k) = G$.

LEMMA 4.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be algebraic groups. Suppose that $\mathcal{G}$ is reduced and that there exists a closed embedding $\varphi : \mathcal{G}(k) \hookrightarrow \mathcal{H}(k)$. Then there exists a closed embedding $\psi : \mathcal{G} \hookrightarrow \mathcal{H}$ of group schemes such that $\psi_k = \varphi$.* □

According to (4.1), there exist uniquely determined reduced subgroups $(\hat{T})_k$, $(\hat{O})_k$ and $(\hat{I})_k$ of SL(2) satisfying $(\hat{T})_k(k) = \hat{T}$, $(\hat{O})_k(k) = \hat{O}$ and $(\hat{I})_k(k) = \hat{I}$, respectively. The proof of Theorem 4.2 below, which extends [**67**, Thm.1] to our context, shows in particular that $N_{(m)} \cong (Q_m)_k$ whenever $(p, 2m) = 1$.

Recall that the McKay graph $\bar{\Theta}_L(\mathcal{G})$ relative to a two-dimensional self-dual module L is the graph associated to the defining generalized Cartan matrix, cf. (V.5).

THEOREM 4.2. *Let $\mathcal{G} \subseteq \mathrm{SL}(2)$ be a finite, linearly reductive subgroup scheme of characteristic $p \geq 3$. Then there exists $g \in \mathrm{SL}(2)(k)$ such that $g\mathcal{G}g^{-1}$ and the McKay graph $\bar{\Theta}_L(\mathcal{G})$ of $\mathcal{G}$ relative to its standard module L belong to the following list:*

$g\mathcal{G}g^{-1}$	$\bar{\Theta}_L(\mathcal{G})$
e_k	$\tilde{A}_0$
$T_{(np^r)}$	$\tilde{A}_{np^r-1}$
$N_{(np^r)}$	$\tilde{D}_{np^r+2}$
$(\hat{T})_k$	$\tilde{E}_6$
$(\hat{O})_k$	$\tilde{E}_7$
$(\hat{I})_k$	$\tilde{E}_8$,

where $(n,p) = 1$, $r := \mathrm{ht}(\mathcal{G}^0)$, *and* $n + r \neq 1$.

PROOF. According to [**80**, (II.2.5)], the two-dimensional, simple SL(2)-module $L = L(1)$ is self-dual. Consequently, L is also a faithful, self-dual $\mathcal{G}$-module.

Recall from (I.3.4) that $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$ is a semidirect product of an infinitesimal group $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{\mathrm{red}}$. Thanks to Nagata's Theorem (1.2), the connected component $\mathcal{G}^0$ is diagonalizable and p does not divide the order of $\mathcal{G}(k) = \mathcal{G}_{\mathrm{red}}(k)$. Since $\mathcal{G}^0$ is diagonalizable, there exists an element $g \in \mathrm{SL}(2)(k)$ such that $g\mathcal{G}^0g^{-1} \subseteq T$. If $\mathrm{ht}(\mathcal{G}^0) = r$, then we have $g\mathcal{G}^0g^{-1} \subseteq T_r \cong \mu_{(p^r)}$. On the other hand, $\mathrm{ord}(\mathcal{G}^0) \geq p^r$ (cf. [**123**, (2.2)]), so that $g\mathcal{G}^0g^{-1} = T_r$. Thus, replacing $\mathcal{G}$ by a suitable conjugate group, we may assume that $\mathcal{G}^0 = T_r = T_{(p^r)}$.

If $r = 0$, then the group $\mathcal{G}$ is reduced, and $\mathcal{G}$ is completely determined by the binary polyhedral group $\mathcal{G}(k) \subseteq \mathrm{SL}(2)(k)$. Directly from [**112**, (4.4)] we infer that $\mathcal{G}(k)$ is conjugate to e_k, $T_{(n)}(k)$ for $n \geq 2$, $N_{(n)}(k)$ for $n \geq 2$, $\hat{T}$, $\hat{O}$, or $\hat{I}$. As all groups involved are associated to reduced group schemes, we obtain the left-hand column of our list. Thanks to (V.5.4) (see also [**67**, Thm.1]), the McKay graphs of these groups have the asserted structure.

