Indecomposables live in all smaller lengths.

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Abstract. Let Λ be a finite-dimensional k-algebra with k algebraically closed. Bongartz has recently shown that the existence of an indecomposable Λ-module of length $n>1$ implies that also indecomposable Λ-modules of length $n-1$ exist. Using a slight modification of his arguments, we strengthen the assertion as follows: If there is an indecomposable module of length n, then there is also an accessible one. Here, the accessible modules are defined inductively, as follows: First, the simple modules are accessible. Second, a module of length $n\geq 2$ is accessible provided it is indecomposable and there is a submodule or a factor module of length $n-1$ which is accessible.

Let k be an algebraically closed field. Let Λ be a finite-dimensional k-algebra, we may (and will) assume that Λ is basic. We are interested in (finite-dimensional left) Λ-modules. A recent preprint [B3] of Bongartz with the same title is devoted to a proof of the following important result:

Theorem (Bongartz 2009). *Let Λ be a finite-dimensional k-algebra with k algebraically closed. If there exists an indecomposable Λ-module of length $n > 1$, there there exists an indecomposable Λ-module of length $n - 1$.*

Unfortunately, the statement does not assert any relationship between the modules of length n and those of length $n - 1$. There is the following open problem: *Given an indecomposable Λ-module M of length $n \geq 2$. Is there an indecomposable submodule or factor module of length $n - 1$*? This is the case for Λ being representation-finite or tame concealed, as Bongartz [B1, B2] has shown already in 1984 and 1996, respectively. Two remarks should be added:

(1) It is definitely necessary to look both for submodules and factor modules, since for suitable algebras Λ, there are indecomposable modules M which have no maximal submodules which are indecomposable. Any local module of length at least 3 and Loewy length 2 is such an example. And dually, there are indecomposable modules M of length $n \geq 3$ such that all factor modules of length $n - 1$ are decomposable.

(2) In case we weaken the assumption on the base field k, then we may find counterexamples. For instance, let k be the field with 2 elements, Q the 3-subspace quiver (this is the quiver of type $\mathbb{D}_4$ with one sink and 3 sources) and M the (unique) indecomposable kQ-module of length 5. There is also only one indecomposable kQ-module of length 4. Now N cannot be a submodule of M, since we even have $\mathrm{Hom}(N, M) = 0$. But N is also not a factor module of M, since $\mathrm{Hom}(M, N)$ is a 2-dimensional k-space and the three non-zero elements in $\mathrm{Hom}(M, N)$ all have images of length 3.

The present note modifies slightly the arguments of Bongartz in [B3] in order to strengthen his assertion. We define inductively *accessible* modules: First, the simple modules are accessible. Second, a module of length $n \geq 2$ is accessible provided it is indecomposable and there is a submodule or a factor module of length $n - 1$ which is accessible. The

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problem mentioned above can be reformulated as follows: Are all indecomposable modules accessible?

Theorem. *Let Λ be a finite-dimensional k-algebra with k algebraically closed. If there is an indecomposable module of length n, then there is an accessible one of length n.*

As we have mentioned, for a representation-finite algebra all the indecomposable modules are accessible, thus we can assume that Λ is representation-infinite. According to Roiter's solution of the first Brauer-Thrall conjecture, a representation-infinite algebra has indecomposable modules of arbitrarily large length, thus we have to show that Λ has accessible modules of any length, and we can assume that Λ is minimal representation-infinite (this means that Λ is representation-infinite and that any proper factor algebra is representation-finite).

According to Bongartz [B3, section 3.2] we only have to consider algebras with non-distributive ideal lattice: Namely, if Λ is minimal representation-infinite and the ideal lattice of Λ is distributive, then the universal covering is interval-finite and the fundamental group is free. Using covering theory, the problem is reduced to representation-directed and to tame concealed algebras, but for both classes all the indecomposable modules are accessible.

It seems to be surprising that here we deal with a question not yet settled for algebras with non-distributive ideal lattice. After all, the class of algebras with non-distributive ideal lattice was the first major class of representation-infinite algebras studied in representation theory, see Jans [J], 1957.

