3G-inequality for planar domains

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Abstract

The 3G-inequality for Green functions g_D on arbitrary bounded domains in $\mathbb{R}^2$, which R.F. Bass and K. Burdzy obtained in 1995 by a genuinely probabilistic proof (using loops of Brownian motion around the origin), is proven (in a more precise form) employing elementary properties of harmonic measures only. Since harmonic measures are hitting distributions of Brownian motion, this purely analytic proof can be viewed as well as being probabilistic. A spin-off is an upper estimate of g_D on subdisks B' of an open disk B in terms of g_B divided by the capacity of $B' \setminus D$ with respect to B.

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1 Introduction and main results

The main result of [3] is the following 3G-inequality for Green functions g_D on domains D in $\mathbb{R}^2$ (which implies that the conditional gauge theorem holds for every bounded domain in $\mathbb{R}^2$; see [3, Theorem 1.2]).

THEOREM 1.1. *Suppose D is a bounded domain in $\mathbb{R}^2$. Then there exists a constant c_1 such that*

$$\frac{g_D(x,y)g_D(y,z)}{g_D(x,z)} \le c_1\big(1+\log^+(1/|x-y|)+\log^+(1/|y-z|)\big).$$

Here $\log^+ x = 0 \vee \log x$ and c_1 depends only on the diameter of D.

An essential part of the proof given in [3] is genuinely probabilistic (see [3, Proposition 2.4] providing a lower estimate for the probability that Brownian motion $X(t)$ starting at a point x of an annulus A centered at y makes a loop around y before exiting $A \cap D$). The purpose of this paper is a proof of the 3G-inequality which, without being more complicated, instead uses the following elementary properties of harmonic measures μ_x^U for points x in open sets U:

(i) μ_x^U is supported by the boundary of the connected component of U which contains x.

(ii) If V is an open set such that $x \in V \subset U$, then

$$\mu_x^U = \mu_x^V|_{\partial U} + \int_U \mu_y^U \, d\mu_x^V(y). \tag{1.1}$$

Let $\mu_x^U := \varepsilon_x$ (Dirac measure at x), if $x \in U^c$. Then (1.1) can be rewritten as

$$\mu_x^U = \int \mu_y^U \, d\mu_x^V(y). \tag{1.2}$$

It is well known that μ_x^U is the distribution of the first hitting of U^c by Brownian motion $X(t)$ starting at x, that is, defining

$$\tau_U := \inf\{t \ge 0\colon X(t) \in U^c\},$$

we have, for every Borel set E,

$$\mu_x^U(E) = \mathbb{P}^x(X(\tau_U) \in E).$$

From a probabilistic point of view, (i) reflects the continuity of Brownian trajectories and (1.2) is an immediate consequence of the the strong Markov property (and the trivial relation $\tau_U = \tau_V + \tau_U \circ \theta_{\tau_V}$). Thus the approach presented in this paper can be viewed both as being purely analytic and as being probabilistic.

We recall that a domain D in $\mathbb{R}^2$ is a *Greenian* domain, if it admits a Green function g_D or – equivalently – if $\mathbb{R}^2 \setminus D$ is non-polar (see [2, Theorem 5.3.8]). It will be convenient to extend g_D to $\mathbb{R}^2$ taking the value 0 on $(\mathbb{R}^2 \times \mathbb{R}^2) \setminus (D \times D)$. For $y \in \mathbb{R}^2$ and $r > 0$, let

$$B(y,r) := \{x \in \mathbb{R}^2\colon |x-y| < r\} \quad \text{and} \quad S(y,r) := \partial B(y,r),$$

$$B_r := B(0,r) \quad \text{and} \quad S_r := S(0,r).$$

As in [3] the capacity of parts of D^c with respect to open disks will play an important role. If E is a compact set in a relatively compact open set U in $\mathbb{R}^2$, the *capacitary or equilibrium potential* q_E of E with respect to U is the potential of a measure μ (called *capacitary or equilibrium measure*), that is, $q_E = G_U\mu := \int g_U(\cdot, y)\, d\mu(y)$, and $\mathrm{cap}_U(E)$ is the total mass $\|\mu\|$ of μ (for every $x \in U$, $q_E(x) := \liminf_{y\to x} {}^U R_1^E(y)$, where ${}^U R_1^E$ is the infimum of all positive superharmonic functions on U which are (at least) 1 on E; for details see [5] or [2]).