If $r \geq 1$, then the identity $\mathcal{G}^0 = T_r$ in conjunction with $p \neq 2$ and [**80**, (I.2.6(11))] implies that

$$\mathcal{G}(k) \subseteq \mathrm{Nor}_{\mathrm{SL}(2)}(T_r)(k) = T(k)\langle\omega\rangle = \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k)) = \mathrm{Nor}_{\mathrm{SL}(2)}(T)(k).$$

Consequently, $\mathcal{G}_{\mathrm{red}} \subseteq \mathrm{Nor}_{\mathrm{SL}(2)}(T)$, so that $\mathcal{G} \subseteq \mathrm{Nor}_{\mathrm{SL}(2)}(T)$. There results an embedding $\mathcal{G}/(\mathcal{G} \cap T) \hookrightarrow \mathrm{Nor}_{\mathrm{SL}(2)}(T)/T$, with the latter group being the *Weyl group* of SL(2). In particular, $\mathcal{G}/(\mathcal{G} \cap T)$ has order ≤ 2. The finite algebraic group $\mathcal{G} \cap T$ coincides with $T_{(np^r)}$ for some n not divisible by p. By our observations above, we have $\mathcal{G}^0 \subseteq \mathcal{G} \cap T$. If $\mathcal{G}(k) \subseteq T(k)$, then $\mathcal{G} = \mathcal{G} \cap T = T_{(np^r)}$. Alternatively, some basic considerations allow us to assume that $\mathcal{H}_4(k) \subseteq \mathcal{G}(k)$ (cf. [**41**, §3]). As $\mathcal{H}_4$ is reduced, we conclude that $\mathcal{G} \cap T \subseteq N_{(np^r)} \subseteq \mathcal{G}$, so that $\mathcal{G} = N_{(np^r)}$.

It remains to identify the McKay graphs of $T_{(np^r)}$ and $N_{(np^r)}$. This can be done using the results of [**67**]. □

5. Groups of Tame Representation Type

Throughout, k is assumed to be an algebraically closed field of characteritic $p \geq 3$. In this section, we discuss the determination of the finite group schemes of tame representation type. The theory proceeds progressively by first studying infinitesimal groups of height 1. In view of (I.4.6), this amounts to analyzing restricted enveloping algebras $U_0(\mathfrak{g})$ of finite-dimensional restricted Lie algebras $(\mathfrak{g}, [p])$.

The first example people looked at was the special linear Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$. In early work, Pollack [**98**] showed that $U_0(\mathfrak{sl}(2))$ has infinite representation type. As mentioned in Chapter III, the basic algebra of $U_0(\mathfrak{sl}(2))$ was determined some 15 years later independently by Drozd [**34**], Fischer [**55**] and Rudakov [**109**]. Their work implies that every non-simple block of $U_0(\mathfrak{sl}(2))$ is Morita equivalent to the trivial extension $T(\tilde{A}_1)$ of the Kronecker algebra $k[\bullet \rightrightarrows \bullet]$. In particular, each of these blocks is special biserial and hence tame. At the same time, Pfautsch [**95**] showed that the tame blocks of the distribution algebras $k\,\mathrm{SL}(2)_r$ are also of this type.

For quite some time, the blocks of the first Frobenius kernel $\mathrm{SL}(2)_1$ remained essentially the only examples of a tame blocks of infinitesimal group. (In characteristic 2, the enveloping algebra of the unipotent Heisenberg algebra is isomorphic to the group algebra of the dihedral group of order 8 and hence tame.) In retrospect, this is not a surprise: We have seen in (III.4.1) that tame blocks of Frobenius kernels of smooth groups are Morita equivalent to the tame blocks of $U_0(\mathfrak{sl}(2))$, so finding new examples amounts to working with non-classical restricted Lie algebras.

The following result, which does not hold for fields of characteristic 2, is fundamental for the following.

THEOREM 5.1 ([**48, 49**]). *Let $\mathcal{G}$ be an infinitesimal group with tame principal block. Then the following statements hold:*

(1) *$\mathcal{G}$ is not solvable.*
(2) *If $\mathcal{G}$ is semi-simple, then* $\mathrm{Lie}(\mathcal{G}) \cong \mathfrak{sl}(2)$.

PROOF. (1) One picks a counter-example of minimal order and uses (3.2) to collect enough structural information to compute the quiver and the relations of $\mathcal{B}_0(\mathcal{G})$. These occur in the lists of [**120**] and are thus wild.

(2) The solvable radical $\mathcal{R}(\mathcal{G}_1)$ of $\mathcal{G}_1$ is a solvable normal subgroup of $\mathcal{G}$ and therefore trivial. As a result, the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ is also semi-simple. Since $\mathcal{B}_0(\mathcal{G})$ is tame, a consecutive application of (III.3.1), (I.1.6), (II.2.2) and (III.2.11) gives $\dim \widehat{\mathcal{V}}_{\mathfrak{g}} = \mathrm{cx}_{U_0(\mathfrak{g})}(k) \leq 2$. In view of (1) and (3.1), we actually have $\dim \widehat{\mathcal{V}}_{\mathfrak{g}} = 2$, and the assertion now follows from (IV.2.6). □

5.1. Restricted Lie Algebras. We begin by studying the easiest case concerning restricted Lie algebras. A priori, it is not clear why this strategy should be fruitful. Contrary to the modular representation theory of finite groups, we don't automatically have a good descent theory at our disposal. As we shall see, subgroups of tame infinitesimal groups may be wild! Fortunately, passage to Frobenius kernels turns out to be much better behaved, and the first Frobenius kernel will be seen to encapsulate most structural features of the group $\mathcal{G}$.

Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra, and denote by $\mathcal{B}_0(\mathfrak{g})$ the principal block of its restricted enveloping algebra $U_0(\mathfrak{g})$.

PROBLEM. We shall be concerned with the following two interrelated tasks:

- Determine those Lie algebras $\mathfrak{g}$ for which $\mathcal{B}_0(\mathfrak{g})$ is tame.
- Find the presentation of $\mathcal{B}_0(\mathfrak{g})$ by its Ext-quiver and relations in case $\mathcal{B}_0(\mathfrak{g})$ is tame.

A few comments are in order. By focussing on the principal block, we make the determination of $\mathfrak{g}$ more tractable. If $T(\mathfrak{g}) \subseteq \mathfrak{g}$ is the maximal toral p-ideal of $\mathfrak{g}$, then $\mathcal{B}_0(\mathfrak{g}) \cong \mathcal{B}_0(\mathfrak{g}/T(\mathfrak{g}))$, so that we work up to extensions by toral ideals. As we shall see below, the presence of toral ideals may complicate the representation theory of $U_0(\mathfrak{g})$ significantly.

Recall that $C(\mathfrak{g}) := \{x \in \mathfrak{g} \ ; \ [x, y] = 0 \ \ \forall\, y \in \mathfrak{g}\}$ is the center of $\mathfrak{g}$. Given $n \in \mathbb{N}_0$, we let

$$\mathfrak{n}_n := \bigoplus_{i=0}^{n-1} k x^{[p]^i} \quad ; \quad x^{[p]^n} = 0 \neq x^{[p]^{n-1}}$$

be the n-dimensional, nil-cyclic Lie algebra.

Let $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$. If $\mathcal{B}_0(\mathfrak{g})$ is tame, then $\mathcal{B}_0(\mathfrak{g}/\mathfrak{r})$ is tame or representation-finite. Owing to (5.1) the algebra $\mathfrak{g}$ is not solvable, so that $\mathfrak{g}/\mathfrak{r}$ is not trivial. Thus, by (3.1), the block $\mathcal{B}_0(\mathfrak{g}/\mathfrak{r})$ is tame and (5.1) implies $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{sl}(2)$. A detailed analysis of the radical then yields:

Proposition 5.2. *Suppose that $\mathcal{B}_0(\mathfrak{g})$ has tame representation type. Then $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$ and $C(\mathfrak{g}) \cong \mathfrak{n}_n \oplus T(\mathfrak{g})$ for some $n \in \mathbb{N}_0$.* □

Let us record one important consequence: Since $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$, we have $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] + C(\mathfrak{g})$, so that $\mathrm{tr}(\mathrm{ad}\, x) = 0$ for all $x \in \mathfrak{g}$. By Schue's result (cf. [**110**] and (I.4)), the restricted enveloping algebra $U_0(\mathfrak{g})$ is symmetric.

The foregoing result marks the extent to which our geometric techniques will take us. We are left with the problem of determining those central extensions of $\mathfrak{sl}(2)$, whose principal blocks are tame.

To obtain the structure of $\mathfrak{g}$, one first observes that the exact sequence

$$(0) \longrightarrow C(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{sl}(2) \longrightarrow (0)$$

splits when considered a sequence of ordinary Lie algebras. Thus, $\mathfrak{g} = \mathfrak{sl}(2) \oplus C(\mathfrak{g})$, and there exists a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})$ such that

$$(x,c)^{[p]} = (x^{[p]}, \psi(x) + c^{[p]}) \qquad \text{for } (x,c) \in \mathfrak{sl}(2) \oplus C(\mathfrak{g}).$$

Let us consider the case where $C(\mathfrak{g}) = kv_0 \neq (0)$ and $v_0^{[p]} = 0$. The group $\mathrm{SL}(2)(k) \times k^\times$ operates on the space of p-semilinear forms such that the orbits of this action correspond to isomorphism classes of restricted Lie algebras. There are exactly three orbits, whose representatives are given in terms of the standard basis $\{e,h,f\} \subseteq \mathfrak{sl}(2)$ by

$$\psi_0 = 0 \;\; ; \;\; \psi_n(e) = 0 = \psi_n(h) \;,\; \psi_n(f) = v_0 \;\; ; \;\; \psi_s(e) = 0 = \psi_s(f) \;,\; \psi_s(h) = v_0.$$

Thus, the corresponding central extensions are the algebras $\mathfrak{sl}(2)_0$, $\mathfrak{sl}(2)_n$, and $\mathfrak{sl}(2)_s$, which were introduced in (I.4). Since $\dim \widehat{\mathcal{V}}_{\mathfrak{sl}(2)_0} = 3$, Theorem III.3.1 tells us that we only have to consider the latter two types.

