

Relations between $O(n)$ -invariants of several matrices

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Abstract

A linear group $G < GL(n)$ acts on d -tuples of $n \times n$ matrices by simultaneous conjugation. In [Adv. Math. 19 (1976), 306–381] Procesi established generators and relations between them for G -invariants, where G is $GL(n)$, $O(n)$, and $Sp(n)$ and the characteristic of base field is zero. We continue generalization of the mentioned results to the case of positive characteristic originated by Donkin in [Invent. Math. 110 (1992), 389–401]. We investigate relations between generators for $O(n)$ -invariants.

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1 Introduction

1.1 Relations

We work over an infinite field \mathbb{F} of arbitrary characteristic $\text{char } \mathbb{F}$. All vector spaces, algebras, and modules are over \mathbb{F} unless otherwise stated. By algebra we always mean an associative algebra.

We consider the ring of polynomials

$$R = R_n = \mathbb{F}[x_{ij}(k) \mid 1 \leq i, j \leq n, 1 \leq k \leq d],$$

in n^2d variables. It is convenient to organize these variables in so-called *generic matrices*

$$X_k = \begin{pmatrix} x_{11}(k) & \cdots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \cdots & x_{nn}(k) \end{pmatrix}.$$

Denote coefficients in the characteristic polynomial of an $n \times n$ matrix X by $\sigma_t(X)$, i.e.,

$$\det(X + \lambda E) = \sum_{t=0}^n \lambda^{n-t} \sigma_t(X). \quad (1)$$

So, $\sigma_0(X) = 1$, $\sigma_1(X) = \text{tr}(X)$ and $\sigma_n(X) = \det(X)$.

Let G be a group from the list $GL(n)$, $O(n)$, $Sp(n)$, where we assume that $\text{char } \mathbb{F} \neq 2$ in case $G = O(n)$. The algebra of *matrix G -invariants* R^G is the subalgebra of R generated by $\sigma_t(A)$, where $1 \leq t \leq n$ and A ranges over all monomials in

- X_1, \dots, X_d , if $G = GL(n)$;
- $X_1, \dots, X_d, X_1^T, \dots, X_d^T$, if $G = O(n)$;
- $X_1, \dots, X_d, X_1^*, \dots, X_d^*$, if $G = Sp(n)$. (Here X^* stands for the symplectic transpose of X).

Moreover, we can assume that A is *primitive*, i.e., is not equal to the power of a shorter monomial. If the characteristic of \mathbb{F} is zero, then it is enough to take traces instead of σ_t , $1 \leq t \leq n$, in order to obtain R^G .

For a field of characteristic zero generators for matrix invariants of $G \in \{GL(n), O(n), Sp(n)\}$ were described by Sibirskii in [19] and Procesi in [17]. Generators for $G = SO(n)$ were calculated by Aslaksen et al. in [2]. Over a field of arbitrary characteristic generators for matrix $GL(n)$ -invariants were found by Donkin in [5]. In [21] Zubkov described generators for matrix $O(n)$ - and $Sp(n)$ -invariants, where in the orthogonal case he assumed that $\text{char } \mathbb{F} \neq 2$. Generators for matrix $SO(n)$ -invariants were calculated by Lopatin in [15].

In case of characteristic zero relations between generators for matrix $GL(n)$ -, $O(n)$ - and $Sp(n)$ -invariants were established by Procesi in [17]. Independently, relations for $R^{GL(n)}$ were found by Razmyslov in [18]. Over a field of arbitrary characteristic relations for matrix $GL(n)$ -invariants were established by Zubkov in [20] (see also Theorem 1.5).

In this paper we describe relations for matrix $O(n)$ -invariants modulo free relations in case $\text{char } \mathbb{F} \neq 2$ (see Theorem 1.1). As a corollary, we explicitly establish relations between indecomposable invariants (see Corollary 6.6), where an element is called *decomposable* if it belongs to \mathbb{F} -span of products of homogeneous invariants of positive degree.

The above mentioned systems of generators are infinite, but it can be showed that they contain finite systems of generators. Moreover, the goal of the constructive theory of invariants is to find out a minimal (by inclusion) homogeneous system of generators of $K[V]^G$ explicitly. It is an important problem, which arose as early as the theory of invariants itself. To find out a minimal system of generators we need a description of relation only between indecomposable invariants. And Corollary 6.6 gives us such description for $R^{O(n)}$. As an application, in the upcoming paper [16] some estimations are given on the highest degree of elements of a minimal system of generators for $R^{O(3)}$ (see Theorem 6.7 below).

For more information on finite generating systems for matrix invariants see overviews [8] and [9] by Formanek. For recent developments in characteristic zero case see [7] and in positive characteristic case see [3]. For results concerning generators and relations between them for mixed representations of quivers see survey [14].

Note that it is possible to define $O(n)$ and $SO(n)$ in characteristic two case. But in this case even generators for invariants of several vectors are not known (for the latest developments see [4]).

We introduce the following notions.

- Let \mathcal{M} be the monoid (without unity) freely generated by *letters* $1, \dots, d, 1^T, \dots, d^T$.
- Let $\mathcal{N} \subset \mathcal{M}$ be the subset of primitive elements, where the notion of a primitive element is defined as above.
- Let $\mathcal{M}_{\mathbb{F}}$ be the vector space with the basis \mathcal{M} .

Assume that $\alpha = \alpha_1 \cdots \alpha_p$ and β are elements of \mathcal{M} , where $\alpha_1, \dots, \alpha_p$ are letters.

- Introduce an involution on \mathcal{M} as follows. Define $\beta^{TT} = \beta$ for a letter β and $\alpha^T = \alpha_p^T \cdots \alpha_1^T \in \mathcal{M}$. We extend the introduced map to an involution $T : \mathcal{M}_{\mathbb{F}} \rightarrow \mathcal{M}_{\mathbb{F}}$ by linearity.
- We say that α and β are *equivalent* and write $\alpha \sim \beta$ if there exists a cyclic permutation $\pi \in S_p$ such that $\alpha_{\pi(1)} \cdots \alpha_{\pi(p)} = \beta$ or $\alpha_{\pi(1)} \cdots \alpha_{\pi(p)} = \beta^T$.
- Let \mathcal{M}_{σ} be a ring with unity of (commutative) polynomials over \mathbb{F} freely generated by “symbolic” elements $\sigma_t(\alpha)$, where $t > 0$ and $\alpha \in \mathcal{M}_{\mathbb{F}}$.
- Let \mathcal{N}_{σ} be a ring with unity of (commutative) polynomials over \mathbb{F} freely generated by “symbolic” elements $\sigma_t(\alpha)$, where $t > 0$ and $\alpha \in \mathcal{N}$ ranges over \sim -equivalence classes. We can also define \mathcal{N}_{σ} as a factor of \mathcal{M}_{σ} by some ideal (see Lemma 3.1). In particular, we can consider $\sigma_t(\alpha)$ with $\alpha \in \mathcal{M}_{\mathbb{F}}$ as an element of \mathcal{N}_{σ} .

We will use the following convention:

$$\text{tr}(\alpha) = \sigma_1(\alpha)$$

for any $\alpha \in \mathcal{M}_{\mathbb{F}}$. For a letter $\beta \in \mathcal{M}$ define

$$X_{\beta} = \begin{cases} X_i, & \text{if } \beta = i \\ X_i^T, & \text{if } \beta = i^T \end{cases} .$$

Given $\alpha = \alpha_1 \cdots \alpha_p \in \mathcal{M}$, where α_i is a letter, we assume that $X_{\alpha} = X_{\alpha_1} \cdots X_{\alpha_p}$.

Consider a surjective homomorphism

$$\Psi_n : \mathcal{N}_{\sigma} \rightarrow R^{O(n)}$$

defined by $\sigma_t(\alpha) \rightarrow \sigma_t(X_{\alpha})$, if $t \leq n$, and $\sigma_t(\alpha) \rightarrow 0$ otherwise. Its kernel K_n is the ideal of *relations* for $R^{O(n)}$. Elements of $\bigcap_{i>0} K_i$ are called *free relations*. For $\alpha, \beta, \gamma \in \mathcal{M}_{\mathbb{F}}$ and $t, r \in \mathbb{N}$, an element $\sigma_{t,r}(\alpha, \beta, \gamma) \in \mathcal{N}_{\sigma}$ is defined in Section 3 (see Definition 3.2), where \mathbb{N} stands for the set of non-negative integers.

Theorem 1.1. *If $\text{char } \mathbb{F} \neq 2$, then the ideal of relations K_n for $R^{O(n)} \simeq \mathcal{N}_{\sigma}/K_n$ is generated by*

(a) $\sigma_{t,r}(\alpha, \beta, \gamma)$, where $t + 2r > n$ and $\alpha, \beta, \gamma \in \mathcal{M}_{\mathbb{F}}$;

(b) *free relations.*

Since \mathbb{F} is infinite, in the formulation of Theorem 1.1 we can take the following elements

(a') $\sigma_{\underline{t}, \underline{r}, \underline{s}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w)$ for $t + 2r > n$, where $\underline{t} = (t_1, \dots, t_u) \in \mathbb{N}^u$, $\underline{r} = (r_1, \dots, r_v) \in \mathbb{N}^v$, $\underline{s} = (s_1, \dots, s_w) \in \mathbb{N}^w$, $t = t_1 + \cdots + t_u$, $r = r_1 + \cdots + r_v = s_1 + \cdots + s_w$, and $x_1, \dots, z_w \in \mathcal{M}$

instead of (a), where $\sigma_{\underline{t}, \underline{r}, \underline{s}}$ is given by Definition 4.1.

In Lemma 6.5 we show that the ideal of free relations lays in the ideal of \mathcal{N}_{σ} generated by $\sigma_t(\alpha)^{\text{char } \mathbb{F}}$, where α ranges over \mathcal{N} and $t > 0$. The following hypothesis is valid in characteristic zero case.

Hypothesis 1.2. In case $G = O(n)$ there are no non-zero free relations.

For $f \in R$ denote by $\text{mdeg } f$ its *multidegree*, i.e., $\text{mdeg } f = (t_1, \dots, t_d) \in \mathbb{N}^d$, where t_k is the total degree of the polynomial f in $x_{ij}(k)$, $1 \leq i, j \leq n$. The algebra $R^{O(n)}$ is homogeneous with respect to \mathbb{N}^d -grading. Denote the degree of $\alpha \in \mathcal{M}$ by $\text{deg } \alpha$, the degree of α in a letter β by $\text{deg}_\beta \alpha$, and the multidegree of α by

$$\text{mdeg}(\alpha) = (\text{deg}_1 \alpha + \text{deg}_{1T} \alpha, \dots, \text{deg}_d \alpha + \text{deg}_{dT} \alpha).$$

For $t > 0$ we assume that $\text{deg } \sigma_t(\alpha) = t \text{deg } \alpha$ and $\text{mdeg } \sigma_t(\alpha) = t \text{mdeg } \alpha$. Therefore \mathcal{N} , \mathcal{M} , and \mathcal{N}_σ have \mathbb{N} -gradings by degrees and \mathbb{N}^d -gradings by multidegrees. Note that the given above generating system for $R^{O(n)}$ as well as relations (a') are \mathbb{N}^d -homogeneous.

1.2 The known results on relations

In this section we explicitly formulate the results on relations that have been mentioned in Section 1. Let us remark that modulo free relations, Theorem 1.1 generalizes Theorem 1.3 in the same manner as Theorem 1.5 generalizes Theorem 1.4.

Theorem 1.3. (*Procesi [17]*) *If $\text{char } \mathbb{F} = 0$, then the ideal of relations K_n for $R^{O(n)} \simeq \mathcal{N}_\sigma / K_n$ is generated by $\sigma_{t,r}(\alpha, \beta, \gamma)$, where $t + 2r = n + 1$ and $\alpha, \beta, \gamma \in \mathcal{M}_\mathbb{F}$.*

We denote by \mathcal{M}' the monoid (without unity) freely generated by letters $1, \dots, d$ and denote by $\mathcal{N}' \subset \mathcal{M}'$ the subset of primitive elements. We define $\mathcal{M}'_\mathbb{F}$ (\mathcal{M}'_σ , respectively) similarly to $\mathcal{M}_\mathbb{F}$ (\mathcal{M}_σ , respectively). Let \mathcal{N}'_σ be a ring of (commutative) polynomials over \mathbb{F} freely generated by “symbolic” elements $\sigma_t(\alpha)$, where $t > 0$ and $\alpha \in \mathcal{N}'$ ranges over all primitive *cycles* (i.e., equivalence classes with respect to cyclic permutations). An analogue of Lemma 3.1 is also valid for \mathcal{N}'_σ . Hence we can consider $\sigma_t(\alpha)$ with $\alpha \in \mathcal{M}'_\mathbb{F}$ as an element of \mathcal{N}'_σ .