The following estimate for g_D could easily be formulated in a way which does not require any a priori knowledge on the existence of g_D, but yields its existence as a consequence.

PROPOSITION 1.2. *Let D be a Greenian domain in $\mathbb{R}^2$, $e_1 := (1,0)$, $y \in D$, $0 < r < R$, and $B := B(y,R)$. Then*

$$\sup_{|x-y|=r} g_D(x,y) \le \frac{1}{\mathrm{cap}_B(D^c \cap \overline{B}(y,r))\, g_{B(0,R)}(re_1, -re_1)} \sup_{|x-y|=r} g_{D\cap B}(x,y) \tag{1.3}$$

(*where* $\sup_{|x-y|=r} g_{D\cap B}(x,y) \le g_B(x,y)$, *for every* $x \in S_r$).

Using (1.3) and results of Section 3, we shall obtain the following (see Section 4).

THEOREM 1.3. *There exist $c, \delta_0 \in (0,\infty)$ such that, for all Greenian domains D in $\mathbb{R}^2$, distinct points $x, y, z \in D$, and $r > 0$,*

$$\frac{g_D(x,y)g_D(y,z)}{g_D(x,z)} \le c\, \frac{1 + \log^+\bigl(18\, r/(|x-y| \wedge |y-z|)\bigr)}{\delta_0 \wedge \mathrm{cap}_{B(y,6r)}(D^c \cap \overline{B}(y,4r))}. \tag{1.4}$$

If r is sufficiently large, $D^c \cap \overline{B}(y,4r)$ is non-polar, $\mathrm{cap}_{B(y,6r)}(D^c \cap \overline{B}(y,4r)) > 0$, and the right side of (1.4) is finite. Moreover, if D is bounded and M is the diameter $\mathrm{diam}(D)$ of D, then $D \subset B(y,M)$, for every $y \in D$, and hence

$$\mathrm{cap}_{B(y,6M)}(D^c \cap \overline{B}(y,4M)) \ge \mathrm{cap}_{B(y,6M)}(\overline{B}(y,4M) \setminus B(y,M)) = \mathrm{cap}_{B_6}(\overline{B}_4 \setminus B_1).$$

Thus Theorem 1.3 implies the following strengthening of Theorem 1.1.

COROLLARY 1.4. 1. *For every $\delta > 0$, there exists $c > 0$ such that the following holds. If D is a Greenian domain in $\mathbb{R}^2$ and $r\colon D \to (0,\infty)$ such that*

$$\mathrm{cap}_{B(y,6r(y))}(D^c \cap \overline{B}(y,4r(y))) \ge \delta, \qquad y \in D,$$

then, for all distinct points $x,y,z \in D$,

$$\frac{g_D(x,y)g_D(y,z)}{g_D(x,z)} \le c\left(1 + \log^+ \frac{18\,r(y)}{|x-y| \wedge |y-z|}\right).$$

2. *In particular, there exists $c > 0$ such that, for every bounded domain D in $\mathbb{R}^2$ and all distinct points $x,y,z \in D$,*

$$\frac{g_D(x,y)g_D(y,z)}{g_D(x,z)} \le c\left(1 + \log \frac{\mathrm{diam}(D)}{|x-y| \wedge |y-z|}\right).$$

Finally, let us recall that the 3G-inequality may fail for arbitrary bounded domains in $\mathbb{R}^d$, $d \ge 3$. See [4] for simple counterexamples. If, however, D is a uniformly John domain, then g_D satisfies the 3G-inequality (see [1]).