In our situation the restricted enveloping algebras $U_0(\mathfrak{g})$ and $U_0(\mathfrak{sl}(2))$ have the same simple modules and almost the same Ext-quiver. In particular, there is a correspondence between the blocks, so that each block has either one or two simple modules. It turns out that the representation type depends on the structure of the hearts $\mathrm{Ht}(P) = \mathrm{Rad}(P)/\mathrm{Soc}(P)$ of the principal indecomposable $U_0(\mathfrak{g})$-modules. Their structure can be analyzed by means of filtrations by Verma modules. We denote the (Jacobson) radical of $U_0(\mathfrak{g})$ by J.

The following result uses techniques from the abstract representation theory of quivers.

Proposition 5.3 ([**51**]). *Let P be a principal indecomposable $U_0(\mathfrak{g})$-module belonging to a block $\mathcal{B} \subseteq U_0(\mathfrak{g})$ with two simple modules.*

(1) *If $\mathfrak{g} = \mathfrak{sl}(2)_n$, then JP/J^3P is indecomposable of length 4, and $\mathcal{B}$ is wild.*
(2) *If $\mathfrak{g} = \mathfrak{sl}(2)_s$, then JP/J^3P is a sum of two uniserial modules, and $\mathcal{B}$ is special biserial.*

Proof. The first part follows from the fact that a Galois covering of $\mathcal{B}/J^3\mathcal{B}$ contains as a subcategory the module category of a one-point extension of the path algebra of the Kronecker quiver given by a regular module of quasi-length 2 in a homogeneous tube.

The second part involves a delicate analysis by which one extends the information on $\mathcal{B}/J^3\mathcal{B}$ to $\mathcal{B}$. Here the fact that $\mathcal{B}$ is symmetric plays an important rôle. □

The result has an interesting consequence: contary to the modular representation theory of finite groups, subalgebras of tame restricted Lie algebras may be wild!

EXAMPLE. Let $\mathfrak{h} := ke \oplus kv_0 \subseteq \mathfrak{sl}(2)_s$. Then $U_0(\mathfrak{h}) \cong k[X,Y]/(X^p, Y^p)$ is wild even though $U_0(\mathfrak{sl}(2)_s)$ is tame .

Let $\mathrm{Rad}_p(\mathfrak{g})$ be the *p-unipotent radical* of $(\mathfrak{g}, [p])$, that is, the largest p-ideal $\mathfrak{n} \unlhd \mathfrak{g}$ such that $x^{[p]^{\dim_k \mathfrak{g}}} = 0$ for every $x \in \mathfrak{n}$. We put $n(\mathfrak{g}) := \dim_k \mathrm{Rad}_p(\mathfrak{g})$.

For $n \in \mathbb{N}_0$, let $\mathcal{K}(n)$ be the bound quiver algebra of the quiver with underlying vertex set $\mathbb{Z}/(2)$ and arrows $\alpha_i, \beta_i : i \to i+1$ $(i = 0, 1)$ subject to the relations

$$\alpha_{i+1}\alpha_i = 0 = \beta_{i+1}\beta_i \ \ ; \ \ (\beta_{i+1}\alpha_i - \alpha_{i+1}\beta_i)^{p^n} = 0.$$

Note that $\mathcal{K}(0) \cong T(\tilde{A}_1)$ is the trivial extension of the Kronecker algebra.

THEOREM 5.4 ([**51**]). *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathfrak{g})$ is tame.*

(2) *Each block of the algebra $U_0(\mathfrak{g}/T(\mathfrak{g}))$ is Morita equivalent to $k[X]/(X^{p^{n(\mathfrak{g})}})$ or to $\mathcal{K}(n(\mathfrak{g}))$, with $\mathcal{B}_0(\mathfrak{g}) \sim_M \mathcal{K}(n(\mathfrak{g}))$.* □

The question of the tameness of $U_0(\mathfrak{g})$ involves the study of representation types of the family $(U_\chi(\mathfrak{g}))_{\chi \in \mathfrak{g}^*}$ of the so-called *reduced enveloping algebras* of $\mathfrak{g}$. In [**51**, §8], precise criteria for $U_0(\mathfrak{g})$ being tame were given. In particular, one encounters yet another phenomenon, which has no analogue in the modular representation theory of finite groups:

