

Abstract

We consider nonlinear parabolic equations with possibly unbounded drift for probability measures on finite-dimensional spaces. Under broad assumptions, the existence of solutions is established.

The goal of this work is to prove the existence of a solution to the following transport equation:

$$\partial_t \mu_t + \operatorname{div}_x (b(\mu, \cdot, \cdot) \mu_t) = 0, \quad \mu_0 = \nu, \quad (1)$$

where

$$b = (b^i)_{i=1}^d: \mathcal{M}_0(\mathbb{R}^d \times [0, 1]) \times \mathbb{R}^d \times [0, 1] \rightarrow \mathbb{R}^d$$

is a Borel mapping and $\mathcal{M}_0(\mathbb{R}^d \times [0, 1])$ is the space of finite Borel measures on $\mathbb{R}^d \times [0, 1]$ equipped with the weak topology.

We shall say that a family $\mu := (\mu_t)_{t \in [0, 1]}$ of finite Borel measures (regarded also as the measure $\mu_t(dx) dt$ on $\mathbb{R}^d \times [0, 1]$) satisfies equation (1) if $b(\mu, \cdot, \cdot) \in L^1(S \times [0, 1], \mu_t(dx) dt)$ for every compact set $S \subset \mathbb{R}^d$, that is, the function $|b(\mu, \cdot, \cdot)|$ is integrable with respect to $|\mu|$ on every compact set in $\mathbb{R}^d \times [0, 1]$, and for all $t \in [0, 1]$ the following identity holds:

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) = \\ & = \int_0^t \int_{\mathbb{R}^d} (b(\mu, x, s), \nabla \varphi(x)) \mu_s(dx) ds \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d). \end{aligned} \quad (2)$$

This equation has been an object of intensive studies over the past decade. For recent surveys, see [1], [2], [3], and [4]; in particular, nonlinear equations are considered in [4]. A typical condition on b in the linear case is the inclusion $b \in L^1([0, 1], W^{1, \infty}(\mathbb{R}^d))$ or the requirement that b is a BV function with respect to x (see [1], [2]). Our aim is to prove the existence assuming only some conditions on the growth of b . In this paper we only consider probability measures. The spaces of probability measures on $\mathbb{R}^d \times [0, 1]$ and \mathbb{R}^d equipped with the weak topology will be denoted by $\mathcal{P}(\mathbb{R}^d \times [0, 1])$ and $\mathcal{P}(\mathbb{R}^d)$, respectively. Our main result is the following theorem.

Theorem 1. *Let ν be a probability measure on \mathbb{R}^d . Suppose that*

(i) *for every fixed measure $\mu \in \mathcal{P}(\mathbb{R}^d \times [0, 1])$, the mapping $x \mapsto b(\mu, x, t)$ is continuous for almost every t and one has uniform convergence $b(\mu_j, x, t) \rightarrow b(\mu, x, t)$ on compact sets whenever $\mu_j \rightarrow \mu$ weakly;*

(ii) *there exist numbers $c \in (0, \infty)$ and $k \in \mathbb{N}$ such that for all $(x, t) \in \mathbb{R}^d \times [0, 1]$ and all $\mu \in \mathcal{P}(\mathbb{R}^d \times [0, 1])$ one has*

$$(b(\mu, x, t), x) \leq c(1 + |x|^2),$$

$$|b(\mu, x, t)| \leq c(1 + |x|^{2k}), \quad \int_{\mathbb{R}^d} |x|^{2k} \nu(dx) < \infty.$$

Then there exists a family $\mu = (\mu_t)_{t \in [0, 1]}$ of probability measures satisfying (1).

Under condition (ii), condition (i) can be reformulated as follows: for every fixed measure μ , the mapping $x \mapsto b(\mu, x, t)$ is continuous for a.e. t and for each compact set $S \subset \mathbb{R}^d$, the mapping b generates a continuous mapping

$$F: \mathcal{P}(\mathbb{R}^d \times [0, 1]) \rightarrow L^\infty([0, 1], C(S))$$

defined by $F(\mu)(t)(x) := b(\mu, x, t)$.