2 Proof of Proposition 1.2

By translation invariance, we may assume that $y = 0$. Let $a := g_B(re_1, -re_1)$, $E := D^c \cap \overline{B}_r$ such that $\mathrm{cap}_B(E) > 0$, $\alpha := a\,\mathrm{cap}_B(E)$, $\tilde{R} > R$, and $\tilde{B} := B_{\tilde{R}}$. To prove (1.3) it suffices to show that

$$\sup_{|x|=r} g_{D\cap\tilde{B}}(x,0) \le \alpha^{-1} \sup_{|x|=r} g_{D\cap B}(x,0). \tag{2.1}$$

Let $S = S_R$ and let ν denote the capacitary measure of E with respect to B. Then, for every $z \in S_r \cap D$, $\mu_z^{B\setminus E}(E) = G_B\nu(z) \ge a\|\nu\| = \alpha$, and hence

$$\mu_z^{B\setminus E}(S) = 1 - \mu_z^{B\setminus E}(E) \le 1 - \alpha. \tag{2.2}$$

Let $\eta \in (0,1)$. There exists $0 < \delta < r$ such that $\overline{B}_\delta \subset D$ and, for every $z \in S_\delta$,

$$g_{D\cap B}(z,0) \ge (1-\eta)\log(1/\delta) \quad \text{and} \quad g_{D\cap\tilde{B}}(z,0) \le (1+\eta)\log(1/\delta).$$

Let $U := (D \cap B) \setminus \overline{B}_\delta$ and $\tilde{U} := (D \cap \tilde{B}) \setminus \overline{B}_\delta$. Then, for every $x \in S_r \cap D$,

$$\begin{aligned} g_{D\cap B}(x,0) = \mu_x^U(g_{D\cap B}(\cdot,0)) &\ge (1-\eta)\log(1/\delta)\mu_x^U(S_\delta), \\ g_{D\cap\tilde{B}}(x,0) = \mu_x^{\tilde{U}}(g_{D\cap\tilde{B}}(\cdot,0)) &\le (1+\eta)\log(1/\delta)\mu_x^{\tilde{U}}(S_\delta). \end{aligned}$$

To obtain (2.1) it hence suffices to prove that

$$\sup_{x\in S_r\cap D} \mu_x^{\tilde U}(S_\delta) \le \alpha^{-1} \sup_{x\in S_r\cap D} \mu_x^U(S_\delta). \tag{2.3}$$

For the moment, let us fix $x \in S_r \cap D$. Then, by (1.1),

$$\mu_x^{\tilde U}(S_\delta) = \mu_x^U(S_\delta) + \int_{S\cap D} \mu_w^{\tilde U}(S_\delta)\, d\mu_x^U(w), \tag{2.4}$$

where, by (1.1) and (2.2), $\mu_x^U(S\cap D) \le \mu_x^{B\setminus E}(S) \le 1-\alpha$. Let $V := D \cap A(0, r, \tilde R)$. Then, by (1.1), for every $w \in S \cap D$,

$$\mu_w^{\tilde U}(S_\delta) = \int_{S_r\cap D} \mu_z^{\tilde U}(S_\delta)\, d\mu_w^V(z) \le \sup_{z\in S_r\cap D} \mu_z^{\tilde U}(S_\delta).$$

Hence, by (2.4),

$$\mu_x^{\tilde U}(S_\delta) \le \mu_x^U(S_\delta) + (1-\alpha) \sup_{z\in S_r\cap D} \mu_z^{\tilde U}(S_\delta).$$

Taking the supremum on all $x \in S_r \cap D$, we finally see that (2.3) holds.

REMARK 2.1. As the proof shows, (1.3) holds as well for any domain D in $\mathbb{R}^d$, $d \ge 3$ (it suffices to replace $\log(1/\delta)$ by δ^{d-2}).