EXAMPLE. Consider the 5-dimensional restricted Lie algebra $\mathfrak{g} := \mathfrak{sl}(2) \oplus kv_0 \oplus kt_0$ with center $C(\mathfrak{g}) = kv_0 \oplus kt_0$ and p-map given by

$$e^{[p]} := 0 \ ; \ h^{[p]} := h + v_0 \ ; \ f^{[p]} := t_0 \ ; \ v_0^{[p]} := 0 \ ; \ t_0^{[p]} := t_0.$$

According to (5.4), the principal block $\mathcal{B}_0(\mathfrak{g})$ is tame. By the results of [**51**, §8], the algebra $U_0(\mathfrak{g})$ is wild.

5.2. Infinitesimal Groups. Our above example seems to portend trouble. Since subgroups of tame infinitesimal groups are not necessarily tame, it is a priori not clear how the results on Lie algebras may be employed in studying infinitesimal groups. In fact, the general theory of algebraic groups tells us that the Lie algebra $\mathrm{Lie}(\mathcal{G})$ usually does not capture much information of $\mathcal{G}$. All Frobenius kernels $(\mathcal{G}_r)_{r \geq 1}$ of an algebraic group $\mathcal{G}$ have the same Lie algebra.

However, our situation is rather special. Passage to subgroups does at least “preserve” the complexity (see (II.2.2)), so that the tameness of $\mathcal{B}_0(\mathcal{G})$ still implies

$$\dim \hat{\mathcal{V}}_{\mathrm{Lie}(\mathcal{G})} \leq 2.$$

If $\mathcal{G}$ is semi-simple, then (5.1) yields $\mathrm{Lie}(\mathcal{G}) \cong \mathfrak{sl}(2)$, so that we obtain:

- $\mathcal{G} \subseteq \mathrm{SL}(2)$ is an infinitesimal subgroup of $\mathrm{SL}(2)$, and
- $\mathcal{G}_1 = \mathrm{SL}(2)_1$.

To illustrate the utility of the foregoing observations, we elaborate on the following result, which also exemplifies some of the methods relevant for the later developments. We let $T \subseteq \mathrm{SL}(2)$ be the standard maximal torus of diagonal matrices.

Proposition 5.5 ([**49**]). *Let $\mathcal{G}$ be a semi-simple infinitesimal group such that $\mathcal{B}_0(\mathcal{G})$ is tame. Then there exists $r \in \mathbb{N}$ with $\mathcal{G} \cong \mathrm{SL}(2)_1 T_r$.*

Proof. By the above, we have $\mathcal{G} \subseteq \mathrm{SL}(2)$ and $\mathcal{G}_1 = \mathrm{SL}(2)_1$. There results an embedding

$$\mathcal{G}/\mathcal{G}_1 \hookrightarrow \mathrm{SL}(2)/\mathrm{SL}(2)_1.$$

As the group SL(2) is smooth, the Frobenius endomorphism of SL(2) induces an isomorphism $\mathrm{SL}(2)/\mathrm{SL}(2)_1 \xrightarrow{\sim} \mathrm{SL}(2)$, so that $\mathcal{G}/\mathcal{G}_1$ can also be considered a subgroup of SL(2). Since $\mathcal{B}_0(\mathcal{G})$ is tame, the principal block $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_1)$ is tame or representation-finite. In the former case, our above observations imply $\mathrm{SL}(2)_2 \subset \mathcal{G}$, so that

$$\mathrm{cx}_{\mathcal{G}}(k) \geq \mathrm{cx}_{\mathrm{SL}(2)_2}(k) = 3.$$

Since this contradicts (III.3.1), the principal block $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_1)$ has finite representation type and (3.1) can be brought to bear. A detailed analysis then shows that there exists $r \in \mathbb{N}$ with

$$\mathcal{G} \cong \mathcal{A}_{[r]} \;\text{ or }\; \mathcal{G} \cong \mathrm{SL}(2)_1 T_r,$$

where $\mathcal{A}_{[r]}$ is given by

$$\mathcal{A}_{[r]}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \; ; \; a^{p^r} = 1 = d^{p^r} \; , \; b^{p^2} = 0 = c^p\}.$$

It thus remains to rule out the former group, so we assume $\mathcal{G} = \mathcal{A}_{[r]}$.