Theorem 1.4. (*Razmyslov [18], Procesi [17]*) *If $\text{char } \mathbb{F} = 0$, then the ideal of relations K'_n for $R^{GL(n)} \simeq \mathcal{N}'_\sigma / K'_n$ is generated by $\sigma_{n+1}(\alpha)$, where $\alpha \in \mathcal{M}'_\mathbb{F}$.*

Theorem 1.5. (*Zubkov, [20]*) *The ideal of relations K'_n for $R^{GL(n)} \simeq \mathcal{N}'_\sigma / K'_n$ is generated by $\sigma_t(\alpha)$, where $t > n$ and $\alpha \in \mathcal{M}'_\mathbb{F}$.*

Remark 1.6. In case $G = GL(n)$ there are no non-zero free relations (see [6]).

1.3 Historical remarks on $\sigma_{t,r}$

The complete linearization of $\sigma_{t,r}$ was introduced by Procesi (see [17], Section 8 of Part I), where it was denoted by $F_{k,n+1}$. It is not difficult to see that

$$\sigma_{t,r}^{\text{lin}}(x_1, \dots, x_t, y_1, \dots, y_r, z_1, \dots, z_r) = F_{r,t+2r}(y_1, \dots, y_r, x_1, \dots, x_t, z_1, \dots, z_r) \text{ in } \mathcal{N}_\sigma, \quad (2)$$

where $\sigma_{t,r}^{\text{lin}}$ stands for the complete linearization of $\sigma_{t,r}$ (see Definition 4.1) and $x_1, \dots, z_r \in \mathcal{M}_\mathbb{F}$. Then $\sigma_{t,r}$ was introduced by Zubkov in [24]. Note that the definition from [24] is different from our definition and their equivalence is established in Lemma 7.14.

Another way to define $\sigma_{t,r}$ is via the determinant-pfaffian $\text{DP}_{r,r}(X, Y, Z)$ that was introduced in [13] as a “mixture” of the determinant of X and pfaffians of Y and Z (see Example 7.5). Lemma 4.9 and Example 7.5 imply that $\text{DP}_{r,r}$ relates to $\sigma_{t,r}$ in the same way as the determinant relates to σ_t , i.e., for $n \times n$ matrices X, Y, Z we have

$$\text{DP}_{r,r}(X + \lambda E, Y, Z) = \sum_{t=0}^{t_0} \lambda^{t_0-t} \sigma_{t,r}(X, Y, Z), \quad (3)$$

where $n = t_0 + 2r$ and $t_0 \geq 0$. Since $\sigma_{t,0}(X, Y, Z) = \sigma_t(X)$ (see Remark 3.4), Formula (3) turns into (1) for $r = 0$. Note that this approach gives us $\sigma_{t,r}(X, Y, Z)$ as a polynomial in entries of matrices X, Y, Z . But for our purposes we have to present $\sigma_{t,r}(X, Y, Z)$ in a different way, namely, as a polynomial in $\sigma_t(\alpha)$, where t ranges over positive integers and α ranges over monomials in X, Y, Z, X^T, Y^T, Z^T .

1.4 The structure of the paper

The paper is organized as follows. In Section 2 notations are given that are used throughout the paper. Key notion $\sigma_{t,r}$ is introduced in Section 3. In Section 4 we obtain some results on $\sigma_{t,r}$. In Lemma 4.2 a formula for partial linearization of $\sigma_{t,r}$ is presented, which is similar to Formula (10) from the definition of $\sigma_{t,r}$. In characteristic zero case Lemma 4.4 allows us to work with complete linearization of $\sigma_{t,r}$ instead of $\sigma_{t,r}$ itself.

In Section 5 Theorem 1.1 is proven. The proof is based on results from paper [24] by Zubkov, where an approach for computation of relations for $R^{O(n)}$ was given (see Sections 3 and 4 of [24]). To complete Zubkov’s approach we used the decomposition formula from [12], which can be considered as a generalization of Amitsur’s formula for $\sigma_{t,r}$.

In Section 6 we establish some restrictions on free relations (see Lemma 6.5). In particular, we show that there is no non-zero linear free relations between indecomposable invariants (see Corollary 6.6).

In Section 7 the decomposition formula is formulated (see (21)). Then it is shown that our definition of $\sigma_{t,r}$ coincides with the original definition from [24] (see Lemma 7.14). Since Section 7 contains numerous notions that have limited usage in the paper, we put it at the end of the paper.

2 Notations

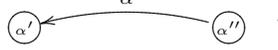
In what follows, \mathbb{Q} stands for the quotient field of the ring of integers \mathbb{Z} . For a vector $\underline{t} = (t_1, \dots, t_u) \in \mathbb{N}^u$ we write $\underline{t}! = t_1! \cdots t_u!$. The length of p -tuple $\underline{c} = (c_1, \dots, c_p)$ we denote by $\#\underline{c} = p$. For short, we write 1^p for $(1, \dots, 1) \in \mathbb{N}^p$. We assume that E_m stands for the identity $m \times m$ matrix.

Consider $n, k \geq 0$. If $k \leq n$, then we write $\binom{n}{k}$ for the binomial coefficient; otherwise, we define $\binom{n}{k} = 0$. We also define $\binom{n}{-1} = 0$. Let us remark that $\binom{n}{0} = 1$.

By substitution $1 \rightarrow \alpha_1, \dots, d \rightarrow \alpha_d$ in $\alpha \in \mathcal{M}$, where $\alpha_1, \dots, \alpha_d \in \mathcal{M}$, we mean the substitution $1 \rightarrow \alpha_1, \dots, d \rightarrow \alpha_d, 1^T \rightarrow \alpha_1^T, \dots, d^T \rightarrow \alpha_d^T$ and denote it by $\alpha|_{1 \rightarrow \alpha_1, \dots, d \rightarrow \alpha_d}$. Similar convention we also use for substitutions of elements of \mathcal{N}_σ and so on.

Denote by $\mathcal{B} = \mathbb{F}[x_1, \dots, x_m]$ the polynomial ring in x_1, \dots, x_m over \mathbb{F} , i.e., \mathcal{B} is a commutative \mathbb{F} -algebra with unity generated by algebraically independent elements x_1, \dots, x_m . Given an \mathbb{N} -graded algebra $\mathcal{A} = \sum_{i \geq 0} \mathcal{A}_i$ and $a = \sum_{i \geq 0} a_i \in \mathcal{A}$, where almost all $a_i \in \mathcal{A}_i$ are zero, we write $\text{hc}^s(\mathcal{A})$ for $\sum_{0 \leq i \leq s} \mathcal{A}_i$ and $\text{hc}_s(a) = a_i$ for the homogeneous component of degree s .

A *quiver* $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a finite oriented graph, where \mathcal{Q}_0 is the set of vertices and \mathcal{Q}_1 is the set of arrows. Multiple arrows and loops in \mathcal{Q} are allowed. For an arrow α , denote by α' its head and by α'' its tail, i.e.,



We say that $\alpha = \alpha_1 \cdots \alpha_s$ is a *path* in \mathcal{Q} (where $\alpha_1, \dots, \alpha_s \in \mathcal{Q}_1$), if $\alpha_1'' = \alpha_2', \dots, \alpha_{s-1}'' = \alpha_s'$, i.e.,



The head of the path α is $\alpha' = \alpha_1'$ and the tail is $\alpha'' = \alpha_s''$. A path α is called *closed* if $\alpha' = \alpha''$. A closed path α is called *incident* to a vertex $v \in \mathcal{Q}_0$ if $\alpha' = v$. Similarly, closed paths β_1, \dots, β_s in \mathcal{Q} are called *incident* to v if $\beta_1' = \dots = \beta_s' = v$.

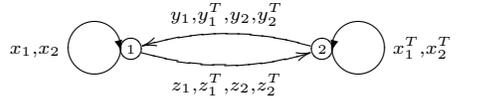
Definition 2.1 (of a mixed quiver). A quiver \mathcal{Q} is called *mixed* if two maps $T : \mathcal{Q}_0 \rightarrow \mathcal{Q}_0$ and $T : \mathcal{Q}_1 \rightarrow \mathcal{Q}_1$ are defined in such a way that

- $v^{TT} = v, \beta^{TT} = \beta$;
- $(\beta^T)' = (\beta'')^T, (\beta^T)'' = (\beta')^T$

for all $v \in \mathcal{Q}_0$ and $\beta \in \mathcal{Q}_1$.

Assume that \mathcal{Q} is a mixed quiver. Denote by $\mathcal{M}(\mathcal{Q})$ the set of all closed paths in \mathcal{Q} and denote by $\mathcal{N}(\mathcal{Q}) \subset \mathcal{M}(\mathcal{Q})$ the subset of primitive paths. Given a path α in \mathcal{Q} , we define the path α^T and introduce \sim -equivalence on $\mathcal{M}(\mathcal{Q})$ in the same way as in Section 1. Moreover, we define $\mathcal{M}_{\mathbb{F}}(\mathcal{Q})$, $\mathcal{M}_{\sigma}(\mathcal{Q})$, and $\mathcal{N}_{\sigma}(\mathcal{Q})$ in the same way as $\mathcal{M}_{\mathbb{F}}$, \mathcal{M}_{σ} , and \mathcal{N}_{σ} have been defined in Section 1. The notions of degree and multidegree for elements of $\mathcal{M}(\mathcal{Q})$ and $\mathcal{N}_{\sigma}(\mathcal{Q})$ are introduced as above.

Example 2.2. Consider the quiver \mathcal{Q} :



where there are four arrows from vertex 1 to vertex 2, there are four arrows in the opposite direction, and each of its vertices has two loops. Here letters $x_i, x_i^T, y_i, y_i^T, z_i, z_i^T$ ($i = 1, 2$) stand for arrows of \mathcal{Q} . We define $1^T = 2$ for vertex 1, so \mathcal{Q} is a mixed quiver. Then

- $(x_1 y_1^T z_1)^T = z_1^T y_1 x_1^T, y_1 z_1 \sim y_1^T z_1^T, x_1 y_1 x_1^T z_1 \sim x_1 y_1^T x_1^T z_1^T$;
- $\deg(y_1 z_1^T) = 2, \deg_{y_1}(y_1 z_1) = 1, \deg_{y_1^T}(y_1 z_1) = 0$, and $\text{mdeg}(x_1 x_1 x_2 y_1 x_1^T x_2^T z_2^T) = (3, 2, 1, 0, 0, 1)$.

3 The definition of $\sigma_{t,r}$

In this section we assume that \mathcal{A} is a commutative unitary algebra over the field \mathbb{F} and all matrices are considered over \mathcal{A} .

Let us recall some formulas. In what follows A, A_1, \dots, A_p stand for $n \times n$ matrices and $1 \leq t \leq n$. Amitsur's formula states [1]:

$$\sigma_t(A_1 + \dots + A_p) = \sum (-1)^{t-(j_1+\dots+j_q)} \sigma_{j_1}(\gamma_1) \dots \sigma_{j_q}(\gamma_q), \quad (4)$$

where the sum ranges over all pairwise different primitive cycles $\gamma_1, \dots, \gamma_q$ in letters A_1, \dots, A_p and positive integers j_1, \dots, j_q with $\sum_{i=1}^q j_i \deg \gamma_i = t$. Denote the right hand side of (4) by $F_{t,p}(A_1, \dots, A_p)$. As an example,

$$\sigma_2(A_1 + A_2) = \sigma_2(A_1) + \sigma_2(A_2) + \text{tr}(A_1) \text{tr}(A_2) - \text{tr}(A_1 A_2). \quad (5)$$

Note that for $a \in \mathcal{A}$ we have

$$\sigma_t(aA) = a^t \sigma_t(A). \quad (6)$$

For $l \geq 2$ we have the following well-known formula:

$$\sigma_t(A^l) = \sum_{i_1, \dots, i_{tl} \geq 0} b_{i_1, \dots, i_{tl}}^{(t,l)} \sigma_1(A)^{i_1} \dots \sigma_{tl}(A)^{i_{tl}}, \quad (7)$$

where we assume that $\sigma_i(A) = 0$ for $i > n$. Denote the right hand side of (7) by $P_{t,l}(A)$. In (7) coefficients $b_{i_1, \dots, i_{tl}}^{(t,l)} \in \mathbb{Z}$ do not depend on A and n . If we take a diagonal matrix $A = \text{diag}(a_1, \dots, a_n)$, then $\sigma_t(A^l)$ is a symmetric polynomial in a_1, \dots, a_n and $\sigma_i(A)$ is the i^{th} elementary symmetric polynomial in a_1, \dots, a_n , where $1 \leq i \leq n$. Thus the coefficients $b_{i_1, \dots, i_{tl}}^{(t,l)}$ with $tl \leq n$ can easily be found. As an example,

$$\text{tr}(A^2) = \text{tr}(A)^2 - 2\sigma_2(A). \quad (8)$$

Lemma 3.1. *We have $\mathcal{N}_\sigma \simeq \mathcal{M}_\sigma/L$ for the ideal L generated by*

- (a) $\sigma_t(\alpha_1 + \dots + \alpha_p) - F_{t,p}(\alpha_1, \dots, \alpha_p)$,
- (b) $\sigma_t(a\alpha) - a^t \sigma_t(\alpha)$,
- (c) $\sigma_t(\alpha^l) - P_{t,l}(\alpha)$,
- (d) $\sigma_t(\alpha\beta) - \sigma_t(\beta\alpha)$,
- (e) $\sigma_t(\alpha) - \sigma_t(\alpha^T)$,

where $p > 1$, $\alpha, \alpha_1, \dots, \alpha_p \in \mathcal{M}_\mathbb{F}$, $a \in \mathbb{F}$, $t > 0$, and $l > 1$.