Our approach is based on the well-known method of “vanishing viscosity” (see, e.g., [5, Theorem 4]) combined with the Schauder theorem. We replace equation (1) by the parabolic equation

$$\partial_t \mu_t - \varepsilon \Delta \mu_t + \operatorname{div}_x (b(\mu, \cdot, \cdot) \mu_t) = 0, \quad \mu_0 = \nu, \quad (3)$$

understood as the following integral identity for all $t \in [0, 1]$:

$$\int_{\mathbb{R}^d} \varphi d\mu_t - \int_{\mathbb{R}^d} \varphi d\nu = \int_0^t \int_{\mathbb{R}^d} [\varepsilon \Delta \varphi + (b, \nabla \varphi)] d\mu_s ds \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d). \quad (4)$$

Under our assumptions on b in the case where b is independent of μ , the next proposition follows from the results established in [6] and [7] (see [6, Corollary 3.3, Lemma 2.1, Lemma 2.2] and [7, Theorem 3.3]).

Proposition 1. *Suppose that a probability measure ν on \mathbb{R}^d has a density $\varrho_0 \in C_0^\infty(\mathbb{R}^d)$, the coefficient b does not depend on μ , and there exist numbers $c \in (0, \infty)$ and $k \geq 1$ such that for all $(x, t) \in \mathbb{R}^d \times [0, 1]$ one has*

$$(b(x, t), x) \leq c(1 + |x|^2),$$

$$|b(x, t)| \leq c(1 + |x|^{2k}), \quad \int_{\mathbb{R}^d} |x|^{2k} \nu(dx) < \infty.$$

Then there exists a unique family $\mu = (\mu_t)_{t \in [0, 1]}$ of probability measures on \mathbb{R}^d solving equation (3). Moreover, there exists a number N depending only on c, k and $\int_{\mathbb{R}^d} |x|^{2k} d\nu$ such that

$$\sup_{t \in [0, 1]} \int_{\mathbb{R}^d} |x|^{2k} d\mu_t < N.$$

Certainly, the same is true for any interval $[0, T]$ in place of $[0, 1]$.

First we prove our main result in the case of b independent of μ by letting $\varepsilon \rightarrow 0$. This case is simpler than the general one, but the proof can be extended to the infinite-dimensional case. So we assume that b does not depend on μ , the mapping $x \mapsto b(x, t)$ is continuous for almost every t , and condition (ii) of Theorem 1 is fulfilled.

Let us fix a Borel probability measure ν on \mathbb{R}^d and a sequence of probability measures $\nu^n = \varrho^n(x) dx$, where $\varrho^n \in C_0^\infty(\mathbb{R}^d)$, such that $\{\nu^n\}$ converges weakly to ν and one has

$$\sup_n \int_{\mathbb{R}^d} |x|^{2k} d\nu^n < \infty.$$

According to Proposition 1, for each n , there exists a unique family $\mu^n = (\mu_t^n)_{t \in [0, 1]}$ of probability measures on \mathbb{R}^d satisfying the equation

$$\partial_t \mu_t^n - n^{-1} \Delta \mu_t^n + \operatorname{div}_x (b \mu_t^n) = 0, \quad \mu_0^n = \nu^n$$

in the sense that for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $t \in [0, 1]$ one has

$$\int_{\mathbb{R}^d} \varphi d\mu_t^n - \int_{\mathbb{R}^d} \varphi d\nu^n = \int_0^t \int_{\mathbb{R}^d} [n^{-1} \Delta \varphi + (b, \nabla \varphi)] d\mu_s^n ds. \quad (5)$$

Moreover, there exists a number C independent of n such that

$$\sup_{t \in [0, 1]} \int_{\mathbb{R}^d} |x|^{2k} d\mu_t^n < C,$$

where k is the number from condition (ii) in Theorem 1. Therefore, the set of measures

$$\{\mu_t^n : t \in [0, 1], n \in \mathbb{N}\}$$

is uniformly tight on \mathbb{R}^d . Now we fix a countable dense set $\mathcal{F} \in C_0^\infty(\mathbb{R}^d)$ (the latter space is equipped with the topology of uniform convergence of all derivatives on compact

sets) and take a countable dense set $T \subset [0, 1]$. We can find a subsequence $\{\mu_t^{n_k}\}$ which converges weakly to a probability measure μ_t for each $t \in T$. Let us prove that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi(x) d\mu_t^{n_k} = \int_{\mathbb{R}^d} \varphi(x) d\mu_t \quad (6)$$

for all $t \in [0, 1]$ and all $\varphi \in \mathcal{F}$. Let us fix a function $\varphi \in \mathcal{F}$. Then there is a number B such that $|b(x, t)| \leq B$ for all $x \in \text{supp } \varphi$ and $t \in [0, 1]$, so we have

$$\left| \int_{\mathbb{R}^d} \varphi d\mu_t^{\varepsilon_k} - \int_{\mathbb{R}^d} \varphi d\mu_s^{\varepsilon_k} \right| \leq (\|\Delta\varphi\|_\infty + B\|\nabla\varphi\|_\infty)|t - s|.$$

