On the Picard principle for $\Delta + \mu$

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Abstract

Given a (local) Kato[1] measure μ on $\mathbb{R}^d \setminus \{0\}$, $d \geq 2$, let $\mathcal{H}_0^{\Delta+\mu}(U)$ be the convex cone of all continuous real solutions $u \geq 0$ to the equation $\Delta u + u\mu = 0$ on the punctured unit ball U satisfying $\lim_{|x|\to 1} u(x) = 0$. It is shown that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if the operator $f \mapsto \int_U G(\cdot, y) f(y)\, d\mu(y)$, where G denotes the Green function on U, is bounded on $\mathcal{L}^2(U, \mu)$ and has a norm which is at most one. Moreover, extremal rays in $\mathcal{H}_0^{\Delta+\mu}(U)$ are characterized and it is proven that $\Delta + \mu$ satisfies the Picard principle on U, that is, that $\mathcal{H}_0^{\Delta+\mu}(U)$ consists of one ray, provided there exists a suitable sequence of shells in U such that, on these shells, μ is either small or not too far from being radial. Further, it is shown that the verification of the Picard principle can be localized.

Several results on L^2-(sub)eigenfunctions and $3G$-inequalities which are used in the paper, but may be of independent interest, are proved at the end of the paper.

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Contents

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[1] A natural assumption: otherwise (1.1) does not admit *continuous* solutions $u \neq 0$; see [16, 10].

1 Introduction

For every open set W in $\mathbb{R}^d$, $d \geq 2$ (W relatively compact, if $d = 2$), let G_W denote the (classical) Green function on W, normalized such that $\Delta G_W(\cdot, y) = -\varepsilon_y$, $y \in W$. Throughout this paper we fix a Kato measure μ on $\mathbb{R}^d \setminus \{0\}$, that is, μ is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that, for every open set W which is relatively compact in $\mathbb{R}^d \setminus \{0\}$, the potentials $G_W\mu \colon x \mapsto \int_W G_W(x, y)\, d\mu(y)$ are continuous and real. It suffices to verify this for a covering of $\mathbb{R}^d \setminus \{0\}$ (see the discussion at the beginning of Section 2).

We recall that Kato measures may be singular with respect to Lebesgue measure λ^d, but that any measure $V\lambda^d$, where $V \in \mathcal{L}^p_{\rm loc}(\lambda^d)$, $p > d/2$, is a Kato measure (cf. [15], [1, Proposition 4.3], [7, Proposition 7.1]).

Once and for all, we fix $R > 0$ and define

$$B := \{x \in \mathbb{R} \colon |x| < R\}, \qquad U := B \setminus \{0\}.$$

Let $\mathcal{H}_0^{\Delta+\mu}(U)$ be the convex cone of all continuous real solutions $u \geq 0$ to the Schrödinger equation

$$\Delta u + u\mu = 0 \tag{1.1}$$

on U which tend to 0 at the boundary ∂B of B. Here solution is meant in the sense of distributions, that is,

$$\int u \Delta\varphi\, d\lambda^d + \int u\varphi\, d\mu = 0$$

for all $\mathcal{C}^\infty$-functions φ with compact support in U.

By definition, $\dim_U(\Delta + \mu)$, the *Picard dimension of* $\Delta + \mu$ *on* U, is the number of extremal rays in $\mathcal{H}_0^{\Delta+\mu}(U)$. Of course, $\dim_U(\Delta + \mu) = 0$ if $\mathcal{H}_0^{\Delta+\mu}(U) = \{0\}$ and $\dim_U(\Delta + \mu) = 1$ if $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ h_0$, $h_0 > 0$. If $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ and x_0 is a reference point in U, then $\{h \in \mathcal{H}_0^{\Delta+\mu}(U) \colon h(x_0) = 1\}$ is a compact base of $\mathcal{H}_0^{\Delta+\mu}(U)$ (see (3.4)), $\dim_U(\Delta + \mu)$ is the number of extreme points of this set, and hence $\dim_U(\Delta + \mu) > 0$.

We say that $\Delta + \mu$ satisfies the *Picard principle on* U provided

$$\dim_U(\Delta + \mu) \leq 1. \tag{1.2}$$

In [18] it is shown that (1.2) holds, if $d = 2$. Moreover, it is satisfied if μ has a rotationally invariant density which is locally Hölder continuous ([17]). However, the problem seems to be open for $d \geq 3$ and general measures μ.

In this paper, we prove that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if the mapping

$$K \colon f \mapsto G(f\mu) = \int G(\cdot, y) f(y)\, d\mu(y),$$

where $G := G_U = G_B|_{U\times U}$, operates on $\mathcal{L}^2(U,\mu)$ with $\|K\|_2 \le 1$ (Corollary 4.3). In particular, (1.2) holds trivially, unless K operates on $\mathcal{L}^2(U,\mu)$ and $\|K\|_2 \le 1$.

Since $\mathcal{H}_0^\Delta(U)$ is the set of all positive multiples of $G_0 := G_B(\cdot,0)|_U$ (cf. [2, Exercise 2.11]) and hence $\dim_U \Delta = 1$, it would be sufficient to consider the case $\mu(U) > 0$. We shall see first that, whatever μ is, every function in an extremal ray of $\mathcal{H}_0^{\Delta+\mu}(U)$ is either a multiple of $g_0 := \sum_{n=0}^\infty K^n G_0$ or a continuous strictly positive K-invariant function h (Proposition 3.3). If $g_0 \ne \infty$, then $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$ and g_0 is extremal (Proposition 3.5). If g_0 is even bounded by a multiple of G_0, then $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ g_0$ (Proposition 5.1). On the other hand, if K operates on $\mathcal{L}^2(U,\mu)$, $\|K\|_2 = 1$, and 1 is an eigenvalue of K, then $g_0 = \infty$ and $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ h_0$, where $Kh_0 = h_0 \in \mathcal{L}^2(U,\mu)$, $h_0 > 0$ (Proposition 5.2).

Further, we shall prove that $\Delta+\mu$ satisfies the Picard principle on U provided that there are arbitrarily small shells A_n of constant relative width such that the potentials $\int_{A_n} G(\cdot,y)\,d\mu(y)$ are small enough (Theorem 5.8). In particular, μ satisfies the Picard principle, if there are arbitrarily small shells A_n of constant relative width such that the restriction μ' of μ on the union $\bigcup_{n\in\mathbb{N}} A_n$ (considered as a measure on $\mathbb{R}^d$) is a Kato measure on $\mathbb{R}^d$ (Theorem 5.9) or μ' is not too far from being invariant under rotations (Theorem 5.13), or μ' is the sum of two such measures (Corollary 5.15).

Moreover, we shall see that the verification of the Picard principle can be localized in different ways (Theorem 6.1).

To prove these results, we present in Section 2, in a self-contained way, various facts on the potential theory of $\Delta+\mu$ which mostly could be retrieved in the literature (sometimes under much more general assumptions and hence not easily accessible; see, for example, [7, 14, 9, 10]).

Further, a first appendix contains a general result on (sub)eigenfunctions, which is crucial for the L^2-result on the Picard principle stated above. In a second appendix, which is used for the discussion of the Picard principle in connection with g_0 and which is of independent interest, too, we study, in a general measurable space setting, relations between conditional $3G$-inequalities on punctured sets $X\setminus\{a\}$. As a consequence we obtain that $3G$-inequalities involving some real function $w > 0$ imply $3G$-inequalities based on $\min\{G(\cdot,a),1\}$.

2 Some potential theory for $\Delta + \mu$

Given an open set W in $\mathbb{R}^d$, $d \ge 2$, let $\mathcal{B}(W)$, $\mathcal{C}(W)$, $S(W)$, $\mathcal{H}(W)$ denote the set of all functions on W which are Borel measurable, continuous and real, superharmonic, harmonic, respectively. Of course, given a set $\mathcal{F}$ of functions, $\mathcal{F}^+$, $\mathcal{F}_b$ will be the set of all functions in $\mathcal{F}$ which are positive, bounded, respectively. It will be convenient to extend functions defined on a subset of $\mathbb{R}^d$ taking the value 0 elsewhere.

For every regular set V in $\mathbb{R}^d$, let H_V denote the harmonic kernel for V, that is, for every continuous function φ, which is defined on a set containing ∂V, $H_V\varphi$ is equal to the solution to the Dirichlet problem for V and $\varphi|_{\partial V}$ and equal to φ outside V. If W is an open set in $\mathbb{R}^d$ (relatively compact, if $d=2$) and V is a regular set such that $\overline{V} \subset W$, then, for all $y \in V$,

$$G_V(\cdot,y) = G_W(\cdot,y) - H_V G_W(\cdot,y). \tag{2.1}$$

In this section, let X be an arbitrary non-empty open set in $\mathbb{R}^d$, $d \geq 2$, and let μ be a Kato measure on X, that is, μ is a (positive) Radon measure on X such that $G_W\mu := \int G_W(\cdot, y)\, d\mu(y) \in \mathcal{C}(W)$, for every open set W which is relatively compact in X (because of (2.2 below), it suffices to verify this for a covering of X). The measure μ does not charge points, since $G_W\mu \geq \mu(\{z\})G_W(\cdot, z)$, $z \in W$ (in fact, it does not charge any polar set).

Given an open set W in X (relatively compact, if $d = 2$), let K_W denote the kernel given by

$$K_W f := G_W(f\mu) = \int G_W(\cdot, y) f(y)\, d\mu(y), \qquad f \in \mathcal{B}^+(W).$$

Then (2.1) implies that, for every regular set V such that $\overline{V} \subset W$,

$$K_W = K_V + H_V K_W. \tag{2.2}$$

If V is a regular set such that $\overline{V} \subset X$, the continuous function $K_V 1$ tends to 0 at the boundary ∂V (this follows from (2.2) choosing any open set W such that $\overline{V} \subset W$ and W is relatively compact in X). Since every function $K_V f$, $f \in \mathcal{B}^+(V)$, is lower semicontinuous, we then see that K_V maps $\mathcal{B}_b(V)$ into the subspace $\mathcal{C}_0(V)$ of all continuous real functions on V which tend to 0 at ∂V.

LEMMA 2.1. *If V is a regular set, relatively compact in X, and (A_n) is a sequence of Borel sets in V such that $A_n \downarrow A_\infty$, then $K_V 1_{A_n} \downarrow K_V 1_{A_\infty}$* uniformly *on V.*

In particular, for all $x \in X$ and $\varepsilon > 0$, there exists an open ball A centered at x such that $K_A 1 \leq \varepsilon$.