Proof. Consider a homomorphism $\rho: \mathcal{M}_\sigma \rightarrow \mathcal{N}_\sigma$ defined by $\sigma_t(\alpha) \rightarrow f$, where we apply equalities (a), (b), and (c) one after another to $\sigma_t(\alpha)$, $\alpha \in \mathcal{M}_\mathbb{F}$, to obtain f . Obviously, ρ is well defined. Since ρ is surjective, to complete the proof it is enough to show that $\rho(f) = 0$, where $f \in L$ ranges over elements (a)–(e).

Assume that f is not element (e) and $\rho(f) \neq 0$. We define \mathcal{M}' , \mathcal{M}'_σ , and \mathcal{N}'_σ in the same way as in Section 1.2 with the only difference that now we start with the monoid \mathcal{M}' freely generated by letters $1, \dots, 2d$. Note that $\mathcal{M}' \simeq \mathcal{M}$, where the isomorphism is given by $i \rightarrow i$ and $i + d \rightarrow i^T$ for all $1 \leq i \leq d$. Hence $\mathcal{M}'_\sigma \simeq \mathcal{M}_\sigma$ and f can be considered as an element of \mathcal{M}'_σ . Define $\rho' : \mathcal{M}'_\sigma \rightarrow \mathcal{N}'_\sigma$ in the same way as ρ . Since $\rho(f) \neq 0$, we have $\rho'(f) \neq 0$. But $\rho'(f)$ is a free relation for $R^{GL(n)}$; a contradiction to Remark 1.6.

We assume that f is element (e), i.e.,

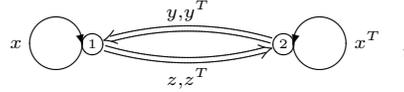
$$f = \sigma_t \left(\sum_{i=1}^s a_i \alpha_i \right) - \sigma_t \left(\sum_{i=1}^s a_i \alpha_i^T \right),$$

where $a_i \in \mathbb{F}$ and $\alpha_i \in \mathcal{M}$. Denote the result of application of (a) and (b) to f by $h \in \mathcal{M}_\sigma$. Let $a^{(i_1, \dots, i_s)}$ stand for $a_1^{i_1} \cdots a_s^{i_s}$. Then

$$h = \sum a^{j_1 \text{ mdeg } \gamma_1 + \dots + j_q \text{ mdeg } \gamma_q} (-1)^{t - (j_1 + \dots + j_q)} (\sigma_{j_1}(\gamma_1^{(1)}) \cdots \sigma_{j_q}(\gamma_q^{(1)}) - \sigma_{j_1}(\gamma_1^{(2)}) \cdots \sigma_{j_q}(\gamma_q^{(2)})), \quad (9)$$

where the sum ranges over j_1, \dots, j_q and cycles $\gamma_1, \dots, \gamma_q$ in letters x_1, \dots, x_s as in (4), $\gamma_l^{(1)}$ is the result of substitution $x_1 \rightarrow \alpha_1, \dots, x_s \rightarrow \alpha_s$ in γ_l , and $\gamma_l^{(2)}$ is the result of substitution $x_1 \rightarrow \alpha_1^T, \dots, x_s \rightarrow \alpha_s^T$ in γ_l ($1 \leq l \leq q$). Note that for any pair $(\underline{j}, \underline{\gamma})$ from (9) there exists a pair $(\underline{j}, \underline{\beta})$ from (9) such that $\gamma_l^{(1)} \sim \beta_l^{(2)}$ for all $1 \leq l \leq q$. Applying (c) to $\sigma_{j_1}(\gamma_1^{(1)}) \cdots \sigma_{j_q}(\gamma_q^{(1)})$ and $\sigma_{j_1}(\beta_1^{(2)}) \cdots \sigma_{j_q}(\beta_q^{(2)})$, we obtain equal elements in \mathcal{N}_σ . Thus, $\rho(f) = 0$. \square

Let $t, r \in \mathbb{N}$. In order to define $\sigma_{t,r}$, we consider the quiver \mathcal{Q}



where there are two arrows from vertex 1 to vertex 2 and there are two arrows in the opposite direction. We define $1^T = 2$ for vertex 1 to turn \mathcal{Q} into a mixed quiver.

Definition 3.2 (of $\sigma_{t,r}(x, y, z)$). Denote by $\mathcal{I} = \mathcal{I}_{t,r}$ the set of pairs $(\underline{j}, \underline{\alpha})$ such that

- $\#\underline{j} = \#\underline{\alpha} = p$ for some p ;
- $\alpha_1, \dots, \alpha_p \in \mathcal{N}(\mathcal{Q})$ belong to pairwise different \sim -equivalence classes and $j_1, \dots, j_p \geq 1$;
- $j_1 \text{ mdeg}(\alpha_1) + \dots + j_p \text{ mdeg}(\alpha_p) = (t, r, r)$.

Then we define $\sigma_{t,r}(x, y, z) \in \mathcal{N}_\sigma(\mathcal{Q})$ by

$$\sigma_{t,r}(x, y, z) = \sum_{(\underline{j}, \underline{\alpha}) \in \mathcal{I}} (-1)^\xi \sigma_{j_1}(\alpha_1) \cdots \sigma_{j_p}(\alpha_p), \quad (10)$$

where $p = \#\underline{j} = \#\underline{\alpha}$ and $\xi = \xi_{\underline{j}, \underline{\alpha}} = t + \sum_{i=1}^p j_i (\text{deg}_y \alpha_i + \text{deg}_z \alpha_i + 1)$. For $t = r = 0$ we define $\sigma_{0,0}(x, y, z) = 1$. Moreover,

- if $\alpha, \beta, \gamma \in \mathcal{M}_{\mathbb{F}}$, then we define $\sigma_{t,r}(\alpha, \beta, \gamma) \in \mathcal{N}_{\sigma}$ as the result of substitution $x \rightarrow \alpha, y \rightarrow \beta, z \rightarrow \gamma$ in (10);
- if X, Y, Z are $n \times n$ matrices, then we define $\sigma_{t,r}(X, Y, Z) \in \mathcal{A}$ as the result of substitution $x \rightarrow X, y \rightarrow Y, z \rightarrow Z$ in (10), where we assume that $\sigma_j(A) = 0$ for $t > n$ and any $n \times n$ matrix A .

Example 3.3. 1. If $t = 0$ and $r = 1$, then \sim -equivalence classes on $\mathcal{N}(\mathcal{Q})$ are yz, yz^T, \dots . Hence, $\sigma_{0,1}(x, y, z) = -\text{tr}(yz) + \text{tr}(yz^T)$.

2. If $t = r = 1$, then \sim -equivalence classes on $\mathcal{N}(\mathcal{Q})$ are

$$x, yz, yz^T, xyz, xyz^T, xy^T z, xy^T z^T, \dots$$

and we can see that $\sigma_{1,1}(x, y, z) =$

$$-\text{tr}(x) \text{tr}(yz) + \text{tr}(x) \text{tr}(yz^T) + \text{tr}(xyz) - \text{tr}(xyz^T) - \text{tr}(xy^T z) + \text{tr}(xy^T z^T).$$

Remark 3.4. $\sigma_{t,0}(x, y, z) = \sigma_t(x)$.

4 Properties of $\sigma_{t,r}$

In this section all matrices have entries in a commutative unitary algebra \mathcal{A} over the field \mathbb{F} .

Definition 4.1 (of a partial linearization of $\sigma_{t,r}$). We assume that $\underline{t} = (t_1, \dots, t_u) \in \mathbb{N}^u$, $\underline{r} = (r_1, \dots, r_v) \in \mathbb{N}^v$, $\underline{s} = (s_1, \dots, s_w) \in \mathbb{N}^w$ satisfy

$$s_1 + \dots + s_w = r_1 + \dots + r_w,$$

and $x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w$ belong to $\mathcal{M}_{\mathbb{F}}$. Consider $\sigma_{t,r}(a_1 x_1 + \dots + a_u x_u, b_1 y_1 + \dots + b_v y_v, c_1 z_1 + \dots + c_w z_w) \in \mathcal{N}_{\sigma}$ as a polynomial in $a_1, \dots, a_u, b_1, \dots, b_v, c_1, \dots, c_w$, where $a_1, \dots, c_w \in \mathbb{F}$. We denote the coefficient of $a_1^{t_1} \dots a_u^{t_u} b_1^{r_1} \dots b_v^{r_v} c_1^{s_1} \dots c_w^{s_w}$ in this polynomial by

$$\sigma_{\underline{t}, \underline{r}, \underline{s}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w) \in \mathcal{N}_{\sigma}. \quad (11)$$

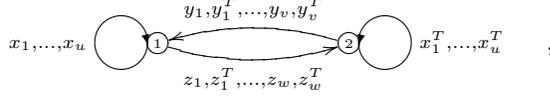
In other words, this coefficient is equal to the homogeneous component of $\sigma_{t,r}(x_1 + \dots + x_u, y_1 + \dots + y_v, z_1 + \dots + z_w)$ of multidegree $(\underline{t}, \underline{r}, \underline{s})$. In multilinear case we have $v = w$ and write

$$\sigma_{u,v}^{\text{lin}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w)$$

for $\sigma_{1^u, 1^v, 1^w}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w)$, where the definition of 1^u was given in Section 2.

Given for $n \times n$ matrices X_1, \dots, Z_w , we define $\sigma_{\underline{t}, \underline{r}, \underline{s}}(X_1, \dots, X_u, Y_1, \dots, Y_v, Z_1, \dots, Z_w) \in \mathcal{A}$ as the result of substitution $x_1 \rightarrow X_1, \dots, z_w \rightarrow Z_w$ in element (11) of \mathcal{N}_{σ} and use the assumption that $\sigma_j(A) = 0$ for $j > n$ and any $n \times n$ matrix A .

In this section we will show that an analogue of Formula (10) from the definition of $\sigma_{t,r}$ is valid for $\sigma_{\underline{t},\underline{r},\underline{s}}$ (see Lemma 4.2 below). Generalize the construction from Section 3 as follows. Using notation from Definition 4.1, we define the mixed quiver $\mathcal{Q}_{\underline{t},\underline{r},\underline{s}}$:



where there are $2w$ arrows from vertex 1 to vertex 2, there are $2v$ arrows in the opposite direction, and $1^T = 2$. Here by abuse of notation x_1, \dots, z_w stand for the arrows of $\mathcal{Q}_{\underline{t},\underline{r},\underline{s}}$ as well as for elements of $\mathcal{M}_{\mathbb{F}}$ from the definition of $\sigma_{t,r}$. Hence we can assume that $\mathcal{N}_{\sigma}(\mathcal{Q}_{\underline{t},\underline{r},\underline{s}}) \subset \mathcal{N}_{\sigma}$.

Given a path α of $\mathcal{Q}_{\underline{t},\underline{r},\underline{s}}$, denote the total degree of α in x_1, \dots, x_u by $\deg_x \alpha$, and the total degree of α in x_1^T, \dots, x_u^T by $\deg_{x^T} \alpha$. Similar notation we also use for y and z . As an example, $\deg_{x^T}(x_1 y_1 x_1^T x_2^T z_2^T) = 2$.

Assume that $\mathcal{I} = \mathcal{I}_{\underline{t},\underline{r},\underline{s}}$ is the set of pairs $(\underline{j}, \underline{\alpha})$ such that

- $\# \underline{j} = \# \underline{\alpha} = p$ for some p ;
- $\alpha_1, \dots, \alpha_p \in \mathcal{N}(\mathcal{Q}_{\underline{t},\underline{r},\underline{s}})$ are representatives of pairwise different \sim -equivalence classes and $j_1, \dots, j_p \geq 1$;
- $j_1 \text{mdeg}(\alpha_1) + \dots + j_p \text{mdeg}(\alpha_p) = (\underline{t}, \underline{r}, \underline{s})$.

Lemma 4.2. *We have*

$$\sigma_{\underline{t},\underline{r},\underline{s}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w) = \sum_{(\underline{j}, \underline{\alpha}) \in \mathcal{I}} (-1)^{\eta} \sigma_{j_1}(\alpha_1) \cdots \sigma_{j_p}(\alpha_p),$$

where $p = \# \underline{j} = \# \underline{\alpha}$ and $\eta = \eta_{\underline{j}, \underline{\alpha}} = t_1 + \dots + t_u + \sum_{i=1}^p j_i (\deg_y \alpha_i + \deg_z \alpha_i + 1)$.