Since every simple $\mathrm{SL}(2)_1$-module has the structure of an SL(2)-module, the simple $\mathrm{SL}(2)_1$-modules are restrictions of simple $\mathcal{A}_{[r]}$-modules, which we denote $L(0), \cdots, L(p-1)$. By (2.1), the simple $\mathcal{A}_{[r]}$-modules are therefore tensor products $L(i) \otimes_k M_j$, with the simple modules M_j of the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{G}_1$. Direct computation shows that $\mathcal{G}' \cong \mathbb{G}_{a(1)} \rtimes \mu_{(p^{r-1})}$ is trigonalizable, so that $\mathcal{G}'$ has p^{r-1} simple modules, each having dimension 1. Using Lemma 2.2(1) one can now compute the Ext-quiver $Q_{\mathcal{B}_0(\mathcal{G})}$ of $\mathcal{B}_0(\mathcal{G})$. It has vertex set $V := \mathbb{Z}/(p^{r-1}) \times \mathbb{Z}/(2)$ and arrows

$$\alpha_{(i,j)} : (i,j) \mapsto (i+1,j) \quad ; \quad \beta_{(i,j)} : (i,j) \mapsto (i+\ell, j+1),$$

where $\ell := \frac{p^{r-1}-1}{2}$. In particular, two arrows start and end at each vertex.

Let $\Lambda = k[Q_{\mathcal{B}_0(\mathcal{G})}]/I$ be the basic algebra of $\mathcal{B}_0(\mathcal{G})$. The Nakayama algebra $k\mathcal{G}'$ has a uniserial module of length 3 with trivial top. We consider its pull-back to $k\mathcal{G}$, which turns out to be a $\mathcal{B}_0(\mathcal{G})$-module. Thus, Λ also possesses a uniserial module N of length 3, whose composition factors can be computed. Using this information, one now shows that neither $\alpha_{(1,0)}\alpha_{(0,0)}$ nor $\beta_{(1,0)}\alpha_{(0,0)}$ annihilate the Λ-module M corresponding to N. This then implies that these paths are not summands of relations of the bound quiver algebra Λ, and (V.3.2) shows that $\mathcal{B}_0(\mathcal{G})$ is wild. □

Note that the group $\mathcal{G} = \mathrm{SL}(2)_1 T_r$ satisfies the technical conditions given in Section 2. In particular, $\mathcal{G}/\mathcal{G}_1 \cong \mu_{(p^{r-1})}$ is diagonalizable and the Ext-quiver of $k\mathcal{G}$ is closely related to the McKay quivers of the linearly reductive subgroup $\mathcal{G}/\mathcal{G}_1 \subseteq \mathrm{SL}(2)$. This information suffices to compute the quiver and the relations of $\mathcal{B}_0(\mathcal{G})$ (cf. [**49**]). In [**52**] these results were extended to arbitrary infinitesimal groups by a detailed analysis of the solvable radical $\mathcal{R}(\mathcal{G})$ of $\mathcal{G}$ and techniques of Galois extensions (see [**7**]). Let me quote a result that is easy to state:

Theorem 5.6 ([**52**]). *Let $\mathcal{G}$ be an infinitesimal group of odd characteristic. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is tame.*

(2) *$\mathcal{B}_0(\mathcal{G}_1)$ is tame and $\mathcal{G}/\mathcal{G}_1$ is diagonalizable.*

In either case, we have $\mathcal{B}_0(\mathcal{G}_1) = \mathcal{B}_0(\mathcal{G})^{X(\mathcal{G}/\mathcal{G}_1)}$. □

The invariants in the last part are given relative to the standard action of $X(\mathcal{G})$ on $k\mathcal{G}$, which associates to $\gamma \in X(\mathcal{G})$ the convolution

$$\mathrm{id}_{k\mathcal{G}} * \gamma : k\mathcal{G} \longrightarrow k\mathcal{G} \ ; \ \ h \mapsto \sum_{(h)} h_{(1)}\gamma(h_{(2)}).$$

Moreover, $X(\mathcal{G}/\mathcal{G}_1)$ is identified with those characters that vanish on the ideal of $k\mathcal{G}$ that is generated by $k\mathcal{G}_1^\dagger$.

The above result implies in particular, that tameness is preserved under passage to Frobenius kernels, thereby providing an important remedy for the failure of descent that we have observed earlier. In [**52**], the infinitesimal groups with tame principal blocks are classified modulo their multiplicative centers and the quiver and relations for the blocks are given. It turns out that all blocks are special biserial. More precisely, the basic algebras are generalizations of trivial extensions $T(\tilde{A}_{2p^{r-1}-1})$ of the radical square zero tame hereditary algebras of type $\tilde{A}_{2p^{r-1}-1}$ (see (5.4) for an example of the nature of these generalizations). Here $r = \mathrm{ht}(\mathcal{G}/\mathcal{M}(\mathcal{G}))$ is the height of the factor group $\mathcal{G}/\mathcal{M}(\mathcal{G})$ of $\mathcal{G}$ by its multiplicative center $\mathcal{M}(\mathcal{G})$. The effect of $\mathcal{M}(\mathcal{G})$ on arbitrary blocks of $k\mathcal{G}$ is investigated in [**53**].

