

# Asymptotic distribution of singular values of powers of random matrices

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**Abstract:** Let  $x$  be a complex random variable such that  $\mathbf{E}x = 0$ ,  $\mathbf{E}|x|^2 = 1$ ,  $\mathbf{E}|x|^4 < \infty$ . Let  $x_{ij}$ ,  $i, j \in \{1, 2, \dots\}$  be independent copies of  $x$ . Let  $\mathbf{X} = (N^{-1/2}x_{ij})$ ,  $1 \leq i, j \leq N$  be a random matrix. Writing  $\mathbf{X}^*$  for the adjoint matrix of  $\mathbf{X}$ , consider the product  $\mathbf{X}^m \mathbf{X}^{*m}$  with some  $m \in \{1, 2, \dots\}$ . The matrix  $\mathbf{X}^m \mathbf{X}^{*m}$  is Hermitian positive semi-definite. Let  $\lambda_1, \lambda_2, \dots, \lambda_N$  be eigenvalues of  $\mathbf{X}^m \mathbf{X}^{*m}$  (or squared singular values of the matrix  $\mathbf{X}^m$ ). In this paper we find the asymptotic distribution function

$$G^{(m)}(x) = \lim_{N \rightarrow \infty} \mathbf{E} F_N^{(m)}(x)$$

of the empirical distribution function

$$F_N^{(m)}(x) = N^{-1} \sum_{k=1}^N \mathbb{I}\{\lambda_k \leq x\},$$

where  $\mathbb{I}\{A\}$  stands for the indicator function of event  $A$ . The moments of  $G^{(m)}$  satisfy

$$M_p^{(m)} = \int_{\mathbb{R}} x^p dG^{(m)}(x) = \frac{1}{mp+1} \binom{mp+p}{p}.$$

In Free Probability Theory  $M_p^{(m)}$  are known as Fuss–Catalan numbers. With  $m = 1$  our result turns to a well known result of Marchenko–Pastur 1967.

**Keywords and phrases:** Random matrices, Fuss-Catalan numbers, Semi-circular law, Marchenko–Pastur distribution.

## 1. Introduction

Let  $\mathbf{X} = (N^{-1/2}x_{ij}^{(N)})$ ,  $1 \leq i, j \leq N$  be a random matrix. We assume that  $x_{ij} \equiv x_{ij}^{(N)}$  are independent complex random variables such that

$$\mathbf{E}x_{ij} = 0, \quad \mathbf{E}|x_{ij}|^2 = 1, \quad \mathbf{E}|x_{ij}|^4 \leq B \quad (1.1)$$

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with some  $B < \infty$  independent of  $N$ . We assume additionally that

$$L_N(\alpha) = N^{-2} \sum_{1 \leq i, j \leq N} \mathbf{E} |x_{ij}|^4 \mathbb{I}\{|x_{ij}| > \alpha\sqrt{N}\} \rightarrow 0 \text{ as } N \rightarrow \infty \quad (1.2)$$

for all  $\alpha > 0$ . Note that  $x_{ij} \equiv x_{ij}^{(N)}$  and  $\mathbf{X} \equiv \mathbf{X}^{(N)}$  can depend on  $N$ , which is not reflected in our further notation.

Writing  $\mathbf{X}^*$  for the adjoint matrix of  $\mathbf{X}$ , consider the product

$$\mathbf{W}^{(m)} = \mathbf{X}^m \mathbf{X}^{*m}$$

with some  $m \in \{1, 2, \dots\}$ . The matrix  $\mathbf{W}^{(m)}$  is Hermitian positive semi-definite. Let  $\lambda_1, \lambda_2, \dots, \lambda_N$  be eigenvalues of  $\mathbf{W}^{(m)}$  (or squared singular values of the matrix  $\mathbf{X}^m$ ). In this paper we find the asymptotic distribution function

$$G^{(m)}(x) = \lim_{N \rightarrow \infty} \mathbf{E} F_N^{(m)}(x)$$

of the empirical distribution function

$$F_N^{(m)}(x) = N^{-1} \sum_{k=1}^N \mathbb{I}\{\lambda_k \leq x\},$$

where  $\mathbb{I}\{A\}$  stands for the indicator function of event  $A$ .

**Theorem 1.1.** *Assume that (1.1) and (1.2) hold. Then the limit  $G^{(m)}(x) = \lim_{N \rightarrow \infty} \mathbf{E} F_N^{(m)}(x)$  exists. The function  $G^{(m)}(x)$  is a distribution function and it has moments*

$$M_p^{(m)} = \int_{\mathbb{R}} x^p dG^{(m)}(x) = \frac{1}{mp + 1} \binom{pm + p}{p}. \quad (1.3)$$

**Corollary 1.2.** *Let  $x_{ij}$  be independent copies of a random variable, say  $x$ , such that*

$$\mathbf{E} x = 0, \quad \mathbf{E} |x|^2 = 1, \quad \mathbf{E} |x|^4 < \infty.$$

*Let  $\mathbf{X} = (N^{-1/2}x_{ij})$ ,  $1 \leq i, j \leq N$ . Then the limit  $\lim_{N \rightarrow \infty} \mathbf{E} F_N^{(m)}(x)$  exists and it is equal to  $G^{(m)}(x)$ .*

Gessel and Xin 2006 [4] showed that for any natural  $m$  the sequence  $M_1^{(m)}, M_2^{(m)}, \dots$  is a sequence of moments of some probability measure. Hence,  $G^{(m)}$  is a probability distribution for any natural  $m$ . Since  $M_p^{(m)} \leq c_m^p$  with some  $c_m < \infty$ , by Carleman's Theorem in [3] the measure  $G^{(m)}$  is uniquely

determined by its moments. The support of the measure  $G^{(m)}$  is the interval  $[0, m^{-m}(m+1)^{m+1}]$ .