3 Further preparations

Let us recall the following. If U, V are Greenian domains in $\mathbb{R}^2$ such that $V \subset U$ and E is a compact set in V, then

$$\operatorname{cap}_U(E) \le \operatorname{cap}_V(E). \tag{3.1}$$

Indeed, let ν be the capacitary measure for E with respect to U. Since $g_V \le g_U$, we see that $G_V\nu = \int g_V(\cdot, y)\, d\nu(y) \le \int g_U(\cdot, y)\, d\nu(y) = G_U\nu \le 1$ on V. Therefore, by [5, Lemma 7.19], $\operatorname{cap}_V(E) \ge \|\nu\| = \operatorname{cap}_U(E)$.

The following modification of [3, Proposition 2.1] is suitable for our purposes.

LEMMA 3.1. *Let D be a Greenian domain in $\mathbb{R}^2$, $y \in D$, $\eta > 0$, and $r_0 > 0$ such that $\operatorname{cap}_{B(y,6r_0)}(D^c \cap \overline{B}(y, 4r_0)) \ge \eta$. Then there exists $r \in (0, r_0)$ such that $\operatorname{cap}_{B(y,6r)}(D^c \cap \overline{B}(y, 5r)) \ge \eta$ and, for all $0 < s \le r$,*

$$\operatorname{cap}_{B(y,6s)}(D^c \cap \overline{B}(y, 4s)) < \eta.\text{[1]}$$

Proof. We may assume without loss of generality that $y = 0$. For every $R > 0$, let $\operatorname{cap}_R := \operatorname{cap}_{B(0,R)}$. We define

$$s_0 := \inf\{s > 0\colon\ \operatorname{cap}_{6s}(D^c \cap \overline{B}_{4s}) \ge \eta\}, \quad q := \sqrt{5/4}, \quad r := s_0/q.$$

Since D is open, we know that $D^c \cap \overline{B}_{4s} = \emptyset$, if s is sufficiently small. Therefore $0 < r < s_0 \le r_0$. Of course, $\operatorname{cap}_{6s}(D^c \cap \overline{B}_{4s}) < \eta$, if $0 < s \le r$. Moreover, there exists $t \in [s_0, qs_0]$ such that $\operatorname{cap}_{6t}(D^c \cap \overline{B}_{4t}) \ge \eta$. Then $4t \le 5r$ and hence, by (3.1), $\operatorname{cap}_{6r}(D^c \cap \overline{B}_{5r}) \ge \operatorname{cap}_{6t}(D^c \cap \overline{B}_{4t}) \ge \eta$. □

[1] As the proof will show, the natural numbers $4, 5, 6$ may be replaced by any $a, b, c \in (0, \infty)$ such that $a < b < c$.

PROPOSITION 3.2. *Given* $\eta \in (0,1)$ *and* $0 < \alpha < \beta < 1$*, there exists* $\delta \in (0,1)$ *such that the following holds. If* $x \in \mathbb{R}^2$*,* $R > 0$*, and* $G_{B(x,R)}\nu$ *is the equilibrium potential of a compact set* E *in* $B(x,R)$ *satisfying* $\mathrm{cap}_{B(x,R)}(E) < \delta$*, then there exists* $t \in [\alpha R, \beta R]$ *such that*

$$G_{B(x,R)}\nu \le \eta \quad \text{on } S(x,t). \tag{3.2}$$

Proof. (Cf. the proof of [3, Proposition 2.3].) By translation and scaling invariance, it suffices to consider the case $x = 0$ and $R = 1$. Let $B := B_1$ and let σ_0 denote linear measure on $I := [\alpha, \beta] \times \{0\}$. Then $c_0 := \sup\{G_B\sigma_0(z) \colon z \in B\} < \infty$. Let

$$\delta := \frac{\eta(\beta - \alpha)}{c_0}.$$

We define $r(x) := (|x|, 0)$, $x \in \mathbb{R}^2$, and fix a compact E in B such that $\mathrm{cap}_B(E) < \delta$. Let $q = G_B\nu$ be the equilibrium potential of E with respect to B, and let us suppose that, for every $t \in [\alpha, \beta]$, (3.2) does not hold.

