AUSLANDER-REITEN THEORY FOR MODULES OF FINITE COMPLEXITY OVER SELFINJECTIVE ALGEBRAS.

OTTO KERNER AND DAN ZACHARIA

Abstract. In this paper we describe the shapes of the stable components containing modules with finite complexity over a selfinjective finite dimensional algebra over an algebraically closed field. We prove that the associated orbit graph of each such component, is either a Dynkin diagram (finite or infinite), or an extended Dynkin diagram.

Peter Webb has described in [16] the Auslander-Reiten quiver of a group algebra $\Bbbk G$ of a finite group over a field. Specifically, if $\mathcal{C}_s$ is a stable component of the Auslander-Reiten quiver, then its tree class is a Dynkin diagram (finite or infinite), or an extended Dynkin diagram. The purpose of this paper is to extend Webb's theorem to a more general setting, namely to components of selfinjective algebras containing modules of finite complexity.

The complexity of a finitely generated module measures the growth of the terms in a minimal projective resolution of the module. To be more precise, let Λ be a finite dimensional algebra over a field $\Bbbk$ and let

$$P^{\bullet}: \quad \cdots \to P^2 \xrightarrow{\delta^2} P^1 \xrightarrow{\delta^1} P^0 \xrightarrow{\delta^0} M \to 0$$

be a minimal projective resolution of a finitely generated Λ-module M. The *i-th Betti number* of M, $\beta_i(M)$, is defined as the number of indecomposable summands of P^i. The *complexity of* M is defined as

$$\operatorname{cx} M = \inf\{n \in \mathbb{N} | \beta_i(M) \leq ci^{n-1} \text{ for some positive } c \in \mathbb{Q} \text{ and all } i \geq 0\}$$

If no such n exists, then we say that the complexity of M is infinite. It is well-known that, if Λ is a selfinjective algebra and $\mathcal{C}$ is a component of the Auslander-Reiten quiver of Λ, then all nonprojective modules in $\mathcal{C}$ have the same complexity [7, 2.2]. Note that over the group algebra $\Bbbk G$ of a finite group, every module has finite complexity by the Alperin-Venkov theorem.

Let Λ be a selfinjective algebra, and let τ denote the Auslander-Reiten translate, see [2]. An indecomposable non projective Λ-module M is called τ-periodic, if $\tau^m M \cong M$ for some $m > 0$. If an Auslander-Reiten component $\mathcal{C}$ contains a τ-periodic module, then its stable part $\mathcal{C}_s$, is a τ-periodic component, that is, every module in $\mathcal{C}_s$ is τ-periodic.The tree class of $\mathcal{C}_s$ is either a finite Dynkin diagram, or is of type A_∞, which means that $\mathcal{C}_s$ is a tube [9]. The complexity of nonprojective

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modules in periodic components clearly is 1. Therefore, for our classification purpose, it is enough to study stable components, which are not τ-periodic. The main result of the paper is the following:

Main Theorem. *Let $\mathcal{C}$ be a non τ-periodic component of the Auslander-Reiten quiver of a selfinjective algebra Λ over an algebraically closed field, containing a module of finite complexity. Then the stable component $\mathcal{C}_s$ is of type $\mathbb{Z}\Delta$, where Δ is either an extended Dynkin diagram of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_i$ $(i = 6, 7, 8)$, or of the type $\mathbb{Z}A_\infty, \mathbb{Z}A_\infty^\infty$ or $\mathbb{Z}D_\infty$. If the component is a regular component, then only the infinite Dynkin trees can occur.*

The paper is organized as follows. In the first section, we study stable components of type $\mathbb{Z}T$, where T is a finite graph. We prove that if $\mathcal{C}_s$ is a non τ-periodic stable component of the Auslander-Reiten quiver, whose orbit graph is a finite tree of wild type, then $\mathcal{C}_s$ must contain only modules of infinite complexity. This is done by analyzing the behavior of the Coxeter transformation on wild trees.

In the second section, we prove the main theorem. We do this by working with a class of very well-behaved modules called Ω-perfect modules, introduced in [7, 8]. Specifically, we obtain explicit descriptions of the tree class of a component, by analyzing Auslander-Reiten sequences ending at certain Ω-perfect modules belonging to that component. This is achieved by using results from [8] describing the types of Auslander-Reiten sequences ending at Ω-perfect modules of finite complexity.

We should also mention Zhang's paper [17] in this introduction. Using her results, the classification would follow immediately if one would also assume that finite growth for minimal projective resolutions is equivalent to the finite growth for minimal injective resolution as well. This is the case in the group algebras case, but does not always happen, not even in the local commutative selfinjective case, see [10] for instance.

Throughout this paper Λ will denote a selfinjective algebra over an algebraically closed field . If $P^\bullet = (P^i, \delta^i)$ is a minimal projective resolution of the module M, then $\Omega^i M$, denotes the i-th syzygy of M; that is, $\mathrm{Im}\delta^i = \Omega^i M$. For a finitely generated Λ-module M, we denote by $\ell(M)$ its composition length, and by $\tau M = \nu\Omega^2 M$ its Auslander-Reiten translate where ν is the Nakayama equivalence. For basic results on Auslander-Reiten sequences and quivers, on Coxeter transformations, and other representation theory concepts, we refer the reader to [2, 3, 11].

1. Finite orbit graphs and finite complexity

In this section we study the case when a stable component $\mathcal{C}_s$ containing modules of finite complexity is of the type $\mathbb{Z}T$ where T is a finite oriented tree with k vertices. We show first how one can use Coxeter matrices to compute dimensions, and hence estimate the growth of the the Betti numbers, of the modules in a "stable" part of $\mathcal{C}$ from the dimensions of the modules located on an "initial" slice $\mathcal{S}$, parallel to T. Therefore, we may write $\mathcal{S} = (M_1, \ldots, M_k)$, where the M_i are the indecomposable modules in $\mathcal{C}_s$ corresponding to the vertices of $\{m\} \times T$ in $\mathbb{Z}T$, for some integer m. We choose $\mathcal{S}$ such that neither indecomposable projective nor simple modules are predecessors of $\mathcal{S}$ in $\mathcal{C}$. Let $\mathcal{S}^{\geq 0} = \bigcup_{i \geq 0} \tau^i \mathcal{S}$. By our choice of $\mathcal{S}$, the sets $\mathcal{S}^{\geq 0}$, respectively $\Omega\mathcal{S}^{\geq 0}$ are the vertices of convex subquivers of $\mathcal{C}$, respectively $\Omega\mathcal{C}$. The dimension function d, defined by $d(M) = \dim M$ is additive on short exact sequences, therefore, its restriction $d\colon \mathcal{S}^{\geq 0} \to \mathbb{Z}$ is additive on Auslander-Reiten sequences. This means that if $0 \to \tau M \to \oplus E_i \to M \to 0$ is an Auslander-Reiten sequence with $M \in \mathcal{S}^{\geq 0}$, then we have

$$d(M) + d(\tau M) = \sum_i d(E_i)$$

Denoting by Φ the Coxeter matrix corresponding to the directed tree T, we have the following formula ([16], Proposition 2.3.):

$$\Phi(d(M_1), \ldots, d(M_k))^t = (d(\tau M_1), \ldots, d(\tau M_k))^t$$

where $(\cdots)^t$ denotes the matrix transpose.