With  $m = 1$  Theorem 1.1 turns to a well known result of Marchenko–Pastur 1967 [6]. Namely, the asymptotic distribution  $G^{(1)}$  of eigenvalues of the matrices  $\mathbf{XX}^*$  has the moments  $M_p^{(1)} = \frac{1}{p+1} \binom{2p}{p}$ . Note that in the case  $m = 1$  our fourth moment assumption is stronger than assumptions in Theorem 2.5 and Theorem 2.8 in Bai 1999 [1]. The question of the weakest sufficient conditions in the case  $m > 1$  remains an open problem.

In Free Probability Theory  $M_p^{(m)}$  are known as Fuss–Catalan numbers. Combinatorial properties of this sequence have been studied by Nica and Speicher 2006 [8]. Młotkowski 2009 [7] investigated a family of distributions, say  $G^{(m,r)}$ , with real  $m \geq 0$  and  $0 \leq r \leq m$ , such that  $G^{(m,r)}$  has moments  $\frac{r}{mp+p+r} \binom{mp+p+r}{p}$ . It is easy to check that  $G^{(m,1)} = G^{(m)}$ . Oravecz 2001 [9] proved that powers of Voiculescu’s circular element have distribution  $G^{(m)}$ . This distribution belongs to the class of Free Bessel Laws (see Banica et al 2008 [2]).

Let  $\mathcal{M}_m(x) = \sum_{p=0}^{\infty} M_p^{(m)} x^p$  be the generating function of the sequence  $M_p^{(m)}$ . It satisfies the following functional equation (see equation (7.68) on p. 347 Graham et al 1988 [5])

$$\mathcal{M}_m(x) = 1 + x\mathcal{M}_m^{m+1}(x). \tag{1.4}$$

Equation (1.4) allows us to describe  $G^{(m)}$  in the framework of Free Probability Theory. In Free Probability Theory the free multiplicative convolution  $\xi \boxtimes \eta$  is defined for any positive random variables  $\xi$  and  $\eta$  (see Nica and Speicher 2006 [8], p. 287). The  $S$ -transform is a homomorphism with respect to free multiplicative convolution, i.e. if  $\xi$  and  $\eta$  are free independent positive variables, then  $S_{\xi \boxtimes \eta}(z) = S_{\xi}(z)S_{\eta}(z)$ . Recall that the  $S$ -transform, say  $S(z)$ , of a distribution  $\mu$  is defined as follows. Let

$$M_p = \int_{\mathbb{R}} x^p d\mu(x), \quad u(z) = \sum_{p=1}^{\infty} M_p z^p.$$

Then

$$S(z) = \frac{z+1}{z} u^{-1}(z), \tag{1.5}$$

where  $u^{-1}$  denotes the inverse function of  $u$ .

Equation (1.4) allows to calculate the  $S$ -transform, say  $S^{(m)}(z)$ , of  $G^{(m)}$ , and

$$S^{(m)}(z) = \frac{1}{(1+z)^m}. \tag{1.6}$$

It means that the family  $G^{(m)}$  has the following property: if a random variable  $\xi$  has distribution  $G^{(m)}$  then the  $r$ -th power of the  $S$ -transform of  $\xi$  is equal to the  $S$ -transform of multiplicative free power  $\xi^{\boxtimes r}$ . This property holds for this family of distributions only.

To prove Theorem 1.1 we use truncation and the method of moments. Truncation means that we can replace (see Section 2.1 for details)  $\mathbf{X}$  by the matrix  $\tilde{\mathbf{X}} = (\tilde{X}_{ij})$  with truncated entries (here and below  $X_{ij} = N^{-1/2}x_{ij}$  denote entries of matrix  $\mathbf{X}$ )

$$\tilde{X}_{ij} = X_{ij}\mathbb{I}\{|X_{ij}| < \alpha_N\}, \tag{1.7}$$

where  $\alpha_N$  is some sequence of positive numbers such that  $\alpha_N \rightarrow 0$  as  $N \rightarrow \infty$ . Lemma 2.1 (see Section 2.1) reduces the proof of Theorem 1.1 to the proof of the next proposition.