Then there exist $z_t \in S_t$, $t \in [\alpha, \beta]$, such that $q(z_t) > \eta$, and hence, by lower semicontinuity, $q > \eta$ on some open radial line segment L_t containing z_t. The open line segments $r(L_t)$, $\alpha \le t \le \beta$, cover I. By compactness and taking differences, we obtain Borel sets $L_1', \cdots, L_m'$, each contained in some L_t, such that the sets $I_j := r(L_j')$, $1 \le j \le m$, form a partition of I. Let σ denote linear measure on $L := L_1' \cup \cdots \cup L_m'$. Of course, $r(\sigma) = \sigma_0$. For all $y, z \in B$, $g_B(y,z) \le g_B(r(y), r(z))$. Therefore

$$\begin{aligned} & \eta(\beta - \alpha) < \int q(z)\,\sigma(dz) = \int \Big(\int g_B(y,z)\,d\sigma(z)\Big)\,d\nu(y) \\ \le\ & \int \Big(\int g_B(r(y), r(z))\,d\sigma(z)\Big)\,d\nu(y) = \int \Big(\int g_B(y,z)\,d\sigma_0(z)\Big)\,dr(\nu)(y) \\ \le\ & c_0\|r(\nu)\| = c_0\|\nu\| < c_0\delta, \end{aligned}$$

which is impossible by our choice of δ. This contradiction finishes the proof. □

The next proposition will be a good substitute for the crucial [3, Proposition 2.4] which provided a lower estimate for the probability that, given $x, y \in D$, $x \ne y$, paths of Brownian motion starting at x contain a loop within an annulus centered at y before exiting D. We define

$$A(y,s,t) := \{x \in \mathbb{R}^2 \colon s < |x - y| < t\}, \qquad y \in \mathbb{R}^d,\ 0 < s < t < \infty,$$

and first prove a simple lemma.

LEMMA 3.3. *Let* $A := A(0,1,4)$ *and* $K := \overline{A(0,2,3)}$*. There exists* $\eta_0 \in (0, 1/2)$ *such that, for every* $x \in K$ *and every polygonal arc* C *in* $\mathbb{R}^2 \setminus \{x\}$ *which intersects both* S_2 *and* S_3*,*

$$\mu_x^{A \setminus C}(C) \ge 2\eta_0. \tag{3.3}$$

Proof. Let $B := B_4$ and $I := [2,3] \times \{0\}$. Then

$$\alpha := \inf\{\mu_x^{B\setminus I}(I) \colon x \in -I\} \in (0,1).$$

Moreover, there exists $\beta \in (0,1)$ such that $\beta g_B \le g_A$ on $K \times K$. We define

$$\eta_0 := \alpha\beta/2.$$

By invariance under rotation, it is sufficient to consider $x \in -I$. Let C be a polygonal arc in $\mathbb{R}^2 \setminus \{x\}$ which intersects both S_2 and S_3. Clearly, C contains an arc $\tilde{C}$ in K which connects a point in S_2 with a point in S_3. Let σ denote the capacitary measure of $\tilde{C}$ with respect to A. Then $G_A^\sigma = 1$ on $\tilde{C}$, since $\tilde{C}$ is not thin at any of its points. As in the proof of Proposition 3.2, let $r(z) := (|z|, 0)$, $z \in \mathbb{R}^2$.

For the moment, let us fix $z_0 \in I$. By connectedness, $r(\tilde{C}) = I$. So there exists $z \in \tilde{C}$ such that $r(z) = z_0$. Since $g_B(r(z), r(z')) \ge g_B(z, z')$, $z' \in B$, we see that $\sigma_0 := r(\sigma)$ satisfies

$$\begin{aligned} G_B^{\sigma_0}(z_0) &= \int g_B(r(z), z')\, d\sigma_0(z') = \int g_B(r(z), r(z'))\, d\sigma(z') \\ &\ge \int g_B(z, z')\, d\sigma(z') \ge \int g_A(z, z')\, d\sigma(z') = G_A^\sigma(z) = 1. \end{aligned}$$

Thus $G_B^{\sigma_0} \ge 1$ on I. This implies that $G_B^{\sigma_0}(x) \ge \mu_x^{B\setminus I}(I) \ge \alpha$.

