Transverse Laplacian for Substitution Tilings[*]

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Abstract

Pearson and Bellissard recently built a *spectral triple* — the data of Riemanian noncommutative geometry — for ultrametric Cantor sets. They derived a family of Laplace–Beltrami like operators on those sets. Motivated by the applications to specific examples, we revisit their work for the transversals of tiling spaces, which are particular self-similar Cantor sets. We use *Bratteli diagrams* to encode the self-similarity, and *Cuntz–Krieger algebras* to implement it. We show that the abscissa of convergence of the ζ-function of the spectral triple gives indications on the exponent of complexity of the tiling. We determine completely the spectrum of the Laplace–Beltrami operators, give an explicit method of calculation for their eigenvalues, compute their Weyl asymptotics, and a Seeley equivalent for their heat kernels.

Contents

1 Introduction and summary of the results

In a recent article [36], Pearson and Bellissard defined a spectral triple — the data of Riemanian noncommutative geometry (NCG) [11] — for ultrametric Cantor sets. They used a construction due to Michon [35]: any ultrametric Cantor set (C, d) can be represented isometrically as the set of infinite paths on a weighted rooted tree. The tree defines the topology, and the weights encode the distance. The spectral triple is then given in terms of combinatorial data on the tree.

With this spectral triple, they could define several objects, including a ζ-function, a measure μ, and a one-parameter family of operators on $L^2(C, \mu)$, which were interpreted as Laplace–Beltrami operators. They showed the abscissa of convergence of the ζ-function to be a fractal dimension of the Cantor set (the upper box dimension).

The goal of Pearson and Bellissard was to build a spectral triple for the transversals of tiling dynamical systems. This opened a new, geometrical approach to the theory of tilings. Until now, all the operator-algebraic machinery used to study tilings and tiling spaces was coming from noncommutative *topology*. Striking application of noncommutative topology were the study of the K-theory for tiling C^*-algebras [3, 5, 30, 31], and namely [1, 13, 15] for computations applied to substitution tilings. A follow-up of this study was the gap-labeling theorems for Schröedinger operators [4, 6, 7, 25, 42] — a problem already appearing in some of the previously cited articles. Other problems include cyclic cohomology and index theorems (and applications to the quantum Hall effect) [2, 8, 26–29, 32]... These problems were tackled using mainly *topological* techniques. The construction of a spectral triple is a proposition for bringing *geometry* into play.

In this article, we revisit the construction of Pearson and Bellissard for the transversals of some tiling spaces. For this purpose, we use the formalism of Bratteli diagrams instead of Michon trees. This approach is equivalent, and applies in general to any ultrametric Cantor set. But in some cases, the diagram conveniently encodes the self-similarity, and is very well suited to handle explicit computations. For this reason, we will focus on diagrams arising from substitution tilings, a class of tilings which is now quite well studied [1, 33, 38, 41].

Bratteli diagrams were introduced in the seventies for the classification of AF-algebras [9]. They were used by Veršik to encode measurable $\mathbb{Z}$-actions, as a tool to approach some dynamical systems by a sequence of periodic dynamical systems [43]. They were then adapted in the topological setting to encode $\mathbb{Z}$-actions on the Cantor set [13, 20, 22]. Then, Bratteli diagrams were used to represent the orbit equivalence relation arising from an action of $\mathbb{Z}^2$ [18] (and recently $\mathbb{Z}^d$ [19]) on a Cantor set. Yet, it is not well understood how the *dynamics* itself should be represented on the diagram. The case we will look at is when the Cantor set is the transversal of a tiling space, and the action is related to the translations. The idea of parametrizing tilings combinatorialy dates back to the work of Grünbaum and Shephard in the seventies, and the picture of a Bratteli diagram can be found explicitly in the book of Connes [10] for the Penrose tiling. However, it took time to generalize these ideas, and to understand the topological and dynamical underlying questions.

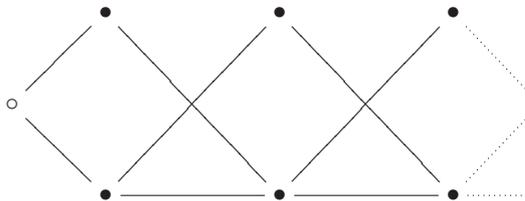Figure 1: A self-similar Bratteli diagram associated with the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (root on the left).

In the self-similar case (for example when the Cantor set is the transversal of a substitution tiling space), the diagram only depends on an adjacency (or Abelianization) matrix. There is a natural C^*-algebra associated with this matrix, called a Cuntz–Krieger algebra [12]. Its generators

implement recursion relations and therefore provide a method of computation for the eigenvalues of the Laplace–Beltrami operators.

While Bratteli diagrams are suited to facilitate computations for any self-similar Cantor set, we focus on diagrams associated with substitution tilings. Indeed, a tiling space comes with a convenient distance, which is encoded (up to Lipschitz-equivalence) in a natural way by weights on the diagram.

Results of the Paper

Let $\mathcal{B}$ be a weighted Bratteli diagram (Definition 2.2, and 2.9), and $(\partial\mathcal{B}, d_w)$ the ultrametric Cantor set of infinite paths in $\mathcal{B}$ (Proposition 2.10).

The Dixmier trace associated with the spectral triple gives a measure $\mu_{\mathrm{D}ix}$ on $\partial\mathcal{B}$. The construction of Pearson–Bellissard gives a family of Laplace–Beltrami operators Δ_s, $s \in \mathbb{R}$, on $L^2(\partial\mathcal{B}, d\mu_{\mathrm{Dix}})$, see Definition 4.12. For all s, Δ_s is a nonpositive, self-adjoint, and unbounded operator. For a path γ, we denote by $[\gamma]$ the clopen set of infinite paths with prefix γ, and by χ_γ its characteristic function. And we let $\mathrm{ext}_1(\gamma)$ denote the set of ordered pairs of edges that extend γ one generation further. The operator Δ_s was shown to have pure point spectrum in [36]. In this paper, we determine all its eigenelements explicitly.

Theorem 4.3. *The eigenspaces of Δ_s are given by the subspaces*

$$E_\gamma = \left\langle \frac{1}{\mu_{\mathrm{Dix}}[\gamma \cdot a]}\chi_{\gamma \cdot a} - \frac{1}{\mu_{\mathrm{Dix}}[\gamma \cdot b]}\chi_{\gamma \cdot b} \,:\, (a,b) \in \mathrm{ext}_1(\gamma) \right\rangle$$

for any finite path γ in $\mathcal{B}$. We have $\dim E_\gamma = n_\gamma - 1$ where n_γ is the number of edges in $\mathcal{B}$ extending γ one generation further.

The associated eigenvalues λ_γ are also calculated explicitly, see equation (4.13). An eigenvector of Δ_s is simply a weighted sum of the characteristic functions of two paths of the same lengths that agree apart from their last edge, see Figure 2 for an example.
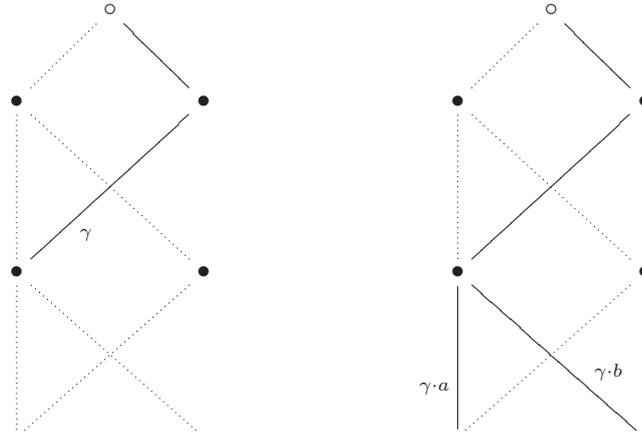
Figure 2: Example of eigenvectors of Δ_s for the Fibonacci diagram.

The transversal Ξ to a substitution tiling space (Ω, ω) of $\mathbb{R}^d$, can be described by a stationary Bratteli diagram, like the Fibonacci diagram shown in Figure 1. There is a "natural map", called the Robinson map in [30], $\psi : \partial\mathcal{B} \to \Xi$, which under some technical conditions (primitivity, recognizability, and border forcing) is a homeomorphism (Theorem 2.22). We endow Ξ with the *combinatorial metric d_Ξ*: two tilings are ε-close if they agree on a ball of radius $1/\varepsilon$ around the origin. Let A be the Abelianization matrix of the substitution, and Λ_{PF} its Perron–Frobenius eigenvalue. We denote by $w(\gamma)$ the weight of a finite path γ in $\mathcal{B}$ (Definition 2.9).

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Theorem 2.25. *If there are constants $c_+ > c_- > 0$ such that $c_- \Lambda_{PF}^{-n/d} \leq w(\gamma) \leq c_+ \Lambda_{PF}^{-n/d}$ for all paths γ of lengths n, then the homeomorphism $\psi : \partial\mathcal{B} \to \Xi$ is $(d_w$–$d_\Xi)$ bi-Lipschitz.*

In our case of substitution tiling spaces, we have a ζ-function which is given as in [36] by

$$\zeta(s) = \sum_{\gamma \in \Pi} w(\gamma)^s\,,$$

where Π is the set of finite paths in $\mathcal{B}$. It is proven in [36] that, when it exists, the abscissa of convergence s_0 of ζ is the upper box dimension of the Cantor set. For self-similar Cantor sets it always exists and is finite.

Theorem 3.7. *For a weighted Bratteli diagram associated with a substitution tiling space of dimension d, the abscissa of convergence of the ζ-function is $s_0 = d$.*

We also have an interpretation of s_0 which is not topological. We link s_0 to the exponent of the complexity function. This function p, associated with a tiling, counts the number of distinct patches: $p(n)$ is the number of patches of radius n (up to translation). We present two results.

Theorem 3.12. *For the transversal of* any *minimal aperiodic tiling space with a well-defined complexity function, the box dimension, when it exists, is given by the following limit:*

$$\dim \Xi = \limsup_{n \to +\infty} \frac{\ln(p(n))}{\ln(n)}\,.$$

And we deduce the following.

Corollary 3.14. *Let Ξ be the transversal of a substitution tiling of dimension d, with complexity function p. Then there exists a function ν such that:*

$$p(n) = n^{\nu(n)}, \qquad \text{with} \ \lim_{n \to +\infty} \nu(n) = d\,.$$

With the above choice of weights we can compute the Dixmier trace μ_{Dix}. Furthermore, there is a uniquely ergodic measure on $(\Omega, \mathbb{R}^d)$, which was first described by Solomyak [41]. It restricts to a measure μ_Ξ^{erg} on the transversal, and we have the following.

Theorem 3.8. *With the above weights one has $\psi_*(\mu_{Dix}) = \mu_\Xi^{erg}$.*

As $\mathcal{B}$ is stationary, the sets of edges between two generations (excluding the root) are isomorphic. Let us denote by $\mathcal{E}$ this set, and by $\mathcal{E}_0$ the set of edges linking to the root. Thanks to the self-similar structure we can define affine maps u_e, $e \in \mathcal{E}$, that act on the eigenvalues of Δ_s as follows (see section 5.1)

$$u_e(\lambda_\eta) = \lambda_{U_e\eta} = \Lambda_{PF}^{(d+2-s)/d}\lambda_\eta + \beta_e\,,$$

where $U_e\eta$ is an extension of the path η (see Definition 5.16), and β_e a constant that only depends on e. Those maps $U_e, e \in \mathcal{E}$, form a representation of a Cuntz–Krieger algebra associated with the matrix A. If $\gamma = (\varepsilon, e_1, e_2, \cdots e_n)$, $\varepsilon \in \mathcal{E}_0$, $e_i \in \mathcal{E}$, is a path of length n, let u_γ be the map $u_{e_1} \circ u_{e_2} \circ \cdots u_{e_n}$. Let us denote by $\lambda_\varepsilon, \varepsilon \in \mathcal{E}_0$, the eigenvalues of Δ_s corresponding to paths of length 1. For any other eigenvalue λ_γ, there is a (unique) λ_ε such that

$$\lambda_\gamma = u_\gamma(\lambda_\varepsilon) = \Lambda_s^n \lambda_\varepsilon \ + \ \sum_{j=1}^{n} \Lambda_s^{j-1} \beta_{e_j}\,,$$

where $\Lambda_s = \Lambda_{PF}^{(d+2-s)/d}$. That is, the Cuntz–Krieger algebra allows to calculate explicitly the full spectrum of Δ_s from the finite data of the $\lambda_\varepsilon, \varepsilon \in \mathcal{E}_0$, and $\beta_e, e \in \mathcal{E}$ — which are immediate to compute, see Section 5.1.