**Proposition 1.3.** *Assume that  $\alpha_N \rightarrow 0$  and  $\beta_N \rightarrow 0$ . Then Theorem 1.1 holds if*

$$\left|X_{ij}^{(N)}\right| \leq \alpha_N, \quad \max_{1 \leq i, j \leq N} \left|\mathbf{E} X_{ij}^{(N)}\right| \leq \beta_N N^{-3/2}, \quad \left|\mathbf{E} \left|X_{ij}^{(N)}\right|^2 - 1/N\right| \leq \beta_N N^{-3/2}. \tag{1.8}$$

Let us explain our proof of Proposition 1.3. Denote by  $\xi_m(N)$  a random variable with distribution  $\mathbf{E} F_N^{(m)}$ . We show that the moments  $\mathbf{E} \xi_m^p(N)$  converge to  $M_p^{(m)}$ . In order to simplify the notation assume for a while that  $X_{ij}$  are real random variables. Then one can represent  $\mathbf{E} \xi_m^p(N)$  as

$$\mathbf{E} \xi_m^p(N) = \sum^{(2mp)} N^{-1} \mathbf{E} \prod_{j=0}^{2mp-1} X_{i_j i_{j+1}}^{\varepsilon(j)},$$

where the sum  $\sum^{(2mp)}$  is taken over  $i_0, \dots, i_{2mp} \in \{1, \dots, N\}$  such that  $i_{2mp} = i_0$ . The notation  $X_{i_j i_{j+1}}^{\varepsilon(j)}$  means  $X_{i_j i_{j+1}}^+$  :=  $X_{i_j i_{j+1}}$  in case of  $\varepsilon(j) = +$  and  $X_{i_{j+1} i_j}^-$  :=  $X_{i_{j+1} i_j}$  in case of  $\varepsilon(j) = -$  (see Section 2.2 for a precise definition of the spin variable  $\varepsilon(j)$ ). We investigate properties of paths  $(i_0, \dots, i_{2mp})$  by combinatorial methods. The moment  $\mathbf{E} \xi_m^p(N)$  converges to the number of paths of a special type. Namely, one can describe such paths as follows: the cardinality of  $\{i_0, \dots, i_{2mp}\}$  is equal to  $mp + 1$  and each factor  $X_{i_j i_{j+1}}$  appears in the product  $\prod_{j=0}^{2mp-1} X_{i_j i_{j+1}}^{\varepsilon(j)}$  twice. In Section 2.4 we count the number of these paths.

## 2. The proof of the main result.

### 2.1. Truncation.

Recalling that  $X_{ij} = N^{-1/2}x_{ij}$ , we can rewrite  $L_N(\alpha)$  as

$$L_N(\alpha) = \sum_{1 \leq i, j \leq N} \mathbf{E} |X_{i,j}|^4 \mathbb{I}\{|X_{i,j}| > \alpha\}.$$

Since for all  $\alpha > 0$  the ratio  $L_N(\alpha)/\alpha^4$  tends to 0, one can find a sequence  $\alpha_N \downarrow 0$  such that  $L_N(\alpha_N)/\alpha_N^4 \rightarrow 0$  and  $N^\delta \alpha_N^{-1} \rightarrow \infty$  for any  $\delta > 0$  as  $N \rightarrow \infty$ . Let  $\tilde{F}_N^{(m)}(t)$  denote the empirical spectral distribution function of the matrix  $\tilde{\mathbf{X}}^m \tilde{\mathbf{X}}^{*m}$ .

**Lemma 2.1.** *The limit behaviors of  $\mathbf{E} \tilde{F}_N^{(m)}(t)$  and  $\mathbf{E} F_N^{(m)}(t)$  are the same, that is*

$$\sup_{t \in \mathbb{R}} |\mathbf{E} \tilde{F}_N^{(m)}(t) - \mathbf{E} F_N^{(m)}(t)| \rightarrow 0.$$

*Proof.* Since by definition  $|\tilde{F}_N^{(m)}(t) - F_N^{(m)}(t)| \neq 0$  only if there exist  $i, j \in \{1, \dots, N\}$  such that  $|X_{ij}| \geq \alpha_N$ , we have

$$|\mathbf{E} \tilde{F}_N^{(m)}(t) - \mathbf{E} F_N^{(m)}(t)| \leq \sum_{1 \leq i, j \leq N} \mathbf{P}(|X_{ij}| \geq \alpha_N). \quad (2.1)$$

Estimating  $\mathbf{P}(|X_{ij}| \geq \alpha_N) \leq \alpha_N^{-4} \mathbf{E} |X_{ij}|^4 \mathbb{I}\{|X_{ij}| > \alpha_N\}$  and using inequality (2.1) we obtain

$$\begin{aligned} \sup_{t \in \mathbb{R}} |\mathbf{E} \tilde{F}_N^{(m)}(t) - \mathbf{E} F_N^{(m)}(t)| &\leq \alpha_N^{-4} \sum_{i, j=1}^N \mathbf{E} |X_{ij}|^4 \mathbb{I}\{|X_{ij}| > \alpha_N\} \\ &= \alpha_N^{-4} L_N(\alpha_N) \rightarrow 0. \end{aligned} \quad (2.2)$$

□

Note, that the lower order moments of the truncated variables are asymptotically equal to the moments of the original variables. Writing for a while  $X = X_{ij}$  we have for  $k \leq 3$

$$|\mathbf{E} \tilde{X}^k - \mathbf{E} X^k| \leq \mathbf{E} |X|^k \mathbb{I}\{|X| > \alpha_N\}. \quad (2.3)$$

The right hand side of (2.3) can be estimated as

$$\mathbf{E} |X|^k \mathbb{I}\{|X| > \alpha_N\} \leq \alpha_N^{k-4} \mathbf{E} |X|^4 \leq \beta_N N^{-3/2}, \quad (2.4)$$

where  $\beta_N = B\alpha_N^{k-4} N^{-1/2} \rightarrow 0$  as  $N \rightarrow \infty$ .