Proof. The first part follows immediately using Dini's lemma. Given $x \in X$ and $\varepsilon > 0$, let V be a regular neighborhood of x, $\overline{V} \subset X$, and let (A_n) be a sequence of open balls in V such that $A_n \downarrow \{x\}$. Since $K_V 1_{\{x\}} = 0$, we see that $K_V 1_{A_n} \leq \varepsilon$, if n is sufficiently large. The proof is finished, since $K_{A_n} 1 \leq K_V 1_{A_n}$. □

PROPOSITION 2.2. *Let V be a regular set, relatively compact in X. Then K_V is a compact operator on $\mathcal{B}_b(V)$.*

Proof. Let $\varepsilon > 0$ and $x \in V$. By Lemma 2.1, there exists an open ball A in V, centered at x, such that $K_V 1_A \leq \varepsilon$. There exists an open ball $\tilde{A} \subset A$, centered at x, such that, for all harmonic functions g on A satisfying $|g| \leq K_V 1$ and all $y \in \tilde{A}$,

$$|g(y) - g(x)| < \varepsilon \tag{2.3}$$

(see, for instance, [2, Lemma 1.5.6]). Now let $f \in \mathcal{B}_b(V)$, $|f| \leq 1$. Then

$$K_V f = K_V(1_{V\setminus A} f) + K_V(1_A f),$$

where $g := K_V(1_{V\setminus A} f)$ is harmonic on A, $|g| \leq K_V 1$, and $|K_V(1_A f)| \leq K_V 1_A \leq \varepsilon$. Hence, by (2.3), for all $y \in \tilde{A}$,

$$|K_V f(y) - K_V f(x)| \leq |g(y) - g(x)| + 2\varepsilon < 3\varepsilon.$$

Therefore the set $\{K_V f\colon f \in \mathcal{B}_b(V), |f| \leq 1\}$ is equicontinuous at x. Since $K_V(\mathcal{B}_b(V))$ is contained in $\mathcal{C}_0(V)$, the claim follows by Arzèla-Ascoli's theorem. □

DEFINITION 2.3. *Given an open set W in X, let $\mathcal{H}^{\Delta+\mu}(W)$ be the set of all* $(\Delta+\mu)$-harmonic functions *on W, that is, of all $u \in \mathcal{C}(W)$ such that $\Delta u + u\mu = 0$ (in the sense of distributions).*

An open set V which is relatively compact in X will be called $(\Delta+\mu)$-regular, *if, for every continuous function φ on ∂V, there exists a unique function $h \in \mathcal{H}^{\Delta+\mu}(V)$ which tends to φ at ∂V, and is positive provided $\varphi \geq 0$.*

A function $s \in \mathcal{C}(W)$ is said to be $(\Delta+\mu)$-superharmonic, *if $(\Delta+\mu)s \leq 0$ (in the sense of distributions).*

Obviously, $\mathcal{H}^{\Delta+\mu}(W)$ is stable under locally uniform convergence and every positive $(\Delta+\mu)$-(super)harmonic function on W is superharmonic.

LEMMA 2.4. *Let V be a regular set such that $\overline{V} \subset X$.*

1. *A function $f \in \mathcal{C}_b(V)$ is $(\Delta+\mu)$-superharmonic if and only if $f - K_V f$ is superharmonic.*
2. *If h is a function which is continuous and real on $\overline{V}$, then h is $(\Delta+\mu)$-harmonic on V if and only if $h - K_V h = H_V h$.*

Proof. 1. It suffices to note that $K_V f \in \mathcal{C}_b(V)$ and $\Delta f + f\mu = \Delta(f - K_V f)$.

2. An immediate consequence of (1), since $h - K_V h$ is continuous on $\overline{V}$ and equal to h on ∂V. □

To be able to prove that every regular set V is $(\Delta+\mu)$-regular provided that K_V is sufficiently small, we recall the following from [14] and [11]. Its proof is so short that we may just as well include it.

LEMMA 2.5. *Let L be a bounded kernel on a measurable space $(E, \mathcal{E})$. Then the following statements are equivalent.*

(i) *The function $\sum_{n=0}^{\infty} L^n 1$ is bounded.*

(ii) *The operator $I - L$ on $(\mathcal{E}_b, \|\cdot\|_\infty)$ is invertible and $(I-L)^{-1} = \sum_{n=0}^{\infty} L^n$.*

(iii) *The operator $I - L$ on $(\mathcal{E}_b, \|\cdot\|_\infty)$ is invertible and its inverse is positive.*

(iv) *There exists $g \in \mathcal{E}_b^+$ such that $1 + Lg \leq g$.*

(v) *There exist $c > 0$ and $\gamma \in (0,1)$ such that, for every $n \in \mathbb{N}$, $L^n 1 \leq c\gamma^n$.*

(vi) *The spectral radius $\rho(L) := \inf \|L^n\|^{1/n}$ is strictly less than 1.*

Proof. (i) ⇒ (ii) ⇒ (iii): Trivial.

(iii) ⇒ (iv): The function $g := (I-L)^{-1}1 \in \mathcal{E}_b^+$ satisfies $1 + Lg = g$.

(iv) ⇒ (v): Of course, $c := \|g\|_\infty \geq 1$. Since $c^{-1}g + Lg \leq 1 + Lg \leq g$, and hence $Lg \leq (1-c^{-1})g$, we obtain that, for every $n \in \mathbb{N}$, $L^n g \leq (1-c^{-1})^n g \leq c(1-c^{-1})^n$.

(v) ⇒ (vi): Trivial, since $\|L^n\| = \|L^n 1\|_\infty$, $n \in \mathbb{N}$.

(vi) ⇒ (i): Let $\gamma \in (\rho(L), 1)$. Then there exists $m \in \mathbb{N}$ such that $L^m 1 \leq \gamma$, and hence $\sum_{n=0}^{\infty} L^n 1 \leq (1-\gamma)^{-1} \sum_{j=0}^{m-1} L^j 1$ is bounded. □

DEFINITION 2.6. An open set V will be called *μ-admissible* in X, if it is relatively compact in X, $I - K_V$ is invertible on $\mathcal{B}_b(V)$, and the inverse of $I - K_V$ is positive.

LEMMA 2.7. *Every open subset W of a μ-admissible open set V in X is μ-admissible. For every $x \in X$, there exists an open ball A centered at x which is μ-admissible.*

Proof. We know that $K_W 1 \le K_V 1$ and, by Lemma 2.1, we may choose A such that $K_A 1 \le 1/2$. So both statements follow from Lemma 2.5. □

LEMMA 2.8. *Let V be a μ-admissible regular set in X, let $s \in \mathcal{C}_b(V)$ be $(\Delta+\mu)$-superharmonic, and, for every $z \in \partial V$, $\liminf_{x\to z} s(x) \ge 0$. Then $s \ge 0$.*

Proof. By Lemma 2.4.1, $\tilde{s} := s - K_V s$ is superharmonic on V. Since $K_V s \in \mathcal{C}_0(V)$, we know that, for every $z \in \partial V$, $\liminf_{y\to z} \tilde{s}(y) = \liminf_{y\to z} s(y) \ge 0$. By the minimum principle, $\tilde{s} \ge 0$. Thus $s = (I - K_V)^{-1}\tilde{s} \ge 0$. □

PROPOSITION 2.9. *Let V be an open set which is relatively compact in X.*

1. *The set V is $(\Delta+\mu)$-regular if and only if it is regular and μ-admissible.*
2. *If V is $(\Delta+\mu)$-regular, then the corresponding harmonic kernel is*

$$H_V^{\Delta+\mu} = (I - K_V)^{-1} H_V = \sum_{n=0}^{\infty} (K_V)^n H_V. \tag{2.4}$$

Proof. 1. Let us suppose first that V is regular and μ-admissible, and let φ be a continuous function on X. Then $h := (I - K_V)^{-1} H_V \varphi$ satisfies $h - K_V h = H_V \varphi$. So, by Lemma 2.4.2, h is $(\Delta+\mu)$-harmonic on V. Since $K_V h \in \mathcal{C}_0(V)$, we see that h is continuous on X and equal to φ on $X \setminus V$. In particular, $\lim_{x\to z, x\in V} h(x) = \varphi(z)$ for every $z \in \partial V$. Moreover, $h \ge 0$ if $\varphi \ge 0$. By Lemma 2.8, $h|_V$ is the only $(\Delta+\mu)$-harmonic function on V which tends to φ at ∂V. So V is $(\Delta+\mu)$-regular and (2.4) holds.

Next let us assume that V is $(\Delta+\mu)$-regular and let φ be a continuous function on ∂V, $0 \le \varphi \le 1$. Let g, h denote the $(\Delta+\mu)$-harmonic function on V which, at ∂V, tends to 1, φ, respectively. Since $0 \le h \le g$, the functions h and $g - h$ are superharmonic. Moreover, the function $t := 1 - (g - h)$ is subharmonic and tends to φ at ∂V. By the minimum principle for superharmonic functions, $t \le h$. So there exists a harmonic function $\tilde{h}$ on V such that $t \le \tilde{h} \le h$. Obviously, $\tilde{h}$ is a solution to the Dirichlet problem for φ. So V is regular.

In particular, $K_V g \in \mathcal{C}_0(V)$. Therefore the harmonic function $g - K_V g$ tends to 1 at ∂V, that is, $g - K_V g = 1$. Hence, by Lemma 2.5, V is μ-admissible. □

PROPOSITION 2.10. *Let U be an open set in X and $s \in \mathcal{C}(U)$. If s is $(\Delta+\mu)$-superharmonic, then $H_V^{\Delta+\mu} s \le s$, for every $(\Delta+\mu)$-regularset V such that $\overline{V} \subset U$.*[2]

[2] In fact, the converse holds as well (see [7, Theorem 3.2] and Lemma 2.4).

Proof. Let V be as indicated. By Proposition 2.9.1, V is regular and μ-admissible. Then $\tilde{s} := (s - H_V^{\Delta+\mu} s)|_V \in \mathcal{C}_0(V)$ and $\tilde{s}$ is $(\Delta + \mu)$-superharmonic on V. Using Lemma 2.4.1, we see that $t := \tilde{s} - K_V \tilde{s} \in \mathcal{C}_0(V) \cap \mathcal{S}(V)$. Therefore $t \geq 0$, by the minimum principle. Since $(I - K_V)^{-1}$ is a positive operator, we finally conclude that $\tilde{s} = (I - K_V)^{-1} t \geq 0$. □

For the next proposition we shall need a triangle inequality for Green functions ($3G$-inequality). For every open ball V in $\mathbb{R}^d$, let V' denote the open ball having the same center as V, but double radius. There exists $C_\Delta > 1$, such that, for all open balls V in $\mathbb{R}^d$ and $x, y, z \in V$, the following triangle inequality holds

$$G_V(x,z) G_V(z,y) \leq C_\Delta G_V(x,y) (G_{V'}(x,z) + G_{V'}(y,z)). \tag{2.5}$$

If $d \geq 3$, this follows from [7, Lemmas 7.5 and 7.4] (since $a_{V'} > 1/4$ on V).

Let us now consider the case $d = 2$ and fix an open disk V of radius r and center x_0. For $x, y \in V$, let

$$\rho(x) := r - |x - x_0|, \quad \psi(x,y) := \frac{\rho(x)\rho(y)}{d(x,y)^2}, \quad F := \frac{1}{4\pi} \ln(1 + \psi).$$

For all $x \in V$, $r\rho(x) \leq r^2 - |x - x_0|^2 \leq 2r\rho(x)$, and hence, by [2, Theorem 4.1.5],

$$F \leq G_V \leq (4\pi)^{-1} \ln(1 + 4\psi) \leq 4F.$$

Having an analogous estimate for $G_{V'}$, we obtain that

$$G_{V'} \geq \frac{1}{4\pi} \ln(1 + \frac{1}{4}) \quad \text{on } V \times V. \tag{2.6}$$

By [13, Proposition 8.6], for all $x, y, z \in V$,

$$F(x,z)F(z,y) \leq 8\, F(x,y) \max\Big\{\frac{\rho(z)}{\rho(x)} F(x,z), \frac{\rho(z)}{\rho(y)} F(y,z)\Big\}.$$

Let $x, z \in V$. Clearly, (2.5) will follow (with $C_\Delta = 16 \cdot 8 \cdot 7$), if we show that

$$\frac{\rho(z)}{\rho(x)} F(x,z) \leq 7\, G_{V'}(x,z).$$

Since $F \leq G_V \leq G_{V'}$, this follows trivially, if $\rho(z) \leq 7\rho(x)$. So let $\rho(z) > 7\rho(x)$. Then $d(x,z) \geq \rho(z) - \rho(x) > (6/7)\rho(z)$ and hence, using (2.6) and $\frac{7}{36} \leq \ln(5/4)$,

$$\frac{\rho(z)}{\rho(x)} F(x,z) \leq \frac{1}{4\pi} \frac{\rho(z)}{\rho(x)} \frac{\rho(z)\rho(x)}{d(x,z)^2} \leq \frac{1}{4\pi} \frac{49}{36} \leq 7\, G_{V'}(x,z).$$

For every open set W in $\mathbb{R}^d$ (relatively compact if $d = 2$) and every Radon measure ν on W, let

$$G_W \nu(x) := \int G_W(x,y)\, d\nu(y), \qquad x \in W. \tag{2.7}$$

The following is known (cf. [7, Proposition 7.6]).