For instance, for the Fibonacci diagram (Figure 1) and $s = s_0 = d$, there are only two such u_e maps, namely $u_a(x) = x \cdot \phi^2 - \phi$, and $u_b(x) = x \cdot \phi^2 + \phi$, where $\phi = (1 + \sqrt{5})/2$ is the golden mean.
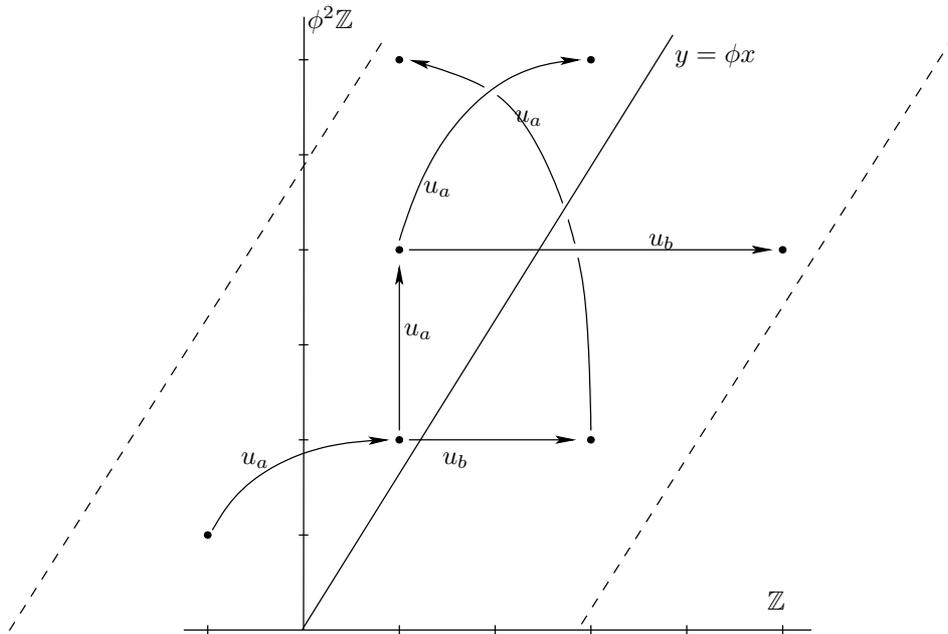
Figure 3: Distribution of the eigenvalues for the Fibonacci diagram.

The eigenvalues of Δ_d are all of the form $p + q\phi^2$ for integers $p, q \in \mathbb{Z}$. They can be represented as points (p, q) in the plane; these points stay within a bounded distance to the line directed by the Perron–Frobenius eigenvector of the Abelianization matrix, see Figure 3. This is an example of a general result, valid for hyperbolic substitutions, see Theorem 5.7. We treat further examples in Section 6, in particular the Thue–Morse and Penrose tilings.

The name Laplace–Beltrami operator for Δ_s can be justified by the following two results. For $s = s_0 = d$, in analogy with the Laplacian on a compact d-manifold, Δ_d satisfies the classical Weyl asymptotics, and the trace of its heat kernel follows the leading term of the classical Seeley expansion.

Let $\mathcal{N}(\lambda)$ be the number of eigenvalues of Δ_d of modulus less than λ.

Theorem 5.5. *There are constants $0 < c_- < c_+$ such that as $\lambda \to +\infty$ one has*

$$c_- \lambda^{d/2} \leq \mathcal{N}(\lambda) \leq c_+ \lambda^{d/2} .$$

Theorem 5.6. *There are constants $0 < c_- < c_+$ such that as $t \downarrow 0$ one has*

$$c_- t^{-d/2} \leq \mathrm{Tr}\left(e^{t\Delta_d}\right) \leq c_+ t^{-d/2} .$$

2 Weighted Bratteli diagrams and substitutions

We first give general the definitions of Cantor sets, Bratteli diagrams, and substitution tilings. We then describe how to associate diagrams to tilings.

Definition 2.1. A Cantor set is a compact, Hausdorff, metrizable topological space, which is totally disconnected and has no isolated points.

An ultrametric d on a topological space X is a metric which satisfies this strong triangle inequality:
$$\forall x, y, z \in X, \quad d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

2.1 Bratteli diagrams

To a substitutive system, one can naturally associate a combinatorial object named a Bratteli diagram. These diagrams were first used in the theory of C^*-algebras, to classify AF-algebras. Then, it was mainly used to encode the dynamics of a minimal action of $\mathbb{Z}$ on a Cantor set.

Definition 2.2. A Bratteli diagram is an oriented graph defined as follows:
$$\mathcal{B} = (\mathcal{V}, \mathcal{E}_{\text{tot}}, r, s),$$

Where $\mathcal{V}$ is the set of vertices, $\mathcal{E}_{\text{tot}}$ is the set of directed edges, and r, s are functions $\mathcal{E}_{\text{tot}} \to \mathcal{V}$ (range and source), which define adjacency. We have a partition of $\mathcal{V}$ and $\mathcal{E}_{\text{tot}}$ in finite sets:
$$\mathcal{V} = \bigcup_{n \geq 0} \mathcal{V}_n \quad ; \quad \mathcal{E}_{\text{tot}} = \bigcup_{n \geq 0} \mathcal{E}_n,$$

$\mathcal{V}_0$ is a single element called the root and noted $\circ$. The edges of $\mathcal{E}_n$ have their source in $\mathcal{V}_n$ and range in $\mathcal{V}_{n+1}$, that is:
$$r : \mathcal{E}_n \to \mathcal{V}_{n+1}, \quad s : \mathcal{E}_n \to \mathcal{V}_n.$$
We ask that $s^{-1}(v) \neq \emptyset$ for all $v \in \mathcal{V}$, and $r^{-1}(v) \neq \emptyset$ for all $v \in \mathcal{V} \setminus \mathcal{V}_0$.

Remark 2.3. If, for all $v \in \mathcal{V} \setminus \{\circ\}$, $r^{-1}(v)$ is a single edge, then $\mathcal{B}$ is a tree. In that sense, the formalism of Bratteli diagrams includes the case of trees, and so is a generalization of the case studied in [36]. However, our goal is to restrict to self-similar diagrams, for which computations are easier.

Definition 2.4. A path γ of length $n \in \mathbb{N} \cup \{+\infty\}$ in a Bratteli diagram is an element
$$(\varepsilon_0, e_1, e_2, \ldots) \in \prod_{i=0}^{n-1} \mathcal{E}_i,$$

which satisfies:
$$\text{for all } 0 \leq i < n, \quad r(e_i) = s(e_{i+1}).$$

We call Π_n the set of paths of length $n < +\infty$, Π the set of all finite paths, and $\partial \mathcal{B}$ the set of infinite paths.

The function r naturally extends to Π: if $\gamma = (\varepsilon_0, \ldots, e_n)$, then $r(\gamma) := r(e_n)$.

In addition to the definition above, we ask that a Bratteli diagram satisfies the following condition.

Hypothesis 2.5. *For all $v \in \mathcal{V}$, there are at least two distinct infinite paths through v.*

Remark 2.6. If, for all $n \geq 1$ and $(v, v') \in \mathcal{V}_n \times \mathcal{V}_{n+1}$, there is at most one edge from v to v', we can simply encode path by vertices: the following map is an homeomorphism onto its image.
$$(\varepsilon_0, e_1, \ldots) \longmapsto (\varepsilon_0, r(e_1), \ldots).$$

Definition 2.7. Given two finite or infinite paths γ and γ', we note $\gamma \wedge \gamma'$ the (possibly empty) longest common prefix of γ and γ'.

The set $\partial\mathcal{B}$ is called the boundary of $\mathcal{B}$. It has a natural topology inherited from the product topology on $\prod_{i=0}^{+\infty} \mathcal{E}_i$, which makes it a compact and totally disconnected set. A basis of neighborhoods is given by the following sets:

$$[\gamma] = \{x \in \partial\mathcal{B} \ ; \ \gamma \text{ is a prefix of } x\}.$$

Hypothesis 2.5 is the required condition to make sure that there are no isolated points. This implies the following.

Proposition 2.8. *With this topology, $\partial\mathcal{B}$ is a Cantor set.*

Definition 2.9. A weight on $\partial\mathcal{B}$ is a function $w : \mathcal{V} \to \mathbb{R}_+^*$, satisfying the following conditions:

  (i) $w(\circ) = 1$;

 (ii) $\sup\{w(v) \ ; \ v \in \mathcal{V}_n\}$ tends to 0 when n tends to infinity;

(iii) $\forall e \in \mathcal{E}_{\text{tot}}, w(s(e)) > w(r(e))$.

A weight extends naturally on paths: by definition, $w(\gamma) := w(r(\gamma))$.

Proposition 2.10. *We define a function d_w on $(\partial\mathcal{B})^2$ by:*

$$d_w(x,y) = \left\{ \begin{array}{ll} w\big(r(x \wedge y)\big) & \text{if } x \neq y; \\ 0 & \text{otherwise} \end{array} \right.$$

It is a ultrametric on $\partial\mathcal{B}$, which is compatible with the topology defined above, and so $(\partial\mathcal{B}, d_w)$ is a ultrametric Cantor set.

A case of interest is when the Bratteli diagram is self-similar. For a diagram, self-similarity means that all the $\mathcal{V}_n$ are isomorphic, and all $\mathcal{E}_n$ are isomorphic for $n \geq 1$, and r, s commute with these identifications. We will focus on self-similarity when the diagram is associated with a substitution.

2.2 Substitution tilings

Let us give the definition of a tiling of $\mathbb{R}^d$. The tilings we are interested in are constructed from a prototile set and an inflation and substitution rule on the prototiles. The notion of substitution dates back to the sixties. For example the self-similarity of the Penrose tiling already appeared in [37]. However, systematic formalisation of substitution tilings and properties of their associated tiling space was done in [33, 40] in the nineties. In this section, we follow the description by Anderson and Putnam [1] with some minor changes. In particular, some non trivial facts are cited along the text. We do not claim to cite the original authors for all of these. The reader can refer to the reviews [39] and [16].

By *tile*, we mean a compact subset of $\mathbb{R}^d$, homeomorphic to the closed unit ball. A tile is *punctured* if it has a distinguished point in its interior. A prototile is the equivalence class under translation of a tile. Let $\mathcal{A}$ be a given set of punctured prototiles. All tilings will be made from these tiles.

A patch p on $\mathcal{A}$ is a finite set of tiles which have disjoint interior. We call $\mathcal{A}^*$ the set of patches modulo translation. A partial tiling is an infinite set of tiles with disjoint interior, and the union of which is connected, and a tiling T is a partial tiling which covers $\mathbb{R}^d$, which means:

$$\text{Supp}(T) := \bigcup_{t \in T} t = \mathbb{R}^d.$$

A substitution rule is a map ω which maps tiles to patches, and such that for all tile $p \in \mathcal{A}$, $\text{Supp}(\omega(p)) = \lambda \text{Supp}(p)$ for some $\lambda > 1$. The factor λ inflates the tiles p, which is then cut into pieces; these pieces are elements of $\mathcal{A}$. The map ω extends to patches, partial tilings and tilings.

Since we will represent tilings spaces by diagrams, we have a specific interest for the combinatorics of the substitution.

Definition 2.11. Given a substitution ω, its *Abelianization matrix* is an integer-valued matrix $A_\omega = (a_{pq})_{p,q \in \mathcal{A}}$ (or simply A) defined by:

$$\forall p, q \in \mathcal{A}, \quad a_{pq} = \text{ number of distinct translates of } p \text{ included in } \omega(q).$$

We now define the tiling space associated with ω.

Definition 2.12. The tiling space Ω is the set of all tilings T such that for all patch p of T, p is also a subpatch of some $\omega^n(t)$ ($n \in \mathbb{N}$, $t \in \mathcal{A}$).

We make the following assumptions on ω:

Hypothesis 2.13. *(i) The matrix A is* primitive*: A^n has non-negative entries for some n.*

(ii) Finite local complexity (FLC): for all $R > 0$, the set of all patches $p \subset T$ which can be included in a ball of radius R is finite up to translation, for all $T \in \Omega$.

Just as every tile has a distinguished point inside it, Ω has a distinguished subset. This is the Cantor set which will be associated with a Bratteli diagram.

Definition 2.14. Let Ξ be the subset of Ω of all tilings T such that 0 is the puncture of one of the tiles of T. This set is called the canonical transversal of Ω.

Note that, by this definition, the substitution ω extends to a map $\Omega \to \Omega$.

We will assume in the following that $\omega : \Omega \to \Omega$ is one-to-one. This condition is equivalent (Solomyak [40]) to the fact that no tiling in Ω has any period. Furthermore, it implies that Ω is not empty, and $\omega : \Omega \to \Omega$ is onto.

The set of interest for us is Ξ. Its topology is given as follows. For any patch p, define the following subset of Ξ:

$$U_p = \{T \in \Xi \; ; \; p \subset T\}.$$

Note that U_p can be empty. Nevertheless, the family of all the U_p's is a basis for a topology on Ξ.

Let us now define a distance d on Ω:

$$d(T, T') = \inf \left(\left\{ \epsilon > 0 \; ; \; \exists x, y \in B(0, \epsilon) \text{ such that} \right.\right.$$

$$\left.\left. B(0, 1/\epsilon) \subset \text{Supp}\left((T - x) \cap (T' - y)\right) \right\} \cup \{1\} \right).$$

Two tilings are d-close when, up to a small translation, they agree on a large ball around the origin. This distance, when restricted to Ξ, is compatible with the topology defined above. With this topology, Ξ is a Cantor set.

Furthermore, the map $\omega : \Omega \to \Omega$ is a homeomorphism, and the dynamical system $(\Omega, \mathbb{R}^d)$ given by translation is continuous and uniquely ergodic [41].

Substitution tiling spaces are *minimal*, which means that every $\mathbb{R}^d$-orbit is dense in Ω. Combinatorially, this is equivalent to the fact that these tilings are repetitive: for all $R > 0$, there is a bound ρ_R, such that every patch of size R appears in the tiling whithin range ρ_R to any tile.

We need an additional assumption — the border forcing condition. It is required in order to give a good representation of Ξ as the boundary of a Bratteli diagram.

Definition 2.15. Let $p \in \mathcal{L}$ be a patch. The *maximal unambiguous extention* of p is the following patch, obtained as an intersection of tilings:

$$\text{Ext}(p) = \bigcap \left\{ T \in \Omega \; ; \; p \subset T \right\}.$$

So any tiling which contains p also contains $\text{Ext}(p)$. The patch $\text{Ext}(p)$ is called the *empire* of the patch p by some authors [21].

Definition 2.16. Assume there is some $m \in \mathbb{N}$ and some $\epsilon > 0$ such that for all $t \in \mathcal{A}$,

$$\text{Ext}\left(\omega^m(t)\right) \text{ contains an } \epsilon 1\text{-neighborhood of } \omega^m(t).$$

Then ω is said to *force its border*.