Proof. We set $t = t_1 + \dots + t_u$ and $r = r_1 + \dots + r_u = s_1 + \dots + s_w$. By definition, $\sigma_{\underline{t},\underline{r},\underline{s}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w)$ is equal to the homogeneous component of multidegree $(\underline{t}, \underline{r}, \underline{s})$ of

$$\begin{aligned} f &= \sigma_{t,r}(x_1 + \dots + x_u, y_1 + \dots + y_v, z_1 + \dots + z_w) \\ &= \sum (-1)^{\xi} \sigma_{j_1}(\alpha_1) \cdots \sigma_{j_p}(\alpha_p) \Big|_{x \rightarrow \sum_{i=1}^u x_i, y \rightarrow \sum_{j=1}^v y_j, z \rightarrow \sum_{k=1}^w z_k}, \end{aligned}$$

where $\xi, \alpha_1, \dots, \alpha_p$, and j_1, \dots, j_p are the same as in the definition of $\sigma_{t,s}$. Let \mathcal{Q} be the quiver from Section 3. For a closed path α in \mathcal{Q} denote the result of substitution $x \rightarrow \sum x_i, y \rightarrow \sum y_j, z \rightarrow \sum z_k$ in α by

$$\sum_{l \in L(\alpha)} \alpha^{(l)},$$

where $\alpha^{(l)}$ is a closed path in $\mathcal{Q}_{\underline{t},\underline{r},\underline{s}}$ for all l . As an example, the result of substitution $x \rightarrow x_1 + x_2, y \rightarrow y_1 + y_2$, and $z \rightarrow z_1 + z_2$ in yz^T is equal to $y_1 z_1^T + y_1 z_2^T + y_2 z_1^T + y_2 z_2^T$.

Applying Amitsur's formula (4) to f , we obtain

$$f = \sum (-1)^{\xi} \cdot (-1)^{\xi_1} \sigma_{j_{11}}(\gamma_{11}) \cdots \sigma_{j_{1q_1}}(\gamma_{1q_1}) * \cdots * (-1)^{\xi_p} \sigma_{j_{p1}}(\gamma_{p1}) \cdots \sigma_{j_{pq_p}}(\gamma_{pq_p}), \quad (12)$$

where $\xi_i = j_i - (j_{i1} + \dots + j_{iq_i})$. Here for any $1 \leq i \leq p$ the sum ranges over pairwise different primitive cycles $\gamma_{i1}, \dots, \gamma_{iq_i}$ in letters $\alpha_i^{(l)}$ (where $l \in L(\alpha_i)$) and $j_{i1}, \dots, j_{iq_i} \geq 1$ such that $j_{i1} \deg(\gamma_{i1}) + \dots + j_{iq_i} \deg(\gamma_{iq_i}) = j_i \deg(\alpha_i)$.

We claim that taking the homogeneous component of f of multidegree $(\underline{t}, \underline{r}, \underline{s})$ we obtain the required formula for $\sigma_{\underline{t}, \underline{r}, \underline{s}}$. We split the proof into several statements.

1. We have $(-1)^{\xi_1 + \dots + \xi_p} = (-1)^\eta$, where $\eta = t + \sum_{i=1}^p \sum_{k=1}^{q_p} j_{ik} (\deg_y(\gamma_{ik}) + \deg_z(\gamma_{ik}) + 1)$.

Proof. For $1 \leq i \leq p$ we have $\deg_y \alpha_i = \deg_y \alpha_i^{(l)}$ for all l . Then

$$\deg_y(\gamma_{ik}) = \deg_y \alpha_i \cdot \frac{\deg \gamma_{ik}}{\deg \alpha_i}$$

for all k . Therefore

$$\begin{aligned} j_i \deg_y \alpha_i &= \frac{\deg_y \alpha_i}{\deg \alpha_i} (j_{i1} \deg(\gamma_{i1}) + \dots + j_{iq_i} \deg(\gamma_{iq_i})) = \\ &= j_{i1} \deg_y(\gamma_{i1}) + \dots + j_{iq_i} \deg_y(\gamma_{iq_i}). \end{aligned}$$

The same formula is also valid for z . Now it is easy to see that the required formula is valid.

Given a closed path γ in $\mathcal{Q}_{\underline{t}, \underline{r}, \underline{s}}$, denote the result of substitution $x_1, \dots, x_u \rightarrow x, y_1, \dots, y_v \rightarrow y$, and $z_1, \dots, z_w \rightarrow z$ in γ by $\psi(\gamma)$. We can consider $\psi(\gamma)$ as a closed path in \mathcal{Q} . Let us remark that if $\alpha^p = \beta^q$ for $\alpha, \beta \in \mathcal{M}(\mathcal{Q})$, $p, q > 0$, then there exists a $\gamma \in \mathcal{M}(\mathcal{Q})$ such that $\alpha = \gamma^i$ and $\beta = \gamma^j$ for some i, j ; in particular, if β is primitive, then $\alpha = \beta^i$ for some i .

2. Let α be a primitive closed path in \mathcal{Q} and γ be a primitive word in letters $\alpha^{(l)}$, where $l \in L(\alpha)$. Then γ is a primitive closed path in $\mathcal{Q}_{\underline{t}, \underline{r}, \underline{s}}$. In particular, $\gamma_{11}, \dots, \gamma_{1q_1}, \dots, \gamma_{p1}, \dots, \gamma_{pq_p}$ are primitive closed paths.

Proof. Assume that there exist $k > 1$ and a primitive closed path β in $\mathcal{Q}_{\underline{t}, \underline{r}, \underline{s}}$ such that $\gamma = \beta^k$. We have $\gamma = \alpha^{(l_1)} \dots \alpha^{(l_m)}$ for some $l_1, \dots, l_m \in L(\alpha)$. Thus $\alpha^m = \psi(\beta)^k$ and applying the above mentioned remark we obtain that there exists a $j > 0$ such that $\psi(\beta) = \alpha^j$. Since $\deg \alpha^{(l)} = \deg \alpha$ for all l , we have $\beta = \alpha^{(l_1)} \dots \alpha^{(l_j)}$. Hence,

$$\gamma = (\alpha^{(l_1)} \dots \alpha^{(l_j)})^k = \alpha^{(l_1)} \dots \alpha^{(l_m)}.$$

Closed paths $\alpha^{(l)}$ (where $l \in L(\alpha)$) are pairwise different. So the last formula implies that $l_i = l_{i+j}$ for any $i \leq m - j$. In other words, γ is not a primitive word in letters $\alpha^{(l)}$, where $l \in L(\alpha)$; a contradiction.

3. Closed paths $\gamma_{11}, \dots, \gamma_{1q_1}, \dots, \gamma_{p1}, \dots, \gamma_{pq_p}$ are pairwise different.

Proof. Assume that $\gamma_{ik} = \gamma_{jm}$. Equalities $\psi(\gamma_{ik}) = \alpha_i \dots \alpha_i$ and $\psi(\gamma_{jm}) = \alpha_j \dots \alpha_j$ together with $\psi(\gamma_{ik}) = \psi(\gamma_{jm})$ imply that $i = j$ (see the above mentioned remark). Since $\gamma_{i1}, \dots, \gamma_{iq_i}$ are pairwise different, we have $k = m$.

To complete the proof of the lemma we apply substitution ψ in the same way as in the proof of Statement 3. \square

Denote by $\mathcal{I}_{u,v}^{\text{lin}}$ the set of $\underline{\alpha}$ such that $(1^p, \underline{\alpha}) \in \mathcal{I}_{1^u, 1^v, 1^v}$, where $p = \#\underline{\alpha}$. The following corollary is a trivial consequence of Lemma 4.2.

Corollary 4.3. *We have*

$$\sigma_{u,v}^{\text{lin}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_v) = \sum_{\underline{\alpha} \in \mathcal{T}_{u,v}^{\text{lin}}} (-1)^{\eta_{\underline{\alpha}}} \text{tr}(\underline{\alpha}),$$

where $\eta_{\underline{\alpha}} = u + \sum_{i=1}^p (\deg_y \alpha_i + \deg_z \alpha_i + 1)$, $p = \#\underline{\alpha}$, and $\text{tr}(\underline{\alpha})$ stands for $\text{tr}(\alpha_1) \cdots \text{tr}(\alpha_p)$.

Lemma 4.4. *Assume that $\mathbb{F} = \mathbb{Q}$. Then for $x, y, z \in \mathcal{M}_{\mathbb{F}}$ we have*

$$\sigma_{t,r}(x, y, z) = \frac{1}{t!(r!)^2} \sigma_{t,r}^{\text{lin}}(\underbrace{x, \dots, x}_t, \underbrace{y, \dots, y}_r, \underbrace{z, \dots, z}_r).$$

Proof. In this proof we use notions formulated in Section 7 (see below). Consider the tableau with substitution $(\mathcal{T}, (X, Y, Z))$ of dimension $(t + 2r, t + 2r)$ from Example 7.2. By Lemma 7.6, we have

$$\sigma_{t,r}(x, y, z) = \sum_{(\underline{j}, \underline{c}) \in \mathcal{I}_{\mathcal{T}}} \text{sgn}(\xi_{\underline{j}, \underline{c}}) \sigma_{j_1}(c_1) \cdots \sigma_{j_p}(c_p)|_{1 \rightarrow x, 2 \rightarrow y, 3 \rightarrow z},$$

where $p = \#\underline{j} = \#\underline{c}$.

Using Lemma 4 from [12], we can reformulate Theorem 3 from [12] for $(\mathcal{T}, (X, Y, Z))$ as follows: if $n = t + 2r$, then

$$\frac{1}{t!(r!)^2} \sum_{\xi \in S_n} \text{sgn}(\xi) \prod_{a \in \mathcal{T}_{c_1}^{\xi}} \text{tr}(X_{\varphi(a)}) = \sum_{(\underline{j}, \underline{c}) \in \mathcal{I}_{\mathcal{T}}} \text{sgn}(\xi_{\underline{j}, \underline{c}}) \sigma_{j_1}(X_{c_1}) \cdots \sigma_{j_p}(X_{c_p}), \quad (13)$$

where $p = \#\underline{j} = \#\underline{c}$. Consider the result of formal substitution $X \rightarrow 1$, $Y \rightarrow 2$, and $Z \rightarrow 3$ in (13), where $1, 2, 3 \in \mathcal{M}^{\infty}$ are letters:

$$\frac{1}{t!(r!)^2} \sum_{\xi \in S_n} \text{sgn}(\xi) \prod_{a \in \mathcal{T}_{c_1}^{\xi}} \text{tr}(\varphi(a)) = \sum_{(\underline{j}, \underline{c}) \in \mathcal{I}_{\mathcal{T}}} \text{sgn}(\xi_{\underline{j}, \underline{c}}) \sigma_{j_1}(c_1) \cdots \sigma_{j_p}(c_p). \quad (14)$$

We claim that (14) is valid equality of two elements from $\mathcal{N}_{\sigma}^{\infty}$. Theorem 3 was proven in Sections 5, 6, and 7 of [12]. We repeat this proof without using Section 5, i.e., we do not apply Lemma 4 from Section 5 in the reasoning several lines before Formula (14) from [12]. Since \mathcal{T} has two columns, we obtain the claim. Lemma 7.12 concludes the proof. \square

Lemma 4.5. *Assume that $\mathbb{F} = \mathbb{Q}$. Then $\sigma_{\underline{t}, \underline{r}, \underline{s}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w) =$*

$$\frac{1}{\underline{t}! \underline{r}! \underline{s}!} \sigma_{\underline{t}, \underline{r}}^{\text{lin}}(\underbrace{x_1, \dots, x_1}_{t_1}, \dots, \underbrace{x_u, \dots, x_u}_{t_u}, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_v, \dots, y_v}_{r_v}, \underbrace{z_1, \dots, z_1}_{s_1}, \dots, \underbrace{z_w, \dots, z_w}_{s_w}),$$

where $t = t_1 + \cdots + t_u$ and $r = r_1 + \cdots + r_v = s_1 + \cdots + s_w$.

Proof. The left hand side of the required formula is equal to the homogeneous component of $\sigma_{t,r}(x_1 + \cdots + x_u, y_1 + \cdots + y_v, z_1 + \cdots + z_w)$ of multidegree $(\underline{t}, \underline{r}, \underline{s})$. Applying Lemma 4.4 to $\sigma_{t,r}$ and using linearity of $\sigma_{t,r}^{\text{lin}}$, we complete the proof. \square

Lemma 4.6. *Assume that $X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r$ are $n \times n$ matrices and there exists a j such that Y_j or Z_j is a symmetric matrix. Then*

$$\sigma_{t,r}^{\text{lin}}(X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r) = 0.$$

Proof. Assume that Y_j is a symmetric matrix. Consider a primitive closed path $\alpha = \beta y_j$ in $\mathcal{Q}_{1^t, 1^r, 1^r}$ such that $\deg_\gamma \alpha + \deg_{\gamma^T} \alpha \leq 1$ for any arrow γ from $\mathcal{Q}_{1^t, 1^r, 1^r}$. Then αy_j^T is also a primitive closed path and $\alpha y_j^T \not\sim \alpha y_j$. Lemma 4.2 concludes the proof. \square

Let us recall that E_n stands for the identity $n \times n$ matrix.