PROPOSITION 2.11. *Let ν be a Radon measure on $\mathbb{R}^d$, let V be an open ball in $\mathbb{R}^d$, and $\varepsilon > 0$ such that $G_{V'}\nu \le \varepsilon/(2C_\Delta)$. Then, for every $s \in \mathcal{S}^+(V)$,*

$$G_V(s\nu) \le \varepsilon s.$$

Proof. For all $x, y \in V$,

$$\begin{aligned} G_V(G_V(\cdot, y)\nu)(x) &= \int G_V(x,z)G_V(z,y)\,d\nu(z) \\ &\le C_\Delta G_V(x,y)\int (G_{V'}(x,z) + G_{V'}(y,z))\,d\nu(z) \\ &= C_\Delta G_V(x,y)(G_{V'}\nu(x) + G_{V'}\nu(y)) \le \varepsilon G_V(x,y). \end{aligned}$$

Integrating with respect to a Radon measure ρ on V, we obtain that $G_V(G_V\rho) \le \varepsilon G_V\rho$. The proof is finished using the fact that every $s \in \mathcal{S}^+(V)$ is the increasing limit of a sequence $(G_V\rho_n)$. □

The next lemma will be very useful.

LEMMA 2.12. *Let V be relatively compact open set in X and let $s \in \mathcal{S}^+(V)$, $g \in \mathcal{H}^+(V)$ such that $s = g + K_V s$, $K_V s \le (1/2)s$, and $K_V g \le (1/2)g$.*
Then s is $(\Delta + \mu)$-harmonic and $g \le s \le 2g$.

Proof. Let $L := K_V$. By induction, for every $m \in \mathbb{N}$,

$$s = \sum_{n=0}^{m-1} L^n g + L^m s,$$

where $L^n g \le 2^{-n} g$ and $L^m s \le 2^{-m} s$. So the sequence $(\sum_{n=0}^{m-1} L^n g)$ converges locally uniformly to a function $\tilde{s} \in \mathcal{C}(V)$ such that $g \le \tilde{s} \le 2g$. Moreover, $s = \tilde{s}$ outside the polar set $\{s = \infty\}$. Since both s and $\tilde{s}$ are finely continuous, we see that, in fact, $s = \tilde{s} \in \mathcal{C}(V)$. We finally observe that $\Delta s = -s\mu$. Thus $s \in \mathcal{H}^{\Delta+\mu}(V)$. □

A first consequence of the inequalities $g \le s \le 2g$ are Harnack's inequalities for positive $(\Delta + \mu)$-harmonic functions (Proposition 2.15; for another application see Proposition 3.1). The following notion will be convenient.

DEFINITION 2.13. *A relatively compact open set V in X is* μ-small, *if*

$$K_V s \le (1/2)s, \qquad \text{for every } s \in \mathcal{S}^+(V).$$

Of course, every μ-small set is μ-admissible.

PROPOSITION 2.14. *Every open set W in X is a union of μ-small balls V such that $\overline{V} \subset W$.*

Proof. Let $x \in W$. By Lemma 2.1, there exists an open ball V such that $x \in V$, the closure of V' is contained in W, and $K_{V'}1 \le 1/(4C_\Delta)$. By Proposition 2.11, V is μ-small. □

PROPOSITION 2.15. *For every connected open set W in X and every compact set A in W, there exists $c > 0$ such that, for every positive $h \in \mathcal{H}^{\Delta+\mu}(W)$,*

$$\sup h(A) \le c \inf h(A). \tag{2.8}$$

Proof. Let V be a μ-small ball, $\overline{V} \subset W$, and let $h \in \mathcal{H}^{\Delta+\mu}(W)$, $h \ge 0$. By Lemma 2.4.2, $h - K_V h = H_V h$. By Lemma 2.12, $H_V h \le h \le 2H_V h$. Using Harnack's inequalities for harmonic functions and Proposition 2.14, the proof is finished by a standard covering argument. □

COROLLARY 2.16. *Let (h_n) be an increasing sequence of $(\Delta + \mu)$-harmonic functions on a connected open set W in X such that $h := \sup h_n$ is finite at some point. Then h is $(\Delta + \mu)$-harmonic on W.*

Proof. By Lemma 2.1, open balls V satisfying $\overline{V} \subset W$ and $K_V 1 \le 1/2$ form a base of W. By Lemma 2.9, these balls are $(\Delta + \mu)$-regular. It follows immediately from (2.8), that the functions $h_n = h_1 + \sum_{k=2}^n (h_k - h_{k-1})$ converge locally uniformly to h as $n \to \infty$. Thus $h \in \mathcal{H}^{\Delta+\mu}(W)$. □

In our situation the Schur test (see [8, Theorem 5.2]) reads as follows.

LEMMA 2.17. *Let W be an open set in X such that $\overline{W}$ is compact, and let $a \ge 0$. If there exists a Borel measurable function $f \colon W \to (0, \infty)$ such that $K_W f \le af$, then K operates on $\mathcal{L}^2(W, \mu)$ and $\|K_W\|_2 \le a$.*

Given an open set V which is relatively compact in X, let Γ_V denote the set of all $\gamma > 0$ such that V is $\gamma\mu$-admissible. We define

$$\alpha_V := \sup \Gamma_V.$$

If $\mu(V) = 0$, then obviously $\Gamma_V = (0, \infty)$ and hence $\alpha_V = \infty$.

PROPOSITION 2.18. *Let V be a connected regular set in X and $\mu(V) > 0$. Then the following holds.*

1. *$0 < \alpha_V < \infty$ and $\Gamma_V = (0, \alpha_V)$.*

2. *There exists a strictly positive $(\Delta+\alpha_V\mu)$-harmonic function $h \in \mathcal{C}_0(V)$. Moreover, $\ker(I - \alpha_V K_V) = \mathbb{R}h$, and every $(\Delta + \alpha_V\mu)$-superharmonic $s \in \mathcal{C}_b^+(V)$ is a multiple of h.*

3. *For every $\beta > \alpha_V$, the constant function 0 is the only function $s \in \mathcal{C}_b^+(V)$ which is $(\Delta + \beta\mu)$-superharmonic.*

4. *If $\beta \ge \alpha_V$, every $(\Delta + \beta\mu)$-superharmonic $s \in \mathcal{C}_b^+(V)$ tends to 0 at ∂V.*

In particular, K_V is a bounded operator on $\mathcal{L}^2(V, \mu)$ and $\|K_V\|_2 = \alpha_V^{-1}$.

Proof. Let C be a compact set in V such that $\mu(C) > 0$. Then $K_V 1_C > 0$ on V. So there exists $\beta \in (0, \infty)$ such that $\beta K_V 1_C \ge 1_C$. By induction, $(\beta K_V)^n 1_C \ge 1_C$ for every $n \in \mathbb{N}$, and hence $\sum_{n=0}^\infty (\beta K_V)^n 1 = \infty$ on C. Therefore, by Lemma 2.5, $0 < \alpha_V \le \beta < \infty$ and Γ_V is an interval from 0 to α_V. We still have to show that

Γ_V is open. To that end let us consider $\gamma \in \Gamma_V$ and $0 < \varepsilon < \|(I - \gamma K_V)^{-1} K_V\|^{-1}$. Then the operator $I - (\gamma + \varepsilon) K_V$ is invertible and its inverse is the operator

$$\sum_{n=0}^{\infty} [\varepsilon (I - \gamma K_V)^{-1} K_V]^n (I - \gamma K_V)^{-1},$$

which, clearly, is positive, hence $\gamma + \varepsilon \in \Gamma_V$. So Γ_V is an open interval, $\Gamma_V = (0, \alpha_V)$.

Let (γ_n) be a sequence in Γ_V which is increasing to α_V. For every $n \in \mathbb{N}$, let

$$g_n := H_V^{\Delta + \gamma_n \mu} 1 \quad \text{and} \quad c_n := \|g_n\|_\infty .$$

By (2.4), for every $n \in \mathbb{N}$, $1 \le g_n \le g_{n+1}$ and

$$g_n - \gamma_n K_V g_n = 1. \tag{2.9}$$

If $\sup c_n < \infty$, then $g := \lim_{n \to \infty} g_n$ is bounded and $g - \alpha_V K_V g = 1$, and hence $\alpha_V \in \Gamma_V$, by Lemma 2.5, a contradiction. So $\sup c_n = \infty$.

Since K_V is a compact operator on $(\mathcal{B}_b(V), \|\cdot\|_\infty)$ which maps $\mathcal{B}_b(V)$ into $C_0(V)$, there exists a subsequence (h_n) of $(c_n^{-1} g_n)$ such that the sequence $(K_V h_n)$ converges uniformly to a function $h \in \mathcal{C}_0^+(V)$. By (2.9), the sequence (h_n) itself converges uniformly to h and $h - \alpha_V K_V h = 0$, that is, $h \in \ker(I - \alpha_V K_V)$. Of course, $\|h\|_\infty = 1$, since $\|h_n\|_\infty = 1$ for every $n \in \mathbb{N}$. Since $h \ge 0$, we hence see that $h = \alpha_V K_V h > 0$ on V. Finally, $\Delta h = -\alpha_V h \mu$. So h is $(\Delta + \alpha_V \mu)$-harmonic.

Let $\beta \ge \alpha_V$ and $s \in \mathcal{C}_b^+(V)$ be $(\Delta + \beta\mu)$-superharmonic. Using Lemma 2.1 we may find a compact set $A \ne \emptyset$ in V such that $W := V \setminus A$ is regular and $\|\alpha_V K_V 1_W\|_\infty < 1$. Then $\alpha_V K_W 1 \le \|\alpha_V K_V 1_W\|_\infty < 1$ and hence $\alpha_V \in \Gamma_W$. Let

$$a := \sup\{\alpha \ge 0 \colon \alpha h \le s \text{ on } A\} \quad \text{and} \quad t := s - ah.$$

Then t is $(\Delta + \alpha_V \mu)$-superharmonic on V, $t \ge 0$ on A, and there exists a point $x_0 \in A$ such that $t(x_0) = 0$. Clearly, $\liminf_{y \to z} t(y) \ge 0$ for every $z \in \partial W$ (recall that $h \to 0$ at ∂V). Therefore, by Lemma 2.8, $t \ge 0$ on W. So $t \ge 0$ on V, and hence t is superharmonic on V. Since V is connected and $t(x_0) = 0$, we conclude that $t = 0$, that is, $s = ah$.

In particular, s is $(\Delta + \alpha_V \mu)$-harmonic and hence $\Delta s + \alpha_V s \mu = 0$. On the other hand, we know that $\Delta s + \beta s \mu \le 0$. So $\beta s \mu \le \alpha_V s \mu$. Since $\mu(V) > 0$, this shows that $s = 0$ if $\beta > \alpha_V$, proving (3).

To finish the proof of (2), let $g \in \ker(I - \alpha_V K_V)$. There exists $b > 0$ such that $\tilde{g} := bh - g \ge 0$ on A. Since $g = a_V K_V g \in \mathcal{C}_0(V)$ and hence $\tilde{g} \in \mathcal{C}_0(V)$, we obtain, by Lemma 2.8, that $\tilde{g} \ge 0$ on W as well. By the preceding considerations, there exists $c > 0$ such that $bh - g = ch$ and therefore $g \in \mathbb{R}h$.