Lemma 2.1 shows that the limit behaviors of  $\tilde{F}_N^{(m)}(t)$  and  $F_N^{(m)}(t)$  are the same. Thus we may replace  $\mathbf{X}$  by  $\tilde{\mathbf{X}}$  in the following arguments and assume that  $\mathbf{X}$  is truncated, that is, that entries of  $\mathbf{X}$  satisfy the assumption (1.8).

### 2.2. Moments of the spectral distribution.

We apply the method of moments. Recall that  $\lambda_1, \lambda_2, \dots, \lambda_N$  denote the eigenvalues of  $\mathbf{X}^m \mathbf{X}^{*m}$ . We can write

$$\mathbf{E} \xi_m^p(N) = N^{-1} \mathbf{E} \sum_{j=1}^N \lambda_j^p = N^{-1} \mathbf{E} \operatorname{Tr}(\mathbf{X}^m \mathbf{X}^{*m})^p. \quad (2.5)$$

We assume that  $m$  and  $p$  are fixed and study the asymptotics of  $\mathbf{E} \xi_m^p(N)$  as  $N \rightarrow \infty$ . In order to simplify notation, hence forth we assume that  $X_{ij}$  are real random variables.

In the Hermitian case, the trace of  $\mathbf{X}^{2k}$  may be rewritten in terms of the entries of  $\mathbf{X}$  via

$$\mathbf{E} \operatorname{Tr} \mathbf{X}^{2k} = \sum^{(2k)} \mathbf{E} \prod_{j=0}^{2k-1} X_{i_j i_{j+1}}, \quad (2.6)$$

where the sum  $\sum^{(s)}$  is taken over  $i_0, \dots, i_s \in \{1, \dots, N\}$  such that  $i_s = i_0$ .

In the non-Hermitian case  $\mathbf{E} \operatorname{Tr}(\mathbf{X}^m \mathbf{X}^{*m})^p$  has a similar representation. An entry of  $\mathbf{X}^m \mathbf{X}^{*m}$  is given by

$$[\mathbf{X}^m \mathbf{X}^{*m}]_{ik} = \sum_{1 \leq i_j \leq N} X_{ii_1} X_{i_1 i_2} \cdots X_{i_{m-1} i_m} X_{i_{m+1} i_m} \cdots X_{ki_{2m-1}} \quad (2.7)$$

(2.8)

We write  $X_{i_j i_{j+1}}^+ := X_{i_j i_{j+1}}$  and  $X_{i_{j+1} i_j}^- := X_{i_{j+1} i_j}$ . Then the right hand side of (2.8) takes the form

$$[\mathbf{X}^m \mathbf{X}^{*m}]_{ik} = \sum_{1 \leq i_j \leq N} \prod_{j=0}^{2m-1} X_{i_j i_{j+1}}^{\varepsilon(j)}, \quad (2.9)$$

where  $i_0 = i, i_{2m} = k$ , and the 'spin' variable  $\varepsilon(j)$  takes values  $\varepsilon(j) = +$  with  $j < m$ , and  $\varepsilon(j) = -$  with  $j \geq m$ . Since  $(\mathbf{X}^m \mathbf{X}^{*m})^p = \mathbf{X}^m \mathbf{X}^{*m} \dots \mathbf{X}^m \mathbf{X}^{*m}$

( $p$  times), one needs to change the order of indices in  $X_{i_j i_{j+1}}^\varepsilon$  if the spin  $\varepsilon = -$  and

$$\varepsilon(j) = \begin{cases} +, & \text{if } j \pmod{2m} \in \{0, \dots, m-1\}, \\ -, & \text{if } j \pmod{2m} \in \{m, \dots, 2m-1\}. \end{cases} \quad (2.10)$$

Using these notions (2.5) takes the form

$$\begin{aligned} \mathbf{E} \xi_m^p(N) &= N^{-1} \mathbf{E} \operatorname{Tr}(\mathbf{X}^m \mathbf{X}^{*m})^p \\ &= \sum^{(2mp)} N^{-1} \mathbf{E} \prod_{j=0}^{2mp-1} X_{i_j i_{j+1}}^{\varepsilon(j)}. \end{aligned} \quad (2.11)$$

A crucial notion in the proof is that of 'paths' of indices of the type  $(i_0, i_1, \dots, i_{2mp-1})$ .