It is always possible to give labels to the tiles, in order to change a non-border forcing substitution into a border-forcing one. See [1].

Example 2.17. Let us give an example in dimension one. Consider the substitution defined on symbols by:

$$\begin{aligned} a &\mapsto baa \\ b &\mapsto ba \end{aligned}$$

It has a geometric realization, with a being associated with an interval of length $\phi = (1 + \sqrt{5})/2$, and b with an interval of length 1. Then the map above is a substitution in the sense of our definitions, with $\lambda = \phi$. The transversal Ξ is naturally identified with a subset of $\{a, b\}^{\mathbb{Z}}$ (the *subshift* associated with ω). The map ω satisfies the border forcing condition. Indeed, if p is a patch of a tiling T (which we identify symbolically with a finite word on $\{a, b\}$ in a bi-infinite word), let $x, y \in \{a, b\}$ be the letters preceding and following p in T respectively. Then $\omega(x)$ ends by an a and $\omega(y)$ begins by a b. Therefore, $\omega(p)$ is always followed by b and preceded by a. This proves that ω forces its border.

2.3 Bratteli diagrams associated with substitution tilings

We show how to associate a Bratteli diagram to a substitution, and identify the transversal with the boundary of the diagram. It is clear that these two are homeomorphic, being Cantor sets. We will give an explicit and somehow natural homeomorphism.

Let ω be a primitive, FLC and border forcing substitution, and $A = (a_{ij})_{i,j \in \mathcal{A}}$ its Abelianization matrix. Let λ be the expansion factor associated with ω. Let Ξ be the transversal of the tiling space associated with ω.

The diagram $\mathcal{B}$ associated with the substitution is defined as follows.

Definition 2.18. The diagram associated with ω is the diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E}_{\text{tot}}, r, s)$, where:

$$\forall n \geq 1, \quad \mathcal{V}_n = \mathcal{A} \times \{n\},$$

for all $n \geq 1$, for all $i, j \in \mathcal{A}$, there are exactly a_{ij} edges of $\mathcal{E}_n$ from (i, n) to $(j, n+1)$, and for all $v \in \mathcal{V}_1$ there is exactly one edge ε_v from the root $\circ$ to v.

We call $\mathcal{E}$ the set of "models" of edges from one generation to another. All $\mathcal{E}_n$ are copies of $\mathcal{E}$ (for example with identification $\mathcal{E}_n = \mathcal{E} \times \{n\}$). Since the models for the vertices are the elements of $\mathcal{A}$, we do not use a specific name. The maps source and range can be restricted as maps $\mathcal{E} \to \mathcal{A}$. When the "depth" of a vertex is not important, we will sometimes consider r, s as functions valued in $\mathcal{A}$.

Remark 2.19. Combinatorially, the diagram only depends on the Abelianization matrix of ω. It is indeed possible to associate a diagram to a primitive matrix A with integer coefficients. We would have similar definitions, with $\mathcal{V}_n = I \times \{n\}$, with I the index set of the matrix.

Proposition 2.20. *This is a Bratteli diagram in the sense of Definition 2.2, except in the case* $A = (1)$.

Example 2.21. Figure 1 is an example of the self-similar diagram associated with the Fibonacci substitution: $a \mapsto ab$; $b \mapsto a$.

There is a correspondence between the paths on $\mathcal{B}$ and the transversal Ξ. It depends on a choice on the edges which remembers the geometry of the substitution: each $e \in \mathcal{E}$ from $a \in \mathcal{A}$ to

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$b \in \mathcal{A}$ corresponds to a different occurrence of a in $\omega(b)$. This correspondence is a homeomorphism $\phi : \Xi \to \partial \mathcal{B}$, called the Robinson map, as defined in [30]. We first give the definition of ϕ, and we will then give a condition the weight function w so that ϕ is bi-Lipschitz.

To construct ϕ, start with $T \in \Xi$. Let $t \in T$ be the tile containing 0, and $[t] \in \mathcal{A}$ the corresponding prototile. Then, the first edge of $\phi(p)$ is the edge from $\circ$ to $([t], 1) \in \mathcal{V}_1$. Assume that the prefix of length n of $\phi(T)$ is already constructed and ends at vertex $([t'], n)$, where t' is the tile of $\omega^{-(n-1)}(T)$ which contains 0. Let t'' be the tile containing 0 in $\omega^{-n}(T)$; then the $(n+1)$-th edge of $\phi(T)$ is the edge corresponding to the inclusion of t' in $\omega(t'')$. This edge ends at $([t''], n+1)$. By induction we construct $\phi(T)$.

Theorem 2.22 (Theorem 4 in [30]). *The function $\phi : \Xi \to \partial \mathcal{B}$ is an homeomorphism.*

We give explicitly the inverse for ϕ as follows. Let us define $\psi : \Pi \to \mathcal{A}^*$ which is increasing in the sense that if a path is a prefix of another, then the patch associated to the first is included in the patch associated to the second. The image of an infinite path will then be defined as the union of the images of its prefixes.

If (ε) is a path of length one, then $\psi(\varepsilon)$ is defined as the tile $r(\varepsilon)$ with puncture at the origin. Now, given a path γ of length n, assume its image by ψ is well defined, and is some translate of $\omega^{n-1}(r(\gamma))$, with the origin at the puncture of one of its tiles. Consider the path $\gamma.e$, with $e \in \mathcal{E}_n$ such that $s(e) = r(\gamma)$. Then, e encodes an inclusion of $s(e)$ inside $\omega(r(e))$. It means that it encodes an inclusion of $\omega^{n-1}(r(\gamma))$ inside $\omega^n(r(e))$. The patch $\psi(\gamma.e)$ is defined as the translate of $\omega^n(r(e))$ such that the inclusion $\psi(\gamma) \subset \psi(\gamma.e)$ is the inclusion defined by the edge e.

Remark that if $x \in \partial \mathcal{B}$, $\psi(x)$ does not *a priori* define more than a partial tiling. The fact that it corresponds to a unique tiling results from the border forcing condition. Then the map ψ, extended to infinite paths, is the inverse of ϕ.

Continuity can be proved directly, but can also be seen as a consequence of Theorem 2.25, which we prove later.

Lemma 2.23. *There exists C_1, C_2 positive constants, such that for all path γ of length n in $\mathcal{B}$,*

$$B(0, C_1 \lambda^n) \subset \mathrm{Supp}\left(\mathrm{Ext}\left(\psi(\gamma)\right)\right),$$
$$B(0, C_2 \lambda^n) \not\subset \mathrm{Supp}\left(\mathrm{Ext}\left(\psi(\gamma)\right)\right),$$

where λ is the expansion factor of ω.

Proof. Let k be the smallest number such that for all $v \in \mathcal{V}$, there exists two distinct paths of length k starting from v. For example, $k = 1$ when there are two elements of $\mathcal{E}_{\mathrm{tot}}$ starting from v, for all v, and $k = 2$ in the case of Fibonacci, pictured in Figure 1. Let C be the maximum diameter of the tiles. Let γ be a path of length n. Then, one can find two distinct extensions of length $n+k$ of γ; call them γ_1 and γ_2. Remember that $\psi(\gamma_i)$ is some translate of $\omega^{n+k-1}(t_i)$ with $t_i \in \mathcal{A}$, and $0 \in \mathrm{Supp}\left(\psi(\gamma)\right)$. So $\psi(\gamma_1)$ and $\psi(\gamma_2)$ differ within range $C\lambda^{n+k-1}$. Therefore, with $C_1 = C\lambda^{k-1}$, we have the second line.

For the first inclusion, let m be the exponent for which ω satisfies the border forcing condition, and γ a path of length m. Let $t \in \mathcal{A}$ be the range of γ. Then $\mathrm{Ext}(\omega^m(t))$ covers an ϵ-neighborhood of $\mathrm{Supp}(\omega^m(t))$. It means that, since $\psi(\gamma)$ is a translate of $\omega^m(t)$ which contains 0,

$$B(0, \epsilon) \subset \mathrm{Ext}(\psi(\gamma)).$$

Similarly, for all path of length $m + k$,

$$B(0, \epsilon \lambda^k) \subset \mathrm{Ext}(\psi(\gamma)).$$

With $n = m + k$ and $C_1 = \epsilon/\lambda^m$, one has the result. For $n < m$, the inequality still holds (up to a reduction of ϵ). $\qquad\square$

Remark 2.24. We can simplify the diagram in the case where a symmetry group $G \subset \mathrm{O}_n(\mathbb{R}^d)$ acts freely on $\mathcal{A}$, such that

$$\forall g \in G, \ \forall p \in \mathcal{A}, \quad \omega(g \cdot p) = g \cdot \omega(p).$$

With this, it is possible to extend the action of G to the edges of $\mathcal{E}_{\mathrm{tot}}$. It induces naturally an action of G on $\partial \mathcal{B}$. The group G also acts on Ω by isometries, and these two actions are conjugate by ψ.

One can "fold" the diagram by taking a quotient as follows. Define: $\mathcal{B}' = (\mathcal{V}', \mathcal{E}'_{\mathrm{tot}}, r', s')$, where $\mathcal{V}'_0 = \mathcal{V}$, $\mathcal{E}'_0 = \mathcal{E}_0 \times G$, and

$$\forall n \geq 1, \quad \mathcal{V}'_n = \mathcal{V}_n / G \text{ and } \mathcal{E}'_n = \mathcal{E}_n / G.$$

Furthermore, for $[e]_G \in \mathcal{E}_n / G$, $r'([e]_G) := [r(e)]_G$ and $s'([e]_G) := [s(e)]_G$. These definitions do not depend on the choice of the representant e. For $(e, g) \in \mathcal{E}'_0 = \mathcal{E}_0 \times G$, define $r'((e, g)) := [r(e)]_G$ and $s'((e, g)) := \circ$. One can check that these two diagrams are "the same" in the sense that the tree structure of their respective sets of paths of finite length is the same.

In terms of substitution, the image by ψ of a path $\gamma = (\varepsilon_0, e_1, \ldots, e_{n-1})$ is the image of $\omega^n(r(\gamma))$ under some element of $\mathbb{R}^d \rtimes G$. The translation part is given by the truncated path $(e_1, \ldots, e_{n-1})$. The rotation part is encoded in the first edge $\varepsilon_0 \in \mathcal{E}'_0$: $\varepsilon_0 = (\varepsilon', g)$, where ε' brings no additional information, but g corresponds to the choice of an orientation for the patch.

There is still an action of G on $\partial \mathcal{B}'$, defined by: $g \cdot ((\varepsilon_0, h), e_1, \ldots) = ((\varepsilon_0, gh), e_1, \ldots)$.

2.4   Weights and metric

We gave in Theorem 2.22 an explicit homeomorphism ϕ between the boundary of the Bratteli diagram, and the transversal of the tiling space. Since we are interested in metric properties of the Cantor set, we now show that a correct choice of weights on the vertices of the Bratteli diagram gives a metric on $\partial \mathcal{B}$ which is Lipschitz equivalent to the usual metric on Ξ (the equivalence being induced by ϕ).

Theorem 2.25. *Let $\mathcal{B}$ the Bratteli diagram associated with a substitution ω. Let λ be the inflation factor of ω. We make the following assumption on the weight function w:*

$$\forall n \geq 1, \ \forall v \in \mathcal{V}, \quad w(v, n+1) = \frac{1}{\lambda} w(v, n). \tag{2.1}$$

Then the function $\phi : (\Xi, d) \to (\partial \mathcal{B}, d_w)$ defined in Proposition 2.22 is a bi-Lipschitz homeomorphism.

Proof. It is enough to show that there are two constants $m, M > 0$, such that for all $\gamma \in \Pi$,

$$m \leq \frac{\mathrm{diam}(U_\gamma)}{\mathrm{diam}(\phi^{-1}(U_\gamma))} \leq M.$$

Since the U_γ are a basis for the topology of $\partial \mathcal{B}$, this will prove the result.

Let $\gamma \in \Pi_n$. By Lemma 2.23, any two tilings in $\phi^{-1}(U_\gamma)$ coincide on a ball of radius at least $C_1 \lambda^n$. Therefore,

$$\mathrm{diam}(\phi^{-1}(U_\gamma)) \leq \frac{1}{C_1} \lambda^{-n}.$$

On the other hand, it is possible to find two tilings in $\phi^{-1}(U_\gamma)$, which disagree on a ball of radius $C_2 \lambda^n$. Therefore,

$$\mathrm{diam}(\phi^{-1}(U_\gamma)) \geq \frac{1}{C_2} \lambda^{-n}.$$

And by definition of the weights,

$$\min\{w(v, 1) \ ; \ v \in \mathcal{V}\} \lambda^{-n+1} \leq \mathrm{diam}(U_\gamma) \leq \max\{w(v, 1) \ ; \ v \in \mathcal{V}\} \lambda^{-n+1}.$$

Together with the previous two inequalities, this proves the result. $\qquad \square$

3 Spectral triple, ζ-function, and complexity

3.1 Spectral triple

Let $\mathcal{B}$ be a weighted Bratteli diagram, and let $(\partial\mathcal{B}, d)$ be the ultrametric Cantor set of infinite rooted paths in $\mathcal{B}$. Pearson and Bellissard built in [36] a *spectral triple* for $(\partial\mathcal{B}, d)$ when $\mathcal{B}$ is a *tree* (that is when for all vertex $v \in \mathcal{V} \setminus \{\circ\}$, the fiber $r^{-1}(v)$ contains a single point). In our setting, their construction is adapted as follows.