Statement (4) is a trivial consequence of (2) and (3).

Since $K_V h = \alpha_V^{-1} h$, we obtain, by Lemma 2.17, that K_V is a bounded operator on $\mathcal{L}^2(V, \mu)$ and $\|K_V\|_2 \le \alpha_V^{-1}$. □

LEMMA 2.19. *Let $V \ne \emptyset$ be a regular set such that $V \subset U$ and $0 \notin \overline{V}$. Suppose that $\mathcal{H}_0^{\Delta+\mu}(U) \ne \{0\}$ or, more generally, there exists a strictly positive $(\Delta + \mu)$-superharmonic function $s \in \mathcal{C}(U)$ which is bounded on V. Then $\alpha_V > 1$, and V is μ-admissible and $(\Delta + \mu)$-regular.*

Proof. Since $s > 0$ on the set $U \cap \partial V \ne \emptyset$, we see by Proposition 2.18.4, that $\alpha_V > 1$ and hence V is μ-admissible. So, by Proposition 2.9.1, V is $(\Delta + \mu)$-regular. □

3 Nature of extremal rays in $\mathcal{H}_0^{\Delta+\mu}(U)$

We now return to the situation considered in the Introduction, where $X = \mathbb{R}^d \setminus \{0\}$, $B = \{x \in \mathbb{R}^d \colon |x| < R\}$, $U = B \setminus \{0\}$, and

$$\mathcal{H}_0^{\Delta+\mu}(U) = \{h \in \mathcal{H}^{\Delta+\mu}(U) \colon h \text{ tends to } 0 \text{ at } \partial B\}.$$

Let

$$K := K_U, \quad G_0 := G_B(\cdot, 0)|_U, \quad g_0 := \sum_{n=0}^{\infty} K^n G_0.$$

Obviously,

$$g_0 = Kg_0 + G_0. \tag{3.1}$$

PROPOSITION 3.1. *If $g_0 \neq \infty$, then $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$.*

Proof. Let $g_0 \neq \infty$. Then $g_0 \in \mathcal{S}^+(U)$. Let V be a μ-small ball, $\overline{V} \subset U$. By (3.1) and (2.2),

$$g_0 = Kg_0 + G_0 = H_V K g_0 + G_0 + K_V g_0, \tag{3.2}$$

where $H_V K g_0 + G_0$ is harmonic on V. Hence, by Lemma 2.12, g_0 is $(\Delta+\mu)$-harmonic on V. By Proposition 2.14, we obtain that $g_0 \in \mathcal{H}^{\Delta+\mu}(U)$. In particular, $g_0 \in C^+(U)$, and hence $p_n := K^n G_0 \in \mathcal{C}^+(U)$, $n \in \mathbb{N}$.

It remains to show that g_0 tends to 0 at ∂B. By Lemma 2.1, there exists $\varepsilon \in (0,1)$ such that $A := \{x \in B \colon |x| > R - \varepsilon\}$ is μ-admissible, and hence $(\Delta+\mu)$-regular. Let $h_0 := H_A^{\Delta+\mu} g_0$ so that h_0 tends to 0 at ∂B (g_0 is extended by the value 0 on ∂B). Of course, $h_0 \le g_0$, by Lemma 2.8.

The proof will be finished if we know that, conversely, $g_0 \le h_0$ on A. To that end we claim first that each potential p_n $n \in \mathbb{N}$, tends to 0 at ∂B. Of course, this is true for $n = 0$. So let us assume that $n \in \{0\} \cup \mathbb{N}$ such that p_n tends to 0 at ∂B, and hence p_n is bounded on A. We have

$$p_{n+1} = K(1_A p_n) + K(1_{U \setminus A} p_n),$$

where $K(1_A p_n) = K_B(1_A p_n)$ on U and $K_B(1_A p_n) \in \mathcal{C}_0(B)$, since $1_A \mu$ is a Kato measure on $\mathbb{R}^d$. Moreover, $K(1_{U\setminus A} p_n)$ is a continuous real potential on U which is harmonic on A, and hence tends to 0 at ∂B. Thus p_{n+1} tends to 0 at ∂B.

Finally, let $m \in \mathbb{N}$ and $q_m := \sum_{n=0}^m K^n G_0$. Then q_m is continuous, $0 \le q_m \le g_0$, and q_m tends to 0 at ∂B. In particular, q_m is bounded on A. Moreover, $q_m - K q_m = G_0 - K^{m+1} G_0$, and hence $\Delta q_m + q_m \mu \ge 0$. By Lemma 2.8, we obtain that $q_m \le h_0$ on A. Thus $g_0 = \sup q_m \le h_0$ on A, and the proof is finished. □

DEFINITION 3.2. *A function $h \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$ is* extremal*, if it is contained in an extremal ray of $\mathcal{H}_0^{\Delta+\mu}(U)$, that is, if every $\tilde{h} \in \mathcal{H}_0^{\Delta+\mu}(U)$ such that $0 \le \tilde{h} \le h$ is a multiple of h.*

For every $n \in \mathbb{N}$, let

$$U_n := \{x \in \mathbb{R}^d \colon R/(n+1) < |x| < R\}.$$

PROPOSITION 3.3. *Let h be an extremal function in $\mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$. Then $Kh = h$ or h is a multiple of g_0 (and hence $Kh < h$).*

Proof. Let

$$g_n := h - K_{U_n} h, \qquad n \in \mathbb{N}.$$

Then the functions g_n are harmonic on U_n, $n \in \mathbb{N}$. Since the sequence $(K_{U_n} h)$ is increasing to Kh, the sequence (g_n) is decreasing to a function $g \in \mathcal{H}^+(U)$ and

$$h = g + Kh. \tag{3.3}$$

To finish the proof it remains to consider the case $g \neq 0$. Since $0 \le g \le h$, we know that g tends to 0 at ∂B. Therefore $g = aG_0$ for some $a > 0$ (see [2, Exercise 2.11]). By (3.3),

$$h = \sum_{n=0}^{m-1} K^n g + K^m h, \qquad m \in \mathbb{N},$$

and therefore $ag_0 = \sum_{n=0}^{\infty} K^n g \le h$. By Proposition 3.1, we see that $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$. Since h is extremal, we get that h is a multiple of g_0, and hence $Kh < h$, by (3.1). □

REMARK 3.4. Let us suppose that we have a function $h \in \mathcal{B}^+(U)$ such that $Kh = h$ and $h \neq \infty$. Then $h \in \mathcal{S}^+(U)$ and, similarly as in the proof of Proposition 3.1 (cf. (3.2)), we obtain that $h \in \mathcal{H}^{\Delta+\mu}(U)$, by Lemma 2.12. In particular, $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ provided h tends to 0 at ∂B. The latter holds if there exists $\varepsilon \in (0,1)$ such that μ does not charge the shell $A := \{x \in U \colon |x| > R - \varepsilon\}$ or h is bounded on A. Indeed, $p := K(1_{U\setminus A} h)$ is a potential on U which is harmonic on A, and hence tends to 0 at ∂B. In the first case, $h = Kh = p$. In the latter case $h = Kh = p + K_B(1_A h)$ on U, where $K_B(1_A h) \in \mathcal{C}_0(B)$, since $1_A \mu$ is a Kato measure on $\mathbb{R}^d$.

Fixing some point $x_0 \in U$, the convex set

$$H_0^{\Delta+\mu} := \{h \in \mathcal{H}_0^{\Delta+\mu}(U) \colon h(x_0) = 1\} \tag{3.4}$$

is compact with respect to locally uniform convergence on U. Indeed, let V be a ball, $\overline{V} \subset U$. By Proposition 2.15, the set $H_0^{\Delta+\mu}$ is bounded on $\overline{V}$. For every $h \in H_0^{\Delta+\mu}$, $h = K_V h + H_V h$, by Lemma 2.4. Hence, by Proposition 2.2 and the equicontinuity of harmonic functions (see, for instance, [2, Lemma 1.5.6]), the functions in $h \in H_0^{\Delta+\mu}$ are equicontinuous on V. Therefore, by Choquet's theorem, for every function $h \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$, there exists a probability measure χ on the set of extreme points of $H_0^{\Delta+\mu}$ such that

$$\frac{h}{h(x_0)} = \int \tilde{h}\, d\chi(\tilde{h}).\text{[3]} \tag{3.5}$$

This leads to the following consequence of Proposition 3.3.

COROLLARY 3.5. *Every function $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ is μ-integrable and satisfies $Kh \le h$, where even $Kh = h$, if $g_0 = \infty$.*

If $g_0 \neq \infty$, then $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$, and g_0 is extremal.

[3] In fact, the measure χ is uniquely determined by h (see [4, pp. 78-79], [6, Corollary 2.2.1], [5, Theorem 28.4]).

Proof. Let us fix $h \in \mathcal{H}_0^{\Delta+\mu}(U)\setminus\{0\}$. Clearly, by (3.5) and Proposition 3.3 , $Kh \le h$, and even $Kh = h$, if $g_0 = \infty$. Of course, h is bounded on $U_1 := \{y \in U : |y| > R/2\}$, and $\mu(U_1) < \infty$. Further, choosing any $x_0 \in U$, $\inf\{G(x_0, y) : y \in U \setminus U_1\} > 0$. Hence the inequality $\int G(x_0, y)h(y)\, d\mu(y) = Kh(x_0) \le h(x_0) < \infty$ implies that h is μ-integrable on $U \setminus U_1$. Thus $h \in \mathcal{L}^1(U, \mu)$.

Finally, let us assume that $g_0 \ne \infty$. Then $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$, by Proposition 3.1, and the measure χ associated with g_0 must charge $g_0/g(x_0)$, since otherwise we would obtain that $Kg_0 = g_0$, contradicting (3.1). So g_0 is extremal. □

4 Characterization of $\mathcal{H}_0^{\Delta+\mu}(U) \ne \{0\}$

In this section we shall see that $\mathcal{H}_0^{\Delta+\mu}(U) \ne \{0\}$ if and only if the μ-eigenvalues α_{U_n} of Δ on the shells $U_n = \{x \in \mathbb{R}^d : R/(n + 1) < |x| < R\}$, $n \in \mathbb{N}$, are at least 1 or – equivalently – if and only if K operates on $\mathcal{L}^2(U, \mu)$ with $\|K\|_2 \le 1$ (Theorem 4.2 and Corollary 4.3). For a useful consequence of $\|K\|_2 \le 1$ see Lemma 5.11.

We first observe the following.

PROPOSITION 4.1. *K is a bounded operator on $\mathcal{L}^2(U, \mu)$ if and only if* $\inf \alpha_{U_n} > 0$, *and then* $\|K\|_2 = (\inf \alpha_{U_n})^{-1}$.

Proof. By Proposition 2.18, $\|K_{U_n}\|_2 = \alpha_{U_n}^{-1}$. Since $G_{U_n} \uparrow G$ as $n \to \infty$, the claim follows immediately. □

THEOREM 4.2. *The following statements are equivalent.*

(i) $\mathcal{H}_0^{\Delta+\mu}(U) \ne \{0\}$.

(ii) *There is a strictly positive $(\Delta + \mu)$-superharmonic function $s \in \mathcal{C}(U)$ which is bounded on each U_n, $n \in \mathbb{N}$.*

(iii) *For every $n \in \mathbb{N}$, $\alpha_{U_n} > 1$.*

(iv) *For every $n \in \mathbb{N}$, $\alpha_{U_n} \ge 1$.*

Proof. Let $\alpha_n := \alpha_{U_n}$, $n \in \mathbb{N}$. If $\mu(U) = 0$, then $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ G_0$ and $\alpha_n = \infty$, $n \in \mathbb{N}$. So let us suppose that $\mu(U) > 0$.