Thus, let Λ be minimal representation-infinite and assume that the ideal lattice of Λ is non-distributive. Let J be the radical of Λ. Then there are (not necessarily different) primitive idempotents e, e' and linearly independent elements ϕ, ψ in eJe' such that $J\phi = J\psi = \phi J = \psi J = 0$.

Let $I(e)$ be the injective envelope of the simple module $\Lambda e/Je$. In $I(e)$, there are elements $x = e'x, y = e'y$ such that

$$\phi x = 0, \quad u := \psi x = \phi y \neq 0, \quad \psi y = 0.$$

Note that u is necessarily an element of the socle $\operatorname{soc} I(e)$. Let $X = \Lambda x$, $Y = \Lambda y$ and $V = X + Y$. Note that $\phi(X + JY) = 0$ as well as $\psi(JX + Y) = 0$.

We consider direct sums of copies $V_{(i)} = V$, say $V^n = \bigoplus_{i=1}^n V_{(i)}$. An element $v \in V$ will be denoted by $v_{(i)}$ when considered as an element of $V_{(i)} \subseteq V^n$; similarly, a submodule $U \subseteq V$ will be denoted by $U_{(i)}$ when considered as a submodule of $V_{(i)} \subseteq V^n$. For $1 \leq i < n$ let $z_i = y_{(i)} + x_{(i+1)}$.

The following three submodules of V^n (with $n \geq 1$) will be of interest:

$$\begin{aligned} M(n-1) &= \sum\nolimits_{i=1}^{n-1} \Lambda z_i, \quad \text{for} \quad n \geq 2, \quad \text{and} \quad M(0) = \Lambda u, \\ R(n) &= X_{(1)} + M(n-1), \\ W(n) &= R(n) + Y_{(n)}. \end{aligned}$$

We want to refine the inclusion $M(n-1) \subset W(n)$ by a chain of indecomposable submodules U_i, say

$$M(n-1) = U_0 \subset U_1 \subset \cdots \subset U_t = W(n),$$

such that U_i/U_{i-1} is simple for $1 \leq i \leq t$.

We call an inclusion of modules $N \subseteq M$ *uniform*, provided any submodule U with $N \subseteq U \subseteq M$ is indecomposable (this is related to the well-accepted notion of a uniform module: a module M is uniform provided it is non-zero and any inclusion $N \subset M$ with $N \neq 0$ is uniform).

Lemma. *The inclusions $M(n-1) \subseteq JX_{(1)} + M(n-1)$ and $R(n) \subseteq R(n) + JY_{(n)}$ are uniform, also the module $W(n)$ is indecomposable.*

Proof. Let $M(n-1) \subseteq U \subseteq JX_{(1)} + M(n-1)$ or $R(n) \subseteq U \subseteq R(n) + JY_{(n)}$, or $U = W(n)$. We have to show that U is indecomposable.

For $n = 1$ this is clear, since $W(1)$ is uniform and $M(0) \neq 0$. Thus, we can assume that $n \geq 2$. We will show: If $U = U' \oplus U''$, then the socle of U is contained in one of the summands, say in U', and therefore $U = U'$.

We claim that

$$(*) \qquad \operatorname{soc} U = \phi U + \psi U$$

First of all, since $U \subseteq V^n$, we see that $\operatorname{soc} U \subseteq \operatorname{soc} V^n = ku_{(1)} + \cdots + ku_{(n)}$. Second, U contains $M(n–1)$, thus the elements $z_1, \ldots, z_{n-1}$. Now ϕU contains the elements $u_{(i)} = \phi z_i$, for $1 \leq i \leq n-1$, whereas ψU contains the elements $u_{(i+1)} = \phi z_i$, for $1 \leq i \leq n-1$. As a consequence, we see that the elements $u_{(1)}, \ldots, u_{(n)}$ are contained in $\phi U + \psi U$ (since $n \geq 2$). Thirdly, $\phi U + \psi U \subseteq \operatorname{soc} U$, sincd $J\phi = 0 = J\psi$. Altogether, we see that

$$\operatorname{soc} U \subseteq ku_{(1)} + \cdots + ku_{(n)} \subseteq \phi U + \psi U \subseteq \operatorname{soc} U,$$

this completes the proof of $(*)$.