A *choice function* on $\mathcal{B}$ is a map

$$\tau : \left\{ \begin{array}{ccc} \Pi & \rightarrow & \partial\mathcal{B} \times \partial\mathcal{B} \\ \gamma & \mapsto & \big(\tau_+(\gamma), \tau_-(\gamma)\big) \end{array} \right. \quad \text{such that} \quad d\big(\tau_+(\gamma), \tau_-(\gamma)\big) = \mathrm{diam}[\gamma], \tag{3.2}$$

and we denote by E the set of choice functions on $\mathcal{B}$. Let $C_{\mathrm{Lip}}(\partial\mathcal{B})$ be the C^*-algebra of Lipschitz continuous functions on $(\partial\mathcal{B}, d)$. Given a choice $\tau \in E$ we define a faithfull $*$-*representation* π_τ of $C_{\mathrm{Lip}}(\partial\mathcal{B})$ by bounded operators on the Hilbert space $\mathcal{H} = l^2(\Pi) \otimes \mathbb{C}^2$ as

$$\pi_\tau(f) = \bigoplus_{\gamma\in\Pi} \left[\begin{array}{cc} f\big(\tau_+(\gamma)\big) & 0 \\ 0 & f\big(\tau_-(\gamma)\big) \end{array} \right]. \tag{3.3}$$

This notation means that for all $\xi \in \mathcal{H}$ and all $\gamma \in \Pi$,

$$\big(\pi_\tau(f) \cdot \xi\big)(\gamma) = \left[\begin{array}{cc} f\big(\tau_+(\gamma)\big) & 0 \\ 0 & f\big(\tau_-(\gamma)\big) \end{array} \right] \cdot \xi(\gamma).$$

A *Dirac operator* D on $\mathcal{H}$ is given by

$$D = \bigoplus_{\gamma\in\Pi} \frac{1}{\mathrm{diam}[\gamma]} \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \tag{3.4}$$

that is D is a self–adjoint unbounded operator such that $(D^2 + 1)^{-1}$ is compact, and the commutator

$$[D, \pi_\tau(f)] = \bigoplus_{\gamma\in\Pi} \frac{f\big(\tau_+(\gamma)\big) - f\big(\tau_-(\gamma)\big)}{\mathrm{diam}[\gamma]} \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right], \tag{3.5}$$

is bounded for all $f \in C_{\mathrm{Lip}}(\partial\mathcal{B})$. Finally a *grading operator* is given by $\Gamma = 1_{l^2(\Pi)} \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]$, and satisfies $\Gamma^2 = \Gamma^* = \Gamma$, and commutes with π_τ and anticommutes with D. The following is Proposition 8 in [36].

Proposition 3.1. $\big(C_{\mathrm{Lip}}(\partial\mathcal{B}), \mathcal{H}, \pi_\tau, D, \Gamma\big)$ *is an* even spectral triple *for all $\tau \in E$.*

In [36] the set of choice functions E is considered an analogue of a tangent bundle over $\partial\mathcal{B}$, so that the above commutator is interpreted as the directional derivative of f along the choice τ. The metric on $\partial\mathcal{B}$ is then recovered from the spectral triple by using Connes formula, *i.e.* by taking the supremum over all directional derivatives

Theorem 3.2 (Thm. 1 in [36])**.** *The following holds:*

$$d(x, y) = \sup\big\{|f(x) - f(y)| \; ; \; f \in C_{\mathrm{Lip}}(\partial\mathcal{B}), \; \sup_{\tau\in E} \|[D, \pi_\tau(f)]\| \leq 1\big\}. \tag{3.6}$$

Definition 3.3. The ζ-*function* associated with the spectral triple is given by:

$$\zeta(s) = \frac{1}{2}\mathrm{Tr}\big(|D|^{-s}\big) = \sum_{\gamma\in\Pi} \mathrm{diam}[\gamma]^s, \tag{3.7}$$

and we will denote by $s_0 \in \overline{\mathbb{R}}$ its *abscissa of convergence*, when it exists.

We now assume that the weight system on $\mathcal{B}$ is such that $s_0 \in \mathbb{R}$.

Definition 3.4. The *Dixmier trace* of a function $f \in C_{\mathrm{Lip}}(\partial \mathcal{B})$ is given by the following limit, when it exists:

$$\mu(f) = \lim_{s \downarrow s_0} \frac{\mathrm{Tr}\left(|D|^{-s} \pi_\tau(f)\right)}{\mathrm{Tr}\left(|D|^{-s}\right)} . \tag{3.8}$$

It defines a probability measure on $\partial \mathcal{B}$ and does not depend on the choice $\tau \in E$ (Theorem 3 in [36]). Furthermore, when $f = \chi_\gamma$ is a characteristic function, the limit above can be rewriten:

$$\mu([\gamma]) := \mu(\chi_\gamma) = \lim_{s \downarrow s_0} \frac{\sum_{\eta \in \Pi_\gamma} w(r(\eta))^s}{\sum_{\eta \in \Pi} w(r(\eta))^s} . \tag{3.9}$$

where Π_γ stands for the set of paths in Π with prefix γ.

3.2   The measure on $\partial \mathcal{B}$

Let (Ω, ω) be a substitution tiling space, A be the Abelianization matrix of ω, and $\mathcal{B}$ be the associated Bratteli diagram, as in previous section (Definition 2.18). We assume that $\mathcal{B}$ comes together with a weight function w, which satisfies the properties given in Definition 2.9, and adapted to the substitution ω. In particular, it satisfies $w(a, n+1) = \lambda^{-1} w(a, n)$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$.

As we assumed A to be primitive, it has a so called *Perron–Frobenius* eigenvalue, denoted Λ_{PF}, which satisfies the following (see for example [23]):

  (i) Λ_{PF} is strictly greater than 0, and equals the spectral radius of A;

 (ii) For all other eigenvalue ν of A, $|\nu| < \Lambda_{PF}$;

(iii) The right and left eigenvectors, v_R and v_L, have strictly positive coordinates;

(iv) If v_R and v_L are normalized so that $\langle v_R, v_L \rangle = 1$, then:

$$\lim_{n \to +\infty} \frac{A^n}{\Lambda_{PF}^n} = v_L v_R^{\mathrm{T}};$$

 (v) Any eigenvector of A with non-negative coordinates corresponds to the eigenvalue Λ_{PF}.

A classical result about linear dynamical systems together with the properties above gives the following result, which will be needed later.

Lemma 3.5. *Let A be a primitive matrix with Perron–Frobenius eigenvalue Λ_{PF}. Let P_M be its minimal polynomial: $P_M = (X - \Lambda_{PF}) \prod_{i=1}^{p} (X - \alpha_i)^{m(i)}$. Then, the coefficients of A^n are given by:*

$$[A^n]_{ab} = c_{ab} \Lambda_{PF}^n + \sum_{i=1}^{p} P_i^{(a,b)}(n) \alpha_i^n ,$$

where the P_i's are polynomials of degree $m(i)$, and $c_{ab} > 0$.

Note that the coefficients (a, b) of A^n gives the number of paths of length n in the diagram between some vertex $(a, k) \in \mathcal{V}_k$ and $(b, n+k)$. This lemma states that this number is equivalent to $c_{ab} \Lambda_{PF}^n$ when n is large.

Proof. We have $c_{ab} > 0$, as it is the (a, b) entry of the matrix $v_L v_R^{\mathrm{T}}$ defined above. The rest is classical, and results from the Jordan decomposition of the matrix A. $\qquad\square$

We assume that Ω is a d-dimensional tiling space. Since ω expands the distances by a factor λ, the volumes of the tiles are dilated by λ^d. This gives the following result:

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Proposition 3.6. *Let Λ_{PF} be the Perron–Frobenius eigenvalue of A. Then $\Lambda_{PF} = \lambda^d$. In particular, $\Lambda_{PF} > 1$.*

Theorem 3.7. *The ζ-function for the weighted Bratteli diagram $(\partial\mathcal{B}, w)$ has abscissa of convergence $s_0 = d$.*

Proof. We have

$$\zeta(s) = \sum_{\gamma \in \Pi} w(\gamma)^s.$$

The quantity $w(\gamma)$ only depends on $r(\gamma)$. Furthermore, $w(a, n)$ tends to zero like $\lambda^n = \left(\Lambda_{PF}^{1/d}\right)^n$ when n tends to infinity. So we have:

$$\sum_{n \in \mathbb{N})} m^s \Lambda_{PF}^{ns/d} \operatorname{Card}(\Pi_n) \leq \zeta(s) \leq \sum_{n \in \mathbb{N})} M^s \Lambda_{PF}^{ns/d} \operatorname{Card}(\Pi_n),$$

where m (resp. M) is the minimum (resp. the maximum) of the $w(a, 1)$, $a \in \mathcal{A}$ Now, since $\operatorname{Card}(\Pi_n)$ grows like Λ_{PF}^n up to a constant (see Lemma 3.5), we have the result. $\qquad\square$

Theorem 3.8. *The measure μ given by the Dixmier trace (Definition (3.4)) is well defined, and given as follows. Let $v = (v_a)_{a \in \mathcal{A}}$ be the (right) eigenvector for A, normalized such that $\sum_{e \in \mathcal{E}_0} v_{r(e)} = 1$. For all $\gamma \in \Pi$, let $(a, n) := r(\gamma) \in \mathcal{V}_n$. Then:*

$$\mu([\gamma]) = v_a \Lambda_{PF}^{-n+1}.$$

In particular, in the case of a substitution tiling, $\psi_*(\mu)$ is the measure given by the frequences *of the patches, and therefore is the restriction to Ξ of the unique ergodic measure on $(\Omega, \mathbb{R}^d)$ (see [41]).*

Proof. Let $\gamma = (\varepsilon_0, e_1, \ldots, e_{n-1}) \in \Pi_n$, and $(a, n) := r(\gamma)$. Let Π_γ the subset of Π of all paths which have γ as a prefix. Define:

$$f(s) = \frac{\displaystyle\sum_{\eta \in \Pi_\gamma} w(\eta)^s}{\displaystyle\sum_{\eta \in \Pi} w(\eta)^s}.$$

Then $\mu([\gamma]) = \lim_{s \downarrow d} f(s)$, when this limit exists. The terms of the sum above can be grouped together: if η is a path of Π_γ, then the quantity $w(\eta)^s$ only depends on the length of η, say $n + k$ ($k \geq 0$), and on $r(\eta) = b \in \mathcal{A}$. Then, if we call $N(a, b; k)$ the number of paths of length k from a to b, we can group the sum and write:

$$f(s) = \frac{\displaystyle\sum_{k \geq 0} \sum_{b \in \mathcal{A}} N(a, b; k) w(b, n + k)^s}{1 + \displaystyle\sum_{k \geq 0} \sum_{\varepsilon \in \mathcal{E}_0} \sum_{b \in \mathcal{A}} N(r(\varepsilon), b; k) w(b, k)^s}.$$

Now, since $w(b, n)^s = \lambda^{(-n+1)s} w(b, 1)^s$, we can write:

$$\sum_{b \in \mathcal{A}} N(a, b; k) w(b, n + k)^s = \lambda^{(-n-k+1)s} E_a^{\mathrm{T}} A^k W(s),$$

where E_a is the vector $(\delta_a(b))_{b \in \mathcal{A}}$, and W is the continuous vector-valued function $s \mapsto (w((b, 1))^s)_{b \in \mathcal{A}}$. Similarly,

$$\sum_{\varepsilon \in \mathcal{E}_0} \sum_{b \in \mathcal{A}} N(r(\varepsilon), b; k) w(b, k)^s = \lambda^{(-k+1)s} E_{\mathcal{E}_0}^{\mathrm{T}} A^k W(s),$$

where $E_{\mathcal{E}}$ is the sum over $\varepsilon \in \mathcal{E}_0$ of all $E_{r(\varepsilon)}$.

Now, by lemma 3.5, we have:

$$E_a^{\mathrm{T}} A^k W(s) = c_a(s) \Lambda_{PF}^n + \sum_{i=1}^{p} P_i(n,s)\alpha_i^n,$$

where the P_i's are polynomial in n for s fixed, with $\Lambda_{PF} > |\alpha_i|$ for all i, and $c_a(s) > 0$. Furthermore, the P_i are continuous in s. Similarly,

$$E_{\mathcal{E}_0}^{\mathrm{T}} A^k W(s) = c_{\mathcal{E}}(s) \Lambda_{PF}^n + \sum_{i=1}^{p} Q_i(n,s)\alpha_i^n,$$

and since $E_{\mathcal{E}_0}$ is a linear combination of the E_a's,

$$c_{\mathcal{E}} = \sum_{\varepsilon \in \mathcal{E}_0} c_{r(\varepsilon)}. \tag{3.10}$$

Then, we write:

$$
\begin{aligned}
f(s) &= \lambda^{-ns} \frac{\displaystyle\sum_{k\geq 0} c_a(\Lambda_{PF}/\lambda^s)^n + \sum_{k\geq 0}\sum_{i=1}^{p} P_i(n,s)(\alpha_i/\lambda^s)^n}{\displaystyle\lambda^s + \sum_{k\geq 0} c_{\mathcal{E}}(\Lambda_{PF}/\lambda^s)^n + \sum_{k\geq 0}\sum_{i=1}^{p} Q_i(n,s)(\alpha_i/\lambda^s)^n} \\[2mm]
&= \lambda^{-ns} \frac{\displaystyle\sum_{k\geq 0} c_a(\Lambda_{PF}/\lambda^s)^n + R_1(s)}{\displaystyle\lambda^s + \sum_{k\geq 0} c_{\mathcal{E}}(\Lambda_{PF}/\lambda^s)^n + R_2(s)},
\end{aligned}
$$

Note that the expression above is defined *a priori* for $s > d$, but R_i ($i = 1,2$) is defined and continuous for $s \geq d$ (the continuity results from the absolute convergence of the sum). The remaining sums above can be computed explicitly, and we have:

$$f(s) = \lambda^{-ns} \frac{c_a(s) + \left(1 - (\Lambda_{PF}/\lambda^s)^n\right) R_1(s)}{c_{\mathcal{E}} + \left(1 - (\Lambda_{PF}/\lambda^s)^n\right)(R_2(s) + \lambda^s)}.$$

Then it is now clear that this expression is continuous when s tends to d, and has limit $\Lambda_{PF}^{-n} c_a(d)/c_{\mathcal{E}}(d)$.