### 2.3. Description of paths.

We consider a path  $\mathbf{i} = (i_0, \dots, i_{2mp-1})$  which corresponds to a product  $\prod_{j=0}^{2mp-1} X_{i_j i_{j+1}}^\varepsilon$ . Let  $\mathcal{P}$  be a set of pairs  $\{(j, j+1)^{\varepsilon(j)}\}_{j=0}^{2m-2} \cup \{(2mp-1, 0)^-\}$ , where  $(j, j+1)^+ := (j, j+1)$ ,  $(j, j+1)^- := (j+1, j)$  and  $\varepsilon(j)$  is given by (2.10). We call pairs  $(j, j+1)^{\varepsilon(j)}$  and  $(k, k+1)^{\varepsilon(k)}$  equivalent (denoted by  $(j, j+1)^{\varepsilon(j)} \sim (k, k+1)^{\varepsilon(k)}$ ) iff  $X_{i_j i_{j+1}}^{\varepsilon(j)} \equiv X_{i_k i_{k+1}}^{\varepsilon(k)}$ . We also call  $(j, j+1)^{\varepsilon(j)}$  an edge of the path  $\mathbf{i}$ . We construct a directed graph  $\mathcal{G}_\mathbf{i}$  as follows. A vertex  $\mathcal{V}$  of  $\mathcal{G}_\mathbf{i}$  is a subset of  $\{0, 1, \dots, 2mp-1\}$  such that  $j \in \mathcal{V}$  and  $k \in \mathcal{V}$  if and only if  $i_j = i_k$ . There exists an edge  $(\mathcal{V}, \mathcal{U})$  if and only if there exist  $l \in \mathcal{V}$  and  $r \in \mathcal{U}$  such that  $(l, r) \in \mathcal{P}$  (note that  $|l-r|=1$ ). Denote by  $V$  the total number of vertices of the graph  $\mathcal{G}_\mathbf{i}$  and by  $E$  its total number of edges. Since the graph  $\mathcal{G}$  is connected  $E \geq V - 1$ . It is clear that  $V$  is a cardinality of  $\{i_0, i_1, \dots, i_{2m-1}\}$  and  $E$  is a cardinality of a quotient set  $\mathcal{P}/\sim$ . Denote by  $k_r$  ( $r = 1, \dots, E$ ) the cardinality of each equivalence class in  $\mathcal{P}$ . Note, that  $k_1 + k_2 + \dots + k_E = 2mp$ .

**Remark 2.2.** Consider paths  $\mathbf{i} = (i_0, \dots, i_{2mp-1})$  and  $\mathbf{k} = (k_0, k_1, \dots, k_{2m-1})$  such that  $\mathcal{G}_\mathbf{i} = \mathcal{G}_\mathbf{k}$ . It is clear that if  $x_{ij}$  are identically distributed then

$$\mathbf{E} \prod_{j=0}^{2mp-1} X_{i_j i_{j+1}}^{\varepsilon(j)} = \mathbf{E} \prod_{j=0}^{2mp-1} X_{k_j k_{j+1}}^{\varepsilon(j)}.$$

We will show, that assuming our conditions the asymptotic products corresponding to equivalent paths are equal as well.

**Definition 2.3.** We define the contribution of a graph  $\mathcal{G}$  to 2.11 as

$$Cont(\mathcal{G}) = \sum_{\mathbf{i}:\mathcal{G}_i=\mathcal{G}} N^{-1} \mathbf{E} \prod_{j=0}^{2mp-1} X_{i_j i_{j+1}}^{\varepsilon(j)}$$

**Lemma 2.4.** Using these notations we have that the contribution of the path  $\mathcal{G}$  is asymptotically given by

$$Cont(\mathcal{G}) \sim N^{V-1} \prod_{r=1}^E \mathbf{E} x_{i_s i_t}^{k_r}, \tag{2.12}$$

when  $N$  tends to infinity.

*Proof.* Since  $X_{ij}$  are independent we have

$$\mathbf{E} \prod_{j=0}^{2mp-1} X_{i_j i_{j+1}}^{\varepsilon} = \prod_{r=1}^E \mathbf{E} X_{i_s i_t}^{k_r}.$$

Furthermore, for any vertex  $\mathcal{V}$  the number of possible values of corresponding indices (indices  $i_j$  such that  $j \in \mathcal{V}$ ) lies between  $N$  and  $N - 2mp \sim N$ . The lower bound  $N - 2mp$  is due to the fact that indices corresponding to this vertex should not coincide with indices corresponding to other vertices and that there are at most  $2mp$  different indices. This yields the multiplicity  $N^V$ . Together with the factor  $N^{-1}$  this finally leads to the formula (2.12).  $\square$

**Definition 2.5.** We call a graph  $\mathcal{G}_i$   $(m, p)$ -regular graph, if it has at least  $mp + 1$  vertices and  $k_r \geq 2$  for all  $r \in \{1, 2, \dots, E\}$ . The path  $\mathbf{i}$  we call  $(m, p)$ -regular path.

**Lemma 2.6.**  $Cont(\mathcal{G}_i)$  does not converge to zero if and only if  $\mathcal{G}_i$  is the regular path.

*Proof.* Since the variables  $X_{ij}$  satisfy conditions (1.8), we have

$$\left| \mathbf{E} X_{ij}^k \right| \leq \mathbf{E} X_{ij}^2 |X_{ij}|^{k-2} \leq N^{-1} \alpha_N^{k-2}. \tag{2.13}$$

Of course, this estimaton holds for  $k = 1$  too. At first we consider that one of  $k_r$  is equal to 1 (without loss of generality  $k_1 = 1$ ). Then we have

$$\begin{aligned} \left| \prod_{r=1}^E \mathbf{E} X_{i(r)j(r)}^{k_r} \right| &= \left| \mathbf{E} X_{i(1)j(1)} \prod_{r=2}^E \mathbf{E} X_{i(r)j(r)}^{k_r} \right| \leq \beta_N N^{-3/2} N^{-E+1} \alpha_N^{\sum_r (k_r-2)} \\ &= \beta_N N^{-3/2} N^{-E+1} \alpha_N^{2mp-1-2(E-1)} \leq N^{-E-1/2}, \end{aligned} \tag{2.14}$$

and the contribution of such a graph is bounded by

$$|Cont(i_0, \dots, i_{2mp-1})| \leq N^{V-1} N^{-E-1/2} = N^{V-E-1} N^{-1/2}. \quad (2.15)$$

Note that  $V-E-1 \leq 0$  since the graph  $\mathcal{G}$  is connected and hence  $N^{V-E-1} N^{-1/2}$  tends to 0.