For a finite dimensional module M we have $d(M)/d(\Lambda) \leq \beta_0(M) \leq d(M)$ and, since Λ is selfinjective

$$d(\tau^m M)/d(\Lambda) \leq \beta_{2m}(M) \leq d(\tau^m M),$$

for all $m \geq 0$. For $r \geq 0$ we hence get $\beta_{r+1}(M) \leq d(\Omega^{r+1}M) < d(P^r) \leq d(\Lambda)\beta_r(M)$, Therefore, we always have the double inequality

$$\beta_{r+2}(M)/d(\Lambda) < \beta_{r+1}(M) < \beta_r(M)d(\Lambda).$$

We start by showing that underlying graphs of wild trees cannot occur as orbit graphs of stable components having modules of finite complexity.

We need first to recall a few facts about properties of Coxeter transformations for wild trees. By [1], (see also [14]), the Coxeter transformation of a wild tree, has a dominant eigenvalue $\rho > 1$ whose geometric multiplicity equals 1, and all the eigenvalues with magnitude different from 1, are positive real numbers. It is also known that in this case, the Jordan canonical form of Φ consists entirely of Jordan blocks of size 1, with the possible exception of some blocks of size 2, that may appear if 1 is also an eigenvalue of Φ ([15]).

The case when $\lambda = 1$ is an eigenvalue of Φ, will require some discussion in the proof to come, so assume that $\lambda = 1$ is an eigenvalue of Φ, and let

$$W = \mathrm{Ker}(\Phi - I)^2$$

be the corresponding generalized eigenspace. Then W has a decomposition

$$W = W_1 \oplus W_2,$$

where $W_1 = \mathrm{Ker}(\Phi - I)$ is the eigenspace of Φ, corresponding to the eigenvalue 1, and $(\Phi - I)(W_2) \subset W_1$. Thus every element $\mathbf{w} \in W$ can be written in the form $\mathbf{w} = \mathbf{z} + \mathbf{z}'$ with $\mathbf{z} \in W_1$, and $\mathbf{z}' \in W_2$. Set $(\Phi - I)\mathbf{z}' = \mathbf{u} \in W_1$. Then we have for each $m \geq 1$ that $\Phi^m \mathbf{w} = \mathbf{w} + m\mathbf{u}$, where $\mathbf{u} \neq 0$ if and only if $\mathbf{z}' \neq 0$, or equivalently, if and only if $\mathbf{w}$ is not an eigenvector of Φ. We may also consider the Coxeter transformation as a linear automorphism of $\mathbb{R}^k$. Denote again by W_1 the real eigenspace of Φ corresponding to the eigenvalue 1. Let $\bar{T}$ be the underlying graph of the quiver T and let $q = q_{\bar{T}} \colon \mathbb{R}^k \to \mathbb{R}$ be the Tits form of $\bar{T}$, again considered as a real quadratic form:

$$q_{\bar{T}}(r_1, \ldots, r_k) = \sum_i r_i^2 - \sum_{i<j} a(i,j) r_i r_j,$$

where $a(i,j)$ denotes the number of edges between the vertices i and j of the graph $\bar{T}$. If rad q denotes the radical of the real Tits form, then it is well known and easy to check, the rad $q = W_1$. Moreover it is well known that rad q contains a nonzero positive vector $\mathbf{x} = (x_1, \ldots, x_k)$ only if $q = q_{\bar{T}}$ is positive semidefinite, that is if $\bar{T}$

is an extended Dynkin diagram. This normally is shown for integral vectors, but the same arguments hold for real vectors.

Proposition 1.1. *Let Λ be a selfinjective algebra and let $\mathcal{C}$ be a non τ-periodic component of the Auslander-Reiten quiver of Λ such that the orbit graph of its stable part $\mathcal{C}_s$ is a finite tree, but neither a Dynkin diagram, nor an extended Dynkin diagram. Then every module in $\mathcal{C}_s$ has infinite complexity.*

Proof. Let $\mathcal{S} = (M_1, \ldots, M_k)$ be a slice in $\mathcal{C}_s$, where we assume that none of the modules M_i has a predecessor in $\mathcal{C}$, which is indecomposable projective or simple. Let $\mathbf{x} = (d(M_1), \ldots, d(M_k))^t$ considered as a vector in $\mathbb{C}^k$. Using the remarks at the beginning of this section, we may write for each $m > 0$

$$\Phi^m \mathbf{x} = (d(\tau^m M_1), \ldots, d(\tau^m M_k))^t$$

and all the entries of $\Phi^m \mathbf{x}$ are positive. Since the Coxeter polynomial is palindromic, λ is an eigenvalue of Φ if and only if λ^{-1} is an eigenvalue of Φ. Using this fact, write all the distinct eigenvalues of the Coxeter transformation in an order of decreasing magnitude:

$$\rho = \lambda_0 > \lambda_1 > \cdots > \lambda_r > 1 = |\lambda_{r+1}| = \cdots = |\lambda_{r+s}| > \lambda_{r+s+1} > \cdots > \lambda_{2r+2+s}$$

and decompose $\mathbb{C}^k$ into a direct sum of generalized eigenspaces corresponding to these eigenvalues:

$$\mathbb{C}^k = V_{\lambda_0} \oplus V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_{2r+2+s}}.$$