(i) ⇒ (ii): Trivial.

(ii) ⇒ (iii): Proposition 2.19.

(iii) ⇒ (i): For every $n \in \mathbb{N}$, U_n is $(\Delta + \mu)$-regular, by Proposition 2.9.1, and

$$g_n := H_{U_n}^{\Delta+\mu} 1_{U\cap\partial U_n}$$

is strictly positive, $(\Delta + \mu)$-harmonic on U_n, and tends to zero at ∂B. Let $x_0 \in U_1$ and $\tilde{g}_n := g_n/g_n(x_0)$, $n \in \mathbb{N}$. Then there exists a subsequence of $(\tilde{g}_n)$ which is locally uniformly convergent to a $(\Delta + \mu)$-harmonic function $g > 0$ on U (see the discussion in connection with (3.4)). The convergence is uniform on U_1, since $\tilde{g}_n = H_{U_1}^{\Delta+\mu}(1_{U\cap\partial U_1}\tilde{g}_n)$, $n \in \mathbb{N}$. Thus $g \in \mathcal{H}_0^{\Delta+\mu}(U)$.

(iii) ⇒ (iv): Trivial.

(iv) ⇒ (iii): Let us fix $n \in \mathbb{N}$ such that $\alpha_n < \infty$, that is, $\mu(U_n) > 0$. Then $\mu(U_{n+1}) > 0$, $\alpha_{n+1} < \infty$. By Proposition 2.18.2, there exists a strictly positive function $h \in \mathcal{C}_0(U_{n+1}) \cap \mathcal{H}^{\Delta+\alpha_{n+1}\mu}(U_{n+1})$. If $\alpha_{n+1} \geq \alpha_n$, then $s := h|_{U_n} \in \mathcal{C}_b^+(U_n)$, s is $(\Delta + \alpha_n\mu)$-superharmonic on U_n, but s does not tend to 0 at $U \cap \partial U_n$. By Proposition 2.18.2. this is impossible. Thus $\alpha_n > \alpha_{n+1} \geq 1$. □

COROLLARY 4.3. *$\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$ if and only if K operates on $\mathcal{L}^2(U,\mu)$ with $\|K\|_2 \leq 1$.*

Let us note that the "only if"-part can also be obtained from Corollary 3.5 and the Schur test (see Lemma 2.17).

5 Sufficient conditions for the Picard principle

In the proof of the following result (and later on) we shall tacitly use the fact that, for every $s \in \mathcal{S}^+(U)$, there exists a unique extension to a superharmonic function $\tilde{s} \in \mathcal{S}^+(B)$ (and $\tilde{s}(0) = \liminf_{y\to 0} s(y)$; see [2, Corollary 5.2.2]). Of course, $\tilde{s} > 0$, if $s > 0$. In particular, by Corollary 3.5, $\mu(U) < \infty$ if $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$.

PROPOSITION 5.1. *Suppose that g_0 is bounded by a multiple of G_0. Then there exists $C > 0$ such that*

$$\sum_{n=0}^{\infty} K^n s \leq Cs \qquad \text{for every } s \in \mathcal{S}^+(U). \tag{5.1}$$

In particular, $\Delta + \mu$ satisfies the Picard principle on U, $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ g_0$.

Proof. By Corollary 8.3 in the Appendix, there exists $C > 0$ such that (5.1) holds. In particular, there is no strictly positive $s \in \mathcal{S}(U)$ such that $Ks = s$. So there is no $h \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$ satisfying $Kh = h$. Thus, by Proposition 3.3, $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ g_0$. □

PROPOSITION 5.2. *Let us suppose that K operates on $\mathcal{L}^2(U,\mu)$, $\|K\|_2 = 1$, and 1 is an eigenvalue of K.*[4] *Then $g_0 = \infty$ and there exists $h_0 \in \mathcal{H}_0^{\Delta+\mu}(U) \setminus \{0\}$ such that $Kh_0 = h_0$ and $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ h_0$. In particular, $\Delta + \mu$ satisfies the Picard principle on U.*

Proof. By Corollary 4.3, there exists a function $h_0 \in \mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$. For every $h \in \mathcal{H}_0^{\Delta+\mu}(U)$, $Kh \leq h$, by Corollary 3.5. Hence, by Proposition 7.1, there exists $u \in \mathcal{L}^2(U,\mu)$ such that $Ku = u$ and every $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ is μ-a.e. equal to a multiple of u.

If $g_0 \neq \infty$, then $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$ and $Kg_0 < g_0$, by Proposition 3.1 and (3.1). Hence g_0 is not μ-a.e. equal to a multiple of u. So $g_0 = \infty$.

Finally, let h be any function in $\mathcal{H}_0^{\Delta+\mu}(U)$. Then, by Corollary 3.5, $Kh = h$. Since both h and h_0 are μ-a.e. equal to a multiple of u, we obtain that there exists $a \geq 0$ such that $h = ah_0$ μ-a.e. This implies that $h = Kh = aKh_0 = ah_0$. □

[4] If K is a *compact* operator on $\mathcal{L}^2(U,\mu)$, then $\|K\|_2$ *is* an eigenvalue of K, since G is symmetric.

We stress that our general assumption on μ does not exclude the possibility that

$$\|K\|_2 = 1, \quad Kh = h, \quad h \notin \mathcal{L}^2(U,\mu), \quad \mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ h,$$

and hence $\mathcal{H}_0^{\Delta+\mu}(U) \cap \mathcal{L}^2(U,\mu) = \{0\}$.

EXAMPLE 5.3. For $n \in \mathbb{N}$, let $r_n := 4^{-n}R$ so that $r_n - r_{n+1} = 3r_{n+1}$. Let W_n be the open ball of diameter r_{n+1} centered at $x_n := ((r_n + r_{n+1})/2, 0, \dots, 0)$. Then the shells $A_n := \{x \in U \colon r_n - r_{n+1} < |x| < r_n + r_{n+1}\}$, $n \in \mathbb{N}$, do not intersect the union of the balls W_m, $m \in \mathbb{N}$. For each $n \in \mathbb{N}$, we fix $a_n \in (n, \infty)$ such that $G(\cdot, x_n)/a_n < 2^{-n}$ on $U \setminus W_n$, and define

$$p_n := \min\{a_n, G(\cdot, x_n)/a_n\}.$$

Then $p_n \in \mathcal{C}(U)$ and $p_n = G\nu_n$ for some measure ν_n which has total mass $1/a_n$ and a support C_n contained in W_n. Let $\nu := \sum_{n=1}^\infty \nu_n$,

$$h := G\nu \quad \text{and} \quad \mu := \frac{1}{h}\,\nu.$$

Then $Kh = G(h\mu) = G\nu = h$. Since the balls W_n, $n \in \mathbb{N}$, are pairwise disjoint and $\sum_{n=1}^\infty 2^{-n} = 1$, we have $h \in \mathcal{C}(U)$ and

$$a_n = p_n \le h \le a_n + 1 \quad \text{on } C_n. \tag{5.2}$$

Obviously, h tends to 0 at ∂B. Hence $h \in \mathcal{H}_0^{\Delta+\mu}(U)$. Moreover, for every $n \in \mathbb{N}$, $\int G\nu_n \, d\nu_n = a_n/a_n = 1$ and hence $\int h^2 \, d\mu = \int G\nu \, d\nu \ge \sum_{n=1}^\infty \int G\nu_n \, d\nu_n = \infty$.

Nevertheless, by Corollary 4.3, K operates on $\mathcal{L}^2(U,\mu)$ and $\|K\|_2 \le 1$. Further, (5.2) implies that, for every $n \in \mathbb{N}$, $K1_{C_n} = G((1/h)\nu_n) \ge a_n/(a_n+1) \ge n/(n+1)$ on C_n, and hence $\|K\|_2 \ge n/(n+1)$. Therefore $\|K\|_2 = 1$.

Moreover, Corollary 5.7 will show that $\Delta + \mu$ satisfies the Picard principle on U, and hence $\mathcal{H}_0^{\Delta+\mu}(U) = \mathbb{R}^+ h$. We could even smear ν a little, add Lebesgue measure on U (leading to a measure μ having a strictly positive $\mathcal{C}^\infty$-density on U), and still have the same result.

For every $r > 0$, let

$$B_r := \{x \in \mathbb{R}^d \colon |x| < r\}, \qquad S_r := \{x \in \mathbb{R}^d \colon |x| = r\}.$$

LEMMA 5.4. *Let $h_1, h_2 \in \mathcal{H}_0^{\Delta+\mu}(U)$ such that $h_2 > 0$, h_2 is extremal, and there exist $r_n \in (0,R)$ and $a \ge 0$ such that $h_1 \le a h_2$ on S_{r_n}. Then h_1 is a multiple of h_2.*

Proof. By Propositions 2.19 and 2.8, $h_1 \le a h_2$ on every shell $\{x \in U \colon |x| > r_n\}$, that is, $h_1 \le a h_2$ on U. So h_1 is a multiple of h_2. □

REMARK 5.5. As we already stated in the introduction, it is proven in [18] that $\Delta + \mu$ satisfies the Picard principle, if $d = 2$. Since the proof is quite involved, let us observe that Lemma 5.4 leads to a simple proof in the case, where $g_0 \ne \infty$.

Indeed, suppose that $d = 2$ and $g_0 \ne \infty$. Then, by Corollary 3.5, $g_0 \in \mathcal{H}_0^{\Delta+\mu}(U)$, and g_0 is an extremal. Since g_0 is superharmonic on U, it can be extended to a superharmonic function on B. Now let $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ and suppose that h is not a multiple of g_0. Then, by Corollary 3.5, $Kh = h$ and hence $K_B h = h$ on U. Hence, by [2, Theorem 7.4.3], the fine limit of h/G_0 at the origin is 0. Therefore, by [2, Theorem 7.3.9], there exist $r_n \in (0, R)$ such that $h \le G_0$ on S_{r_n}, $n \in \mathbb{N}$. Since $G_0 \le g_0$, we conclude by Lemma 5.4, that $h \le g_0$, and hence h is a multiple of g_0, a contradiction.

PROPOSITION 5.6. *Suppose that there exist $c > 0$ and $r_n \in (0, R)$ such that $r_n \downarrow 0$ and, for all $h \in \mathcal{H}_0^{\Delta+\mu}(U)$ and $n \in \mathbb{N}$,*

$$\sup h(S_{r_n}) \le c \inf h(S_{r_n}). \tag{5.3}$$

Then $\Delta + \mu$ satisfies the Picard principle on U.