Lemma 4.7. *Let $X_1, \dots, X_{t-1}, Y_1, \dots, Y_r, Z_1, \dots, Z_r$ be $n \times n$ matrices. Then*

$$\sigma_{t,r}^{\text{lin}}(X_1, \dots, X_{t-1}, E_n, Y_1, \dots, Y_r, Z_1, \dots, Z_r) = a \sigma_{t-1,r}^{\text{lin}}(X_1, \dots, X_{t-1}, Y_1, \dots, Y_r, Z_1, \dots, Z_r),$$

where $a = n - (t + 2r) + 1$.

Proof. Denote $X_t = E_n$ and $\mu = \sigma_{t,r}^{\text{lin}}(X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r)$. Let x_1, \dots, z_r be arrows of $\mathcal{Q}_{1^t, 1^r, 1^r}$. Denote by ψ the substitution $x_i \rightarrow X_i, y_j \rightarrow Y_j$, and $z_j \rightarrow Z_j$ for all $1 \leq i \leq t$ and $1 \leq j \leq r$. By Corollary 4.3, we have

$$\mu = \sum_{\underline{\alpha} \in \mathcal{I}_{t,r}^{\text{lin}}} (-1)^{\eta_{\underline{\alpha}}} \psi(\text{tr}(\underline{\alpha})) = \mu_1 + \mu_2,$$

where μ_1 is the sum of all $(-1)^{\eta_{\underline{\alpha}}} \text{tr}(\underline{\alpha})$ such that $\alpha_j \sim x_t$ for some j . Equalities

$$\mu_1 = \text{tr}(E_n) \sum_{\underline{\beta} \in \mathcal{I}_{t-1,r}^{\text{lin}}} (-1)^{\eta_{\underline{\beta}}} \psi(\text{tr}(\underline{\beta})) \quad \text{and}$$

$$\mu_2 = -(t + 2r - 1) \sum_{\underline{\beta} \in \mathcal{I}_{t-1,r}^{\text{lin}}} (-1)^{\eta_{\underline{\beta}}} \psi(\text{tr}(\underline{\beta}))$$

conclude the proof. \square

The following remark is trivial.

Remark 4.8. Let $p > 0$ be the characteristic of \mathbb{F} and $\mathbb{F}[x_1, \dots, x_m]$ be a polynomial ring. Define the ring homomorphism $\pi : \mathbb{Z} \rightarrow \mathbb{F}$ by $\pi(1) = 1_{\mathbb{F}}$, where $1_{\mathbb{F}}$ stands for the unity of \mathbb{F} . Consider $f = \sum a_i f_i \in \mathbb{F}[x_1, \dots, x_m]$, where $a_i \in \pi(\mathbb{Z})$, and f_i is a monomial in x_1, \dots, x_m for all i . Take $b_i \in \{0, 1, \dots, p-1\} \subset \mathbb{Z}$ such that $\pi(b_i) = a_i$ and set $h = \sum_i b_i f_i \in \mathbb{Q}[x_1, \dots, x_m]$. Then $h = 0$ implies $f = 0$.

Lemma 4.9. *Let X, Y, Z be $n \times n$ matrices and $n \geq t + 2r$. Then*

$$\sigma_{t,r}(X + \lambda E_n, Y, Z) = \sum_{i=0}^t \binom{n - (t + 2r) + i}{i} \lambda^i \sigma_{t-i,r}(X, Y, Z)$$

for any $\lambda \in \mathcal{A}$.

Proof. By Remark 4.8, without loss of generality we can assume $\mathbb{F} = \mathbb{Q}$. Then Lemma 4.5 and linearity of $\sigma_{t,r}^{\text{lin}}$ imply

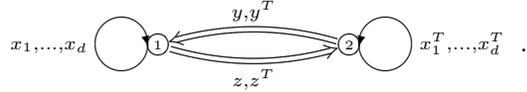
$$\sigma_{t,r}(X + \lambda E_n, Y, Z) = \frac{1}{t!(r!)^2} \sum_{i=0}^t \binom{t}{i} \lambda^i \sigma_{t,r}^{\text{lin}}(\underbrace{X, \dots, X}_{t-i}, \underbrace{E_n, \dots, E_n}_i, Y, \dots, Y, Z, \dots, Z).$$

Lemma 4.7 concludes the proof. \square

5 Proof of Theorem 1.1

Without loss of generality we can assume that \mathbb{F} is an algebraically closed field (cf. Remark 2.1 from [22] and Remark 3.2 from [21]).

Assume that $X_1 = X_{n,1}, \dots, X_d = X_{n,d}, Y = Y_n, Z = Z_n$ are $n \times n$ generic matrices and entries of these matrices are $x_{ij}(1), \dots, x_{ij}(d), y_{ij}, z_{ij}$ ($1 \leq i, j \leq n$), respectively. Consider the mixed quiver \mathcal{G}



where there are d loops in each of its vertices and $1^T = 2$.

Given $\alpha \in \mathcal{M}(\mathcal{G})$ we define $X_{n,\alpha}$ in the same manner as in Section 1, using convention that

$$X_{n,\alpha} = \begin{cases} X_{n,i}, & \text{if } \alpha = x_i \\ Y_n, & \text{if } \alpha = y \\ Z_n, & \text{if } \alpha = z \end{cases}.$$

Denote by $C_n = \mathbb{F}[x_{ij}(k), y_{ij}, z_{ij} \mid 1 \leq i, j \leq n, 1 \leq k \leq d]$ the coordinate ring of the space of mixed representations of \mathcal{G} of dimension vector (n, n) . Then its algebra of *invariants* $J_n \subset C_n$ is generated by $\sigma_t(X_{n,\alpha})$, where $1 \leq t \leq n$ and α is a primitive closed path in \mathcal{G} (see [23]). We define the inclusion $R_n \subset C_n$ in the natural way. Consider a surjective homomorphism

$$\Upsilon_n : \mathcal{N}_\sigma(\mathcal{G}) \rightarrow J_n$$

defined by $\sigma_t(\alpha) \rightarrow \sigma_t(X_{n,\alpha})$, if $t \leq n$, and $\sigma_t(\alpha) \rightarrow 0$ otherwise. Its kernel T_n is the ideal of relations for J_n . Elements of $\bigcap_{i>0} T_i$ are called *free* relations for J_n .

Let (α, β, γ) be a triple of paths in \mathcal{G} . Then it is called *good* if $\alpha' = \alpha'' = 1$, β is a path from 2 to 1, and γ is a path from 1 to 2. Since Lemma 3.1 can be reformulated for $\mathcal{N}_\sigma(\mathcal{G})$, we have

$$\sigma_{t,r} \left(\sum_i a_{1i} \alpha_{1i}, \sum_j a_{2j} \alpha_{2j}, \sum_k a_{3k} \alpha_{3k} \right) \in \mathcal{N}_\sigma(\mathcal{G}), \quad (15)$$

where $a_{1i}, a_{2j}, a_{3k} \in \mathbb{F}$ and $(\alpha_{1i}, \alpha_{2j}, \alpha_{3k})$ is a good triple for all i, j, k .

Theorem 5.1. (Zubkov, [24]) *The ideal of relations T_n for $J_n \simeq \mathcal{N}_\sigma(\mathcal{G})/T_n$ is generated by*

- elements (15) for $t + 2r > n$;
- free relations for J_n .

Given $N > n$, we define two homomorphisms $\tilde{p}_{N,n}, p_{N,n} : C_N \rightarrow C_n$ as follows:

$$\begin{aligned} \tilde{p}_{N,n} : X_{N,k} &\rightarrow \begin{pmatrix} X_{n,k} & 0 \\ 0 & 0 \end{pmatrix}, Y_N \rightarrow \begin{pmatrix} Y_n & 0 \\ 0 & 0 \end{pmatrix}, Z_N \rightarrow \begin{pmatrix} Z_n & 0 \\ 0 & 0 \end{pmatrix}, \text{ and} \\ p_{N,n} : X_{N,k} &\rightarrow \begin{pmatrix} X_{n,k} & 0 \\ 0 & 0 \end{pmatrix}, Y_N \rightarrow \begin{pmatrix} Y_n & 0 \\ 0 & E_{N-n} \end{pmatrix}, Z_N \rightarrow \begin{pmatrix} Z_n & 0 \\ 0 & E_{N-n} \end{pmatrix}, \end{aligned}$$

where $1 \leq k \leq d$ and E_{N-n} stands for the identity matrix. In other words, we assume that

$$\tilde{p}_{N,n}(x_{ij}(k)) = \begin{cases} x_{ij}(k), & \text{if } 1 \leq i, j \leq n \\ 0, & \text{otherwise} \end{cases}$$

and so on. For short, we use the following notations: $\tilde{p} = \tilde{p}_{N,n}$ and $p = p_{N,n}$.

Consider a mapping $\Phi_N : J_N \rightarrow J_N$ defined by:

$$\Phi_N : X_{N,k} \rightarrow X_{N,k}, Y_N \rightarrow E_N, Z_N \rightarrow E_N$$

for all $1 \leq k \leq d$. The following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{N}_\sigma(\mathcal{G}) & \xrightarrow{\Upsilon_N} & J_N & \xrightarrow{\Phi_N} & R^{O(N)} & \xleftarrow{\Psi_N} & \mathcal{N}_\sigma \\ & \searrow \Upsilon_n & \downarrow p & & \downarrow p=\tilde{p} & & \swarrow \Psi_n \\ & & J_n & \xrightarrow{\Phi_n} & R^{O(n)} & & \end{array}$$

The next lemma is a reformulation of Lemmas 3.3 and 3.4 from [24].

Lemma 5.2. *If $N > s > 0$, then*

- the kernel of $\Psi_N : \text{hc}^s(\mathcal{N}_\sigma) \rightarrow \text{hc}^s(R^{O(N)})$ is generated by free relations for $R^{O(n)}$ of degree s ;
- the kernel of $\Upsilon_N : \text{hc}^s(\mathcal{N}_\sigma(\mathcal{G})) \rightarrow \text{hc}^s(J_N)$ is generated by free relations for J_n of degree s .

Lemma 5.2 implies that we can identify $\text{hc}^s(\mathcal{N}_\sigma)$ with $\text{hc}^s(R^{O(N)})$ as well as $\text{hc}^s(\mathcal{N}_\sigma(\mathcal{G}))$ with $\text{hc}^s(J_N)$ modulo free relations. Let us formulate Proposition 3.2 from [24].

Lemma 5.3. *The ideal K_n is generated by elements $\Phi_N(h_f)$, where*

- f ranges over elements (15) with $t + 2r > n$;
- $N > \deg f = s$ and $N \gg n$ is big enough, so we can assume that $f \in J_N$;
- $h = h_f \in J_N$ such that $p(h) = 0$, $\deg h = s$, and $\text{hc}_s(h) = \text{hc}_s(f)$.

Here $\Phi_N(h_f)$ is considered as an element of \mathcal{N}_σ .

Note that in Lemma 5.3 f can be non-homogeneous.

Given $t, r \geq 0$ satisfying $t + 2r > n$, we consider an element (15)

$$f = \sigma_{t,r}(f_1, f_2, f_3),$$

where $f_1 = \sum_i a_{1i} \alpha_{1i}$, $f_2 = \sum_j a_{2j} \alpha_{2j}$, and $f_3 = \sum_k a_{3k} \alpha_{3k}$. There exists an $N \gg n$ such that $N > t + 2r$, $N > t + n$, and we can assume that $f \in J_N$. In what follows we will write f_1 instead of $\sum_i a_{1i} X_{N, \alpha_{1i}}$ and similarly for f_2 and f_3 .

By Theorem 5.1, we have

$$\tilde{p}(f) = 0. \quad (16)$$

But we can not claim that $p(f) = 0$. We will construct $h = h_f \in C_N$ that is “close” to f and $p(h) = 0$ (see Lemma 5.5 below).