Moreover, by ([15]) for each $\lambda_i \neq 1$, the spaces V_{λ_i} consist entirely of eigenvectors of Φ corresponding to λ_i. If 1 is an eigenvalue, say $\lambda_{r+1} = 1$, then $V_{r+1} = \mathrm{Ker}(\Phi - I)^2$, and we get $\Phi^m(\mathbf{x}_{r+1}) = \mathbf{x}_{r+1} + m\mathbf{u}_{r+1}$, where $\mathbf{u}_{r+1}$ is nonzero if and only if $\mathbf{x}_{r+1} \notin \mathrm{Ker}(\Phi - I)$. Write $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_{2r+2+s},$$

where for each $i \geq 0$, $\mathbf{x}_i \in V_{\lambda_i}$. We look now at the growth of the dimensions of the modules $\tau^m M_i$ by analyzing positive powers of Φ applied to the vector $\mathbf{x}$. For each $m > 0$ we have

$$\begin{aligned}(a) \qquad \Phi^m \mathbf{x} &= \Phi^m \mathbf{x}_0 + \Phi^m \mathbf{x}_1 + \cdots + \Phi^m \mathbf{x}_{2r+2+s} \\ &= \lambda_0^m \mathbf{x}_0 + \lambda_1^m \mathbf{x}_1 + \cdots + \lambda_{2r+2+s}^m \mathbf{x}_{2r+2+s}\end{aligned}$$

if Φ is diagonalizable, and

$$\begin{aligned}(b) \qquad \Phi^m \mathbf{x} &= \Phi^m \mathbf{x}_0 + \Phi^m \mathbf{x}_1 + \cdots + \Phi^m \mathbf{x}_{2r+2+s} \\ &= \lambda_0^m \mathbf{x}_0 + \lambda_1^m \mathbf{x}_1 + \cdots + \lambda_{2r+2+s}^m \mathbf{x}_{2r+2+s} + m\mathbf{u}_{r+1}\end{aligned}$$

if V_{r+1} is not an eigenspace, that is, Φ is not diagonalizable.

Let n be smallest such that the component $\mathbf{x_n}$ of $\mathbf{x}$ is non zero.

Assume first that $0 \leq n \leq r$, so $\lambda_n > 1$. Since all the entries of $\Phi^m \mathbf{x}$ are positive and $\mathbf{x}_n \neq 0$, we may assume without loss of generality that the first entry of $\Phi^m \mathbf{x_n}$ is nonzero, and that we have:

$$\begin{aligned} d(\tau^m M_1) &= \lambda_n^m \mathbf{x_n^{(1)}} + \lambda_{n+1}^m \mathbf{x}_{n+1}^{(1)} + \cdots + \lambda_{2r+2+s}^m \mathbf{x}_{2r+2+s}^{(1)} \\ &= \lambda_n^m (\mathbf{x_n^{(1)}} + \frac{\lambda_{n+1}^m}{\lambda_n^m} \mathbf{x}_{n+1}^{(1)} + \cdots + \frac{\lambda_{2r+2+s}^m}{\lambda_n^m} \mathbf{x}_{2r+2+s}^{(1)}) \end{aligned}$$

in case (a) and

$$\begin{aligned}d(\tau^m M_1) &= \lambda_n^m \mathbf{x}_\mathbf{n}^{(\mathbf{1})} + \cdots + \lambda_{2r+2+s}^m \mathbf{x}_{2r+2+s}^{(1)} \\ &= \lambda_n^m (\mathbf{x}_\mathbf{n}^{(\mathbf{1})} + \cdots + \frac{1}{\lambda_n^m}\mathbf{x}_{r+1}^{(1)} + \frac{m}{\lambda_n^m}\mathbf{u}_{r+1}^{(1)} + \cdots + \frac{\lambda_{2r+2+s}^m}{\lambda_n^m}\mathbf{x}_{2r+2+s}^{(1)})\end{aligned}$$

in case (b). Since $\lambda_n > 1$, it follows that the dimension of $\tau^m M_1$ and hence the Betti numbers $\beta_{2m}(M_1)$ grow exponentially yielding a contradiction.

If $n > r+s$, then we obtain that for each $1 \leq i \leq k$,

$$d(\tau^m M_i) = \lambda_n^m \mathbf{x}_\mathbf{n}^{(\mathbf{i})} + \lambda_{n+1}^m \mathbf{x}_{n+1}^{(i)} + \cdots + \lambda_{2r+2+s}^m \mathbf{x}_{2r+2+s}^{(i)}$$

and the eigenvalues $\lambda_n, \lambda_{n+1}, \ldots, \lambda_{2r+s+2}$ are all less than 1, hence

$$\lim_{m\to\infty} \beta_{2m}(M_i) = \lim_{m\to\infty} d(\tau^m M_i) = 0$$

for each module M_i on the slice $\mathcal{S}$, yielding also a contradiction, since none of the modules on the slice is projective.

Assume now that $r+1 \leq n \leq r+s$, so that we have $|\lambda_n| = 1$. In case (a) we have

$$\begin{aligned}\Phi^m \mathbf{x} &= \Phi^m \mathbf{x}_n + \cdots + \Phi^m \mathbf{x}_{r+s} + \cdots + \Phi^m \mathbf{x}_{r+2+s} \\ &= \lambda_n^m \mathbf{x}_n + \cdots + \lambda_{r+s}^m \mathbf{x}_{r+s} + \cdots + \cdots + \lambda_{2r+2+s}^m \mathbf{x}_{2r+2+s}\end{aligned}$$

and the right hand side is bounded in magnitude, hence the even, and consequently all the Betti numbers are also bounded. This yields that the complexity of all M_i is 1, but $\mathcal{C}_s$ is neither a stable tube, nor a $\mathbb{Z}A_\infty$ component, nor its orbit graph is of Dynkin type. So we have a contradiction, see [7].

In case (b) and $n = r+1$ we get

$$\Phi^m(\mathbf{x}) = \mathbf{x}_{r+1} + m\mathbf{u}_{r+1} + \cdots + \lambda_{2r+2+s}^m \mathbf{x}_{2r+2+s}$$

If $\mathbf{u}_{r+1}$ is zero, then we are in the situation just discussed, so assume that $\mathbf{u}_{r+1} \neq 0$ The left hand side is an integral vector, and has only positive entries, for every $m > 0$, so a quick analysis shows that the entries $\mathbf{u}_{r+1}^{(i)}$ are all real and positive. This means that $\mathbf{u}_{r+1}$ is a positive eigenvector of Φ corresponding to the eigenvalue 1. This only can happen, it T is of Euclidean type, hence we get again a contradiction. The proof is now complete. □

If T is a Dynkin diagram, there are no stable components $\mathcal{C}_s$ of type $\mathbb{Z}T$. This follows directly from [4]. Indeed, if T is a Dynkin diagram, there is no positive integral vector $\mathbf{x}$, with $\Phi^m\mathbf{x}$ positive, for all $m \geq 0$, where again Φ denotes the Coxeter transformation. The "tame situation" was studied in Webb's paper, and his proof works in the more general context too with a small adaptation. We present the result and its proof for completeness.

Proposition 1.2. *Let Λ be a selfinjective algebra and let $\mathcal{C}$ be a component of the Auslander-Reiten quiver of Λ. Assume that the stable part of $\mathcal{C}$ is of type $\mathbb{Z}\Delta$ where Δ is an extended Dynkin diagram. Then $\mathcal{C} \neq \mathcal{C}_s$, that is, $\mathcal{C}$ is not a regular component. Moreover, every non projective module in such a component has complexity* 2.