5.3. Finite Group Schemes. Let $\mathcal{G}$ be an infinitesimal group. We have seen in (5.6) that the tameness of $\mathcal{B}_0(\mathcal{G})$ implies the diagonalizability of $\mathcal{G}/\mathcal{G}_1$ along with $\mathcal{G}_1$ being closely related to the first Frobenius kernel $\mathrm{SL}(2)_1$. The Galois correspondence established there heavily relies on the character group $X(\mathcal{G}/\mathcal{G}_1)$, which fully describes the quotient $\mathcal{G}/\mathcal{G}_1$. This will only work for diagonalizable group schemes, so that Galois covering techniques are not suitable for arbitrary finite group schemes.

We thus change our point of view and note that the quivers occurring for infinitesimal groups can be interpreted as McKay quivers. For this approach to be successful in the general case, we need a good supply of linearly reductive groups. Here is the relevant generalization of (5.6), whose proof relies on the identification of the quivers of minimal counter-examples:

PROPOSITION 5.7 ([**41**]). *Let $\mathcal{G}$ be a finite group scheme with tame principal block. Then the following statements hold:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G}_1)$ is tame.*
(2) *The group $\mathcal{G}/\mathcal{G}_1$ is linearly reductive.*

In particular, the order of the finite group $\mathcal{G}(k)$ is not divisible by p. □

To ease on the technical aspects we shall discuss our main result in a simplified context. We thus assume that our group $\mathcal{G}$ and its Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ satisfy the following conditions

- $\mathcal{G}^0 = \mathcal{G}_1$, and
- $C(\mathfrak{g}) = (0)$.

Setting $G := \mathcal{G}(k)$, the first condition implies

$$k\mathcal{G} \cong U_0(\mathfrak{g})G,$$

while the second condition says that the center of $\mathcal{G}_1$ is trivial.

Now assume $\mathcal{B}_0(\mathcal{G})$ to be tame. Thanks to (5.7), the principal block of $U_0(\mathfrak{g})$ is tame. Since $C(\mathfrak{g}) = (0)$, Proposition 5.2 yields $\mathfrak{g} \cong \mathfrak{sl}(2)$. Consequently,

$$k\mathcal{G} \cong U_0(\mathfrak{sl}(2))G \ \text{ and } \ p \nmid \mathrm{ord}(G).$$

Let $N \trianglelefteq G$ denote the kernel of the representation $\varrho : G \longrightarrow \mathrm{Aut}_k(U_0(\mathfrak{sl}(2)))$. Since N is linearly reductive, the principal block $\mathcal{B}_0(\mathcal{G})$ is isomorphic to $\mathcal{B}_0(U_0(\mathfrak{sl}(2))(G/N))$. We thus simplify further by assuming $N = \{1\}$.

The group G acts on $U_0(\mathfrak{sl}(2))$ via automorphisms of Hopf algebras. As a result, ϱ defines an embedding

$$G \hookrightarrow \mathrm{Aut}_k(U_0(\mathfrak{sl}(2))) \cong \mathrm{Aut}_p(\mathfrak{sl}(2)) \cong \mathrm{PSL}(2)(k).$$

Passage to the double cover provides a linearly reductive group $\tilde{G} \subseteq \mathrm{SL}(2)(k)$ such that

$$\mathcal{B}_0(U_0(\mathfrak{sl}(2))G) \cong \mathcal{B}_0(U_0(\mathfrak{sl}(2))\tilde{G}).$$

Thus, we may assume that $G \subseteq \mathrm{SL}(2)(k)$ is a binary polyhedral group. These groups were classified by Klein around 1884 [**82**], and we have already noted in (4.2) that they are uniquely determined by their McKay graphs relative to the two-dimensional standard representation $L(1) = k^2$. Let us recall the "classical" list of groups up to conjugacy in $\mathrm{SL}(2)(k)$:

G	$\Theta_{L(1)}(G)$
$\mathbb{Z}/(n)$	$\tilde{A}_{n-1}$
Q_n	$\tilde{D}_{n+2}$
$\hat{T}$	$\tilde{E}_6$
$\hat{O}$	$\tilde{E}_7$
$\hat{I}$	$\tilde{E}_8$.