We can rewrite f_1 as $f_1 = f_{11} + f_{12}$, where f_{11} is a sum of all $a_{1i} X_{N, \alpha_{1i}}$ such that α_{1i} contains an arrow x_k or x_k^T for some k . Similarly, we can rewrite f_2, f_3 as $f_2 = f_{21} + f_{22}$ and $f_3 = f_{31} + f_{32}$, where f_{21} and f_{31} contain all summands with x_k or x_k^T . Let $\lambda_l \in \mathbb{F}$ be the sum of coefficients of summands of $f_{l,2}$ for $l = 1, 2, 3$. Note that

$$p(f_l) = \tilde{p}(f_l) + \lambda_l E_{N,n} \quad (17)$$

for $l = 1, 2, 3$. Here $E_{N,n}$ stands for the following $N \times N$ matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & E_{N-n} \end{pmatrix}.$$

Lemma 5.4.

$$p(f) = \sum_{i=0}^t \binom{N-n}{i} \tilde{p}(\sigma_{t-i,r}(f_1, f_2, f_3)) \lambda_1^i.$$

Proof. By Remark 4.8, without loss of generality we can assume $\mathbb{F} = \mathbb{Q}$. By (17), we have

$$p(f) = \sigma_{t,r}(\tilde{p}(f_1) + \lambda_1 E_{N,n}, \tilde{p}(f_2) + \lambda_2 E_{N,n}, \tilde{p}(f_3) + \lambda_3 E_{N,n}).$$

Lemmas 4.4, 4.6 together with linearity of $\sigma_{t,r}^{\text{lin}}$ imply $p(f) =$

$$\frac{1}{t!(r!)^2} \sum_{i=0}^t \binom{t}{i} \sigma_{t,r}^{\text{lin}}(\underbrace{\tilde{p}(f_1), \dots, \tilde{p}(f_1)}_{t-i}, \underbrace{E_{N,n}, \dots, E_{N,n}}_i, \tilde{p}(f_2), \dots, \tilde{p}(f_2), \tilde{p}(f_3), \dots, \tilde{p}(f_3)) \lambda_1^i.$$

Since $\tilde{p}(f_l) E_{N,n} = 0$ for all l , applying Lemma 4.2 we obtain

$$p(f) = \frac{1}{(r!)^2} \sum_{i=0}^t \frac{1}{i!(t-i)!} \sigma_{t-i,r}^{\text{lin}}(\tilde{p}(f_1), \dots, \tilde{p}(f_1), \tilde{p}(f_2), \dots, \tilde{p}(f_2), \tilde{p}(f_3), \dots, \tilde{p}(f_3)) \lambda_1^i a,$$

where $a = \sigma_{i,0}^{\text{lin}}(E_{N,n}, \dots, E_{N,n})$. By Lemma 4.4 and Remark 3.4, we have

$$a = i! \sigma_{i,0}(E_{N,n}, 0, 0) = i! \sigma_i(E_{N,n}) = i! \binom{N-n}{i}.$$

Lemma 4.4 concludes the proof. \square

Lemma 5.5. *There exist $b_0, b_1, \dots, b_t \in \mathbb{Z}$ such that*

$$h = h_f = \sum_{i=0}^t b_i \lambda_1^i \sigma_{t-i,r}(f_1, f_2, f_3) \in J_N$$

satisfies the following conditions:

- a) $p(h) = 0$;
- b) $\deg h = s$ and $\text{hc}_s(h) = \text{hc}_s(f)$, where $s = \deg f$.

Proof. We set $b_0 = 1$. By Lemma 5.4, we have

$$p(h) = \sum_{i=0}^t b_i \lambda_1^i \sum_{j=0}^{t-i} \binom{N-n}{j} \tilde{p}(\sigma_{t-i-j,r}(f_1, f_2, f_3)) \lambda_1^j.$$

We substitute k for $t - i - j$ and obtain

$$p(h) = \sum_{k=0}^t \tilde{p}(\sigma_{k,r}(f_1, f_2, f_3)) \lambda_1^{t-k} \sum_{i=0}^{t-k} b_i \binom{N-n}{t-i-k}.$$

The following system of linear equations

$$\sum_{i=0}^{t-k} b_i \binom{N-n}{t-i-k} = 0, \text{ where } 0 \leq k < t \quad (18)$$

with respect to b_1, \dots, b_t is triangular and has 1 on the main diagonal. Hence this system has a solution $b_1, \dots, b_t \in \mathbb{Z}$. Using Formula (16), we can see that the equality $p(h) = 0$ holds. Obviously, $\deg h = s$ and $\text{hc}_s(h) = \text{hc}_s(b_0 \sigma_{t,r}(f_1, f_2, f_3)) = \text{hc}_s(f)$. \square

Lemma 5.6. *Assume that $b_0 = 1$ and b_1, \dots, b_t satisfy system of linear equations (18). Then*

$$\sum_{i=0}^{t-k} b_{t-i-k} \binom{N-(l+2r)}{i} = 0$$

for $0 \leq k \leq l \leq n - 2r$.

Proof. Denote the left hand side of the required formula by $b_{k,l}$.

The proof is by decreasing induction on l . If $l = n - 2r$, then $b_{k,l} = 0$ by (18).
Let $l < n - 2r$. Then formula

$$\binom{q-1}{p} + \binom{q-1}{p-1} = \binom{q}{p},$$

where $0 \leq p < q$, implies

$$b_{k,l} = b_{k,l+1} + \sum_{i=0}^{t-k} b_{t-i-k} \binom{N - (l + 2r + 1)}{i-1}.$$

We substitute j for $i - 1$ and obtain $b_{k,l} = b_{k,l+1} + b_{k+1,l+1}$. Induction hypothesis concludes the proof. \square

Denote by I_N the ideal of $R^{O(N)}$ generated by $\sigma_{p,q}(\sum_i c_{1i}X(\gamma_{1i}), \sum_j c_{2j}X(\gamma_{2j}), \sum_k c_{3k}X(\gamma_{3k}))$, where $X(\gamma)$ stands for $X_{N,\gamma}$, $p + 2q > n$, $\gamma_{1i}, \gamma_{2j}, \gamma_{3k}$ are monomials in $x_1, \dots, x_d, x_1^T, \dots, x_d^T$, and $c_{1i}, c_{2j}, c_{3k} \in \mathbb{F}$.

Lemma 5.7. *Assume that $h \in J_N$ is the element from Lemma 5.5. Then $\Phi_N(h)$ belongs to I_N .*

Proof. For short, denote $\Phi_N(f_{1l}) = g_l$ for $l = 1, 2, 3$. We have $\Phi_N(f_l) = g_l + \lambda_l E_N$ for all l .

To begin with, we assume that $\mathbb{F} = \mathbb{Q}$. Using Lemmas 4.4 and 4.6 together with linearity of $\sigma_{t,r}^{\text{lin}}$ in the same manner as in the proof of Lemma 5.4, we obtain

$$\Phi_N(h) = \sum_{i=0}^t b_i \lambda_1^i \sigma_{t-i,r}(g_1 + \lambda_1 E_N, g_2, g_3).$$

Lemma 4.9 implies

$$\Phi_N(h) = \sum_{k=0}^t \sigma_{k,r}(g_1, g_2, g_3) \lambda_1^{t-k} \sum_{i=0}^{t-k} b_{t-i-k} \binom{N - (k + 2r)}{i}. \quad (19)$$

In the general case Remark 4.8 shows that Formula (19) is also valid.

If $k + 2r \leq n$, then

$$\sum_{i=0}^{t-k} b_{t-i-k} \binom{N - (k + 2r)}{i} = 0 \quad (20)$$

by Lemma 5.6. Thus $\Phi_N(h)$ belongs to I_N . \square

Lemma 5.8. *Any element $g = \sigma_{t,r}(\sum_i c_{1i}\gamma_{1i}, \sum_j c_{2j}\gamma_{2j}, \sum_k c_{3k}\gamma_{3k})$ of \mathcal{N}_σ , where $t + 2r > n$ and $c_{1i}, c_{2j}, c_{3k} \in \mathbb{F}$, $\gamma_{1i}, \gamma_{2j}, \gamma_{3k} \in \mathcal{M}$, belongs to K_n .*

Proof. There exists a $g' \in \mathcal{N}_\sigma(\mathcal{G})$ such that $\Phi_N(\Upsilon_N(g')) = \Psi_N(g)$. Since $\Upsilon_n(g') = 0$ by Theorem 5.1, we obtain

$$\Psi_n(g) = p \circ \Psi_N(g) = p \circ \Phi_N \circ \Upsilon_N(g') = \Phi_n \circ \Upsilon_n(g') = 0.$$

\square

Lemmas 5.3, 5.7, and 5.8 conclude the proof of Theorem 1.1.

6 Free relations

In this section it is convenient to write $x_1, \dots, x_d, x_1^T, \dots, x_d^T$ for letters $1, \dots, d, 1^T, \dots, d^T$ that are free generators of \mathcal{M} . Similarly to Section 1, we denote by \mathcal{M}^∞ the monoid freely generated by letters $x_1, x_2, \dots, x_1^T, x_2^T, \dots$ and define \mathcal{N}^∞ ($\mathcal{N}_\sigma^\infty$, respectively) similarly to \mathcal{N} (\mathcal{N}_σ , respectively).

Definition 6.1 (of $\text{Lin}(f)$). Given \mathbb{N}^d -homogeneous $f \in \mathcal{N}_\sigma$ with $\text{mdeg } f = \underline{t} = (t_1, \dots, t_d)$, we define its *complete linearization* $\text{Lin}(f) \in \mathcal{N}_\sigma^\infty$ as follows. Let $h \in \mathcal{N}_\sigma^\infty$ be the result of substitution $x_i \rightarrow \sum_{j=0}^{t_i-1} a_{ij} x_{i+jd}$ ($1 \leq i \leq d$) in f , where $a_{ij} \in \mathbb{F}$, and we consider h as a polynomial in a_{ij} . We denote the coefficient of $a_{11} \cdots a_{1t_1} \cdots a_{d1} \cdots a_{dt_d}$ in h by $\text{Lin}(f)$.

Definition 6.2 (of $\text{Lin}^{-1}(\alpha)$). Given $\alpha \in \mathcal{M}^\infty$, we define $\text{Lin}^{-1}(\alpha) \in \mathcal{M}$ as the result of substitution $x_{i+jd} \rightarrow x_i$ ($1 \leq i \leq d$ and $j > 0$) in α .

Example 6.3. If $f = \sigma_2(x_1)$, then (5) implies that $\text{Lin}(f) = -\text{tr}(x_1 x_{d+1}) + \text{tr}(x_1) \text{tr}(x_{d+1})$.

If $f = \text{tr}(x_1)^3$, then $\text{Lin}(f) = 6 \text{tr}(x_1) \text{tr}(x_{d+1}) \text{tr}(x_{2d+1})$.

Given $f = \sigma_{j_1}(\alpha_1) \cdots \sigma_{j_p}(\alpha_p) \in \mathcal{N}_\sigma$, where $\alpha_1, \dots, \alpha_p \in \mathcal{N}$, we define c_f and e_f as follows. We consider a subset S of the set of pairs (j_i, α_i) , $1 \leq i \leq p$, such that elements of S are pairwise different with respect to \sim , where $(j_i, \alpha_i) \sim (j_k, \alpha_k)$ if and only if $j_i = j_k$ and $\alpha_i \sim \alpha_k$. Given $(j, \alpha) \in S$, we denote $c_{(j, \alpha)} = \#\{(j_i, \alpha_i) \mid j_i = j, \alpha_i \sim \alpha, 1 \leq i \leq p\}$. We set $c_f = \prod_{(j, \alpha) \in S} c_{(j, \alpha)}$ and $e_f = p$.

The proof of the following lemma is similar to the proof of Lemma 4.2.

Lemma 6.4. Let $f = \sigma_{j_1}(\alpha_1) \cdots \sigma_{j_p}(\alpha_p) \in \mathcal{N}_\sigma$, where $\alpha_1, \dots, \alpha_p \in \mathcal{N}$, and $\underline{t} = \text{mdeg } f$. Then

$$\text{Lin}(f) = (-1)^{j_1 + \cdots + j_p - p} c_f \sum \text{tr}(\gamma_1) \cdots \text{tr}(\gamma_p) + c_f h,$$

where

- $\gamma_1, \dots, \gamma_p \in \mathcal{N}^\infty$ are representatives of pairwise different \sim -equivalence classes;
- we have $\deg_{x_l}(\gamma_1 \cdots \gamma_p) + \deg_{x_l^T}(\gamma_1 \cdots \gamma_p) = \begin{cases} 1, & \text{if } l = i + jd \text{ for } 1 \leq i \leq d \text{ and } 0 \leq j < t_i \\ 0, & \text{otherwise} \end{cases}$ for all $l > 0$;
- $\text{Lin}^{-1}(\gamma_i) = \alpha_i^{j_i}$ for all $1 \leq i \leq p$.
- $h = \sum_k \pm h_k$, where h_k is a product of $p+1$ or more traces.

Lemma 6.5. If $\text{char } \mathbb{F} > 0$, then the ideal of free relations lays in the ideal of \mathcal{N}_σ generated by $\sigma_t(\alpha)^{\text{char } \mathbb{F}}$, where α ranges over \mathcal{N} and $t > 0$. If $\text{char } \mathbb{F} = 0$, then the only free relation is trivial.

Proof. Consider an \mathbb{N}^d -homogeneous free relation $f \in \mathcal{N}_\sigma$.