Proof. Assume that the component is a regular component, and assume that Δ is a an extended Dynkin diagram with k vertices. Let $\mathcal{S} = (M_1, \ldots, M_k)$ be a slice in $\mathcal{C}$. Consider again $d\colon \mathcal{C} \to .\mathbb{Z}$, given by $d(M) = \dim M$. By the remarks at the beginning of this section we have that

$$\Phi^n(d(M_1), \dots, d(M_k))^t = (d(\tau^n M_1), \dots, d(\tau^n M_k))^t$$

for all $m \in \mathbb{Z}$, where Φ is the appropriate Coxeter transformation for the slice $\mathcal{S}$. Setting $\mathbf{x} = (\dim M_1, \dots, \dim M_k)^t$, we have that $\Phi^m(\mathbf{x}) = \mathbf{x} + \alpha\mathbf{n}$ for a positive integer m, and some $\mathbf{n} \in \mathbb{N}^k$, with $\Phi(\mathbf{n}) = \mathbf{n}$ and integral scalar α ([13]). If $\alpha = 0$, then the dimension vectors in each τ-orbit are bounded, hence each module in $\mathcal{C}$ has complexity 1. By [7], $\mathcal{C}$ must be either a stable tube, or a $\mathbb{Z}A_\infty$-component. In either case we obtain a contradiction to the number of τ-orbits being finite. Assume now that $\alpha \neq 0$. Then for a large enough integer t, we have that the vector $\Phi^{-mt}\mathbf{x} = \mathbf{x} - t\alpha\mathbf{n}$ for $\alpha > 0$, respectively $\Phi^{mt}\mathbf{x} = \mathbf{x} + t\alpha\mathbf{n}$, for negative α has negative entries, and this contradicts the fact that $\mathcal{C}$ is a regular component.

It remains to prove that the projective resolutions grow linearly for the modules in $\mathcal{C}_s$. As we have seen above, we have that $\Phi^m(\mathbf{x}) = \mathbf{x} + \alpha\mathbf{n}$ for some positive integer m, where the scalar $\alpha > 0$. Thus, for each $t > 0$ we have

$$\Phi^{mt}\mathbf{x} = \mathbf{x} + t\alpha\mathbf{n}$$

and this shows that the Betti numbers of the modules on the slice $\mathcal{S}$ increase linearly. □

2. Proof of the Theorem

This section is devoted to proving our main theorem. For the proof, we have to distinguish between two cases. First we consider the case, when every module in $\mathcal{C}$ is eventually Ω-perfect, a concept introduced in [7]. Then we will consider the case when $\mathcal{C}$ contains a non projective module that is not eventually Ω-perfect. We recall the following definitions. An irreducible map $g\colon B \to C$ is called *Ω-perfect* if the induced irreducible maps $\Omega^n g\colon \Omega^n B \to \Omega^n C$ are either all monomorphisms, for every $n \geq 0$, or are all epimorphisms, for every $n \geq 0$. An indecomposable non projective module M is called *Ω-perfect*, if and only if all irreducible homomorphisms $g\colon E \to M$ are Ω-perfect. Note that if M is Ω-perfect, then automatically all irreducible maps $f\colon \tau M \to E$ are also Ω-perfect. Finally, M is called *eventually Ω-perfect*, if $\Omega^m M$ is perfect, for some $m \geq 0$. Since the Nakayama functor ν is an exact equivalence, and $\tau = \nu\Omega^2$ for a selfinjective algebra, then M being Ω-perfect implies that τM is also Ω-perfect. We also recall that if there are no simple Ω-periodic Λ-modules, then every indecomposable non projective Λ-module is eventually Ω-perfect [8].

We will prove the following:

Theorem 2.1. *Let $\mathcal{C}$ be a non τ-periodic component of the Auslander-Reiten quiver of some selfinjective algebra Λ over an algebraically closed field, containing a module of finite complexity. If every module in $\mathcal{C}$ is eventually Ω-perfect, then the stable component $\mathcal{C}_s$ is of type $\mathbb{Z}\Delta$, where Δ is either an extended Dynkin diagram of type $\tilde{A}_n, \tilde{D}_n$, $\tilde{E}_i$ $(i = 6, 7, 8)$, or of the the type $\mathbb{Z}A_\infty, \mathbb{Z}A_\infty^\infty$ or $\mathbb{Z}D_\infty$. If the component is a regular component, then only the infinite Dynkin trees can occur.*

The proof of this theorem is organized as follows: We first show that all the indecomposable middle terms in the Auslander-Reiten sequences, are pairwise non-isomorphic, or else $\mathcal{C}_s$ is of type $\mathbb{Z}\tilde{A}_1$, where $\tilde{A}_1$ denotes the Kronecker quiver.

When there are no multiple arrows in $\mathcal{C}_s$ or equivalently, if the indecomposable middle terms in all Auslander-Reiten sequences in $\mathcal{C}$ are non-isomorphic, we can apply Riedtmann's Structure Theorem, [12, 1.4] which states that there exists a directed tree T, such that $\mathcal{C}_s$ is isomorphic to $\mathbb{Z}T/G$, where G is an admissible group of automorphisms of $\mathbb{Z}T$. We will start by considering an Auslander-Reiten sequence $0 \to \tau M \to E \to M \to 0$, such that M is Ω-perfect, without projective predecessors in $\mathcal{C}$ and such that $\alpha(M)$ is maximal in $\mathcal{C}$. We construct the tree T by "knitting" together its directed edges. The possible shapes of those Auslander-Reiten sequences are listed below. It turns out that T is either a finite tree with 3 or 4 arms, or is an infinite tree of type A_∞, A_∞^∞ or D_∞.

It is easy to see that $\langle\tau\rangle$ is a normal subgroup of $\mathrm{Aut}(\mathbb{Z}T)$, where τ corresponds to the Auslander-Reiten translation. Since $\mathcal{C}_s$ contains no τ-periodic modules,

$$G \cap \langle\tau\rangle = 1.$$

The cokernel of the embedding $\langle\tau\rangle \hookrightarrow \mathrm{Aut}(\mathbb{Z}T)$ is $p\colon \mathrm{Aut}(\mathbb{Z}T) \twoheadrightarrow \mathrm{Aut}(\bar{T})$ where $\bar{T}$ denotes the underlying tree, [12, 3].

If T is a finite tree, or A_∞ or D_∞, then $\mathrm{Aut}(\bar{T})$ is finite, possibly trivial. If $a \in G$ and $p(a)$ has finite order m, then $a^m \in \ker p = \langle\tau\rangle$. Hence $a^m = \tau^i$ for some integer i, but this implies that $a^m = 1$, hence a has finite order. By [12, 4.1,Corollary] the identity e is the only element of finite order in G, hence $a = e$, which means $G = \{e\}$. This shows that in all the cases where $T \neq A_\infty^\infty$, the stable component $\mathcal{C}_s = \mathbb{Z}T$. Observe also, that, by the results of the previous section, the tree T can be neither wild nor of finite Dynkin type.