Let u_a be defined as $c_a/c_{\mathcal{E}}$ for all a. Let us show that $u = v$. First, show that $(u_a)_{a \in \mathcal{A}}$ (or equivalently, $(c_a)_{a \in \mathcal{A}}$) is an eigenvector of A associated with Λ_{PF}. We have:

$$
\begin{aligned}
c_a(d) &= \lim_{n \to +\infty} \frac{E_a A^n W(d)}{\Lambda_{PF}^n} \\
&= E_a L W(d),
\end{aligned}
$$

where $L = xy^{\mathrm{T}}$, with x (resp. y) an eigenvector of A (resp. of A^{T}) associated with Λ_{PF}, and $\langle x, y \rangle = 1$. So c_a is the a-coordinate of $LW(d) = \langle y, W(d) \rangle x$, and so is an eigenvector of A associated to Λ_{PF}. Equation (3.10) now proves that u has the good normalization, and that $u = v$. $\qquad\square$

3.3 Complexity and box counting dimension

Definition 3.9. Let (X, d) be a compact metric space. Then, the box counting dimension is defined as the following limit, when it exists:

$$\dim(X, d) = \lim_{t \to 0} -\frac{\ln(N_t)}{\ln(t)},$$

where N_t is the minimal number of balls of radius t needed to cover X.

Theorem 3.10 (Thm. 2 in [36]). *Given an ultrametric Cantor set and its associated ζ-function, let s_0 be the abscissa of convergence of ζ. Then:*

$$s_0 = \dim(X, d).$$

This dimension can be linked to complexity for aperiodic repetitive tilings.

Definition 3.11. The complexity of a tiling T is a function p defined as follows:

$$p(n) = \text{Card} \left\{ q \subset (T - x) \; ; \; x \in \mathbb{R}^d, \; B(0, n) \subset \text{Supp}(q) \text{ and } \forall q' \subset q, \; B(0, n) \not\subset \text{Supp}(q') \right\}.$$

In other words, $p(n)$ is the number of patches q of T of size n.

Note that when T is repetitive (for example when T is a substitution tiling) the complexity is the same for all the tilings which are in the same tiling space.

Theorem 3.12. *Let Ω be a minimal tiling space, and Ξ its canonical transversal, endowed with the metric defined in Section 2.2. Let p be the associated complexity function. Then, for this metric, the box dimension is given by:*

$$\dim(\Omega, d) = \lim_{n \to +\infty} \frac{\ln(p(n))}{\ln(n)}.$$

Proof. Let N_t be the number of balls of diameter smaller than t needed to cover Ξ. Let us first prove that for all $n \in \mathbb{N}$,

$$p(n) = N_{1/n}.$$

Let $\mathcal{L}(n)$ be the set of all patches of size n, so that $p(n) = \text{Card}(\mathcal{L}(n))$. Then, since all the tilings of Ξ have some patch of size n at the origin, the set:

$$\left\{ U_q \; ; \; q \in \mathcal{L}(n) \right\}$$

is a cover of Ξ by sets of diameter smaller than $1/n$. So $p(n) \geq N_{1/n}$. To prove the equality, assume we have a covering of Ξ by open sets $\{V_i \; ; \; i \in I\}$, with $\text{Card}(I) < p(n)$, and $\text{diam}(V_i) \leq 1/n$ for all n. Then, in every V_i, we can find some set of the form U_q, q a patch. This allows us to associate some patch $q(i)$ to all $i \in I$. We claim that for all such $q(i)$, $B(0, n) \subset \text{Ext}(q(i))$, where $\text{Ext}(q)$ is the maximal unambiguous extension of p, as defined in 2.15. Indeed, if it were not the case, $\text{diam}(U_q)$ would be smaller than n. Therefore, by restriction, to each $i \in I$, we can associate a patch $q'(i)$ of size n, such that $U_{q'(i)} \subset V_i$. Since the $\{V_i\}_{i \in I}$ cover Ξ, all patches of size n are obtained this way, and $p(n) \leq \text{Card}(I)$. Since this holds for all cover, $p(n) \leq N_{1/n}$.

Now, N_t is of course an increasing function of t. Therefore,

$$N_{t-1} \leq p\big([1/t]\big) \leq N_t,$$

and so:

$$-\frac{N_{t-1}}{\ln(t)} \leq \frac{p\big([1/t]\big)}{\ln([1/t])} \leq -\frac{N_t}{\ln(t-1)}.$$

Letting t tend to zero proves the theorem. $\qquad\square$

Corollary 3.13. *Let Ξ be the transversal of a minimal aperiodic tiling space, with a complexity function which satisfies $C_1 n^\alpha \leq p(n) \leq C_2 n^\alpha$ for $C_1, C_2, \alpha > 0$. Then:*

$$\dim(\Xi, d) = \alpha.$$

Let now consider a tiling space Ω associated to a substitution ω. Let $\mathcal{B}$ be the weighted Bratteli diagram associated with it. We proved in Section 3.2 that for a Bratteli diagram associated with a substitution tiling of dimension d, the abscissa of convergence is exactly d. It is furthermore true that the box dimension of the transversal Ξ with the usual metric is d; it results from the invariance of the box dimension under bi-Lipschitz equivalence, which is proved in [14, Ch. 2.1].

Therefore, we can deduce the following result:

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Corollary 3.14. *Let Ω be a substitution tiling space satisfying our conditions. Then there exists a function ν such that:*

$$p(n) = n^{\nu(n)}, \quad with \quad \lim_{n \to +\infty} \nu(n) = d.$$

Equivalently, for all $\epsilon > 0$, there exists $C_1, C_2 > 0$ such that for all n large enough,

$$C_1 n^{d-\epsilon} \le p(n) \le C_2 n^{d+\epsilon}.$$

This result is actually weaker than what we can actually expect: in fact, there exist $C_1, C_2 > 0$ such that

$$C_1 n^d \le p(n) \le C_2 n^d.$$

The upper bound was proved by Hansen and Robinson for self-affine tilings (see [39]). The lower bound can be proved by direct analysis for substitution tilings. It would also result from the conjecture that any d-dimensional tiling with low complexity (which means $p(n)/n^d$ tends to zero) has at least one period (see [34]).

However, it is still interesting to see how the apparently abstract fact that the abscissa of convergence s_0 equals the dimension gives in fact a result on complexity.

4 Laplace–Beltrami operator

Let $\mathcal{B}$ be a weighted Bratteli diagram. The Dixmier trace (3.8) induces a probability measure ν on the set E of choice functions (see [36] section 7.2.). The following is [36] Theorem 4.

Proposition 4.1. *For all $s \in \mathbb{R}$ the bilinear form on $L^2(\partial\mathcal{B}, d\mu)$ given by*

$$Q_s(f,g) = \frac{1}{2} \int_E \mathrm{Tr}\left(|D|^{-s}[D, \pi_\tau(f)]^*[D, \pi_\tau(g)]\right) d\nu(\tau), \tag{4.11}$$

with dense domain $\mathrm{Dom}\, Q_s = \langle \chi_\gamma : \gamma \in \Pi \rangle$, is a closable Dirichlet form.

The classical theory of Dirichlet forms [17] allows to identify $Q_s(f,g)$ with $\langle f, \Delta_s g\rangle$ for a non-positive definite self-adjoint operator Δ_s on $L^2(\partial\mathcal{B}, d\mu)$ which is the generator of a Markov semi-group. We have $\mathrm{Dom}\, Q_s \subset \mathrm{Dom}\, \Delta_s \subset \mathrm{Dom}\, \tilde{Q}_s$ where $\tilde{Q}_s$ is the smallest closed extension of Q_s. The following is taken from [36] section 8.3.

Theorem 4.2. *The operator Δ_s is self-adjoint and has pure point spectrum.*

Following Pearson and Bellissard we can calculate Δ_s explicitly on characteristic functions of cylinders. For a path $\eta \in \Pi$ let us denote by $\mathrm{ext}_1(\eta)$ the set of ordered pairs of distinct edges (e, e') which can extend η one generation further.

$$\Delta_s \chi_\gamma = -\sum_{k=0}^{|\gamma|-1} \frac{1}{G_s(\gamma_k)} \Big((\mu[\gamma_k] - \mu[\gamma_{k+1}])\chi_\gamma - \mu[\gamma](\chi_{\gamma_k} - \chi_{\gamma_{k+1}}) \Big) \tag{4.12a}$$

$$\text{with} \quad G_s(\eta) = \frac{1}{2}\, \mathrm{diam}[\eta]^{2-s} \sum_{(e,e')\in \mathrm{ext}_1(\eta)} \mu[\eta \cdot e]\mu[\eta \cdot e'] \tag{4.12b}$$

Note that the term $\chi_{\gamma_k} - \chi_{\gamma_{k+1}}$ is the characteristic function of all the paths which coincide with γ up to generation k and differ afterwards, *i.e.* all paths which split from γ at generation k. And $\mu[\gamma_k] - \mu[\gamma_{k+1}]$ is the measure of this set.

We now state the main theorem which gives explicitly the full spectrum of Δ_s.

Theorem 4.3. *The spectrum of Δ_s is given by the following.*

(i) 0 is a single eigenvalue with eigenspace $\langle 1 = \chi_{\partial\mathcal{B}} \rangle$.

(ii) $\lambda_0 = \frac{1}{G_s(\circ)}$ is eigenvalue with eigenspace $E_0 = \langle \frac{1}{\mu[\varepsilon]} \chi_\varepsilon - \frac{1}{\mu[\varepsilon']} \chi_{\varepsilon'} \; : \; \varepsilon, \varepsilon' \in \mathcal{E}_0 \, , \varepsilon \neq \varepsilon' \rangle$ of dimension $\dim E_0 = n_0 - 1$, where n_0 is the cardinality of $\mathcal{E}_0$.

(iii) For $\gamma \in \Pi$,

$$\lambda_\gamma = \sum_{k=0}^{|\gamma|-1} \frac{\mu[\gamma_{k+1}] - \mu[\gamma_k]}{G_s(\gamma_k)} - \frac{\mu[\gamma]}{G_s(\gamma)} \tag{4.13}$$

is eigenvalue with eigenspace

$$E_\gamma = \Big\langle \frac{1}{\mu[\gamma \cdot e]} \chi_{\gamma \cdot e} - \frac{1}{\mu[\gamma \cdot e']} \chi_{\gamma \cdot e'} \; : \; (e, e') \in \mathrm{ext}_1(\gamma) \Big\rangle \tag{4.14}$$

of dimension $\dim E_\gamma = n_\gamma - 1$, where n_γ is the number of edges extending γ one generation further.

Proof. The formula for the eigenvalues is calculated easily noticing that $\Delta_s \chi_{\gamma \cdot e}$ and $\Delta_s \chi_{\gamma \cdot e'}$ only differ by the last term in the sum in equation (4.12a).

The spectrum of Δ_s is always the closure of its set of eigenvalues (whatever its domain may be). Hence we do not miss any of it by restricting to characteristic functions.

We now show that all the eigenvalues of Δ_s are exactly given by the λ_γ. It suffices to check that the restriction of Δ_s to Π_n has exactly $\dim \Pi_n$ eigenvalues (counting multiplicity). Notice that χ_γ is the sum of $\chi_{\gamma \cdot e}$ over all edges e extending γ one generation further. Hence an eigenfunction in E_γ for $\gamma \in \Pi_k$ can be written as a linear combination of characteristic functions of paths in Π_m, for any $m > k$. The number of eigenvalues λ_γ for $\gamma \in \Pi_{n-1}$ is $\sum_{\gamma \in \Pi_{n-1}} \dim E_\gamma = \sum_{\gamma \in \Pi_{n-1}} (n_\gamma - 1) = \dim \Pi_n - \dim \Pi_{n-1}$. So the number of eigenvalues λ_γ for $\gamma \in \Pi_k$ for all $1 \leq k \leq n-1$ is $\sum_{k=1}^{n-1} \big(\dim \Pi_{k+1} - \dim \Pi_k \big) = \dim \Pi_n - \dim \Pi_1$. And counting 0 and λ_0 adds up $(\dim \Pi_1 - 1) + 1$ to make the count match. $\qquad \square$

Remark 4.4. As noted in Remark 2.3, our formalism with Bratteli diagrams includes as a special case the approach of Pearson–Bellissard for weighted Cantorian Michon trees. Hence Theorem 4.3 gives also the spectrum and eigenvectors of their Laplace–Beltrami operators.

The eigenvectors (4.14) are very simple to picture. Given a path γ, and two extensions $(a, b) \in \mathrm{ext}_1(\gamma)$, an eigenvector is the difference of their characteristic functions weighted by their measures. See Figure 2 in section 1 for an example for the Fibonacci diagram.

5 Cuntz–Krieger algebras and applications

We now consider stationary Bratteli diagrams. We use the self-similar structure to further characterize the operator Δ_s and its spectrum.

5.1 Cuntz–Krieger algebras

Let $\mathcal{B}$ be a stationary Bratteli diagram. Let A be its Abelianization matrix. Let us denote by $\mathcal{E}_0$ the set of edges ε linking the root to generation 1, and by $\mathcal{E}$ the set of edges linking two generations (excluding the root). Let $\tilde{A} = \big(\tilde{a}_{ef} \big)_{e,f \in \mathcal{E}}$ be the square matrix with entries $\tilde{a}_{ef} = 1$ if e can be composed with (or followed down the diagram by) f and $\tilde{a}_{ef} = 0$ else. There is an associated Bratteli diagram $\tilde{\mathcal{B}}$ with Abelianization matrix $\tilde{A}$, which is "dual" to $\mathcal{B}$ in the sens that its vertices corresponds to the edges of $\mathcal{B}$ and its edges to the adjacencies of edges in $\mathcal{B}$. Note that, because the entries of $\tilde{A}$ are zeros or ones, all the edges in $\tilde{\mathcal{B}}$ are simple.