Furthermore, we consider the case  $V < mp + 1$ . Note that  $k_r \geq 2$  for any  $r$  and  $E \leq 2mp/2 = mp$ . Our truncation leads to

$$\left| \prod_{r=1}^E \mathbf{E} X_{i(r)j(r)}^{k_r} \right| \leq N^{-E} \alpha_N^{\sum_r (k_r-2)} = N^{-E} \alpha_N^{2mp-2E}. \quad (2.16)$$

Using inequality (2.16) to estimate the terms in (2.12), we obtain for such a product

$$N^{V-1} \left| \prod_{r=1}^E \mathbf{E} X_{i(r)j(r)}^{k_r} \right| \leq N^{V-E-1} \alpha_N^{2mp-2E}. \quad (2.17)$$

Note that  $E \geq V - 1$  and  $2mp - 2E \geq 0$ . It follows that the right hand side of (2.17) does not converge to 0 only if  $2mp - 2E = 0$  and  $V - E - 1 = 0$ , i.e.  $V = mp + 1$  and the graph  $\mathcal{G}$  is a regular graph.  $\square$

Furthermore, we obtain

**Lemma 2.7.** *A regular graph is a tree and it has exactly  $V = mp + 1$  vertices and exactly  $E = mp$  edges ( each representing an equivalence class of size  $k_r = 2$ ).*

**Remark 2.8.** Due to the fact that  $\mathbf{E} X_{ij}^2 \sim 1/N$  and by the remarks above we can write the contribution of a regular graph  $\mathcal{G}_{reg}$  as

$$Cont(\mathcal{G}_{reg}) \sim 1. \quad (2.18)$$

We now show the connection between the moments of the spectral distribution  $\mathbf{E} F_m^{(N)}$  and the number of regular graphs. Indeed,  $\xi_m(N)$  has distribution  $\mathbf{E} F_m^{(N)}$ . Denote by  $T_{m,p}$  the set of all possible graphs of view  $\mathcal{G}_1$  and by  $T_{m,p}^{reg}$  the set of all  $(m,p)$ -regular graphs. Then

$$\mathbf{E} \xi_m^p(N) = \sum_{\mathcal{S} \in T_{m,p}} Cont(\mathcal{S}) \sim \sum_{\mathcal{S} \in T_{m,p}^{reg}} 1 = \#T_{m,p}^{reg}. \quad (2.19)$$

We can reformulate 2.19 as

**Lemma 2.9.**  $\lim_{N \rightarrow \infty} \mathbf{E} \xi_N^p$  is equal to the number of  $(m,p)$ -regular graphs.

**2.4. Counting of the number of regular graphs.**

**Lemma 2.10.** *The number of all  $(m, p)$ -regular graphs is  $\#T_{m,p}^{reg} = M_p^{(m)}$ .*

*Proof.* The numbers  $M_p^{(m)} = \frac{1}{p+1} \binom{mp+p}{p}$  satisfy to the recurrence (see [5]):

$$M_p^{(m)} = \sum_{p-1} \prod_{i=0}^{m-1} M_{p_i}^{(m)}, \quad M_1^{(m)} = 1, \tag{2.20}$$

where the sum  $\sum_{p-1}$  is taken over all  $p_0 + p_1 + \dots + p_m = p - 1$ . We will show that there is one-to-one correspondence between collections of  $(m, p_k)$ -regular graphs  $(\mathcal{G}_{m,p_0}, \dots, \mathcal{G}_{m,p_m}) : \sum_{i=0}^m p_i = p - 1$  and  $(m, p)$ -regular graphs  $\mathcal{G}_{m,p}$ . It follows that

$$\#T_{m,p}^{reg} = \# \bigcup_{p-1} \bigotimes_{i=0}^m T_{m,p_i}^{reg} = \sum_{p-1} \prod_{i=0}^m \#T_{m,p_i}^{reg} \tag{2.21}$$

and the sequence  $\#T_{m,p}^{reg}$  satisfies to both the same recurrence and initial conditions as the sequence Fuss–Catalan numbers  $M_p^{(m)}$  and, by this reason, these two sequences are equal.