1. Let $\text{mdeg } f = \underline{t}$ satisfy $t_i \leq 1$ for all $1 \leq i \leq d$. We assume that $f \neq 0$. Then $f = \sum_k a_k f_k$, where $a_k \in \mathbb{F}$, $a_k \neq 0$, and f_k are pairwise different products of traces. Denote $n = \text{deg } f$ and denote by $e_{i,j}$ the $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is 1 and any other entry is 0. Let $f_1 = \text{tr}(\alpha_1) \cdots \text{tr}(\alpha_p)$

for some $\alpha_1, \dots, \alpha_p \in \mathcal{N}$. Given $\alpha_1 = y_1 \cdots y_q$, where y_1, \dots, y_q are letters, we set $Y_i = e_{i,i+1}$ for $1 \leq i < q$ and $Y_q = e_{q,1}$. Considering $\alpha_2, \dots, \alpha_p$, we define Y_i for $q < i \leq \deg f_1$. Since f is a free relation, the substitution $y_i \rightarrow Y_i$ ($1 \leq i \leq \deg f_1$) implies that $a_1 = 0$; a contradiction. Thus, $f = 0$.

2. For some q we have $f = \sum_i a_i f_i + \sum_j b_j h_j$, where $a_i, b_j \in \mathbb{F}$, $a_i, b_j \neq 0$, f_i, h_j are pairwise different products of σ_t ($t > 0$) such that $e_{f_i} = q$ and $e_{h_j} > q$ for all i, j . Since $\text{Lin}(f)$ is also a free relation, linearity of Lin , Lemma 6.4, and part 1 of the proof imply that $c_{f_i} = 0$. Hence f_i lays in the required ideal of \mathcal{N}_σ in case $\text{char } \mathbb{F} > 0$ and $f = 0$ in \mathcal{N}_σ in case $\text{char } \mathbb{F} = 0$. Since $\text{Lin}(f_i) = 0$, we can repeat the same reasoning for $h = \sum_j b_j h_j$ and so on. \square

Given an \mathbb{N} -graded algebra \mathcal{A} , denote by \mathcal{A}^+ the subalgebra generated by elements of \mathcal{A} of positive degree. It is easy to see that a set $\{a_i\} \subseteq \mathcal{A}$ is a minimal (by inclusion) homogeneous system of generators (m.h.s.g.) for \mathcal{A} if and only if $\{\overline{a_i}\}$ is a basis for $\overline{\mathcal{A}} = \mathcal{A}/(\mathcal{A}^+)^2$ and $\{a_i\}$ are homogeneous. Let us recall that an element $a \in \mathcal{A}$ is called *decomposable* if it belongs to the ideal $(\mathcal{A}^+)^2$. Therefore the least upper bound for the degrees of elements of a m.h.s.g. for $R^{O(n)}$ is equal to the maximal degree of indecomposable invariants and we denote it by D_{\max} . Theorem 1.1 together with Lemma 6.5 imply the following corollary.

Corollary 6.6. *If $\text{char } \mathbb{F} \neq 2$, then the ideal of relations $\overline{K_n}$ of $R^{O(n)} \simeq \overline{\mathcal{N}_\sigma}/\overline{K_n}$ is generated by $\sigma_{t,r}(\alpha, \beta, \gamma)$, where $t + 2r > n$ and α, β, γ range over $\mathcal{M}_\mathbb{F}$.*

As an application of Corollary 6.6 we obtained the following result in [16].

Theorem 6.7. *Let $n = 3$ and $d \geq 1$. Then*

- *If $\text{char } \mathbb{F} = 3$, then $2d + 4 \leq D_{\max} \leq 2d + 7$.*
- *If $\text{char } \mathbb{F} \neq 2, 3$, then $D_{\max} = 6$.*

As about matrix $GL(n)$ -invariants in case $n = 3$, its minimal system of generators was explicitly calculated in [10] and [11].

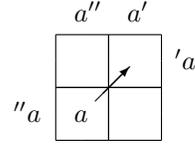
7 Appendix: other definitions of $\sigma_{t,r}$

In this section we assume that \mathcal{A} is a commutative unitary algebra over the field \mathbb{F} and all matrices are considered over \mathcal{A} . In what follows we recall some definitions from [12]. Note that in this section we consider only rectangular tableaux with two columns whereas in [12] tableaux with arbitrary number of columns of any length were defined.

Definition 7.1 (of a tableau with substitution). A pair $(\mathcal{T}, (X_1, \dots, X_d))$ is called a *tableau with substitution* of dimension (n, n) , if

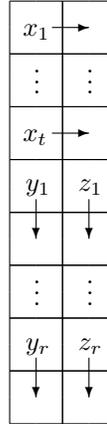
- \mathcal{T} is a rectangular tableau with two columns and n rows. The tableau \mathcal{T} is filled with arrows in such a way that an *arrow* goes from one cell of the tableau into another one, and each cell of the tableau is either the head or the tail of one and only one arrow. We write $a \in \mathcal{T}$ for an arrow a from \mathcal{T} . Given an arrow $a \in \mathcal{T}$, denote by a' and a'' the columns containing the

head and the tail of a , respectively. Similarly, denote by $'a$ the row containing the head of a , and denote by $''a$ the row containing the tail of a . Schematically this is depicted as



- φ is a fixed mapping from the set of arrows of \mathcal{T} onto $\{1, \dots, d\}$ that satisfies the following property:
if $a, b \in \mathcal{T}$ and $\varphi(a) = \varphi(b)$, then $a' = b'$, $a'' = b''$;
- X_1, \dots, X_d are $n \times n$ matrices.

Example 7.2. Let X, Y, Z be $(t + 2r) \times (t + 2r)$ matrices and let $\mathcal{T} = \mathcal{T}_{t,r}$ be the tableau:



We define φ as follows: $\varphi(x_i) = 1$, $\varphi(y_j) = 2$, $\varphi(z_j) = 3$ for $1 \leq i \leq t$ and $1 \leq j \leq r$. Then $(\mathcal{T}, (X, Y, Z))$ is a tableau with substitution of dimension $(t + 2r, t + 2r)$.

Example 7.3. Let $X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r$ be $(t + 2r) \times (t + 2r)$ matrices and let \mathcal{T} be the tableau from Example 7.2. We define φ as follows: $\varphi(x_i) = i$, $\varphi(y_j) = t + j$, $\varphi(z_j) = t + r + j$ for $1 \leq i \leq t$ and $1 \leq j \leq r$. Then $(\mathcal{T}, (X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r))$ is a tableau with substitution of dimension $(t + 2r, t + 2r)$.

Definition 7.4 (of $\text{bpf}_{\mathcal{T}}(X_1, \dots, X_d)$). Let X_1, \dots, X_d be $n \times n$ matrices and let $(\mathcal{T}, (X_1, \dots, X_d))$ be a tableau with substitution of dimension (n, n) . Then we define

$$\text{bpf}_{\mathcal{T}}(X_1, \dots, X_d) = \sum \text{sgn}(\pi_1) \text{sgn}(\pi_2) \prod_{a \in \mathcal{T}} (X_{\varphi(a)})_{\pi_{a''}('a), \pi_{a'}('a)},$$

where $(X_{\varphi(a)})_{ij}$ stands for $(i, j)^{\text{th}}$ entry of the matrix $X_{\varphi(a)}$ and the sum ranges over permutations $\pi_1, \pi_2 \in S_n$ such that for any $a, b \in \mathcal{T}$ the conditions $\varphi(a) = \varphi(b)$ and " $a < b$ " imply that $\pi_i('a) < \pi_i('b)$ for $i = a'' = b''$. For $\mathbb{F} = \mathbb{Q}$ there exists a more convenient formula

$$\text{bpf}_{\mathcal{T}}(X_1, \dots, X_d) = \frac{1}{t!(r!)^2} \sum_{\pi_1, \pi_2 \in S_n} \text{sgn}(\pi_1) \text{sgn}(\pi_2) \prod_{a \in \mathcal{T}} (X_{\varphi(a)})_{\pi_{a''}('a), \pi_{a'}('a)}.$$

Example 7.5. Assume that $(\mathcal{T}, (X, Y, Z))$ is the tableau with substitution from Example 7.2. Then

$$\text{DP}_{r,r}(X, Y, Z) = \text{bpf}_{\mathcal{T}}(X, Y, Z),$$

where the determinant-pfaffian $\text{DP}_{r,r}(X, Y, Z)$ was introduced in [13].

The decomposition formula was formulated in Theorem 3, [12] (see (21) below). Assume that $(\mathcal{T}, (X, Y, Z))$ is the tableau with substitution from Example 7.2. Denote by $\text{dec}_{\mathcal{T}}(X, Y, Z)$ the right hand side of the decomposition formula, applied to the tableau with substitution $(\mathcal{T}, (X, Y, Z))$. Given $x, y, z \in \mathcal{M}_{\mathbb{F}}$, we assume that $\text{dec}_{\mathcal{T}}(x, y, z) \in \mathcal{N}_{\sigma}$ stands for the result of formal substitution $X \rightarrow x, Y \rightarrow y$, and $Z \rightarrow z$ in $\text{dec}_{\mathcal{T}}(X, Y, Z)$.

Lemma 7.6. *We have $\sigma_{t,r}(x, y, z) = \text{dec}_{\mathcal{T}}(x, y, z)$ for $x, y, z \in \mathcal{M}_{\mathbb{F}}$, where $\mathcal{T} = \mathcal{T}_{t,r}$ is defined in Example 7.2.*

In particular, $\sigma_{t,r}(X, Y, Z) = \text{DP}_{r,r}(X, Y, Z)$ for $(t + 2r) \times (t + 2r)$ matrices X, Y, Z .

To prove this lemma we need we need additional definitions from [12].

Let $(\mathcal{T}, (X_1, \dots, X_d))$ be a tableau with substitution of dimension (n, n) . Consider \mathcal{M}^{∞} and $\mathcal{N}_{\sigma}^{\infty}$ from Section 6, where letters are $1, 2, \dots, 1^T, 2^T, \dots$. Given $a \in \mathcal{T}$, we consider $\varphi(a) \in \{1, \dots, d\}$ as an element of \mathcal{M}^{∞} . For $u \in \mathcal{M}^{\infty}$ define the matrix X_u in the same way as in Section 1. Here we assume that $X_i = X_{i^T} = 0$ for $i > d$.

For an arrow $a \in \mathcal{T}$ denote by a^T the *transpose arrow*, i.e., by definition $(a^T)'' = a'$, $(a^T)' = a''$, $''(a^T) = 'a$, $'(a^T) = ''a$, and $\varphi(a^T) = \varphi(a)^T \in \mathcal{M}^{\infty}$. We write $a \stackrel{T}{\in} \mathcal{T}$ if a is an arrow or a transpose arrow of \mathcal{T} .

Definition 7.7 (of paths). We say that $a_1, a_2 \stackrel{T}{\in} \mathcal{T}$ are *successive* in \mathcal{T} , if $a'_1 \neq a''_2$ and $'a_1 = ''a_2$.

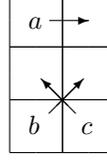
A word $a = a_1 \cdots a_s$, where $a_1, \dots, a_s \stackrel{T}{\in} \mathcal{T}$, is called a *path* in \mathcal{T} , if a_i, a_{i+1} are successive for any $1 \leq i \leq s - 1$. In this case by definition $\varphi(a) = \varphi(a_1) \cdots \varphi(a_s) \in \mathcal{M}^{\infty}$ and $a^T = a_s^T \cdots a_1^T$ is a path in \mathcal{T} ; we denote $a'_s, 'a_s, a''_1, ''a_1$, respectively, by $a', 'a, a'', ''a$, respectively.

A path $a_1 \cdots a_s$ is *closed* if a_s, a_1 are successive.

Consider words $a = a_1 \cdots a_p, b = b_1 \cdots b_q$, where $a_1 \cdots a_p, b_1 \cdots b_q \stackrel{T}{\in} \mathcal{T}$ (in particular, both a, b might be paths in \mathcal{T}). We write $a \sim b$ if there is a cyclic permutation $\pi \in S_p$ such that $a_{\pi(1)} \cdots a_{\pi(p)}$ is equal to b or b^T . We will use similar notations for elements of \mathcal{M}^{∞} , for sets of paths, and for subsets of \mathcal{M}^{∞} .

Denote by \mathcal{T}_{cl} a set of representatives of closed paths in \mathcal{T} with respect to the \sim -equivalence.

Example 7.8. Let



be a fragment of tableau. Then a and bc^T are closed paths.

Definition 7.9 (of \mathcal{T}^τ). Given $\tau \in S_n$, we permute the cells of the 2nd column of \mathcal{T} by τ and denote the resulting tableau by \mathcal{T}^τ . The arrows of \mathcal{T}^τ are $\{a^\tau \mid a \in \mathcal{T}\}$, where $\varphi(a^\tau) = \varphi(a)$, $(a^\tau)'' = a''$, $(a^\tau)' = a'$, and

$$''(a^\tau) = \begin{cases} ''a, & \text{if } a'' = 1 \\ \tau(''a), & \text{if } a'' = 2 \end{cases}, \quad '(a^\tau) = \begin{cases} 'a, & \text{if } a' = 1 \\ \tau('a), & \text{if } a' = 2 \end{cases}.$$

Obviously, $(\mathcal{T}^\tau, (X_1, \dots, X_d))$ is a tableau with substitution. For short we will write $\mathcal{T}_{\text{cl}}^\tau$ instead of $(\mathcal{T}^\tau)_{\text{cl}}$.