Recall that the Kronecker quiver is given by two vertices, say a and b, and two arrows $a \to b$, the representations of the Kronecker quiver are called Kronecker modules, and the classification of the indecomposable Kronecker modules is known (see for example [R]).

Define a functor F from the category of Λ-modules to the category of Kronecker modules, by attaching to a Λ-module M the Kronecker module $F(M) = (M, \operatorname{soc} M; \phi\cdot, \psi\cdot)$. Here again we use that the multiplication by ϕ and ψ maps M into $\operatorname{soc} M$.

The structure of $F(U)$ for our Λ-modules U is easy to determine:

(a) If $M(n-1) \subseteq U \subseteq JX_{(1)} + M(n-1)$, then $F(U)$ is the direct sum of the indecomposable preprojective Kronecker module of dimension $2n-1$ and several copies of the simple injective Kronecker module T.

(b) If $R(n) \subseteq U \subseteq R(n) + JY_{(n)}$, then $F(U)$ is the direct sum of an indecomposable regular module of dimension $2n$ and again several copies of T.

(c) Finally, if $U = W(n)$, then $F(U)$ is the direct sum of the indecomposable preinjective Kronecker module of dimension $2n+1$ and several copies of T.

In all cases we denote by G the indecomposable direct summand of $F(U)$ that is different from T.

Let us calculate the radical of such a Kronecker module $F(U)$. For any Λ-module M, the radical $\operatorname{rad} F(M)$ of $F(M)$ is equal to $(0, \phi M + \psi M; 0, 0)$, thus according to $(*)$,

$\operatorname{rad} F(U) = (0, \operatorname{soc} M; 0, 0)$. On the other hand, $\operatorname{rad} F(U)$ is just $\operatorname{rad} G$ (since the copies of T do not contribute to the radical of $F(U)$)

Now assume that U is decomposable, say $U = U' \oplus U''$. Since F is a functor, $F(U) = F(U') \oplus F(U'')$. We know that $F(U)$ is the direct sum of an indecomposable Kronecker module G with socle of dimension n and several copies of T. According to the Krull-Remak-Schmidt theorem, one of the summands, say $F(U')$ has to contain a direct summand G' isomorphic to G, but then $\operatorname{rad} G = \operatorname{rad} G' \subset F(U')$. This shows that $\operatorname{soc} U \subseteq U'$, and therefore $U = U'$. This concludes the proof of the Lemma.

Note that $R(n)/(M(n-1) + JX_1)$ and $W(n)/(R(n) + JY_n)$ are simple modules. Thus, maximal refinements of the inclusions

$$M(n-1) \subseteq M(n-1) + JX_1 \quad \text{and} \quad R(n) \subseteq R(n) + JY_n$$

yield a submodule chain $M(n-1) = U_0 \subset U_1 \subset \cdots \subset U_t = W(n)$, such that all the factors U_i/U_{i-1} are simple for $1 \leq i \leq t$ and all the modules U_i for $0 \leq i \leq t$ are indecomposable. In particular, we see: *if $M(n-1)$ is accessible, then also $W(n)$ is accessible.*

Using the dual arguments, we similarly see: *if $W(n-1)$ is accessible, then also $M(n)$ is accessible.* But $M(0)$ is a simple module, thus accessible. It follows by induction that all the modules $M(n)$ and $W(n)$ are accessible. This completes the proof of the theorem.

Remark. Note that in general the inclusion $M(n–1) \subset W(n)$ is not uniform. Consider for Λ the Kronecker algebra kQ itself, and look at the submodules N, N' of $W(2)$ generated by the elements $z = x_{(1)} + y_{(1)} + x_{(2)} + y_{(2)}$ and $z' = x_{(1)} - y_{(1)} - x_{(2)} + y_{(2)}$, respectively. We have $\dim N = \dim N' = 2$. Assume now that the characteristic of k is different from 2. Then $N \neq N'$ and even $N \cap N' = 0$. Thus $N \oplus N'$ is a decomposable submodule of $W(2)$. Also, $M(1)$ is contained in $N \oplus N'$ (as the submodule generated by $\frac{1}{2}(z - z')$).

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