**Proposition 2.11.** *The number  $\#T_{m,1}^{reg} = 1$  for all  $m$ . If  $\mathcal{G}_{\mathbf{i}}$  is a  $(m, 1)$ -regular graph then indices  $i_k$  and  $i_l$  are equal iff  $(k + l) = 2m$ .*

*Proof.* By induction. Consider  $m = 1$ . In this case it is clear, that there is only one regular graph  $0 \rightarrow 1$  and Proposition 2.11 holds. Assume, that Proposition 2.11 holds for all  $m < m_0$ . Consider the path  $\mathbf{i}$  and a corresponding graph  $\mathcal{G}_{\mathbf{i}}$  (see fig.1). This path has  $m_0 + 1$  distinct indices and

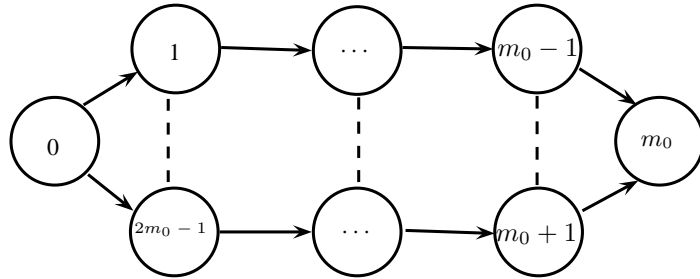


FIG 1. The path  $\mathbf{i}$ . Vertices, that correspond to equal indices, are connected via dotted lines.

it has  $2m_0$  at all. It follows that there exist at least 2 one-element vertices of  $\mathcal{G}_{\mathbf{i}}$ . Let these one-element vertices be  $\{s\}$  and  $\{t\}$ . Consider the

pair  $(i_{t-1}, i_t)^\varepsilon(t)$ . It must have an equal pair, but  $i_t$  is not equal to any other index. It means, that  $(i_{t-1}, i_t)^\varepsilon(t-1) = (i_t, i_{t+1})^\varepsilon(t)$ . It follows, that  $\varepsilon(t-1) \neq \varepsilon(t)$ . There are exactly two possibilities for this:  $t = m_0$  or  $t = 0$ . Assume without loss of generality that  $s = 0$  and  $t = m_0$ . Therefore  $i_{m_0-1} = i_{m_0+1}$  (notice, that  $(m_0 - 1) + (m_0 + 1) = 2m_0$ ). Define  $(m_0 - 1, 1)$ -path  $\mathbf{j}$  as follows:  $j_k := i_k$  if  $k \in \{0, \dots, m_0 - 2\}$ ,  $j_{m_0-1} := i_{m_0-1} = i_{m_0+1}$ ,  $j_k := i_{k+2}$  if  $k \in \{m_0, \dots, 2(m_0 - 1) - 1\}$ . (See fig.2) The path  $\mathbf{j}$  is the

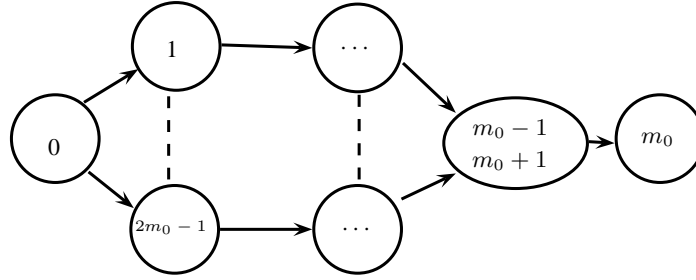


FIG 2. The path  $\mathbf{j}$  is a  $(m_0 - 1, 1)$ -path

$(m_0 - 1, 1)$ -regular path. There is only one such path by inductive hypothesis and  $(i_k = i_l) \Leftrightarrow (j_k = j_{l-2}) \Leftrightarrow (k + (l-2) = 2(m_0 - 1)) \Leftrightarrow (k + l = 2m_0)$ .  $\square$

**Definition 2.12.** Notice, that the vertex of a regular graph has two outgoing edges iff the corresponding index has the form  $i_{2mk}$  (because it should be  $(i_j, i_{j+1})^\varepsilon(j) = (i_j, i_{j+1})$  and  $(i_{j-1}, i_j)^\varepsilon(j-1) = (i_j, i_{j-1})$  and it happens if and only if  $j = 2mk$ ). The distance between such vertex and vertex  $\mathcal{V}$  is called a *type of vertex*  $\mathcal{V}$ . The *type of index*  $i_j$  is the type of a vertex  $\mathcal{V}$  such that  $j \in \mathcal{V}$ . It is clear, that index  $i_j$  has type  $j \pmod{2m}$  if  $j \pmod{2m} \in \{0, \dots, m-1\}$  or type  $-j \pmod{2m}$  in the other case. There are  $m + 1$  types of vertices. Note that only indices of the same type can be equal (this is proved in the case  $p = 1$  in Proposition 2.11 and it will be proved for other cases below).

Consider a collection of  $(m, p_k)$ -regular paths  $(\mathbf{i}_0, \mathbf{i}_1, \dots, \mathbf{i}_m)$  (such that  $\sum_{k=0}^m p_k = p-1$ ) and collection of corresponding regular graphs  $(\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_m)$ . Sum of path's lengths is  $2m(p-1)$ . We indicate the recipe how to obtain the  $(m, p)$ -regular graph from these collections. We take an  $(m, 1)$ -regular graph and attach to its vertices the graphs from the collection in the following way: the graph  $\mathcal{G}_0$  is attached to vertex of type 0, the graph  $\mathcal{G}_1$  is attached to vertex of type 1,  $\dots$ , the graph  $\mathcal{G}_m$  to the vertex of type  $m$ . For a more detailed argument we denote  $\sum_{i=0}^k p_i$  by  $P_k$  ( $P_m = p-1$ ) and the indices of the  $k^{th}$  path  $\mathbf{i}_k$  by  $i_j^{(k)}$ . The resulting  $(m, p)$ -regular graph is denoted by  $\mathcal{G}_j$ .