Proof. Let us assume that we have extremal functions $h_1, h_2 \in \mathcal{H}_0^{\Delta+\mu} \setminus \{0\}$, and let us fix points $x_n \in S_{r_n}$, $n \in \mathbb{N}$. If $h_1(x_n)/h_2(x_n)$ tends to ∞ as $n \to \infty$, we exchange the role of h_1 and h_2. Passing to a subsequence, if necessary, we may hence assume without loss of generality that the sequence $(h_1(x_n)/h_2(x_n))$ is bounded by some $m \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ and $y \in S_{r_n}$,

$$h_1(y) \le c\, h_1(x_n) \le mc\, h_2(x_n) \le mc^2\, h_2(y).$$

Thus, by Lemma 5.4, h_1 is a multiple of h_2. □

By scaling invariance and Harnack's inequalities for harmonic functions, for every $\delta \in (0, 1)$, there exists $c_\delta > 0$, such that

$$\sup g(S_r) \le c_\delta \inf g(S_r), \tag{5.4}$$

whenever $r > 0$ and g is a positive harmonic function on

$$A_{r,\delta} := \{x \in \mathbb{R}^d \colon (1 - \delta)r < |x| < (1 + \delta)r\}.$$

COROLLARY 5.7. *Suppose that there exist $\delta \in (0, 1)$ and $r_n \in (0, R/2)$ such that $\mu(A_{r_n,\delta}) = 0$, $n \in \mathbb{N}$. Then $\Delta + \mu$ satisfies the Picard principle on U.*

Proof. The inequalities (5.3) follow from (5.4), since, for every $n \in \mathbb{N}$, the functions in $\mathcal{H}_0^{\Delta+\mu}(U)$ are harmonic on $A_{r_n,\delta}$. □

THEOREM 5.8. *Let us suppose that there exist $\delta \in (0, 1)$ and $r_n \in (0, R/4)$ such that $r_n \downarrow 0$ and, for every $n \in \mathbb{N}$,*

$$G(1_{A_{r_n,\delta}}\mu) \le 1/(4C_\Delta). \tag{5.5}$$

Then $\Delta + \mu$ satisfies the Picard principle on U.

Proof. Let $V := B_{R/2}$, $n \in \mathbb{N}$, $A := A_{r_n,\delta}$, $S := S_{r_n}$, $h \in \mathcal{H}_0^{\Delta+\mu}(U)$, and $\tilde{h} := H_A h$. Extending by lower limits at 0, these functions are superharmonic on V. Of course, $\tilde{h}$ is harmonic on A and $h = \tilde{h} + K_A h$ (see Lemma 2.4). By Proposition 2.11, $G_V(s1_A\mu) \le (1/2)s$, for every $s \in \mathcal{S}^+(V)$. Therefore $K_A h \le G_V(h1_A\mu) \le (1/2)h$ and $K_A\tilde{h} \le G_V(\tilde{h}1_A\mu) \le (1/2)\tilde{h}$. Thus, by Lemma 2.12, $\tilde{h} \le h \le 2\tilde{h}$, and therefore, by (5.4),

$$\sup h(S) \le 2 \sup \tilde{h}(S) \le 2c_\delta \inf \tilde{h}(S) \le 2c_\delta \inf h(S).$$

An application of Proposition 5.6 finishes the proof. □

COROLLARY 5.9. *Suppose that μ (considered as a measure on $\mathbb{R}^d$) is a Kato measure on $\mathbb{R}^d$ or, more generally, that there are $\delta \in (0,1)$ and $r_n \in (0, R/4)$, $n \in \mathbb{N}$, such that $r_n \downarrow 0$ and the sum ν of the measures $1_{A_{r_n,\delta}}\mu$ is a Kato measure on $\mathbb{R}^d$.*

Then $\Delta + \mu$ satisfies the Picard principle on U.

Proof. By Lemma 2.1, there exists $r > 0$ such that $G(1_{B_r}\nu) = G_B(1_{B_r}\nu) \le 1/(4C_\Delta)$ on U. Then, of course, $G(1_{A_{r_n,\delta}}\mu) \le 1/(4C_\Delta)$, whenever $(1+\delta)r_n \le r$. □

For $r > 0$, let σ_r denote the normalized surface measure on S_r. Given $r \in (0, R)$, we define

$$a_r := \|G\sigma_r\|_\infty.$$

Since the potential $G\sigma_r$ is harmonic on $B \setminus S_r$ and rotationally invariant,

$$a_r = G(0, (r, 0, \ldots, 0)) = G\sigma_r(0) = G\sigma_r \quad \text{on } \overline{B}_r. \tag{5.6}$$

LEMMA 5.10. *For every $C > 2^{d-2}$, there exists $r_0 \in (0, R/4)$ such that, for all $r \in (0, r_0)$ and $x, y \in B_r$,*

$$C^{-1} a_{r/2} \le a_r \le C G(x,y). \tag{5.7}$$

Proof. Let $r \in (0, R/4)$, $V := B_{R-r}$, and $x, y \in B_r$. Then, $x \in y + V \subset B$, $|x - y| < 2r$, and

$$G(x,y) \ge G_{y+V}(x,y) = G_V(x - y, 0). \tag{5.8}$$

If $d \ge 3$, then $G_{B_\rho}(\cdot, 0) = \kappa_d(|\cdot|^{2-d} - \rho^{2-d})$ on B_ρ, hence, by (5.6) and (5.8),

$$\frac{a_r}{G(x,y)} \le \frac{r^{2-d} - R^{2-d}}{(2r)^{2-d} - (R-r)^{2-d}} \quad \text{and} \quad \frac{a_{r/2}}{a_r} = \frac{(r/2)^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}},$$

where the right sides tend to 2^{d-2} as r tends to 0.

If $d = 2$, then $G_{B_\rho}(\cdot, 0) = \kappa_2(\ln|(1/|\cdot|) - \ln(1/\rho))$ on B_ρ, so, by (5.6) and (5.8),

$$\frac{a_r}{G(x,y)} \le \frac{\ln(1/r) - \ln(1/R)}{\ln(1/2r) - \ln(1/(R-r))} \quad \text{and} \quad \frac{a_{r/2}}{a_r} = \frac{\ln(2/r) - \ln(1/R)}{\ln(1/r) - \ln(1/R)},$$

where the right sides tend to 1 as r tends to 0. □

For the remainder of this section, let us fix $C > 2^{d-2}$ and r_0 according to Lemma 5.10. For the convenience of the reader we include a short proof for the following consequence of the inequality $\|K\|_2 \le 1$.

LEMMA 5.11. *Suppose that K operates on $\mathcal{L}^2(U, \mu)$ and $\|K\|_2 \le 1$. Then*

$$\mu(B_r) \le C/a_r \qquad \text{for every } r \in (0, r_0). \tag{5.9}$$

Proof. Let $r \in (0, r_0)$ and $A := B_r \setminus \{0\}$. By Lemma 5.10, for every $x \in A$,

$$K1_A(x) = \int_A G(x,y)\, d\mu(y) \geq C^{-1} a_r \mu(A)$$

and therefore

$$\int (K1_A)^2\, d\mu \geq \int_A (K1_A)^2\, d\mu \geq (C^{-1} a_r \mu(A))^2 \mu(A).$$

On the other hand, since $\|K\|_2 \leq 1$, we know that $\int (K1_A)^2\, d\mu \leq \int (1_A)^2\, d\mu = \mu(A)$. Since $\mu(A) = \mu(B_r)$, (5.9) follows. □

DEFINITION 5.12. *Given $c > 0$, $r \in (0, R/2)$, and $\delta \in (0, 1/2)$, we shall say that μ is* c-radial *on $A_{r,\delta}$, if for every shell $A := \{x \in \mathbb{R}^d \colon s < |x| < t\}$, where $(1-\delta)r \leq s < t \leq (1+\delta)r$,*

$$G(1_A \mu) \leq c a_{r/2} \mu(A).$$

If μ is rotationally invariant on $A := A_{r,\delta}$ then μ is 1-radial on A, since $a_t \leq a_{r/2}$, whenever $(1-\delta)r < t < R$. More generally, if $c > 0$ and

$$1_A \mu = \int_{(1-\delta)r}^{(1+\delta)r} \mu_t\, dt,$$

such that

$$G\mu_t \leq c a_t \mu_t(S_t) \qquad \text{for } (1-\delta)r \leq t \leq (1+\delta)r, \tag{5.10}$$

then μ is c-radial on A. Let us note that (5.10) holds if $\mu_t = \varphi_t \sigma_t$ such that

$$\sup \varphi_t(S_t) \leq c \inf \varphi_t(S_t). \tag{5.11}$$

Indeed, (5.11) obviously implies that

$$G(\varphi_t \sigma_t) \leq \sup \varphi_t(S_t) a_t \leq c \inf \varphi_t(S_t) a_t \leq c a_t \mu_t(S_t).$$

THEOREM 5.13. *Suppose that there exist $c > 0$ and $\delta \in (0, 1/2)$ such that*

$$\inf\{r \in (0, r_0) \colon \mu \textit{ is c-radial on } A_{r,\delta}\} = 0.$$

Then $\Delta + \mu$ satisfies the Picard principle on U.

Proof. By Corollary 4.3, it remains to consider the case, where K operates on $\mathcal{L}^2(U, \mu)$ and $\|K\|_2 \leq 1$. Then, by Lemmas 5.11 and 5.10,

$$\mu(B_r) \leq C/a_r \leq C^2 a_{r/2} \qquad \text{for every } r \in (0, r_0). \tag{5.12}$$

We fix $m \in \mathbb{N}$ and $\eta \in (0, \delta/2)$ such that

$$2^{-m} c\, C^2 \leq 1/(4C_\Delta) \qquad \text{and} \qquad (1-2\eta)^{2^m+1} \geq 1 - \delta. \tag{5.13}$$

Let $r \in (0, r_0)$ such that μ is c-radial on $A := A_{r,\delta}$. By Theorem 5.8, it suffices to prove that there exists $\tilde{r} \in (0, r)$ such that $\tilde{A} := A_{\tilde{r},\eta}$ satisfies

$$G(1_{\tilde{A}}\mu) \leq 1/(4C_\Delta). \tag{5.14}$$

To that end we define, for every $1 \leq j \leq 2^m$,

$$r_j := (1-2\eta)^j r \quad \text{and} \quad A_j := A_{r_j,\eta}.$$

Then $(1+\eta)r_1 < r$, $(1-\eta)r_j \geq (1+\eta)r_{j+1}$, $1 \leq j < 2^m$, and $(1-\eta)r_{2^m} \geq (1-\delta)r$. Therefore the 2^m shells $A_1, A_2, \dots, A_{2^m}$ are pairwise disjoint sets in $A \cap B_r$. Hence, by (5.12), there exists $1 \leq j \leq 2^m$ such that, defining $\tilde{r} := r_j$ and $\tilde{A} := A_j$, we have

$$\mu(\tilde{A}) \leq 2^{-m} C^2 / a_{r/2}.$$

Since μ is c-radial on A, we finally conclude that

$$G(1_{\tilde{A}}\mu) \leq c\, a_{r/2}\, \mu(\tilde{A}) \leq 2^{-m} c\, C^2 \leq 1/(4C_\Delta).$$

This finishes the proof. □

COROLLARY 5.14. *If μ is rotationally invariant on U or, more generally, if there exist $\delta \in (0, 1/2)$ and $r_n \in (0, R/2)$ such that $r_n \downarrow 0$ and μ is rotationally invariant on every $A_{r_n,\delta}$, $n \in \mathbb{N}$, then $\Delta + \mu$ satisfies the Picard principle on U.*

Finally, we observe that we may combine the statements of Corollary 5.9 and Theorem 5.13 in the following way (it suffices to replace $1/(4C_\Delta)$ appearing in both proofs by $1/(8C_\Delta)$).

COROLLARY 5.15. *Let μ be the sum of two Kato measures μ_1 and μ_2 on $\mathbb{R}^d \setminus \{0\}$ and suppose that there exist $\delta \in (0, 1/2)$ and $r_n \in (0, R/2)$ such that $r_n \downarrow 0$, the sum ν of the measures $1_{A_{r_n,\delta}}\mu$ (considered as a measure on $\mathbb{R}^d$) is a Kato measure on $\mathbb{R}^d$, and, for some $c > 0$ and each $n \in \mathbb{N}$, the measure μ_2 is c-radial on the shell $A_{r_n,\delta}$.*

Then $\Delta + \mu$ satisfies the Picard principle.