Moreover, for every $z \in B$, $|x - z| \le |x - r(z)|$, and hence $g_B(x, z) \ge g_B(x, r(z))$. Therefore, using (1.1) for the first inequality,

$$\mu_x^{A\setminus C}(C) \ge \mu_x^{A\setminus \tilde{C}}(\tilde{C}) = G_A^\sigma(x) \ge \beta G_B^\sigma(x) \ge \beta G_B^{\sigma_0}(x) \ge \alpha\beta = 2\eta_0.$$

□

We now fix $\delta_0 \in (0,1)$ according to Proposition 3.2, where we take $\alpha = 1/36$, $\beta = 1/18$, and $\eta = \eta_0$.

PROPOSITION 3.4. *Let $D \subset \mathbb{R}^2$ be a Greenian domain, $y \in D$, $r > 0$, $A := A(y, r, 4r)$, and $x \in D \cap S(y, 5r/2)$. Let C be a polygonal arc in $\mathbb{R}^2 \setminus \{x\}$ which intersects both $S(y, 2r)$ and $S(y, 3r)$, and suppose that $\mathrm{cap}_{B(y,6r)}(D^c \cap \overline{A}) < \delta_0$. Then*

$$\mu_x^{(D\cap A)\setminus C}(C) \ge \eta_0\, \mu_x^{D\cap A}(D). \tag{3.4}$$

In probabilistic terms, (3.4) states that, for $T_C := \inf\{s \ge 0 \colon X(s) \in C\}$,

$$\mathbb{P}^x(T_C \le \tau_{D\cap A}) \ge \eta_0\, \mathbb{P}^x(\tau_A < \tau_D).$$

Proof of Proposition 3.4. By invariance under translation and scaling, it suffices to consider the case, where $y = 0$ and $r = 1$.

Let $E := D^c \cap \overline{A}$. Since $B_6 \subset B(x, 9)$, we see, by (3.1), that $\mathrm{cap}_{B(x,9)}(E) \le \mathrm{cap}_{B_6}(E) < \delta_0$. Let ν be the capacitary measure for E with respect to $B(x, 9)$. By Proposition 3.2, there exists $t \in [1/4, 1/2]$ such that $G_{B(x,9)}\nu \le \eta_0$ on $S(x, t)$.

We temporarily fix $z \in (S(x,t) \cap D) \setminus C$. Clearly, $V := (D \cap A) \setminus C \subset B(x,9) \setminus E$, and hence

$$\mu_z^V(E) \le \mu_z^{B(x,9)\setminus E}(E) = G_{B(x,9)}\nu(z) \le \eta_0.$$

By Lemma 3.3, $\mu_z^{A\setminus C}(C) \ge 2\eta_0$. Since $V \subset A \setminus C$ and $(A \setminus C) \cap \partial V \subset E$, we therefore conclude, by (1.1), that

$$\begin{aligned} \mu_z^V(C) &= \mu_z^{A\setminus C}(C) - \int_{A\setminus C} \mu_{z'}^{A\setminus C}(C)\, d\mu_z^V(z') \\ &\ge \mu_z^{A\setminus C}(C) - \mu_z^V(E) \ge 2\eta_0 - \eta_0 = \eta_0. \end{aligned}$$

If $z \in C$, then $z \in V^c$, $\mu_z^V(C) = 1 > \eta_0$. Defining $W := (D \cap B(x,t)) \setminus C$, we hence know that $\mu_z^V(C) \ge \eta_0$, for every $z \in F := D \cap \partial W$. So, by (1.2),

$$\mu_x^V(C) = \int \mu_z^V(C)\, d\mu_x^W(z) \ge \eta_0 \mu_x^W(F),$$

where $\mu_x^W(F) = 1 - \mu_x^W(\partial D) \ge 1 - \mu_x^{D\cap A}(\partial D) = \mu_x^{D\cap A}(D)$. Thus (3.4) holds. $\square$