Let us fix a point $t \in [0, 1]$. For any $\varepsilon > 0$ there are a number $r \in T$ and a natural number N such that for all numbers $k > N$ one has

$$\left| \int_{\mathbb{R}^d} \varphi d\mu_t^{n_k} - \int_{\mathbb{R}^d} \varphi d\mu_r^{n_k} \right| \leq \varepsilon/3, \quad \left| \int_{\mathbb{R}^d} \varphi d\mu_r^{n_k} - \int_{\mathbb{R}^d} \varphi d\mu_r \right| \leq \varepsilon/6.$$

Then, for all $k, l > N$, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \varphi d\mu_t^{n_k} - \int_{\mathbb{R}^d} \varphi d\mu_t^{n_l} \right| \\ & \leq \left| \int_{\mathbb{R}^d} \varphi d\mu_t^{n_k} - \int_{\mathbb{R}^d} \varphi d\mu_r^{n_k} \right| + \left| \int_{\mathbb{R}^d} \varphi d\mu_r^{n_k} - \int_{\mathbb{R}^d} \varphi d\mu_r^{n_l} \right| + \left| \int_{\mathbb{R}^d} \varphi d\mu_r^{n_l} - \int_{\mathbb{R}^d} \varphi d\mu_t^{n_l} \right| \leq \varepsilon. \end{aligned}$$

Hence the sequence $\mu_t^{n_k}$ on \mathbb{R}^d is weakly fundamental and uniformly tight. Therefore, we obtain (6) for all t and all $\varphi \in \mathcal{F}$. Then equality (6) holds for all continuous functions φ with compact support. Letting $k \rightarrow \infty$ in (5) we obtain the equality

$$\int_{\mathbb{R}^d} \varphi d\mu_t - \int_{\mathbb{R}^d} \varphi d\nu = \int_0^t \int_{\mathbb{R}^d} (b, \nabla\varphi) d\mu_s ds$$

because, for almost every fixed s , by the continuity of $x \mapsto b(x, s)$, we have

$$\int_{\mathbb{R}^d} (b(x, s), \nabla\varphi(x)) d\mu_s^{n_k} \rightarrow \int_{\mathbb{R}^d} (b(x, s), \nabla\varphi(x)) d\mu_s$$

and the left-hand side is uniformly bounded, which enables us to integrate in s and obtain the aforementioned equality. This gives Theorem 1 in the linear case.

Let us proceed to the general case where b may depend on μ . We construct a solution to (1) as a weak limit of solutions to approximating nondegenerate parabolic equations with the extra terms $-\varepsilon\Delta\mu_t$, where the coefficient b satisfies condition (ii) of Theorem 1, but in place of condition (i) we impose the following much weaker condition:

(i)' the mapping b is defined on the space $\mathcal{P}_{\text{abs}} \times \mathbb{R}^d \times [0, 1]$, where the space \mathcal{P}_{abs} consists of all absolutely continuous probability measures on $\mathbb{R}^d \times [0, 1]$ and is equipped with variation distance, and for Lebesgue a.e. (x, t) , the mapping $\mu \mapsto b(\mu, x, t)$ is continuous in the variation distance.

In fact, we need even less: it suffices that b be defined only for μ from the subset in \mathcal{P}_{abs} consisting of measures of the form $\varrho(x, t) dx dt$ such that $x \mapsto \varrho(x, t)$ is a probability density for each $t \in [0, 1]$.

Let us consider the following nonlinear parabolic equation:

$$\partial_t \mu_t - \varepsilon \Delta \mu_t + \text{div}_x (b(\mu, \cdot, \cdot) \mu_t) = 0, \quad \mu_0 = \nu, \quad (7)$$

where $\nu = \varrho_0(x) dx$ and $\varrho_0 \in C_0^\infty(\mathbb{R}^d)$.