Remark 5.1. If $\mathcal{B}$ has only simple edges, we can simply take $\tilde{\mathcal{B}} = \mathcal{B}$ and $\tilde{A} = A$.

The Cuntz-Krieger algebra $\mathcal{O}_{\tilde{A}}$, is the C^*-algebra generated by the partial isometries $U_e, e \in \mathcal{E}$ (on a separable, complex, and infinite dimensional Hilbert space $\mathcal{H}$) that satisfy the following relations.

$$\mathcal{O}_{\tilde{A}} = C^* \langle\, U_e, U_e^*\,, e \in \mathcal{E} \ \mid U_e U_e^*, U_e^* U_e \in \mathcal{P}(\mathcal{H})\,, \ U_e^* U_e = \sum_{f \in \mathcal{E}} \tilde{A}_{ef}\, U_f U_f^* \,\rangle, \qquad (5.15)$$

where $\mathcal{P}(\mathcal{H})$ denotes the set of projections in $\mathcal{H}$: $p \in \mathcal{P}(\mathcal{H}) \iff p^2 = p^* = p$.

By abuse of notation we write the basis elements of $l^2(\Pi \backslash \Pi_1)$ as $\gamma \in \Pi \backslash \Pi_1$. The Cuntz-Krieger algebra $\mathcal{O}_{\tilde{A}}$ is represented on $l^2(\Pi \setminus \Pi_1)$ as follows.

$$U_e(\varepsilon, e_1, e_2, \cdots) = \begin{cases} (\varepsilon', e, e_1, e_2, \cdots) & \text{if } A_{ee_1} = 1 \\ 0 & \text{else} \end{cases} \qquad (5.16a)$$

$$U_e^*(\varepsilon, e_1, e_2, \cdots) = \begin{cases} (\varepsilon', e_2, e_3, \cdots) & \text{if } e_1 = e \\ 0 & \text{else} \end{cases} \qquad (5.16b)$$

where $\varepsilon' \in \mathcal{E}_0$ stands for the (possibly) new edges linking to the root, as illustrated in the case of the Penrose substitution below. The orientation of ε' however is taken to be the same as that of ε: if $\varepsilon = (s(e_1), g)$ then we have $\varepsilon' = (s(e), g)$ (see Remark 2.24). In other words we require that $\psi(\varepsilon', e, e_1, \cdots) = \omega \circ \psi(\varepsilon, e_1, \cdots) + x$ for some $x \in \mathbb{R}^d$, and where $\psi : \partial \mathcal{B} \to \Xi$ is the homeomorphism of Proposition 2.22. See Figure 4 for some examples.
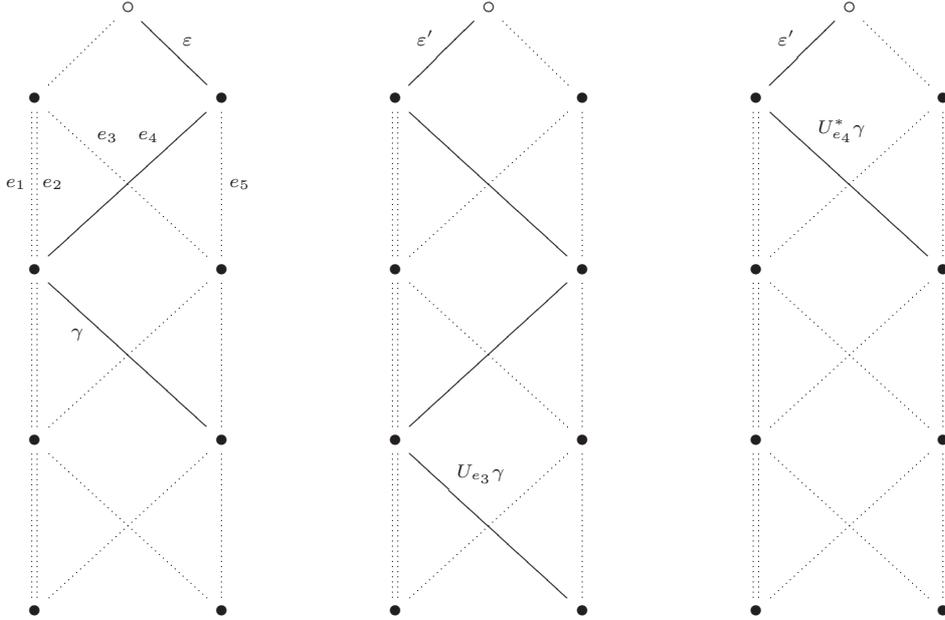

Figure 4: Example of some Cuntz–Krieger operators acting on finite paths.

There is an induced "action" on $\text{Dom} \Delta_s \subset L^2(\partial \mathcal{B}, d\mu)$ defined as follows:

$$U_e^{(*)} \chi_\gamma = \begin{cases} \chi_{U_e^{(*)}\gamma} & \text{if } U_e^{(*)}\gamma \neq 0 \\ 0 & \text{else} \end{cases} \qquad (5.17)$$

for paths in $\Pi_{n \geq 2}$ and by linearity for shorter paths, using the relations $\chi_\gamma = \sum_e \chi_{\gamma \cdot e}$ (the sum running over all edges e extending the path one generation further).

If $\varphi \in E_\gamma$, for $\gamma \in \Pi_{n \geq 2}$, is an eigenfunction of Δ_s, then we see from Equation (5.17) that $U_e \varphi$, if not zero, is another eigenfunction. Since the diagram $\mathcal{B}$ is stationary, we have the following obvious fact.

$$U_e\, E_\gamma = \begin{cases} E_{U_e \gamma} & \text{if } U_e \gamma \neq 0 \\ \{0\} & \text{else}\,. \end{cases} \qquad (5.18)$$

This allows to define an induced "action" of $\mathcal{O}_{\tilde{A}}$ on the set of eigenvalues of Δ_s as follows.

$$u_e(\lambda_\gamma) = \begin{cases} \lambda_{U_e\gamma} & \text{if } U_e\gamma \neq 0 \\ 0 & \text{else}. \end{cases} \qquad (5.19)$$

Those maps u_e are calculated as shown below.

Lemma 5.2. *Let Λ_{PF} be the Perron–Frobenius eigenvalue of A. Let $\gamma = (\varepsilon, e_1, e_2, \cdots) \in \Pi_{n \geq 2}$. Then, if $U_e\gamma \neq 0$, we have*

$$u_e(\lambda_\gamma) = \Lambda_{PF}^{(d+2-s)/d}\left(\lambda_\gamma - \frac{\mu[\varepsilon] - \mu[\circ]}{G_s(\circ)}\right) + \frac{\mu[\varepsilon'] - \mu[\circ]}{G_s(\circ)} + \frac{\mu[\varepsilon'e] - \mu[\varepsilon']}{G_s(\varepsilon')} \qquad (5.20)$$

Proof. By equation (4.13) we have

$$\lambda_{U_e\gamma} = -\sum_{k=0}^{|U_e\gamma|-1} \frac{\mu[(U_e\gamma)_k] - \mu[(U_e\gamma)_{k+1}]}{G_s((U_e\gamma)_k)} - \frac{\mu[U_e\gamma]}{G_s(U_e\gamma)}. \qquad (5.21)$$

The terms for $k = 0, 1$, in the above sum give the last two terms in equation (5.20). For all $k \geq 2$, $\mu[(U_e\gamma)_k] = \Lambda_{PF}^{-1}\mu[\gamma_{k-1}]$, and $G_s((U_e\gamma)_k) = \Lambda_{PF}^{(s-d)/d-2}G_s(\gamma_{k-1})$ (see Theorem 3.8 for the rescaling of the measures). Hence the rest of the sum, over $k = 2, \cdots |U_e\gamma|-1$ (and the last term) in equation (5.21), rescale by a factor $\Lambda_{PF}^{(d-s+2)/d}$ to the sum over $k = 1, \cdots |\gamma|-1$ (and the last term) for the eigenvalue λ_γ (equation (4.13)). We then add the contribution of the root, *i.e.* the term for $k = 0$, to get equation (5.20). $\qquad \square$

Note that the maps u_e are affine, with constant terms, written β_e, that only depend on e. We will write for now on

$$u_e(\lambda_\gamma) = \Lambda_s\lambda_\gamma + \beta_e, \qquad \text{with} \quad \Lambda_s = \Lambda_{PF}^{(d+2-s)/d}. \qquad (5.22)$$

The eigenelements of Δ_s corresponding to E_0, and $E_\varepsilon, \varepsilon \in \mathcal{E}_0$, are immediate to calculate explicitly from equations (4.13) and (4.14). We can therefore calculate explicitly all other eigenelements by action of $\mathcal{O}_{\tilde{A}}$ on those corresponding to the $E_\varepsilon, \varepsilon \in \mathcal{E}_0$. We summarize this in the following.

Proposition 5.3. *For $\gamma = (\varepsilon, e_1, e_2, \cdots e_n) \in \Pi_{n \geq 1}$, set $U_\gamma = U_{e_1}U_{e_2} \cdots U_{e_n}$ and $u_\gamma = u_{e_1} \circ u_{e_2} \circ \cdots u_{e_n}$. For any $\gamma \in \Pi_{n \geq 1}$, we have $E_\gamma = U_\gamma E_{\varepsilon'}$, and*

$$\lambda_\gamma = u_\gamma(\lambda_{\varepsilon'}) = \Lambda_s^n \lambda_{\varepsilon'} + \sum_{j=1}^{n} \Lambda_s^{j-1}\beta_{e_j}. \qquad (5.23)$$

5.2 Bounded case

We consider here the case $s > d+2$. We show that Δ_s is bounded and characterize the boundary of its spectrum.

Proposition 5.4. *For $s > d+2$, Δ_s is a bounded, and we have*

$$\|\Delta_s\|_{\mathcal{B}(L^2(\partial\mathcal{B}, d\mu))} \leq c\, \frac{1}{1 - \Lambda_{PF}^{(d+2-s)/d}},$$

with $c = \max\{\max_{\varepsilon \in \mathcal{E}_0} |\lambda_\varepsilon|, \max_{e \in \mathcal{E}} |\beta_e|\}$.

Proof. By Proposition 5.3, equation (5.23), we see that for $\gamma \in \Pi_n$ we have.

$$|\lambda_\gamma| \leq \max\{\max_{\varepsilon \in \mathcal{E}_0} |\lambda_\varepsilon|, \max_{e \in \mathcal{E}} |\beta_e|\} \sum_{j=0}^{n} \Lambda_s^j.$$

From equation (5.22) we have $\Lambda_s = \Lambda_{PF}^{(d-s+2)/d} < 1$, therefore the above geometric sum converges, and is bounded for all n by its sum $1/(1 - \Lambda_s)$. $\qquad \square$

We define the ω-spectrum of Δ_s as

$$\mathrm{Sp}_\omega(\Delta_s) = \bigcap_{n \in \mathbb{N}} \overline{\mathrm{Sp}(\Delta_s) \setminus \mathrm{Sp}\big(\Delta_s|_{\Pi_n}\big)}\,.$$

In our case here, this is the boundary of the (pure point) spectrum of Δ_s. Under some conditions on A and the $\beta_e, e \in \mathcal{E}$, and for $s > d+2$ large enough, one can show that $\mathrm{Sp}_\omega(\Delta_s)$ is homeomorphic to Ξ, and that this homeomorphism is *Hölder* [24].

5.3 Weyl asymptotics

The following theorem justifies calling Δ_s a Laplace–Beltrami operator. Indeed, for $s = s_0 = d$, Theorem 5.5 shows that the number of eigenvalues of Δ_d of modulus less that λ behaves like $\lambda^{d/2}$ when $\lambda \to \infty$, which is the classical Weyl asymptotics for the Laplacian on a compact d-manifold.

Theorem 5.5. *Let $\mathcal{N}_s(\lambda) = \mathrm{Card}\big\{\lambda' \text{ eigenvalue of } \Delta_s \ : \ |\lambda'| \leq \lambda\big\}$. For $s < d+2$, we have the following Weyl asymptotics*

$$c_- \lambda^{d/(d-s+2)} \leq \mathcal{N}_s(\lambda) \leq c_+ \lambda^{d/(d-s+2)}\,, \tag{5.24}$$

as $\lambda \to +\infty$, for some constants $0 < c_- < c_+$.

Proof. By Proposition 5.3, equation (5.23), there exists constants $x_\pm, y_\pm > 0$, such that for all $\gamma \in \Pi_n$ we have $x_- \Lambda_s^n + y_- \leq |\lambda_\gamma| \leq x_+ \Lambda_s^n + y_+$. For $s < d+2$, $\Lambda_s > 1$ (see equation (5.22)), so there is an integer $k > 0$ (independent of n) such that $x_+ \Lambda_s^n + y_+ \leq x_- \Lambda_s^{n+k} + y_-$. Hence for all $\gamma \in \Pi_l$, $l \leq n$, we have $|\lambda_\gamma| \leq x_+ \Lambda_s^n + y_+$, and for all $\gamma \in \Pi_l$, $l \geq n+k$, we have $|\lambda_\gamma| \geq x_- \Lambda_s^{n+k} + y_-$. Therefore we get the inequalities: $\mathrm{Card}\,\Pi_n \leq \mathcal{N}(x_+ \Lambda_s^n + y_+) \leq \mathrm{Card}\,\Pi_{n+k}$. There are constants $c_1 > c_2 > 0$ such that for all $l \in \mathbb{N}$, $c_1 \Lambda_{PF}^l \leq \mathrm{Card}\,\Pi_l \leq c_2 \Lambda_{PF}^n$, so that we get $c_1 \Lambda_{PF}^n \leq \mathcal{N}(x_+ \Lambda_s^n + y_+) \leq c_2 \Lambda_{PF}^{n+k} = c_3 \Lambda_{PF}^n$. We substitute Λ_s from equation (5.22) to complete the proof. $\square$

5.4 Seeley equivalent

For the case $s = s_0 = d$ we give an equivalent to the trace $\mathrm{Tr}\big(e^{t\Delta_d}\big)$, for $s = s_0 = d$, as $t \downarrow 0$. The behavior of $\mathrm{Tr}\big(e^{t\Delta_d}\big)$ like $t^{-d/2}$ as $t \downarrow 0$ is in accordance with the leading term of the classical Seeley expansion for the heat kernel on a compact d–manifold.