6 Localization

We recall that the sufficient conditions in Theorem 5.8, Theorem 5.13, and Corollary 5.14 depend only on the behavior of μ close to the origin. The following result shows that, even in the most general case, the verification of the Picard principle for $\Delta + \mu$ on U can be localized at 0 in two different ways (which can be combined in an obvious manner; see the proof of Corollary 6.2). To that end let $r \in (0, R)$ and $V := \{x \in U : |x| < r\}$.

THEOREM 6.1. *Let μ' be a measure on $\mathbb{R}^d \setminus \{0\}$ such that $1_V \mu \leq \mu' \leq \mu$. Then $\Delta + \mu$ satisfies the Picard principle on U, if $\Delta + \mu'$ satisfies the Picard principle on U or if $\Delta + \mu$ satisfies the Picard principle on V.*

Proof. 1. Let us assume that $\mathcal{H}_0^{\Delta+\mu}(U) \neq \{0\}$. Then, for every $h \in \mathcal{H}_0^{\Delta+\mu}(U)$, we shall construct corresponding minorants in $\mathcal{H}_0^{\Delta+\mu}(V)$ and $\mathcal{H}_0^{\Delta+\mu'}(U)$. To that end, we define

$$V_n := V \cap U_n, \qquad n \geq R/r.$$

By Proposition 2.18.4 and Proposition 2.9, the open sets U_n, V_n are regular with respect to $\Delta + \mu$ and $\Delta + \mu'$.

2. If $w \in \mathcal{H}_0^{\Delta+\mu}(V)$, then w (extended by 0) is $(\Delta + \mu')$-subharmonic on U, that is, $-w$ is $(\Delta+\mu')$-superharmonic. Indeed, let us fix $n \in \mathbb{N}$ such that $n \geq R/r$. Then $w - K_{V_n} w = H_{V_n} w$, hence, by (2.2),

$$w - K_{U_{n+1}} w = w - K_{V_n} w - H_{V_n} K_{U_{n+1}} w = H_{V_n} w - H_{V_n} K_{U_{n+1}} w.$$

Obviously, $H_{V_n} w$ (which vanishes on $U \setminus V$) is subharmonic on $U_n = V_n \cup (U \setminus V)$, and $H_{V_n} K_{U_{n+1}} w$ is superharmonic on U_{n+1}. Therefore $\Delta(w - K_{U_{n+1}} w) \geq 0$ on U_n, hence $\Delta w + w\mu \geq 0$ on U_n. Since $\Delta w + w\mu = 0$ on V, we finally conclude that $\Delta w + w\mu' = \Delta w + w\mu \geq 0$ on U.

3. Now let us take $h \in \mathcal{H}_0^{\Delta+\mu}(U)$. We first define

$$v_n := H_{V_n}^{\Delta+\mu}(1_{U\setminus V} h), \qquad n \geq R/r, \tag{6.1}$$

so that $v_n \in \mathcal{C}^+(U)$, $v_n = h$ on $U \setminus V$ and $v_n = 0$ on $V \setminus V_n$. The sequence (v_n) is increasing to a function $\tilde{h}$ on U which is $(\Delta + \mu)$-harmonic on V and is majorized by h on U. Since $v_1 \leq \tilde{h} \leq h$ (and $v_1 = h$ on ∂V), we see that $\tilde{h} \in \mathcal{C}^+(U)$. So

$$g := h - \tilde{h} \in \mathcal{C}^+(U) \quad \text{and} \quad g = 0 \ \text{ on } U \setminus V.$$

Thus $g|_V \in \mathcal{H}_0^{\Delta+\mu}(V) = \mathcal{H}_0^{\Delta+\mu'}(V)$.

By the previous consideration, g is $(\Delta + \mu')$-subharmonic on U. Moreover, h is a majorant of g, which is $(\Delta + \mu')$-superharmonic, since $\Delta h + \mu' h \leq \Delta h + \mu h = 0$. Therefore, by Lemma 2.8, the functions

$$u_n := H_{U_n}^{\Delta+\mu'} g, \qquad n \in \mathbb{N},$$

satisfy $g \leq u_n \leq h$ and are increasing to a $(\Delta + \mu')$-harmonic function h' on U such that $g \leq h' \leq h$. Thus, $h' \in \mathcal{H}_0^{\Delta+\mu'}(U)$.

There is a natural way to get g back from h': Let

$$v_n' := H_{V_n}^{\Delta+\mu'}(1_V h'), \qquad n \geq R/r \tag{6.2}$$

(where we could just as well write μ instead of μ', since $V_n \subset V$). The functions v_n' are continuous on U and vanish on $U \setminus V$. We claim that

$$v_n' \downarrow g \qquad \text{as } n \to \infty. \tag{6.3}$$

Indeed, since $g \leq 1_V h' \leq h'$, we obtain that, for every $n \geq R/r$,

$$g = H_{V_n}^{\Delta+\mu'} g \leq v_n' \leq H_{V_n}^{\Delta+\mu'} h' = h'.$$

Then, for every $n \geq R/r$, $v_{n+1}' \leq 1_V h'$, since v_{n+1}' vanishes on $U \setminus V$, and hence

$$v_{n+1}' = H_{V_n}^{\Delta+\mu'} v_{n+1}' \leq H_{V_n}^{\Delta+\mu'}(1_V h') = v_n'.$$

This implies that the sequence (v'_n) is decreasing to a function v' on U which is $(\Delta+\mu')$-harmonic on V and satisfies

$$g \le v' \le 1_V h'.$$

Moreover, v' tends to zero at ∂V, since every v'_n does. So $v'-g \in \mathcal{H}_0^{\Delta+\mu'}(V) = \mathcal{H}_0^{\Delta+\mu}(V)$, and hence $v'-g$ is $(\Delta+\mu')$-subharmonic on U. Therefore

$$0 \le v'-g \le H_{U_n}^{\Delta+\mu'}(v'-g) \le H_{U_n}^{\Delta+\mu'}h' - u_n = h'-u_n,$$

for every $n > R/r$. Letting $n\to\infty$ we see that $v'-g=0$ proving (6.3).

4. Now let h_1, h_2 be extremal functions in $\mathcal{H}_0^{\Delta+\mu}(U)\setminus\{0\}$. Then we have corresponding functions $\tilde h_1, \tilde h_2$, g_1, g_2, and h'_1, h'_2. If $\Delta+\mu'$ satisfies the Picard principle on U, then h'_1, h'_2 are proportional and hence g_1, g_2 are proportional, by (6.2) and (6.3). If $\Delta+\mu$ satisfies the Picard principle on V, then we know immediately that g_1, g_2 are proportional.

5. So let us consider the case that $g_1 = a g_2$ for some $a \ge 0$. Of course, there exists $b>0$ such that $h_1 \le b h_2$ on ∂V and hence $h_1 \le b h_2$ on $U\setminus V$. By (6.1), we see that $\tilde h_1 \le b\tilde h_2$. Having $h_j = g_j + \tilde h_j$, $j\in\{1,2\}$, we obtain that $h_1 \le (a+b)h_2$. Since h_2 is extremal, we finally conclude that h_1 is a multiple of h_2. □

A consequence of Theorem 6.1 is the following result (we note that, of course, (6.4) holds if μ is a Kato measure on $\mathbb{R}^d$).

COROLLARY 6.2. *Let us suppose that*

$$\limsup_{x\to 0} K1(x) < \liminf_{x\to 0} K1(x) + 1^5 \tag{6.4}$$

or, more generally, that K_V is a bounded operator on $(\mathcal{B}_b(V), \|\cdot\|_\infty)$ having a spectral radius $\rho(K_V)<1$. Then $\Delta+\mu$ satisfies the Picard principle on U.

Proof. If (6.4) holds, then $\sup K1(\overline{V}) - \inf K1(\overline{V}) < 1$, if r is sufficiently small, and hence, by (2.2), $\|K_V 1\|_\infty < 1$, since $G = G_U$ on $U\times U$ and $G_V = G_{B_r}$ on $V\times V$. So let us assume that K_V operates on $(B_b(V), \|\cdot\|_\infty)$ and $\rho(K_V) = \inf(\|K_V 1\|_\infty)^{1/n} < 1$. Let $\mu' := 1_{B_{r/2}}\mu$. Of course, the spectral radius of the operator $L\colon f\mapsto \int G_V(\cdot,y) f(y)\,d\mu'(y)$ is at most $\rho(K_V)$.

By Lemma 2.5 and Corollary 8.3 (see the Appendix), and Proposition 5.1, $\Delta+\mu'$ satisfies the Picard principle on V. By Theorem 6.1, applied to V in place of U, we obtain first that $\Delta+\mu$ satisfies the Picard principle on V. Using Theorem 6.1 again, we finally see that $\Delta+\mu$ satisfies the Picard principle on U. □

7 Appendix: An L^2-eigenfunction result

PROPOSITION 7.1. *Let $(E,\mathcal{E},\nu)$ be a measure space, $\nu\ne 0$, and let $\tilde{\mathcal{E}}$ denote the set of all equivalence classes of $\mathcal{E}$-measurable numerical functions.*

Let K be an additive and positively homogeneous mapping of $\tilde{\mathcal{E}}^+$ into $\tilde{\mathcal{E}}^+$ such that $Kv>0$, if $v\ne 0$, and $K(\tilde{\mathcal{E}}^+\cap L^2(E,\nu)) \subset L^2(E,\nu)$. Moreover, we suppose that the induced linear operator on $L^2(E,\nu)$ (which will be denoted by K as well) is bounded, symmetric with respect to the inner product $\langle\cdot,\cdot\rangle$, and $\|K\|_2$ is an eigenvalue of K. Then the following holds.

[5] This implies that $K1$ is bounded.

1. *There exists $u \in L^2(E,\nu)$, $u > 0$, such that $Ku = \|K\|_2\, u$.*

2. *If $v \in L^2(E,\nu)$ and $Kv = \|K\|_2\, v$, or if $v \in \tilde{\mathcal{E}}^+$ and $Kv \le \|K\|_2\, v < \infty$, then v is a multiple of u.*

Proof. Let us note that the statements on $u, v \in L^2(E,\nu)$ are known (see, e.g., [3, Proposition 3.12]). However, since everything can be obtained almost simultaneously, we shall present a complete proof.

We may assume without loss of generality that $\|K_2\| = 1$. By assumption, there exists $u \in L^2(E,\nu)$, $u \neq 0$, such that $Ku = u$. We may assume that $u^+ \neq 0$ (if necessary, we replace u by $-u$).

Next let $v \in L^2(E,\nu) \cup \tilde{\mathcal{E}}^+$ such that $v \ge -|u|$ and $Kv \le v$. Let

$$w := (|u| - v)^+. \tag{7.1}$$

Since $K|u| \ge |Ku| = |u|$, we know that $Kw \ge K(|u| - v) \ge |u| - v$. Hence $Kw \ge w \ge 0$. Further, $w \in L^2(E,\nu)$, since $0 \le w \le 2|u|$, whence $\|Kw\|_2 \le \|w\|_2$. Therefore

$$Kw = w.$$

If $w \neq 0$, then $Kw > 0$, hence $w > 0$, that is, $|u| - v > 0$. If, however, $w = 0$, then $v \ge |u|$. Thus

$$|u| > v \quad \text{or} \quad v \ge |u|. \tag{7.2}$$

If $|u| > u$, then $u^+ = 0$ in contradiction to our assumption on u. Choosing $v := u$ we hence conclude from (7.2) that $u \ge |u| \ge 0$. Thus $u = Ku > 0$, since $u \neq 0$.

In fact, this shows that, for every $u' \in L^2(E,\nu)$ satisfying $Ku' = u'$,

$$u' \ge 0 \quad \text{or} \quad -u' \ge 0. \tag{7.3}$$

So every $v \in L^2(E,\nu)$ satisfying $Kv = v$ is a multiple of u, since otherwise there certainly is a linear combination u' of u and v violating (7.3).