First we prove the existence of a solution to equation (7). Suppose that we are given real numbers $\alpha > 0$ and $c_1 > 0$ and that, for each closed interval $J_r = [r, 1 - r]$ and each closed ball $B_R = \{x \in \mathbb{R}^d: |x| \leq R\}$, where $r, R > 0$ we are given a number $c_2(r, R) > 0$. Let k be the number from condition (ii) and let $C^\alpha(E)$ denote the Banach space of α -Hölder functions on E with its natural norm.

Let us consider the set $\mathcal{K} \subset L^1(\mathbb{R}^d \times [0, 1])$ of all functions ϱ satisfying the following conditions:

$$\varrho \geq 0, \quad \int_{\mathbb{R}^d} \varrho(x, t) dx = 1, \quad \int_{\mathbb{R}^d} |x|^{2k} \varrho(x, t) dx \leq c_1 \quad \forall t \in [0, 1],$$

$$\|\varrho\|_{C^\alpha(J_r \times B_R)} \leq c_2(r, R) \quad \forall r, R > 0,$$

$\varrho(x, 0) = \varrho_0(x)$ a.e. and for each $\varphi \in C_0^\infty(\mathbb{R}^d)$ the function

$$t \mapsto \int_{\mathbb{R}^d} \varphi(x) \varrho(x, t) dx$$

is Lipschitzian with some constant $C(\varphi)$.

Lemma 1. *The set \mathcal{K} is convex and compact in the Banach space $L^1([0, 1] \times \mathbb{R}^d)$.*

Proof. Indeed, given a sequence in \mathcal{K} , by a diagonal argument we choose a subsequence $\{\varrho_n\}$ that converges uniformly on compact sets in $\mathbb{R}^d \times (0, 1)$ (here we use the bounds on the Hölder norms). Since

$$\int_{\mathbb{R}^d} |x|^{2k} \varrho(x, t) dx \leq c_1 \quad \forall t \in [0, 1],$$

the set of probability measures $\varrho(x, t) dx$ on \mathbb{R}^d , where $\varrho \in \mathcal{K}$ and $t \in [0, 1]$, is uniformly tight. Hence, for each fixed $t \in [0, 1]$, the measures $\varrho_n(x, t) dx$ on \mathbb{R}^d converge weakly to a probability measure μ_t on \mathbb{R}^d , where $\mu_0 = \nu$. Locally uniform convergence of densities shows that $\mu = \mu_t dt$ has a density ϱ , which is locally Hölder continuous on $\mathbb{R}^d \times (0, 1)$ and satisfies the equality $\mu_t = \varrho(x, t) dx$. For each fixed $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi(x) \varrho(x, t) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi(x) \varrho_n(x, t) dx, \quad t \in [0, 1],$$

hence the left-hand side is Lipschitzian with constant $C(\varphi)$. Therefore, $\varrho \in \mathcal{K}$. Hence $\varrho_n \rightarrow \varrho$ in the norm of $L^1(\mathbb{R}^d \times [0, 1])$; we recall that pointwise convergence of probability densities to a probability density yields convergence in mean, see [8, Theorem 2.8.9]. Obviously, \mathcal{K} is convex. \square

It should be noted that we obtain a convex compact set even if we omit the last condition in the definition of \mathcal{K} regulating the behavior in t and specifying the value at $t = 0$. However, for subsequent applications to parabolic equations the introduced class turns out to be more convenient. The probability measure with density $\varrho \in \mathcal{K}$ on $\mathbb{R}^d \times [0, 1]$ will be denoted by the same symbol ϱ and ϱ_t will denote both the probability density $x \mapsto \varrho(x, t)$ on \mathbb{R}^d and the measure with this density.

Now we define a mapping $T: \mathcal{K} \rightarrow \mathcal{K}$ as follows:

$$\chi = T(\varrho) \iff \partial_t \chi_t - \varepsilon \Delta \chi_t + \operatorname{div}_x (b(\varrho, \cdot, \cdot) \chi_t) = 0, \quad \chi_0 = \nu.$$

According to Proposition 1 and [9, Corollary 3.9], the mapping T is well-defined. Note that the Lipschitzness of the integral of $\varphi \in C_0^\infty(\mathbb{R}^d)$ with respect to $\chi_t(dx)$ follows from (4) due to the uniform boundedness of $b(\varrho, x, t)$ on $\operatorname{supp} \varphi \times [0, 1]$.