Theorem 5.6. *There exists constants $c_+ \geq c_- > 0$, such that as $t \downarrow 0$*

$$c_- t^{-d/2} \leq \mathrm{Tr}\big(e^{t\Delta_d}\big) \leq c_+ t^{-d/2}\,. \tag{5.25}$$

Proof. Let P_γ be the spectral projection (onto E_γ) for $\gamma \in \Pi_{n \geq 1}$, and P_0 that on E_0. The trace reads

$$\mathrm{Tr}\big(e^{t\Delta_s}\big) = 1 + e^{\lambda_0 t} \mathrm{Tr}(P_0) + \sum_{n=1}^\infty \sum_{\gamma \in \Pi_n} e^{t\lambda_\gamma} \mathrm{Tr}(P_\gamma)\,. \tag{5.26}$$

Now $\mathrm{Tr}(P_\gamma) = n_\gamma - 1$, with n_γ the number of possible extensions of γ one generation further (see equation (4.14) in theorem 4.3), and $\mathrm{Tr}(P_0) = n_0 - 1$. Since the Bratteli diagram of the substitution is stationary, the integers n_γ are bounded, so there are $p_-, p_+ > 0$, such that for all $\gamma \in \Pi$ we have:

$$p_- \leq \mathrm{Tr}(P_\gamma) \leq p_+\,. \tag{5.27}$$

By equation (5.23), the $\lambda_\varepsilon, \beta_e$, being bounded, there exists $\lambda_-, \lambda_+ > 0$, such that for all $\gamma \in \Pi$ we have:

$$\lambda_- \Lambda_{PF}^{nd/2} \leq |\lambda_\gamma| \leq \lambda_+ \Lambda_{PF}^{nd/2}\,. \tag{5.28}$$

The cardinality of Π_n grows like Λ_{PF}^n so there are $\pi_-, \pi_+ > 0$, such that for all $n \geq 0$ we have:

$$\pi_- \Lambda_{PF}^n \leq |\Pi_n| \leq \pi_+ \Lambda_{PF}^n\,. \tag{5.29}$$

We substitute inequalities (5.27), (5.28), and (5.29) into equation (5.26) to get

$$1 + p_-\pi_- \sum_{n=0}^{\infty} \Lambda_{PF}^n e^{-t\lambda_+ \Lambda_{PF}^{nd/2}} \leq \operatorname{Tr}\left(e^{t\Delta_s}\right) \leq 1 + p_+\pi_+ \sum_{n=0}^{\infty} \Lambda_{PF}^n e^{-t\lambda_- \Lambda_{PF}^{nd/2}} . \qquad (5.30)$$

Set $N_t = d\log(1/t)/(2\log(\Lambda_{PF}))$, and split the above sums into two parts: the sum over $n < N_t$, and the remainder. For the finite sum we have:

$$\sum_{n=0}^{N_t-1} \Lambda_{PF}^n e^{-t\lambda_{\pm} \Lambda_{PF}^{nd/2}} = t^{-d/2} \Lambda^{-N_t} \sum_{n=0}^{N_t-1} \Lambda_{PF}^n e^{-\lambda_{\pm} \Lambda_{PF}^{(n-N_t)d/2}} , \qquad (5.31)$$

where we have used $\Lambda_{PF}^{N_t d/2} = 1/t$. With the inequalities $e^{-\lambda_{\pm}} \leq e^{-\lambda_{\pm} \Lambda_{PF}^{(n-N_t)d/2}} \leq 1$, the above sum on the right hand side of (5.31) is bounded by the geometric series $\sum_{n=0}^{N_t-1} \Lambda_{PF}^n = (\Lambda_{PF}^{N_t} - 1)/(\Lambda_{PF} - 1)$. Multiplying by $t^{-d/2}\Lambda^{-N_t}$ we get the inequalities:

$$t^{-d/2} c'_- \leq \sum_{n=0}^{N_t-1} \Lambda_{PF}^n e^{-t\lambda_{\pm} \Lambda_{PF}^{nd/2}} \leq t^{-d/2} c'_+ , \qquad (5.32)$$

for some constants $c'_-, c'_+ > 0$, and t small enough.

For the remainder of the sums in (5.30) we have

$$\sum_{n=N_t}^{\infty} \Lambda_{PF}^n e^{-t\lambda_{\pm} \Lambda_{PF}^{nd/2}} = \Lambda_{PF}^{N_t} \sum_{m=0}^{\infty} \Lambda_{PF}^m e^{-\lambda_{\pm} \Lambda_{PF}^{md/2}} = t^{-d/2} c''_{\pm} , \qquad (5.33)$$

where $c''_{\pm} > 0$ is the sum of the absolutely convergent series. We put together inequalities (5.32) and equation (5.33) into inequalities (5.30) to complete the proof. $\qquad \square$

5.5 Eigenvalues distribution

We now restrict to the case $s = s_0 = d$, so that $\Lambda_d = \Lambda_{PF}^{2/d}$ in equation (5.22). We also suppose here that the weights are simply given by the d-th roots of the measures: $\operatorname{diam}[\gamma] = \mu[\gamma]^{1/d}$.

Let us consider the cases $d = 1$ and $d = 2$ first. One sees from equations (4.12b), (4.13), and (5.20), that the $\lambda_\varepsilon, \varepsilon \in \mathcal{E}_0$, and $\beta_e, e \in \mathcal{E}$, are rational functions of Λ_{PF}. Hence by equation (5.23) all the eigenvalues of Δ_d belong to the field $\mathbb{Q}[\Lambda_{PF}]$. The maps u_e in equation (5.22) become affine maps in this field, whose linear part is the multiplication by $\Lambda_{PF}^{2/d}$ ($= \Lambda_{PF}^2$ or Λ_{PF}).

Let $P_A(X) = X^r + a_{r-1}X^{r-1} + \cdots a_1 X + a_0$ be the minimal polynomial of A. The field $\mathbb{Q}[\Lambda_{PF}]$ is isomorphic to $\mathbb{Q}^r$, so that the eigenvalues of Δ_d can be represented by points in $\mathbb{Q}^r$. The $\lambda_\varepsilon, \beta_e$, are identified with fixed vectors $\vec{\lambda}_\varepsilon, \vec{\beta}_e \in \mathbb{Q}^r$, and the maps u_e become affine maps in $\mathbb{Q}^r$. The multiplication by $\Lambda_{PF}^{2/d}$ (linear parts of the maps u_e) is implemented in the basis $\mathbb{Q} \oplus \Lambda_{PF}^{2/d}\mathbb{Q} \oplus \cdots \Lambda_{PF}^{2(r-1)/d}\mathbb{Q}$ by the companion matrix of P_A:

$$\begin{bmatrix} 0 & & & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -a_{r-2} \\ & & 1 & -a_{r-1} \end{bmatrix} .$$

Note that it is equivalent to A and therefore has the same Perron–Frobenius eigenvector. Since there are finitely many $\vec{\beta}_e, \vec{\lambda}_\varepsilon \in \mathbb{Q}^r$, upon multiplication by a large enough integer, we can actually represent the eigenvalues by points of the lattice $\mathbb{Z}^r$. It is then natural to ask which points correspond to eigenvalues of Δ_s and if one can characterize this set. The answer is surprisingly simple:

Eigenvalues correspond to points of integer coordinates in $\mathbb{Q}^r$ that stay within a bounded strip to the Perron–Frobenius eigenline of A.

See section 6.1 for an example, and Figure 3 in section 1.

For $d \neq 1, 2$ a similar result hold, but the points can no longer be chosen to have integer coordinates. In order to prove this in general – for d generic – we consider a quotient ring of $\mathbb{Q}[\Lambda_{PF}]$ where we can implement the multiplication by $\Lambda_{PF}^{2/d}$. Define $Q_A(x) = P_A(x^{d/2})$ if d is even, and let C be its companion matrix. If d is odd define $Q_A(x) = P_A(x^d)$, and let C be the square of its companion matrix. Let us denote by d' either $d/2$ if d is even, or d if it is odd. The matrix C implements the multiplication by $\Lambda_{PF}^{d'}$ on the ring $\mathbb{Q}[X]/Q_A$, which we embed in the vector space $V = \mathbb{R} \oplus \Lambda_{PF}\mathbb{R} \oplus \Lambda_{PF}^2\mathbb{R} \oplus \cdots \Lambda_{PF}^{rd'-1}\mathbb{R}$.

Because of the factor $\mathrm{diam}[\eta]^{2-s} = \mu[\eta]^{(2-d)/d}$ in equation (4.12b), we have to work over the field $\mathbb{R}$. An eigenvalue of Δ_d can then be written as a vector $\vec{\lambda}_\gamma \in V$. The action of the Cuntz-Krieger algebra $\mathcal{O}_{\tilde{A}}$ on such an eigenvalue, given in equation (5.22), becomes here

$$u_e(\vec{\lambda}_\gamma) = C\,\vec{\lambda}_\gamma + \vec{\beta}_e\,, \tag{5.34}$$

where $\vec{\beta}_e$ is the expression of β_e as a vector in V. The general expression of an eigenvalue given in proposition 5.3, equation (5.23), takes here the form

$$\vec{\lambda}_\gamma = C^n\,\vec{\lambda}_{\varepsilon'} + \sum_{j=1}^{n} C^{j-1}\,\vec{\beta}_{e_j}\,. \tag{5.35}$$

We now assume that C is *Pisot*, *i.e.* it has a single real eigenvalue $\mu_u > 1$, and all its other eigenvalues μ satisfy $|\mu| < 1$. Let us denote by V_u the eigenspace of μ_u, and by V_s its orthogonal complement in V. We can now characterize the distribution of the eigenvalues of Δ_d.

Theorem 5.7. *There exist a constant $m > 0$ such that for any eigenvalue λ_γ of Δ_d, one has*

$$\mathrm{dist}_V\big(\vec{\lambda}_\gamma, V_u\big) \leq \frac{m}{1 - \|C_s\|}\,, \tag{5.36}$$

where C_s is the projection of C to V_s, and dist_V the induced Euclidean distance on V.

Proof. The distance $\mathrm{dist}_V\big(\vec{\lambda}_\gamma, V_u\big)$ equals the norm of the projection of $\vec{\lambda}_\gamma$ to V_s:

$$
\begin{aligned}
\mathrm{dist}_V\big(\vec{\lambda}_\gamma, V_u\big) = \quad & \big\|P^{-1}\big(C_s^n P^{-1}\vec{\lambda}_{\varepsilon'} + \sum_{j=1}^{n} C_s^{j-1} P^{-1}\vec{\beta}_{e_j}\big)\big\| \\
\leq \quad & \|P^{-1}\|^2 \max\{\|\vec{\beta}_e\|, \|\vec{\lambda}_\varepsilon\| : \varepsilon \in \mathcal{E}_0, e \in \mathcal{E}\} \sum_{j=0}^{n} \|C_s^j\| \\
\leq \quad & m \sum_{j=0}^{\infty} \|C_s\|^j = m/(1 - \|C_s\|)\,,
\end{aligned}
$$

where P is an invertible matrix diagonalizing C, $m = \|P^{-1}\|^2 \max\{\|\vec{\beta}_e\|, \|\vec{\lambda}_\varepsilon\|\}$, and the series converges because C_s is diagonal with eigenvalues of moduli strictly less than one. $\qquad\square$

Remark 5.8. If C is no longer Pisot but strictly hyperbolic (i.e. has no eigenvalue of modulus 1), then the above result still holds but with V_u the unstable space (the span of the eigenvectors with eigenvalues $|\mu| > 1$) and V_s its orthogonal complement.

6 Examples

We illustrate here the results of sections 4 and 5 for the classic examples of the Thue–Morse, Fibonacci, Ammann–A2, and Penrose tilings.

6.1 The Fibonacci diagram

The Bratteli diagram for the (uncollared) Fibonacci substitution $\begin{cases} a & \to & ab \\ b & \to & a \end{cases}$ reads
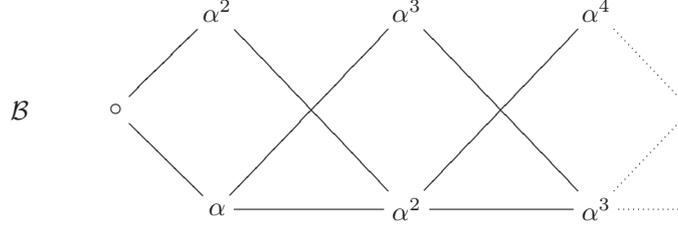
where the top vertices are of "type b" and the bottom ones of "type a", where $\alpha = 1/\phi = (\sqrt{5}-1)/2$ is the inverse of the golden mean, and the term α^n at a vertex is the measure of the cylinder of infinite paths through that vertex.

Note that this substitution does not force the border, so that $\partial\mathcal{B}$ is not the transversal of the Fibonacci tiling space. For illustration purposes it is however worth carrying this example in details. We treat the "real" Fibonacci tiling together with the Penrose tiling in section 6.3.