We use notation $\{\dots\}_m$ for *multisets*, i.e., given an equivalence $=$ on a set S and $a_1, \dots, a_p, b_1, \dots, b_q \in S$, we write $\{a_1, \dots, a_p\}_m = \{b_1, \dots, b_q\}_m$ if and only if $p = q$ and

$$\#\{1 \leq j \leq p \mid a_j = a_i\} = \#\{1 \leq j \leq q \mid b_j = a_i\}$$

for any $1 \leq i \leq p$.

Definition 7.10 (of $\mathcal{I}_\mathcal{T}$). Let $\underline{j} \in \mathbb{N}^p$ and $\underline{c} = (c_1, \dots, c_p)$, where $c_1, \dots, c_p \in \mathcal{M}^\infty$ are primitive and pairwise different with respect to \sim . Then $(\underline{j}, \underline{c})$ is called a \mathcal{T} -pair.

A \mathcal{T} -pair $(\underline{j}, \underline{c})$ is called \mathcal{T} -admissible if for some $\xi = \xi_{\underline{j}, \underline{c}} \in S_n$ the following equivalence of multisets holds:

$$\varphi(\mathcal{T}_{\text{cl}}^\xi) \sim \underbrace{\{c_1, \dots, c_1\}}_{j_1}, \dots, \underbrace{\{c_p, \dots, c_p\}}_{j_p}.$$

We write $(\underline{j}^0, \underline{c}^0) \sim (\underline{j}, \underline{c})$ and say that these pairs are equivalent if and only if

$$\underbrace{\{c_1^0, \dots, c_1^0\}}_{j_1^0}, \dots, \underbrace{\{c_p^0, \dots, c_p^0\}}_{j_p^0} \sim \underbrace{\{c_1, \dots, c_1\}}_{j_1}, \dots, \underbrace{\{c_p, \dots, c_p\}}_{j_p}.$$

If $(\underline{j}^0, \underline{c}^0) \sim (\underline{j}, \underline{c})$ and $(\underline{j}, \underline{c})$ is \mathcal{T} -admissible, then the pair $(\underline{j}^0, \underline{c}^0)$ also has the same property, since we can take $\xi_{\underline{j}^0, \underline{c}^0} = \xi_{\underline{j}, \underline{c}}$. Denote by $\mathcal{I}_\mathcal{T}$ a set of representatives of \mathcal{T} -admissible pairs with respect to the \sim -equivalence.

The decomposition formula states

$$\text{bpf}_\mathcal{T}(X_1, \dots, X_d) = \sum_{(\underline{j}, \underline{c}) \in \mathcal{I}_\mathcal{T}} \text{sgn}(\xi_{\underline{j}, \underline{c}}) \sigma_{j_1}(X_{c_1}) \cdots \sigma_{j_p}(X_{c_p}), \quad (21)$$

where $p = \#j = \#\underline{c}$. Note that $\xi_{\underline{j}, \underline{c}}$ is not unique for a representative of the \sim -equivalence class of a \mathcal{T} -admissible pair $(\underline{j}, \underline{c})$, but $\text{sgn}(\xi_{\underline{j}, \underline{c}})$ is unique (see Lemma 7.11 below). Hence

$$\text{dec}_{\mathcal{T}}(x, y, z) = \sum_{(\underline{j}, \underline{c}) \in \mathcal{I}_{\mathcal{T}}} \text{sgn}(\xi_{\underline{j}, \underline{c}}) \sigma_{j_1}(c_1) \cdots \sigma_{j_p}(c_p) |_{1 \rightarrow x, 2 \rightarrow y, 3 \rightarrow z} \quad (22)$$

for $x, y, z \in \mathcal{M}_{\mathbb{F}}$.

Proof of Lemma 7.6. Given an arrow or transpose arrow a of \mathcal{T} , we can consider $\varphi(a) \in \mathcal{M}^{\infty}$ as an arrow of the quiver \mathcal{Q} from Section 3 as follows: $\varphi(x_i), \varphi(x_i^T)$ are loops in vertices 1 and 2 of \mathcal{Q} , respectively, $\varphi(y_j)$ goes from vertex 2 to vertex 1 and so on. Similarly, for any word a in arrows and transpose arrows of \mathcal{T} we can consider $\varphi(a)$ as a word in arrows of \mathcal{Q} . Note that

- if a is a path in \mathcal{T} , then $\varphi(a)$ is a path in \mathcal{Q} ;
- if $a \in \mathcal{T}_{\text{cl}}$, then $\varphi(a)$ is a closed path in \mathcal{Q} .

Hence for any \mathcal{T} -admissible pair $(\underline{j}, \underline{c})$ we can consider c_1, \dots, c_p as closed paths in \mathcal{Q} . Therefore it is not difficult to see that $\mathcal{I}_{\mathcal{T}} = \mathcal{I}_{t,r}$. Formula (22) together with Lemma 7.11 (see below) concludes the proof. \square

Lemma 7.11. *We use notations from Lemma 7.6 and its proof. For any $(\underline{j}, \underline{c}) \in \mathcal{I}_{\mathcal{T}}$ we have*

$$\text{sgn}(\xi_{\underline{j}, \underline{c}}) = (-1)^{\xi},$$

where $\xi = t + \sum_{i=1}^{\#\underline{c}} j_i (\deg_y c_i + \deg_z c_i + 1)$. Here c_i is considered as a path in \mathcal{Q} ; therefore $\deg_y c_i$ and $\deg_z c_i$ are well defined.

Proof. We start the proof with the definition. For a permutation $\pi \in S_{t+2r}$ and $a = a_1 \cdots a_s \in \mathcal{T}_{\text{cl}}^{\pi}$, where a_1, \dots, a_s are arrows or transpose arrows of \mathcal{T} , there is a permutation $\tau \in S_{t+2r}$ such that

- $\tau(i) = i$ for any $i \in \{1, \dots, t+2r\} \setminus \{a_1, \dots, a_s\}$;
- for any $1 \leq i \leq s$ there exists a $b \in \mathcal{T}$ such that a_i^{τ} and b coincides, i.e., $(a_i^{\tau})'' = b''$, $(a_i^{\tau})' = b'$, $''a_i^{\tau} = ''b$, $'a_i^{\tau} = 'b$.

Denote $\text{sgn}(a) = \text{sgn}(\tau)$. In spite of non-uniqueness of τ , $\text{sgn}(a)$ is well defined, it does not depend on π , and $\text{sgn}(a) = \text{sgn}(a^T)$. For any $1 \leq i \leq \#j$ we consider $a_i \in \mathcal{T}_{\text{cl}}^{\xi_{\underline{j}, \underline{c}}}$ such that $\varphi(a_i) = c_i$. Then we have

$$\text{sgn}(\xi_{\underline{j}, \underline{c}}) = \prod_{i=1}^{\#j} \text{sgn}(a_i)^{j_i}. \quad (23)$$

Arrows of the tableau \mathcal{T} are x_i, y_j, z_j , where $1 \leq i \leq t$ and $1 \leq j \leq r$. Let $\pi \in S_{t+2r}$, $a \in \mathcal{T}_{\text{cl}}^{\pi}$, and let b, c be some paths in \mathcal{T}^{π} .

1. If $a = x_i$, then $\text{sgn}(a) = 1$.
2. If $a = bx_i c$, then $\text{sgn}(a) = -\text{sgn}(x_i) \text{sgn}(bc) = -\text{sgn}(bc)$.

3. If $a = y_j z_k$, then $\text{sgn}(a) = -1$.
4. If $a = y_j z_k^T$, then $\text{sgn}(a) = 1$.
5. If $a = b y_j z_k c$, then $\text{sgn}(a) = -\text{sgn}(y_j z_k) \text{sgn}(bc) = \text{sgn}(bc)$.
6. If $a = b y_j z_k^T c$, then $\text{sgn}(a) = -\text{sgn}(y_j z_k^T) \text{sgn}(bc) = -\text{sgn}(bc)$.
7. If $a = b z_k y_j c$, then $\text{sgn}(a) = -\text{sgn}(y_j z_k) \text{sgn}(bc) = \text{sgn}(bc)$.
8. If $a = b z_k^T y_j c$, then $\text{sgn}(a) = -\text{sgn}(y_j z_k^T) \text{sgn}(bc) = -\text{sgn}(bc)$.

For $a \in \mathcal{T}_{\text{cl}}^\pi$ we have two possibilities:

- If $\deg_y a + \deg_z a = 0$, then $\text{sgn}(a) = (-1)^{\deg_x a + \deg_{x^T} a + 1}$ (see items 1 and 2).
- If $\deg_y a + \deg_z a > 0$, then by item 2 we have

$$\text{sgn}(a) = (-1)^{\deg_x a + \deg_{x^T} a} \text{sgn}(a_0),$$

where $a_0 = a|_{x_i \rightarrow 1, x_i^T \rightarrow 1}$ is the result of elimination of letters x_i, x_i^T ($1 \leq i \leq t$) from a . By items 3–8, $\text{sgn}(a_0) = (-1)^{\deg_{y^T} a_0 + \deg_{z^T} a_0 + 1} = (-1)^{\deg_{z^T} a + \deg_{z^T} a + 1}$.

Thus,

$$\text{sgn}(a) = (-1)^{\deg_x a + \deg_{x^T} a + \deg_{y^T} a + \deg_{z^T} a + 1}.$$

Formula (23) concludes the proof. \square

Lemma 7.12. *Assume that $(\mathcal{T}, (X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r))$ is the tableau with substitution from Example 7.3 and $x_1, \dots, x_t, y_1, \dots, y_r, z_1, \dots, z_r$ belong to $\mathcal{M}_{\mathbb{F}}$. Then*

$$\sigma_{t,r}^{\text{lin}}(x_1, \dots, x_t, y_1, \dots, y_r, z_1, \dots, z_r) = \sum_{\xi \in S_{t+2r}} \text{sgn}(\xi) \prod_{a \in \mathcal{T}_{\text{cl}}^\xi} \text{tr}(\varphi(a))|_{i \rightarrow x_i, (t+j) \rightarrow y_j, (t+r+j) \rightarrow z_j},$$

where the substitution is applied for all $1 \leq i \leq t$ and $1 \leq j \leq t$. In particular, $\sigma_{t,r}^{\text{lin}}(X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r) = \text{bpf}_{\mathcal{T}}(X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r)$ for $n = t + 2r$.

Proof. Using the fact that $\mathcal{T}_{\text{cl}}^\xi \sim \mathcal{T}_{\text{cl}}^\tau$ for $\xi, \tau \in S_{t+2r}$ if and only if $\xi = \tau$, we prove this lemma similar to Lemma 7.6. \square

Remark 7.13. Lemma 7.11 can also be generalized for $\sigma_{\underline{t}, \underline{r}, \underline{s}}(x_1, \dots, x_u, y_1, \dots, y_v, z_1, \dots, z_w)$, where $x_1, \dots, z_w \in \mathcal{M}_{\mathbb{F}}$.

We assume that $\sigma'_{t,r}(x, y, z)$ stands for the element of \mathcal{N}_σ defined by Zubkov (see p. 292 of [24]), where $t \geq 2r$ and $x, y, z \in \mathcal{M}_{\mathbb{F}}$.

Lemma 7.14. *For $t, r \geq 0$ and $x, y, z \in \mathcal{M}_{\mathbb{F}}$ we have $\sigma_{t,r}(x, y, z) = \sigma'_{t+2r,r}(x, z, y)$.*

Proof. By Remark 4.8, we can assume that $\mathbb{F} = \mathbb{Q}$. Consider the tableau with substitution $(\mathcal{T}, (X_1, \dots, X_t, Y_1, \dots, Y_r, Z_1, \dots, Z_r))$ from Example 7.3. By definition, we have

$$\sigma'_{t+2r,r}(x, z, y) = \frac{1}{t!(r!)^2} \sum_{\sigma \in S_{t+2r}} \text{sgn}(\sigma) f(\text{tr}^*(\sigma)),$$

where $\text{tr}^*(\sigma) \in \mathcal{N}_\sigma^\infty$ is defined on p. 291 of [24] and f stands for the substitution $i \rightarrow x_i$, $(t+j) \rightarrow y_j$, $(t+r+j) \rightarrow z_j$ for all $1 \leq i \leq t$ and $1 \leq j \leq t$. Considering all 18 possibilities from the definition of $\text{tr}^*(\sigma)$ we can see that

$$\text{tr}^*(\sigma) = \prod_{a \in \mathcal{T}^{\sigma^{-1}}} \text{tr}(\varphi(a)).$$

Lemmas 4.4 and 7.12 conclude the proof. \square

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