Lemma 2. *The mapping T is continuous.*

Proof. Let $\varrho^n, \varrho \in \mathcal{K}$, $\|\varrho^n - \varrho\|_{L^1} \rightarrow 0$ and $\chi^n = T(\varrho^n)$. Since \mathcal{K} is compact, we can find a convergent subsequence $\{\chi^{n_k}\}$. We prove that $\chi := \lim_{k \rightarrow \infty} \chi^{n_k}$ satisfies equation (7) with $b = b(\varrho, \cdot, \cdot)$. For every $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have the identity

$$\int_{\mathbb{R}^d} \varphi(x) \chi^{n_k}(x, t) dx - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) = \varepsilon \int_0^t \int_{\mathbb{R}^d} \Delta \varphi(x) \chi^{n_k}(x, s) dx ds +$$

$$+ \int_0^t \int_{\mathbb{R}^d} (b(\varrho^{n_k}, x, s), \nabla \varphi(x)) \chi^{n_k}(x, s) dx ds. \quad (8)$$

Set $S := \text{supp } \varphi$. Since $|b(\varrho^{n_k}, x, t)|$ is uniformly bounded on $S \times [0, 1]$ and

$$\|\chi^{n_k} - \chi\|_{L^1} \rightarrow 0, \quad \|\varrho^{n_k} - \varrho\|_{L^1} \rightarrow 0, \quad |b(\varrho^{n_k}, x, t) - b(\varrho, x, t)| \rightarrow 0 \quad \text{a.e.},$$

we can let $k \rightarrow \infty$ in (8) and obtain for all $t \in [0, 1]$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \chi(x, t) dx - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) &= \varepsilon \int_0^t \int_{\mathbb{R}^d} \Delta \varphi(x) \chi(x, s) dx ds + \\ &+ \int_0^t \int_{\mathbb{R}^d} (b(\varrho, x, s), \nabla \varphi(x)) \chi(x, s) dx ds. \end{aligned}$$

This shows that $\chi = T(\varrho)$. According to the uniqueness assertion in Proposition 1 we conclude that each subsequence in $\{\chi^n\}$ contains a subsequence convergent to χ . This yields that $\chi^n \rightarrow \chi$, hence T is continuous. \square

Applying Schauder's fixed point theorem we conclude that there exists $\varrho \in \mathcal{K}$ such that $\varrho = T(\varrho)$ and the family of measures $\mu_t = \varrho(x, t) dx$ satisfies equation (7). Hence we arrive at the following assertion.

Proposition 2. *Suppose that a probability measure ν has a density from $C_0^\infty(\mathbb{R}^d)$ and b and ν satisfy condition (ii) of Theorem 1 and condition (i)' above. Then there exists a family $\mu = (\mu_t)_{t \in [0, 1]}$ of probability measures on \mathbb{R}^d satisfying (7). Moreover, there exists a number N depending only on c, k and $\int_{\mathbb{R}^d} |x|^{2k} d\nu$ such that*

$$\sup_{t \in [0, 1]} \int_{\mathbb{R}^d} |x|^{2k} d\mu_t < N.$$

Now we prove Theorem 1. Let us fix a probability measure ν . We can find probability measures $\nu^n = \varrho_n(x) dx$ weakly convergent to the measure ν such that $\varrho_n \in L^\infty(\mathbb{R}^d)$ and

$$\sup_n \int_{\mathbb{R}^d} |x|^{2k} d\nu^n < \infty.$$

For each $\varepsilon = n^{-1}$ we take the solution $(\mu_t^n)_{t \in [0, 1]}$ to equation (7) with $\mu_0^n = \nu^n$. Repeating our reasoning from the linear case we find a sequence $\{n_k\}$ such that $\{\mu_t^{n_k}\}$ converges weakly to μ_t for all $t \in [0, 1]$. Denote $b(\mu, \cdot, \cdot)$ and $b(\mu^{n_k}, \cdot, \cdot)$ by b and b_k respectively. We have the identity

$$\int_{\mathbb{R}^d} \varphi d\mu_t^{n_k} - \int_{\mathbb{R}^d} \varphi d\nu^{n_k} = \int_0^t \int_{\mathbb{R}^d} [n_k^{-1} \Delta \varphi + (b_k, \nabla \varphi)] d\mu_s^{n_k} ds.$$