Since the Bratteli diagram has only simple edges, as noted in Remark 2.6, the paths can be indexed by the vertices they go through. The paths in Π_1 are thus written a and b, and the paths in Π_2 are written aa, ab, and ba (note that these are *not* orthonormal bases for the dot product given by the Dixmier trace , so that the Laplacians written below will not be symmetric). The restrictions of the Laplace operator (4.12) for $s = s_0 = 1$ to Π_1 and Π_2 are given below together with their eigenelements.

$$\Delta|_{\Pi_1} = \begin{bmatrix} 1-\phi^2 & -1+\phi^2 \\ \phi^2 & -\phi^2 \end{bmatrix}, \text{ with eigenelements } \left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \quad \left(1+2\phi^2, \begin{bmatrix} 1 \\ 1-\phi^2 \end{bmatrix}\right).$$

$$\Delta|_{\Pi_2} = \begin{bmatrix} 2-3\phi^2 & -1+2\phi^2 & -1+\phi^2 \\ -1+3\phi^2 & 2-4\phi^2 & -1+\phi^2 \\ -1+\phi^2 & 1 & -\phi^2 \end{bmatrix}, \qquad \text{with eigenelements}$$

$$\left(0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right), \ \left(1+2\phi^2, \begin{bmatrix} 1 \\ 1 \\ 1-\phi^2 \end{bmatrix}\right), \ \left(3+6\phi^2, \begin{bmatrix} 1 \\ 1-\phi^2 \\ 0 \end{bmatrix}\right).$$

Using the identities $\chi_b = \chi_{ba}$, and $\chi_a = \chi_{aa} + \chi_{ab}$, we see that the first two eigenvectors of $\Delta|_{\Pi_2}$ are exactly those of $\Delta|_{\Pi_1}$ expressed in Π_2. Note that $1-\phi^2 = -\phi$ and that the above eigenvectors of Δ are $\chi_a + \chi_b$, $\chi_a - \phi\chi_b$, and $\chi_{aa} - \phi\chi_{ab}$. And all other eigenvectors are given by $\chi_{\gamma aa} - \phi\chi_{\gamma ab}$ for $\gamma \in \Pi$.

Since the Bratteli diagram has only simple edges, as noted in Remark 5.1, we can take $\tilde{\mathcal{B}} = \mathcal{B}$ and $\tilde{A} = A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The action of the two Cuntz–Krieger operators U_a and U_b on the eigenvalues of Δ as in equation (5.19) is given in here by

$$u_a(\lambda_\gamma) = \phi^2\lambda_\gamma + 1 - \phi^2, \qquad u_b(\lambda_\gamma) = \phi^2\lambda_\gamma - 1 + \phi^2, \tag{6.37}$$

if γ is compatible with their action, and $u_a(\lambda_\gamma) = 0$ or $u_b(\lambda_\gamma) = 0$ otherwise.

Over the ring $\mathbb{Z} \oplus \phi^2\mathbb{Z}$, the companion matrix of A is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and the operators (6.37) become the affine maps

$$u_a(\vec{\lambda}_\gamma) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{\lambda}_\gamma - \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad u_b(\vec{\lambda}_\gamma) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{\lambda}_\gamma + \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

when γ is compatible with the corresponding action.

Figure 3 illustrates Theorem 5.7 that characterizes the repartition of the eigenvalues of $-\Delta$ as point of integer coordinates that stay within a bounded strip to the Perron-Froebenius eigenline of A (slope ϕ) in $\mathbb{Z} \oplus \phi^2 \mathbb{Z}$. Note that the repartition of points in the strip is not "homogeneous", *i.e.* the number of points within a distance r to the origin is not linear in r, but rather follows the Weyl asymptotics in $\sqrt{r}$ (Theorem 5.5).

6.2 The dyadic Cantor set and the Thue–Morse tiling

Those examples have enough symmetries to allow easy and direct calculations (without using the operators of the Cuntz–Krieger algebra). The Bratteli diagram $\mathcal{B}$ of the dyadic Cantor set is the dyadic odometer,

$$\mathcal{B} \qquad \circ \overset{0}{\underset{1}{=\!=\!=\!=}} \tfrac{1}{2} \overset{0}{\underset{1}{=\!=\!=\!=}} \tfrac{1}{4} \overset{0}{\underset{1}{=\!=\!=\!=}} \tfrac{1}{8} \cdots\cdots$$

and its associated diagram for its Cuntz-Krieger algebra $\tilde{\mathcal{B}}$ is the Bratteli diagram of the (uncollared) Thue-Morse substitution $\begin{cases} 0 & \to & 01 \\ 1 & \to & 10 \end{cases}$:
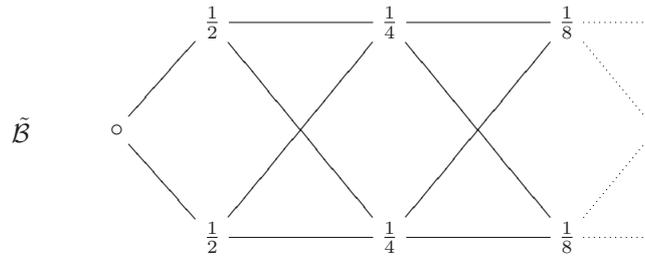
where the term $\frac{1}{2^n}$ at a vertex is the measure of the cylinder of infinite paths through that vertex. The top vertices are of "type 0", the bottom ones of "type 1".

We label the paths in $\mathcal{B}$ by sequences of 0's and 1's labeling the edges they go through from the root: $\gamma \in \Pi_n$ is written $\gamma = (\varepsilon_1, \cdots, \varepsilon_n)$. The Laplacian on $\mathcal{B}$ commutes with the following operators:

$$\tau_i \chi_{(\varepsilon_1, \cdots, \varepsilon_n)} = \begin{cases} \chi_{(\varepsilon_1, \cdots, \varepsilon_i+1, \cdots \varepsilon_n)} & \text{if } 1 \leq i \leq n \,, \\ \chi_{(\varepsilon_1, \cdots, \varepsilon_n)} & \text{else}, \end{cases} \tag{6.38}$$

where the addition is taken mod 2. The operators τ_i commute with each other and square up to the identity. One can therefore choose an eigenbasis for Δ made of eigenelements of the τ_i's: that is *Haar functions* on the dyadic Cantor set $\partial\mathcal{B}$. We recover this way the example treated in [36] and we refer the reader there for the details.

For the Thue–Morse diagram $\tilde{\mathcal{B}}$, we can also index paths by sequences of 0's and 1's labeling the vertices they go through from the root. The Laplacian is also commuting with the operators $\tilde{\tau}_i$ defined like the τ_i defined in equation (6.38). A basis of eigenvectors of Δ for $s = s_0 = 1$ is given by the constant function $\chi_{\partial\tilde{\mathcal{B}}}$ (with eigenvalue 0), and the functions

$$\varphi_{n,\gamma} = \chi_\gamma - \tilde{\tau}_n \chi_\gamma \,, \quad \gamma \in \tilde{\Pi}_n \,,$$

for $n \in \mathbb{N}$, with eigenvalues $\lambda_n = -\frac{2}{3}\left(7 \cdot 4^{n-1} - 1\right)$ of degeneracy $2^{n-1} = \text{Card}\,\Pi_n$. The eigenvalues satisfy the induction formula $\lambda_{n+1} = 4\lambda_n - 2$.

The Weyl asymptotics of Theorem 5.5 reads here $\frac{1}{2}\sqrt{\frac{6}{7}\lambda + \frac{10}{7}} \leq \mathcal{N}(\lambda) \leq \sqrt{\frac{6}{7}\lambda + \frac{4}{7}}$.

6.3 The Penrose tiling

The Fibonacci, Penrose, and Ammann–A2 [21] tilings have formally the "same" substitution on prototiles modulo their symmetry groups, with Abelianization matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The Penrose and Ammann–A2 substitutions force the border. And for the Fibonacci tiling, one considers the conjugate substitution $a \to baa, b \to ba$, which is primitive, recognizable, and forces the border as noted in Example 2.17. Those three substitution tilings have the same Perron–Frobenius eigenvalue, namely $\Lambda_{PF} = \phi^2$, where $\phi = (1+\sqrt{5})/2$ is the golden mean. In conclusion, the transversals of those tiling spaces can be described by the set of infinite paths in the same Bratteli diagram $\mathcal{B}$ illustrated below:
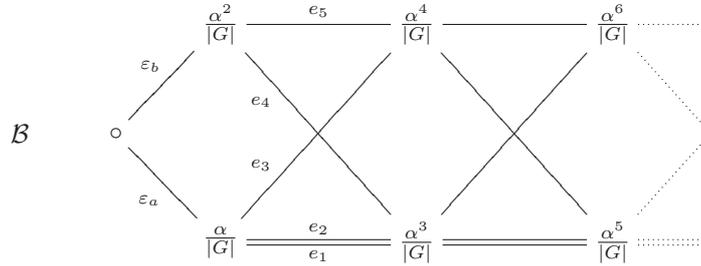
where $\alpha = 1/\phi = (\sqrt{5} - 1)/2$ is the inverse of the golden mean, and the term $\alpha^n/|G|$ at a vertex is the measure of the set of infinite paths through that vertex. And where G is the symmetry group of the tiling introduced in section 2.24, and $|G|$ the cardinality of G. That is $G = \{1\}$ is the trivial group (so $|G| = 1$) for the Fibonacci tiling, $G = C_2 \times C_2$ for the Ammann–A2 tiling (symmetries of "vertical and horizontal" reflections, $|G| = 4$), and $G = D_{10}$ for the Penrose tiling (10-fold rotational symmetries, and reflections, $|G| = 20$).

Let us denote by $U_i, i = 1, \cdots 5$, the generators of the Cuntz–Krieger algebra (5.15) associated with the Abelianization matrix of $\tilde{\mathcal{B}}$. The induced action on the eigenvalues of $\Delta = \Delta_{s_0}$ as in equation (5.19) reads here for Penrose and Ammann–A2 (for Fibonacci, $\Lambda_{PF}^{2/d} = \phi^4$ has to replace ϕ^2 in the following equations):

$$u_1(\lambda_\gamma) = \phi^2 \lambda_\gamma + (1 - \phi^2)\frac{\mu[\varepsilon_a] - \mu(\circ)}{G(\circ)} + \frac{\mu[\varepsilon_a e_1] - \mu[a]}{G(\varepsilon_a)}$$

$$u_2(\lambda_\gamma) = \phi^2 \lambda_\gamma + (1 - \phi^2)\frac{\mu[\varepsilon_a] - \mu(\circ)}{G(\circ)} + \frac{\mu[\varepsilon_a e_2] - \mu[a]}{G(\varepsilon_a)}$$

$$u_3(\lambda_\gamma) = \phi^2 \lambda_\gamma + -\phi^2\frac{\mu[\varepsilon_a] - \mu(\circ)}{G(\circ)} + \frac{\mu[\varepsilon_b] - \mu(\circ)}{G(\circ)} + \frac{\mu[\varepsilon_b e_3] - \mu[a]}{G(\varepsilon_b)}$$

$$u_4(\lambda_\gamma) = \phi^2 \lambda_\gamma + -\phi^2\frac{\mu[\varepsilon_b] - \mu(\circ)}{G(\circ)} + \frac{\mu[\varepsilon_a] - \mu(\circ)}{G(\circ)} + \frac{\mu[\varepsilon_a e_4] - \mu[a]}{G(\varepsilon_a)}$$

$$u_5(\lambda_\gamma) = \phi^2 \lambda_\gamma + (1 - \phi^2)\frac{\mu[\varepsilon_b] - \mu(\circ)}{G(\circ)} + \frac{\mu[\varepsilon_b e_5] - \mu[\varepsilon_b]}{G(\varepsilon_b)}$$

if λ_γ is compatible with the operators. Here $G = G_{s_0}$ as in equation (4.12b), so we have $G(\circ) = |G|(|G|-1)\left(\left(\frac{\alpha}{|G|}\right)^2 + \left(\frac{\alpha^2}{|G|}\right)^2\right) + |G|^2\frac{\alpha}{|G|}\frac{\alpha^2}{|G|}$, $G(\varepsilon_a) = 2\left(\left(\frac{\alpha^2}{|G|}\right)^2 + \left(\frac{\alpha^3}{|G|}\right)^2\right) + 4\frac{\alpha^3}{|G|}\frac{\alpha^4}{|G|}$ and $G(\varepsilon_b) = 2\frac{\alpha^3}{|G|}\frac{\alpha^4}{|G|}$. The eigenelements of $\Delta|_{\Pi_2}$ are 0 for $\chi_{\partial\mathcal{B}}$, λ_0 for $\chi_{\varepsilon_a} - \phi\chi_{\varepsilon_b}$, λ_{ε_a} for $\chi_{\varepsilon_a e} - \phi\chi_{\varepsilon_a f}$, $e, f \in \text{ext}_1(\varepsilon_a)$, and λ_{ε_b} for $\chi_{\varepsilon_b e} - \phi\chi_{\varepsilon_b f}$, $e, f \in \text{ext}_1(\varepsilon_b)$, where

$$\lambda_0 = \frac{-2|G|(|G| + 1 - 4\phi^2)}{|G|^2 - 10|G| + 5}, \quad \lambda_{\varepsilon_a} = \frac{\mu[\varepsilon_a] - \mu[\circ]}{G(\circ)} - \frac{\mu[\varepsilon_a]}{G(\varepsilon_a)}, \quad \lambda_{\varepsilon_b} = \frac{\mu[\varepsilon_b] - \mu[\circ]}{G(\circ)} - \frac{\mu[\varepsilon_b]}{G(\varepsilon_b)}.$$

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