If $T = A_\infty^\infty$, then $\mathrm{Aut}(\bar{T})$ contains many elements of infinite order. Hence G can be nontrivial in this case.

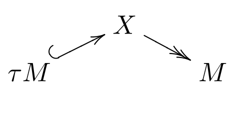
Throughout this section $\mathcal{C}_s$ will be the stable part of some infinite, non τ-periodic, Auslander-Reiten component $\mathcal{C}$. We will also assume that $\mathcal{C}$ contains a module of finite complexity, and also, unless explicitly stated, that every module in $\mathcal{C}_s$ is eventually Ω-perfect.

For an indecomposable nonprojective module M, the number of indecomposable nonprojective direct summands in the middle term of the Auslander-Reiten sequence ending in M, is denoted by $\alpha(M)$. Denote by

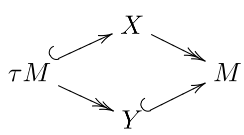
$$\alpha(\mathcal{C}) = \sup\{\alpha(M) | M \text{ indecomposable } \Omega\text{-perfect in } \mathcal{C}\}$$

By [8, section 3], $\alpha(\mathcal{C}) \leq 4$, and the possible shapes of the Auslander-Reiten sequences ending at an Ω-perfect module M belong to the following list:

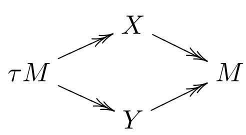
Proposition 2.2. (a) *If $\alpha(M) = 1$ or $\alpha(M) = 2$, then the Auslander-Reiten sequences ending in M have the following shapes:*



(1)

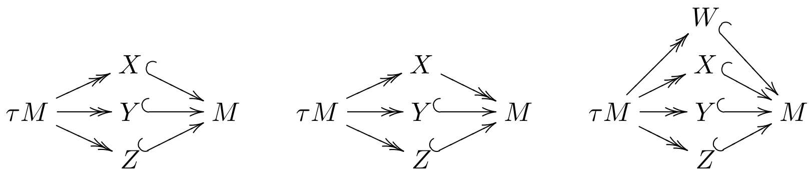


(2a)



(2b)

(b) *If $\alpha(M) = 3$ and $\alpha(M) = 4$, the possible shapes are:*



(3a) **(3b)** **(4)**

We have the following:

Lemma 2.3. *Let* $0 \to \tau M \xrightarrow{[\,f_1, f_2\,]^T} U_1 \oplus V \xrightarrow{[\,g_1, g_2\,]} M \to 0$ *be the Auslander-Reiten sequence ending at M, and assume that the module U_1 is indecomposable.*

(a). *If g_1 is a monomorphism, and f_1 is an epimorphism, then either $\alpha(U_1) = 1$, or $\alpha(U_1) = 2$ and the Auslander-Reiten sequence ending at U_1 is of type* **(2a)**.

(b). *If f_1 and g_1 are both epimorphisms, and U_1 is Ω-perfect, then we either have that $\alpha(U_1) = 2$ and the Auslander-Reiten sequence ending at U_1 is of type* **(2b)**, *or $\alpha(U_1) = 3$, and the Auslander-Reiten sequence ending at U_1 is of type* **(3b)**.

Proof. (a). Since the module M is perfect, the induced irreducible map $\tau U_1 \to \tau M$ is again a monomorphism. Since g_1 is an irreducible monomorphism, the U_1 is also Ω-perfect, by [8], hence the Auslander-Reiten sequence ending at U_1 has the shape **(1)** or **(2a)** from our list, and the first part follows.

(b) Since M is perfect, the map $\tau g_1 \colon \tau U_1 \to \tau M$ is again an epimorphism, and since U_1 is also Ω-perfect, the statement follows from an examination of our list. □

A repeated application of part (a) of this lemma yields immediately the following consequence:

Corollary 2.4. *If the Auslander-Reiten sequence ending at the Ω-perfect module M has the form*

$$0 \to \tau M \xrightarrow{[\,f_1, f_2\,]^T} U_1 \oplus V \xrightarrow{[\,g_1, g_2\,]} M \to 0$$

with U_1 indecomposable, and where f_1 is an epimorphism and g_1 is a monomorphism, then there exists a finite chain of irreducible monomorphisms

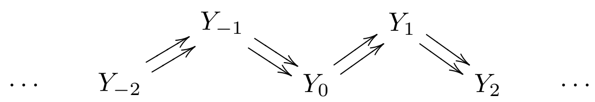
$$U_r \hookrightarrow U_{r-1} \hookrightarrow \cdots \hookrightarrow U_2 \hookrightarrow U_1$$

with $r \geq 1$, $\alpha(U_r) = 1$, and $\alpha(U_i) = 2$ for all $1 \leq i < r$. □

We show now that the possibility of having multiple arrows in the Auslander-Reiten components containing modules of finite complexity is rather slim. We have the following:

Proposition 2.5. *Let $M \in \mathcal{C}_s$ be a module of finite complexity, and let $0 \to \tau M \to \bigoplus_{1 \leq i \leq k} X_i \to M \to 0$ be an Auslander-Reiten sequence ending at M. Then, for all $i \neq j$, $X_i \not\cong X_j$, unless $k = 2$ and the component $\mathcal{C}_s$ is of type $\mathbb{Z}\tilde{A}_1$.*

Proof. We prove the proposition using a case by case analysis on the allowable possibilities from our list. There is nothing to prove if $k = 1$. Assume that $k = 2$. The result is clear if we are in the situation **(2a)** from our list since the two summands in the middle have distinct lengths, so assume that we are in situation **(2b)**. This means that there exists an Auslander-Reiten sequence of the form $0 \to \tau M \to X \oplus X \to M \to 0$. Without loss of generality, we may assume that X is also Ω-perfect, and, using the list, we have an Auslander-Reiten sequence $0 \to \tau X \to \tau M \oplus \tau M \to X \to 0$, and, continuing in this way we obtain a "flat" component



This means that the entire component $\mathcal{C}_s$ is of type $\mathbb{Z}\tilde{A}_1$.

Consider now the situation when we have an Auslander-Reiten sequence of the type $0 \to \tau M \to X \oplus X \oplus X \to M \to 0$. This means that we must be in situation **(3a)**, and since X is Ω-perfect. by [8], $\alpha(X) \geq 3$. But each of the three irreducible maps $\tau M \to X$ is an epimorphism, and therefore the type of the Auslander-Reiten sequence ending at X cannot belong to our list, obtaining a contradiction.