Let $S := \text{supp } \varphi$. Note that

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^d} (b_k, \nabla \varphi) d\mu_s^{n_k} ds - \int_0^t \int_{\mathbb{R}^d} (b, \nabla \varphi) d\mu_s ds \right| &\leq \\ &\leq \|b_k - b\|_{L^\infty(S \times [0, 1])} \|\nabla \varphi\|_\infty + \\ &+ \left| \int_0^t \int_{\mathbb{R}^d} (b, \nabla \varphi) d\mu_s^{n_k} ds - \int_0^t \int_{\mathbb{R}^d} (b, \nabla \varphi) d\mu_s ds \right|. \end{aligned}$$

Letting $k \rightarrow \infty$ we obtain

$$\int_{\mathbb{R}^d} \varphi d\mu_t - \int_{\mathbb{R}^d} \varphi d\nu = \int_0^t \int_{\mathbb{R}^d} (b(\mu, \cdot, \cdot), \nabla \varphi) d\mu_s ds$$

since by our assumption the mapping $x \mapsto b(\mu, x, s)$ is continuous for a.e. $s \in [0, 1]$ and the function $(x, s) \mapsto |(b(\mu, x, s), \nabla \varphi(x))|$ is uniformly bounded. This completes the proof of Theorem 1.

It is worth noting that, according to (2), since the function $(b(\mu, x, t), \nabla\varphi(x))$ has bounded support and is uniformly bounded for each $\varphi \in C_0^\infty(\mathbb{R}^d)$, the function

$$t \mapsto \int_{\mathbb{R}^d} \varphi(x) \mu_t(dx)$$

is Lipschitzian. This function is continuously differentiable if b is continuous in t .

Finally, we observe that positivity of measures is essential for our a priori estimates employed in the proof. The same techniques apply to much more general second order elliptic operators in place of the Laplacian: we only need the assumptions from [6], [7], and [9]. Extensions to the infinite-dimensional case in the spirit of [10] will be considered in a forthcoming paper.

The authors are grateful to Thomas Lorenz for fruitful discussions and helpful remarks. This work has been supported by the projects RFBR 07-01-00536, 08-01-91205-JF, 08-01-90431-Ukr, and the SFB 701 at Bielefeld University.

References

- [1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* 2004. V. 158, n 2. P. 227–260.
- [2] L. Ambrosio, Transport equation and Cauchy problem for non-smooth vector fields. “Calculus of Variations and Non-Linear Partial Differential Equations” (CIME Series, Cetraro, 2005). B. Dacorogna, P. Marcellini eds., pp. 2–41. *Lecture Notes in Mathematics*. V. 1927. Springer, New York, 2008.
- [3] S. Maniglia, Probabilistic representation and uniqueness results for measure-valued solutions of transport equation. *J. Math. Pures Appl.* 2007. V. 87. P. 601–626.
- [4] T. Lorenz, Radon measures solving the Cauchy problem of the nonlinear transport equation. IWR Preprint, Heidelberg University, 2007; <http://www.uni-heidelberg.de/archiv/7252>.
- [5] R.J. Di Perna, P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* 1989. V. 98. P. 511–548.
- [6] V.I. Bogachev, G. Da Prato, M. Röckner, On parabolic equations for measures. *Comm. Partial Diff. Eq.* 2008. V. 33, n 1–3. P. 397–418.
- [7] V.I. Bogachev, G. Da Prato, M. Röckner, W. Stannat, Uniqueness of solutions to weak parabolic equations for measures. *Bull. London Math. Soc.* 2007. V. 39. P. 631–640.
- [8] V.I. Bogachev, *Measure theory*. V. 1, 2. Berlin: Springer, 2007.
- [9] V.I. Bogachev, N.V. Krylov, M. Röckner, On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. *Comm. Partial Diff. Equations*. 2001. V. 26, n 11–12. P. 2037–2080.
- [10] V.I. Bogachev, G. Da Prato, M. Röckner, Parabolic equations for measures on infinite-dimensional spaces. *Dokl. Russian Acad. Sci.* 2008. V. 421, n 4. P. 439–444; English transl.: *Dokl. Math.* 2008. V. 78, n 1. P. 544–549.