Finally, let us consider the case, where $v \in \tilde{\mathcal{E}}^+$ and $Kv \le v$. To show that v is a multiple of u we may suppose that $(u - v)^+ \neq 0$ (we replace v by εv taking $\varepsilon > 0$ sufficiently small). Then, by (7.2), $u > v$. In particular, $v \in L^2(E,\nu)$. If $v - Kv \neq 0$, then, by the symmetry of K,

$$\langle u, v\rangle > \langle u, Kv\rangle = \langle Ku, v\rangle = \langle u, v\rangle,$$

a contradiction. So $Kv = v$. Thus v is a multiple of u. □

8 Appendix: Triangle property on punctured sets

Let us recall the generalized triangle property. Given an arbitrary set X and functions $w, w^*\colon X \to (0,\infty)$, a function $F\colon X \times X \to [0,\infty]$ has the (w,w^*)-*triangle property*, if there exists $C > 0$ such that, for all $x, y, z \in X$,

$$F(x,z)F(z,y) \le C F(x,y) \max\Big\{\frac{w(z)}{w(x)} F(x,z), \frac{w^*(z)}{w^*(y)} F(z,y)\Big\}$$

or – equivalently – that the function $F_{w,w^*}\colon (x,y) \mapsto F(x,y)/(w(x)w^*(y))$ satisfies the *triangle property*, that is, for all $x,y,z \in X$,

$$F_{w,w^*}(x,z)F_{w,w^*}(z,y) \le CF_{w,w^*}(x,y)\max\bigl\{F_{w,w^*}(x,z), F_{w,w^*}(z,y)\bigr\}, \tag{8.1}$$

which, in turn, can be rewritten as

$$\min\bigl\{F_{w,w^*}(x,z), F_{w,w^*}(z,y)\bigr\} \le CF_{w,w^*}(x,y). \tag{8.2}$$

The following results are of independent interest.

PROPOSITION 8.1. *Let X be an arbitrary set, $a \in X$, $X^a := X \setminus \{a\}$. Suppose that $G\colon X \times X \to [0,\infty]$ is symmetric, $0 < G^a := G(\cdot,a)|_{X^a} < \infty$, and, for some $w\colon X \to (0,\infty)$, G has the (w,w)-triangle property.*

Then $G|_{X^a\times X^a}$ has the (G^a,G^a)-triangle property.

Proof. 1. Let us suppose first that $w = 1$, that is, there exists $C \ge 1$ such that, for all $x,y,z \in X$,

$$\min\bigl\{G(x,z), G(z,y)\bigr\} \le CG(x,y). \tag{8.3}$$

We define $\tilde G\colon X^a \times X^a \to [0,\infty]$ by

$$\tilde G(x,y) := G_{G^a,G^a}(x,y) = \frac{G(x,y)}{G^a(x)G^a(y)}\,.$$

Let us fix $x,y,z \in X^a$. We claim that $\min\{\tilde G(x,z), \tilde G(z,y)\} \le C^2\tilde G(x,y)$, that is,

$$\min\bigl\{G^a(y)G(x,z), G^a(x)G(z,y)\bigr\} \le C^2G^a(z)G(x,y). \tag{8.4}$$

By symmetry, we may assume that $G^a(x) \le G^a(y)$. If $G^a(y) \le CG^a(z)$, then

$$\begin{aligned}\min\bigl\{&G^a(y)G(x,z), G^a(x)G(z,y)\bigr\}\\ &\le CG^a(z)\min\bigl\{G(x,z), G(z,y)\bigr\} \le C^2G^a(z)G(x,y).\end{aligned}$$

So let us suppose $CG^a(z) < G^a(y)$. Since, by (8.3), $\min\bigl\{G^a(y), G(y,z)\bigr\} \le CG^a(z)$, we see that $G(y,z) \le CG^a(z)$. In addition, $G^a(x) = \min\bigl\{G^a(x), G^a(y)\bigr\} \le CG(x,y)$. Therefore

$$G^a(x)G(y,z) \le C^2G^a(z)G(x,y)$$

whence (8.4). Thus $G|_{X^a\times X^a}$ has the (G^a,G^a)-triangle property.

2. To reduce the general case to the special one, where $w = 1$, it suffices to note that $G_{w,w}$ is symmetric and that, for all $x,y \in X^a$,

$$\frac{G_{w,w}(x,y)}{G_{w.w}(x,a)G_{w,w}(y,a)} = w(a)^2\frac{G(x,y)}{G^a(x)G^a(y)}.$$

□

For a better understanding of the first corollary, let us recall that, given an inner product space $(V, \langle\cdot,\cdot\rangle)$,

$$\rho\colon (x,y) \mapsto \frac{\|x-y\|}{\|x\|\|y\|}$$

(where, of course, $\|z\| := \langle x,x\rangle^{1/2}$) is known to define a metric on $V \setminus \{0\}$, since $\rho(x,y) = \|\|x\|^{-2}x - \|y\|^{-2}y\|$ (see [19, Lemma A.1]).

If X is an arbitrary set and $\rho\colon X\times X \to \mathbb{R}^+$ is symmetric and vanishes on the diagonal, but nowhere else, then ρ is a quasi-metric if and only if ρ^{-1} has the triangle property (see e.g. [13, p. 646, Remark 2.1.2]). So Proposition 8.1 has an immediate consequence for quasi-metrics (and is more or less equivalent to it).

COROLLARY 8.2. *Let ρ be a quasi-metric on a set X. Let $a \in X$, $X^a := X\setminus\{a\}$, and*

$$\rho^a(x,y) := \frac{\rho(x,y)}{\rho(x,a)\rho(y,a)} \qquad (x,y \in X^a).$$

Then ρ^a is a quasi-metric on X^a.

Let us note that the following corollary has obvious analogues in the more general situations considered in [13, Section 9] and [12].

COROLLARY 8.3. *Let ν be any measure on B and $Lf := \int G_B(\cdot,y)f(y)\,d\nu(y)$, $f \in \mathcal{B}^+(B)$. Let us consider the following statements.*

(i) *There exists $a \in B$ and $c > 0$ such that $\nu(\{a\}) = 0$ and*

$$\sum_{n=0}^{\infty} L^n G_B(\cdot,a) \le cG_B(\cdot,a). \tag{8.5}$$

(ii) *There exists $c > 0$ such that, for every $s \in \mathcal{S}^+(B)$, $\sum_{n=0}^\infty L^n s \le cs$.*

(iii) *The function $\sum_{n=0}^\infty L^n 1$ is bounded.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *. If ν has compact support in B, then also* (iii) $\Rightarrow$ (ii).

Proof. Let $w := \min\{G_B(\cdot,0),1\}$. The function G_B has the (w,w)-triangle property (see e.g. [13, Proposition 9.3]). So, by Proposition 8.1, $G_B|_{B^a\times B^a}$ has the (G^a,G^a)-triangle property.

(i) $\Rightarrow$ (ii): For $f \in \mathcal{B}^+(B^a)$ and $x \in B^a$, let

$$L_a f(x) := \int_{B^a} G_{B^a}(x,y)f(y)\,d\nu(y) \quad \text{and} \quad \tilde{L}_a f(x) := (1/G_B^a(x))L_a(fG_B^a)(x).$$

By (8.5), we know that $\sum_{n=0}^\infty L_a^n G_B^a \le cG_B^a$, and hence $\sum_{n=0}^\infty \tilde{L}_a^n 1 \le c$. So, by Lemma 2.5, $\rho(\tilde{L}_a) < 1$. Therefore by [13, Proposition 2.3 and Corollary 3.3], we infer that there exists $C > 0$ such that, for every $s \in \mathcal{S}^+(B^a)$, $\sum_{n=0}^\infty L_a^n s \le Cs$. Since $\nu(\{a\}) = 0$ and $G_B|_{B^a\times B^a} = G_{B^a}$, (iii) follows.

(ii) $\Rightarrow$ (i),(iii): Trivial, since $G_B(\cdot,a), 1 \in \mathcal{S}^+(B)$.

Finally, let us suppose that ν is supported by a compact set A in B and that $\sum_{n=0}^\infty L^n 1$ is bounded. Since $\inf w(A) > 0$ and $\sup w(A) < \infty$, we know that $G_B|_{A\times A}$ has the triangle property. Thus, by [11, Proposition 3.10], (ii) follows. ☐

Finally, let us note another consequence of Proposition 8.1 (where we shall not try to achieve the utmost generality).

COROLLARY 8.4. *Let G be a symmetric Green function for a connected Brelot space $(X,\mathcal{H})$ and $a \in X$ such that the following holds.*

(i) *$G(a,a)=\infty$ and $\limsup_{x\to\infty} G(x,a)<\infty$.*

(ii) *G has the local triangle property.*

(iii) *G has the (w,w)-triangle property for some function $w\colon X\to(0,\infty)$.*

Let $g:=\min\{G^a,1\}$. Then G has the (g,g)-triangle property.

Proof. By (i), there exist a relatively compact open neighborhood V of a and $M\ge 1$ such that $G^a\ge 1$ on $\overline{V}$, $G^a\le M$ on V^c. Then $g=1$ on $\overline{V}$ and $G^a\le Mg$ on V^c. Let L be a compact neighborhood of $\overline{V}$ and W a relatively compact open neighborhood of L. We may assume without loss of generality that $g\ge 1/M$ on W. By (ii), (iii), and Proposition 8.1, there exists $C\ge 1$ such that, for all $x,y,z\in W$,

$$\min\{G(x,z),G(z,y)\}\le CG(x,y), \tag{8.6}$$

and, for all $x,y,z\in X^a$,

$$\min\{G^a(y)G(x,z),G^a(x)G(z,y)\}\le CG^a(z)G(x,y). \tag{8.7}$$

Moreover, there exists $c\ge 1$ such that

$$h(z)\le ch(\tilde z), \tag{8.8}$$

whenever $h\ge 0$ is a harmonic function on W or on the interior of L and $z,\tilde z\in L$, $z,\tilde z\in\overline{V}$, respectively.

We claim that, for all $x,y,z\in X$,

$$\min\{g(y)G(x,z),g(x)G(z,y)\}\le McCg(z)G(x,y). \tag{8.9}$$

If $x,y,z\in W$, then (8.9) follows from (8.6), since $g\le 1\le Mg(z)$. So we may assume that $W\ne X$.

Suppose next that $z\in V^c$. If $x,y\in X^a$, then, by (8.7),

$$\min\{g(y)G(x,z),g(x)G(z,y)\}\le MCg(z)G(x,y), \tag{8.10}$$

since $g\le G^a$ and $G^a(z)\le MCg(z)$. Then, by continuity, (8.10) holds as well, if $x\ne a$, but $y=a$. Analogously, if $y=a$ and $x\ne a$. If $x=y=a$, then (8.10) holds trivially.

So we may and shall assume from now on that $z\in V$ and $(x,y)\notin W\times W$. If $x\in L$ and $y\notin W$, then we may apply (8.8) and obtain that $g(x)G(z,y)\le cG(x,y)\le Mcg(z)G(x,y)$. Analogously if $y\in L$ and $x\notin W$. Hence (8.9) holds in these two cases.

Therefore it remains to consider the case, where $x,y\in L^c$. Fixing any point $\tilde z\in\partial V\subset V^c$ we know, by (8.10), that

$$\min\{g(y)G(x,\tilde z),g(x)G(\tilde z,y)\}\le MCg(\tilde z)G(x,y).$$

By (8.8), $G(x,z)\le cG(x,\tilde z)$, $G(y,z)\le cG(y,\tilde z)$. Since $g(\tilde z)=1=g(z)$, (8.9) holds as well in our last case. □

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