Let us also consider, the possibility that the Auslander-Reiten sequence ending at M is of the form $0 \to \tau M \to X \oplus X \oplus Y \to M \to 0$ where $X \not\cong Y$. We are now either in situation **(3a)**, or **(3b)**. In either case, $\alpha(X) \geq 2$, and there are two irreducible epimorphisms $\tau M \to X$. Using our list, it follows that we must have $\alpha(X) = 2$, case **(2b)**.. But then the irreducible maps $\tau X \to \tau M$ have to be epimorphisms, whereas the irreducible maps $X \to M$ are monomorphisms, so we obtain a contradiction.

It remains to examine the situation when $\alpha(M) = 4$. We cannot have a summand X appearing with multiplicity 3 or 4, in the middle of the Auslander-Reiten sequence ending at M, since then we would obtain that $\alpha(X) \geq 3$ and three or more irreducible epimorphisms $\tau M \to X$ which would contradict the restrictions imposed by our list. The only case remaining is when the Auslander-Reiten sequence ending at M has the form $0 \to \tau M \to X \oplus X \oplus Y \oplus Z \to M \to 0$ where $Y \not\cong X \not\cong Z$, but as above, this case cannot exist either. □

The proof of the following lemma is immediate.

Lemma 2.6. *If $\alpha(\mathcal{C}) = 2$ and M is an Ω-perfect module in $\mathcal{C}$ having no simple or projective predecessors in $\mathcal{C}$. Then all the predecessors of M are Ω-perfect.* □

For the remaining part of the proof of theorem 2.1 we assume that the indecomposable middle terms of the Auslander-Reiten sequences in $\mathcal{C}$ are pairwise non-isomorphic.

First we characterize the shapes of the stable components $\mathcal{C}$ with $\alpha(\mathcal{C}) = 2$.

Lemma 2.7. *Assume that $\alpha(\mathcal{C}) = 2$ and that*

$$0 \to \tau M \xrightarrow{[f_1, f_2]^T} X_1 \oplus Y_1 \xrightarrow{[g_1, g_2]} M \to 0$$

is an Auslander-Reiten sequence ending at the Ω-perfect module $M \in \mathcal{C}$, with X_1 and Y_1 indecomposable. Then

(a) *If f_1, g_2 are monomorphisms and f_2, g_1 are epimorphisms, then the stable component $\mathcal{C}_s$ is of type $\mathbb{Z}A_\infty$.*

(b) *If each of the irreducible maps $f_1, f_2.g_1, g_2$ is an epimorphism, then $\mathcal{C}_s$ is of the type $\mathbb{Z}A_\infty^\infty$, or $\mathbb{Z}\tilde{A}_n$ for some $n > 1$.*

Proof. (a). Since f_2 is an epimorphism and g_1 is a monomorphism, we may apply lemma 2.2 and corollary 2.3 to obtain the existence of a (necessarily) finite chain of irreducible monomorphisms

$$Y_r \hookrightarrow Y_{r-1} \hookrightarrow \cdots \hookrightarrow Y_2 \hookrightarrow Y_1$$

where $\alpha(Y_r) = 1$, and $\alpha(Y_i) = 2$ for all $1 \leq i < r$. A similar analysis allows us to construct an arbitrary long chain of irreducible epimorphisms ending at M:

$$\cdots \twoheadrightarrow X_s \twoheadrightarrow X_{s-1} \twoheadrightarrow \cdots \twoheadrightarrow X_2 \twoheadrightarrow X_1 \twoheadrightarrow M$$

where for each $i \geq 1$, we have $\alpha(X_i) = 2$. Therefore, by Riedtmann, $\mathcal{C}_s$ is of type $\mathbb{Z}A_\infty$.

(b). Since $\alpha(\mathcal{C}) = 2$, the previous results show that every predecessor of M has an Auslander-Reiten sequence ending at it of type **(2b)**. If $\mathcal{C}_s$ has infinitely many τ–orbits, then $\mathcal{C}_s = \mathbb{Z}A_\infty^\infty$, so assume that $\mathcal{C}_s$ has finitely many orbits, that is $\mathcal{C}_s$ is of the form $\mathbb{Z}A_\infty^\infty/G$ where G is an admissible group of automorphisms of $\mathbb{Z}A_\infty^\infty$. Assume that, by taking sufficient powers of τ, M is isomorphic to a predecessor in $\mathbb{Z}A_\infty^\infty$. This means that there are two different vertices x and y in different τ-orbits of $\mathbb{Z}A_\infty^\infty$, a finite path from x to y, an element $g \in G$ such that $y = x^g$. and both vertices represent the module M. By taking sufficiently large powers of τ, we can assume that all the modules, corresponding to the vertices on this path are Ω-perfect. Since the Auslander-Reiten sequence ending at each predecessor of M, is of type **(2b)**, it follows that there must be a proper surjection $M \to M$. This clearly cannot happen. Therefore we may not identify M with predecessors, so we get that $\mathcal{C}_s = \mathbb{Z}A_\infty^\infty/G$ is of type $\mathbb{Z}\tilde{A}_n$ for some $n > 1$. The proof of the lemma is now complete. □

We turn our attention to the components having the property that $\alpha(\mathcal{C}) = 3$.

Lemma 2.8. *Assume that $\alpha(\mathcal{C}) = 3$ and let*

$$0 \to \tau M \xrightarrow{[f_1, f_2, f_3]^T} X \oplus Y \oplus Z \xrightarrow{[g_1, g_2, g_3]} M \to 0$$

be an Auslander-Reiten sequence with three indecomposable non projective summands in the middle.

(a) *If g_1, g_2, g_3 are monomorphisms, the $\mathcal{C}_s$ is of type $\mathbb{Z}\tilde{E}_i$, $i = 6, 7, 8$.*

(b) *If g_1 is an epimorphism, and g_2, g_3 are monomorphisms, then $\mathcal{C}_s$ is either of type $\mathbb{Z}D_\infty$, or of type $\mathbb{Z}\tilde{D}_n$ for some $n \geq 5$.*

Proof. (a) It follows from our list that, in this case, f_1, f_2 and f_3 are all epimorphisms. Therefore, by 2.4., the tree type of $\mathcal{C}_s$ is T_{rst} where T_{rst} is a rooted tree with three arms of lengths r, s and t, respectively. Therefore, $\mathcal{C}_s \cong \mathbb{Z}T_{rst}$, and, since $\mathcal{C}_s$ is not τ-periodic, the tree T_{rst} cannot be of Dynkin type. It cannot be of wild type either by our results in the previous section. Therefore, it must be an extended Dynkin diagram $\tilde{E}_i$, $i = 6, 7, 8$.