4 Proof of Theorem 1.3

Let us recall that we introduced a constant $\eta_0 \in (0, 1/2)$ in Lemma 3.3 and a constant $\delta_0 \in (0,1)$ before formulating Proposition 3.4. We now define

$$a := g_{B_6}(5e_1, -5e_1)^{-1}, \qquad c := 4a/\eta_0.$$

Let D be a Greenian domain in $\mathbb{R}^2$, $g := g_D$, and let x, y, z be distinct points in D. Since g is symmetric, we may assume without loss of generality that $|x-y| \le |y-z|$. Suppose that $r > 0$ such that $\mathrm{cap}_{B(y,6r)}(D^c \cap \overline{B}(y,4r)) > 0$. Let

$$\delta := \delta_0 \wedge \mathrm{cap}_{B(y,6r)}(D^c \cap \overline{B}(y,4r)).$$

By Lemma 3.1, there exists $s \in (0,r)$ such that, for all $0 < t \le s$,

$$\mathrm{cap}_{B(y,6t)}(D^c \cap \overline{B}(y,4t)) < \delta, \quad \text{whereas} \quad \mathrm{cap}_{B(y,6s)}(D^c \cap \overline{B}(y,5s)) \ge \delta. \tag{4.1}$$

Let

$$\gamma := \log^+ \frac{18\, r}{|x-y|} \quad \text{and} \quad t := s \wedge \frac{|x-y|}{3}.$$

To prove (1.4), it suffices to show, by the minimum principle, that

$$g(\cdot,y) g(y,z) \le c\, \frac{1+\gamma}{\delta}\, g(\cdot,z) \quad \text{on } D \cap S(y, 5t/2). \tag{4.2}$$

So let us fix $w \in D \cap S(y, 5t/2)$ and define

$$A := A(y,t,4t) \quad \text{and} \quad \beta := \mu_w^{D\cap A}(D).$$

Further, let $\lambda := g(y,z)/2$. The points y, z are contained in the same connected component of the open set $\{g(\cdot,y) > \lambda\}$. So there exists a polygonal arc C connecting y and z in D such that

$$g(\cdot,z) > \lambda \quad \text{on } C. \tag{4.3}$$

Since $3t \le |x-y| \le |y-z|$, the arc C intersects both $S(y,2t)$ and $S(y,3t)$. Since $g(\cdot,z)$ is superharmonic on D, we hence obtain, by (4.3), (4.1), and Proposition 3.4, that

$$g(w,z) \ge \mu_w^{(D\cap A)\setminus C}(g(\cdot,z)) \ge \lambda \mu_w^{(D\cap A)\setminus C}(C) \ge \lambda\eta_0\beta. \tag{4.4}$$

Moreover, $g(\cdot,y)$ is a potential on D, harmonic on $D\setminus\{y\}$, and $y\notin A$. Hence

$$g(w,y) = \mu_w^{D\cap A}(g(\cdot,y)). \tag{4.5}$$

We suppose first that $t=s$. Since $g_{B_6}(e_1,0)=\ln 6 \le 2$, we see, by Proposition 1.2, that $g(\cdot,y)\le 2a/\delta$ on $D\cap\partial A$. So $g(w,y)\le 2a\beta/\delta$, by (4.5), and hence, by (4.4),

$$\frac{g(w,y)g(y,z)}{g(w,z)} \le \frac{4a\beta\lambda}{\delta\lambda\eta_0\beta} = \frac{c}{\delta}\,.$$

Finally, we assume that $t=|x-y|/3<s$. By Proposition 1.2,

$$g(\cdot,y)\le (a/\delta)g_{B(y,6s)}(\cdot,y)\le (a/\delta)\gamma \quad \text{on } D\cap\partial A.$$

So, by (4.5), $g(w,y)\le (a/\delta)\gamma\beta$, and therefore

$$\frac{g(w,y)g(y,z)}{g(w,z)} \le \frac{2a\gamma\beta\lambda}{\delta\lambda\eta_0\beta} = \frac{c\gamma}{2\delta}\,.$$

Thus (4.2) holds, and the proof is finished.

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