Here, each quiver is obtained from the given graph by replacing a bond $\bullet$ — $\bullet$ is by a pair of arrows $\bullet \leftrightarrows \bullet$. Thus, modulo our simplifications, we know the groups that can occur, i.e., we understand the Hopf algebra structure. Moreover, the affine quivers describing the tame hereditary algebras also appear. It remains to relate the Ext-quiver of $B_0(\mathcal{G})$ to the McKay quivers of G.

Let S, T be simple $\mathcal{G}$-modules. Then G acts on $\mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(S,T)$, and since G is linearly reductive, we have

$$\mathrm{Ext}^1_{\mathcal{G}}(S,T) \cong \mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(S,T)^G.$$

Every simple $U_0(\mathfrak{sl}(2))$-module is the restriction of a simple $\mathrm{SL}(2)(k)$-module. Consequently, the same holds for the canonical restriction map

$$\mathrm{Res} : \mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,U_0(\mathfrak{sl}(2)).$$

Let $L(0), \ldots, L(p-1)$ be the corresponding simple $\mathcal{G}$-modules, $M_1, \ldots, M_n$ be the simple kG-modules, which we view as simple $k\mathcal{G}$-modules. According to (2.1), $(L(i) \otimes_k M_j)_{ij}$ is a complete set of representatives for the simple $\mathcal{G}$-modules, and (2.2(2)) implies

$$\mathrm{Ext}^1_{\mathcal{G}}(L(i) \otimes_k M_r, L(j) \otimes_k M_s) \cong \mathrm{Hom}_G(M_r, \mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(L(i), L(j)) \otimes_k M_s).$$

Accordingly, the Ext-groups count the multiplicity of M_r in $\mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(L(i), L(j)) \otimes_k M_s$. By virtue of [**96**], the $\mathrm{SL}(2)(k)$-module $\mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(L(i), L(j))$ has the following structure

$$\mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(L(i), L(j)) \cong \begin{cases} L(1)^{[1]} & i + j = p - 2 \\ (0) & \text{else}, \end{cases}$$

where $L(1)^{[1]}$ is the Frobenius twist of the standard module. There results a close relationship between the Ext-quiver $Q_{k\mathcal{G}}$ and the McKay quiver of $\Theta_{L(1)}(G)$ of G. In this fashion, one obtains complete information on the block structure of $k\mathcal{G}$:

THEOREM 5.8 ([**41**]). *Let $\mathcal{G}$ be a finite algebraic group such that $\mathcal{B}_0(\mathcal{G})$ tame. Then the following statements hold:*

(1) *There exists a finite, linearly reductive group scheme $\tilde{\mathcal{G}} \subseteq \mathrm{SL}(2)$ such that the* Ext*-quiver of $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the McKay quiver $\Theta_{L(1)}(\tilde{\mathcal{G}})$.*

(2) *The block $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to a generalization of $T(Q_{B_0(\mathcal{G})})$.* □

The Morita equivalence given in (2) follows from the structure of the principal indecomposable $\mathcal{G}$-modules, provided in (2.3). Recall that $T(Q_{\mathcal{B}_0(\mathcal{G})})$ is the trivial extension of the radical square zero hereditary algebra of type $Q_{\mathcal{B}_0(\mathcal{G})}$. The generalization involves the "lengthening" of commutativity relations analogous to the procedure delineated in (5.4).

Example. Let us consider the case, where $G = Q_n$ is the generalized quaternion group of order $4n$ with $p \nmid n$. Thus, $\tilde{\mathcal{G}} = N_{(n)}$ is the reduced group with $\tilde{\mathcal{G}}(k) = Q_n$, and $\mathcal{G} = \mathrm{SL}(2)_1 \rtimes N_{(n)}$. Theorem 5.8 provides a Morita equivalence

$$\mathcal{B}_0(\mathcal{G}) \sim_M T(k[\tilde{D}_{n+2}]).$$

In particular, $\mathcal{B}_0(\mathcal{G})$ has $n+3$ simple modules, whose dimensions can be read off from the subadditive function d_C defined in (V.5.4). Since Q_n has 4 one-dimensional simple modules, located at the ends of $\tilde{D}_{n+2}$, and $n-1$ two-dimensional simple modules, (2.1,2.3) yield the dimensions of the simple and principal indecomposable $\mathcal{B}(\mathcal{G})$-modules. For $n = 3$, $\mathcal{B}_0(\mathcal{G})$ has two 1-dimensional simple modules, two $(p-1)$-dimensional simple modules, one 2-dimensional simple module and one $(2p-2)$-dimensional simple module. By the same token, the block has two principal indecomposable modules of dimension $4p$ and four principal indecomposables of dimension $2p$. Consequently, $\dim_k \mathcal{B}_0(\mathcal{G}) = 8p^2$.

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