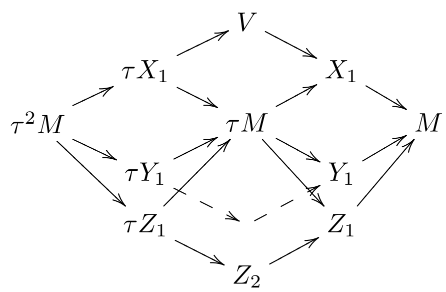
(b) We are now in the case **(3b)** of our list. Applying 2.4 to $Y_1 = Y$ and $Z_1 = Z$, we obtain chains of irreducible monomorphisms $Y_r \hookrightarrow Y_{r-1} \hookrightarrow \cdots \hookrightarrow Y_2 \hookrightarrow Y_1$, and $Z_s \hookrightarrow Z_{s-1} \hookrightarrow \cdots \hookrightarrow Z_2 \hookrightarrow Z_1$, with $\alpha(Y_r) = \alpha(Z_s) = 1$, and $\alpha(Y_i) = \alpha(Z_j) = 2$, for $1 \le i < r$ and $1 \le j < s$. Note also, that by taking enough powers of the Auslander-Reiten translate τ, we may also assume that $X_1 = X$ is Ω-perfect. The Auslander-Reiten sequence ending at X_1 has the form

$$0 \to \tau X_1 \xrightarrow{[h_1, \tau g_1]^T} V \oplus \tau M \xrightarrow{[h, f_1]} X_1 \to 0$$

where $V \neq 0$ but possibly decomposable, and both h_1 and h are epimorphisms. We show first that $r = s = 1$. Assume not, and say, $s > 1$, and let

$$0 \to \tau Z_1 \to Z_2 \oplus \tau M \to Z_1 \to 0$$

be the Auslander-Reiten sequence ending at Z_1. We have the following diagram representing the Auslander-Reiten sequences ending at M, and its immediate predecessors:



We now compute lengths along these Auslander-Reiten sequences, and if we denote by $\ell(C)$ the length of a module C, we obtain

$$\begin{aligned}
\ell(\tau M) + \ell(M) &= \ell(X_1) + \ell(Y_1) + \ell(Z_1) \\
\ell(\tau M) + \ell(\tau^2 M) &= \ell(\tau X_1) + \ell(\tau Y_1) + \ell(\tau Z_1) \\
\ell(X_1) + \ell(\tau X_1) &= \ell(V) + \ell(\tau M) \\
\ell(Z_1) + \ell(\tau Z_1) &= \ell(Z_2) + \ell(\tau M) \\
\ell(Y_1) + \ell(\tau Y_1) &\ge \ell(\tau M)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\ell(M) + \ell(\tau^2 M) + 2\ell(\tau M) &= \ell(X_1) + \ell(Y_1) + \ell(Z_1) + \ell(\tau X_1) + \ell(\tau Y_1) + \ell(\tau Z_1) \\
&\ge 3\ell(\tau M) + \ell(Z_2) + \ell(V)
\end{aligned}$$

and consequently, we get

$$\ell(M) + \ell(\tau^2 M) \ge \ell(\tau M) + \ell(Z_2) + \ell(V).$$

Since we have proper epimorphisms $V \twoheadrightarrow X_1 \twoheadrightarrow M$ and $\tau M \twoheadrightarrow X_1 \twoheadrightarrow M$, we obtain $\ell(\tau^2 M) > \ell(M) + \ell(Z_2)$. Since we have irreducible monomorphisms $Z_2 \to Z_1$ and $Z_1 \to M$, we have (see [2], chapter V, for instance) that $\ell(Z_2) \ge \dfrac{1}{(1 + D^2)^2}\ell(M)$, where $D = \dim_{\Bbbk} \Lambda$. Letting

$$c = 1 + \frac{1}{(1 + D^2)^2} > 1$$

we get as in [8] that $\ell(\tau^2(M)) \ge c\ell(M)$, hence, for all $m \ge 1$, we have

$$\ell(\tau^{2m}(M)) \ge c^m \ell(M)$$

which means that the complexity of M is infinite, contradicting our assumption. Therefore, $r = s = 1$.

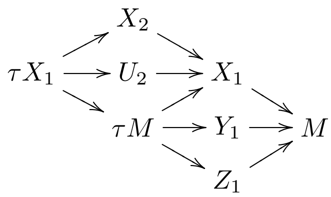
We turn now our attention to the Ω-perfect module X_1. Clearly $\alpha(X_1) > 1$.

Assume first that all the indecomposable modules Y with $\alpha(Y) = 3$ are in the τ-orbit of M. This means that $\alpha(X_1) = 2$, and the Auslander-Reiten sequence ending at X_1 is of type **(2b)**. Letting $X_2 = V$, we obtain for each $l \geq 0$, epimorphisms $\tau^l X_2 \to \tau^l X_1$. In this way we construct inductively an infinite sectional path of irreducible epimorphisms

$$\cdots \twoheadrightarrow X_s \twoheadrightarrow X_{s-1} \twoheadrightarrow \cdots \twoheadrightarrow X_2 \twoheadrightarrow X_1 \twoheadrightarrow M$$

where for each i, $\alpha(X_i) = 2$. This means that in this case $T = D_\infty$ and therefore the stable component $\mathcal{C}_s$ is of type $\mathbb{Z}D_\infty$.

Assume now that there are more τ-orbits in $\mathcal{C}_s$ containing modules Y such that $\alpha(Y) = 3$. If $\alpha(X_1) = 3$, we get the following picture:



where $0 \to \tau X_1 \xrightarrow{[h_{11}, h_{12}, \tau f_1]^T} X_2 \oplus \oplus U_2 \oplus \tau M \xrightarrow{[h, h', g_1]^T} X_1 \to 0$ is the Auslander-Reiten sequence ending at X_1. Since g_1 and τf_1 are epimorphisms, we are again in the case **(3b)** in our list. The same argument used to show that $\alpha(Y_1) = \alpha(Z_1) = 1$ can be used now to prove that $\alpha(U_2) = \alpha(X_2) = 1$, and then $\mathcal{C}_s$ is of type $\mathbb{Z}\tilde{D}_5$. If $\alpha(X_1) = 2$, then we have a sectional path of epimorphisms $X_r \twoheadrightarrow X_{r-1} \twoheadrightarrow \cdots \twoheadrightarrow X_1$, where each X_i is Ω-perfect, $\alpha(X_i) = 2$ for all $1 \leq i < r$, and $\alpha(X_r) = 3$. Thus, the Auslander-Reiten sequences ending at each X_i for $i < r$ are of type **(2b)**. Let $0 \to \tau X_r \to X_{r-1} \oplus U \oplus V \to X_r \to 0$ be the Auslander-Reiten sequence ending at X_r. Then again, the maps $V \to X_r$ and $U \to X_r$ are monomorphisms, and we also have $\alpha(U) = \alpha(V) = 1$. This means that $\mathcal{C}_s$ is of type $\mathbb{Z}\tilde{D}_n$ where $n = r + 4$, and the proof of the lemma is complete. ☐

Lemma 2.9. *Assume that $\mathcal{C}_s$ contains a module M such that $\alpha(M) = 4$. Then the component $\mathcal{C}_s$ is of type $\mathbb{Z}\tilde{D}_4$.*

Proof. Without loss of generality we may assume that the module M is Ω-perfect. Let $0 \to \tau M \to \bigoplus_{1 \leq i \leq 4} X_i \to M \to 0$ be the Auslander-Reiten sequence ending at M. We are in the case **(4)** of our list, so each map $X_i \to M$ is a monomorphism, and each $\tau M \to X_i$ is an epimorphism. By 2.4., we obtain for each $1 \leq i \leq 4$, the existence of a finite sectional path of monomorphisms $X_i^{(r_i)} \hookrightarrow \cdots \hookrightarrow X_i^{(1)} = X_i$, with $\alpha(X_i^{(r_i)}) = 1$. This means that our component is of type $\mathbb{Z}T_{r_1 r_2 r_3 r_4}$ where $T_{r_1 r_2 r_3 r_4}$ is star with four arms of lengths $r_1, \ldots, r_4$. If $r_i > 1$ for some i, then the tree is of wild type, so $\mathcal{C}_s$ cannot contain a module of finite complexity, by the results of the previous section. Therefore, we must have $T_{r_1 r_2 r_3 r_4} = \tilde{D}_4$. ☐

Theorem 2.1. follows now from the preceding lemmas.

It remains to prove that our classification result holds also in the case where a stable component $\mathcal{C}_s$ contains a module M that is not eventually Ω-perfect. It turns out that this is the easy case. Moreover, the assumption that $\mathcal{C}$ contains an indecomposable module M with finite complexity is not needed. The proof is based on the following result from [7, 2.6] and [8, 2.3]:

Lemma 2.10. *Let $\mathcal{C}_s$ be a stable component that is not τ-periodic and contains a module M that is not eventually Ω-perfect. Let $g\colon E \to \tau^m M$ be an irreducible epimorphism, such that $\Omega g\colon \Omega E \to \Omega\tau^m M$ is injective. Then the kernel S of g is simple and Ω-periodic.*

We have the following:

Theorem 2.11. *Let $\mathcal{C}_s$ be a stable component that is not τ-periodic and contains a module that is not eventually Ω-perfect. Then $\mathcal{C}_s \cong \mathbb{Z}\Delta$ where Δ is either an extended Dynkin diagram, or one of the infinite Dynkin trees A_∞^∞ or D_∞.*

Proof. If the component contains a module of complexity 1, then it must be a $\mathbb{Z}A_\infty$ by [7], so let C be an indecomposable module in $\mathcal{C}_s$ that is not eventually Ω-perfect and assume that $\operatorname{cx} C > 1$. This means that there exists an irreducible epimorphism $B \to C$ whose kernel is isomorphic to some simple Ω-periodic module S. Let $V = S \oplus \Omega S \oplus \cdots \oplus \Omega^{n-1}S$, where n is the Ω-period of S, by 2.10. Clearly $\Omega V \cong V$. But the Nakayama functor ν also has finite order on every simple module, so let m be such that $\nu^m S \cong S$. Set $W = V \oplus \nu V \oplus \cdots \oplus \nu^{m-1}V$, so we also have $\nu W \cong W$. Since the syzygy functor commutes with the Nakayama functor, we have $\tau W \cong W$. Each indecomposable direct summand of W has complexity 1, and every module in $\mathcal{C}_s$ has complexity greater than 1, so no summand of W is either in $\mathcal{C}_s$ or in $\Omega\mathcal{C}_s$. Following [5, 6], let $d_W : \mathcal{C}_s \to \mathbb{N}$ be the function

$$d_W(M) = \dim_{\Bbbk} \underline{\operatorname{Hom}}_\Lambda(W, M).$$

Since $\underline{\operatorname{Hom}}_\Lambda(S, B) \cong \operatorname{Hom}_\Lambda(S, B) \neq 0$, $d_W \neq 0$. This function satisfies the conditions of Lemma 3.2. of [6], hence it is an additive function on the stable component. Moreover, since $\tau W \cong W$, we also have

$$d_W(\tau M) = \dim_{\Bbbk} \underline{\operatorname{Hom}}_\Lambda(W, \tau M) = \dim_{\Bbbk} \underline{\operatorname{Hom}}_\Lambda(\tau W, \tau M) = d_W(M)$$

If X is indecomposable and $f\colon X \to B$ is an irreducible map, the additivity of d_W implies $2d_W(X) \geq d_W(B) > 0$, which shows that d_W a positive additive function, hence it induces a positive additive function on the orbit graph $\bar{\Delta}$ of $\mathcal{C}_s \cong \mathbb{Z}\Delta$. Hence Δ is either an extended Dynkin diagram, or one of the infinite Dynkin quivers A_∞, A_∞^∞ or D_∞ by [9]. Since $\mathcal{C}_s$ contains a module that is not eventually Ω-perfect, the case A_∞ can be excluded. Indeed, if $\mathcal{C}_s$ is of type $\mathbb{Z}A_\infty$, take an indecomposable module M in $\mathcal{C}_s$, such that no predecessor in $\mathcal{C}$ is projective or simple, and such that $\alpha(M) = 1$. Then also $\alpha(\Omega^m M) = 1$ and $\alpha(\tau^m M) = 1$, for all $m \geq 0$, hence all irreducible maps ending in $\Omega^m M$ are epimorphisms. Therefore M is Ω-perfect, and by lemma 2.6., all modules in $\mathcal{C}_s$ are eventually Ω-perfect. □

Theorem 2.11 has the following unexpected direct consequence

Corollary 2.12. *Let Δ be a finite or infinite quiver which is not of finite Dynkin type, or of extended Dynkin type, or A_∞^∞ or D_∞. If $\mathcal{C}$ is a non-periodic Auslander-Reiten component of a selfinjective algebra, such that $\mathcal{C}_s$ is of type $\mathbb{Z}\Delta$, then every nonprojective module in $\mathcal{C}$ is eventually Ω-perfect.* □

Finally, our main result follows by putting together theorems 2.1 and 2.11.

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Mathematisches Institut, Heinrich-Heine-Universität, 40225 Düsseldorf, Germany
E-mail address: `kerner@math.uni-duesseldorf.de`

Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA
E-mail address: `zacharia@syr.edu`