Define the map  $\Delta : \Delta(\mathbf{i}_0, \mathbf{i}_1, \dots, \mathbf{i}_m) = \mathbf{j}$  as follows

$$\begin{aligned}
 j_0 &:= i_0^{(0)}, j_1 = i_1^{(0)}, \dots, j_{2mP_0-1} := i_{2mp_0-1}^{(0)}, j_{2mP_0} := j_0; \\
 j_{2mP_0+1} &:= i_1^{(1)}, \dots, j_{2mP_1-1} := i_{2mp_1-1}^{(1)}, j_{2mP_1} := i_0^{(1)}, j_{2mP_1+1} := j_{2mP_0+1}; \\
 j_{2mP_1+2} &:= i_2^{(2)}, \dots, j_{2mP_2-1} := i_{2mp_2-1}^{(2)}, j_{2mP_2} := i_0^{(2)}, j_{2mP_2+1} := i_1^{(2)}, \\
 j_{2mP_2+2} &:= j_{2mP_1+2}; \\
 &\dots \\
 j_{2mP_{m-1}+m} &:= i_m^{(m)}, \dots, j_{2m(p-1)-1} := i_{2mp_m-1}^{(m)}, j_{2m(p-1)} := i_0^{(m)}, \\
 j_{2mp-m+1} &:= j_{2mP_{m-1}+m-1}, \dots, j_{2mp-k} := j_{2mP_k+k}, \dots, j_{2mp-1} := j_{2mP_1+1}.
 \end{aligned} \tag{2.22}$$

Let  $\tilde{\Delta}$  is the corresponding map  $\tilde{\Delta}(\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_m) = \mathcal{G}_{\mathbf{j}}$ . Graphically the constructon (2.22) looks as follows: (fig. 3).

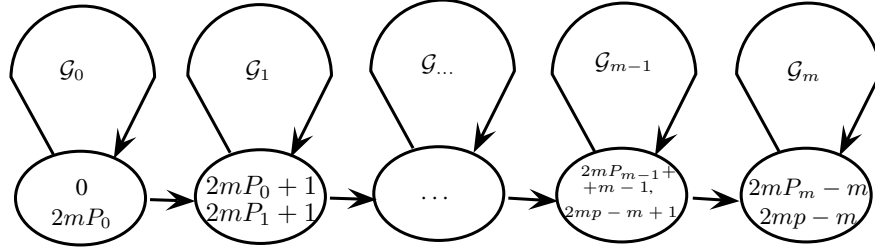


FIG 3. The regular graph  $\mathcal{G}_{\mathbf{j}}$ , obtained from the collection of  $(m, p_i)$ -regular graphs  $(\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_m)$

**Example.** For example, we consider for  $m = 2$  the collection of  $(2, p_k)$ -regular graphs  $(\mathcal{G}_{2,2}, \mathcal{G}_{2,0}, \mathcal{G}_{2,1})$  (see fig. 4) and we obtain from it a  $(2, 4)$ -regular graph  $\mathcal{G}_{2,4}$ (see fig.5).

**Proposition 2.13.** *Using the above construction we get an  $(m, p)$ -regular graph.*

*Proof.* Indeed, the graph  $\mathcal{G}_{\mathbf{j}}$  has exactly  $mp$  edges and  $k_r = 2$  for all  $r = 1, 2, \dots, mp$ . Furthermore, there are exactly  $mp+1$  vertices (there is no new-introduced vertex and there are exactly  $\sum_{i=0}^m (mp_i + 1) = m(p-1)+m+1 = mp+1$  vertices of graphs  $\mathcal{G}_k$ ).  $\square$

Note, that the map  $\Delta$  is the injection.

Now we consider the arbitrary  $(m, p)$ -regular path  $\mathbf{i}$  and try to construct

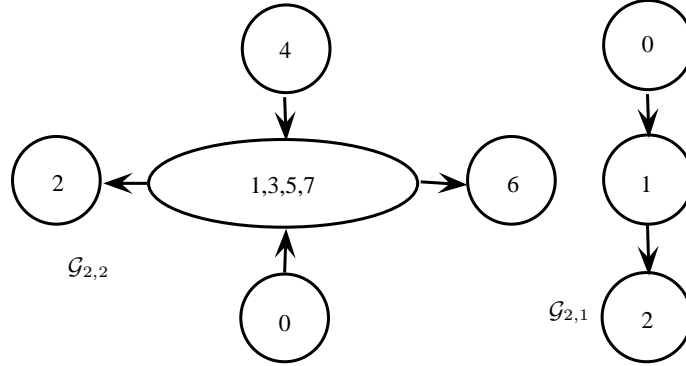


FIG 4. Graphs  $\mathcal{G}_{2,2}$  and  $\mathcal{G}_{2,1}$ . Graph  $\mathcal{G}_{m,0}$  is empty

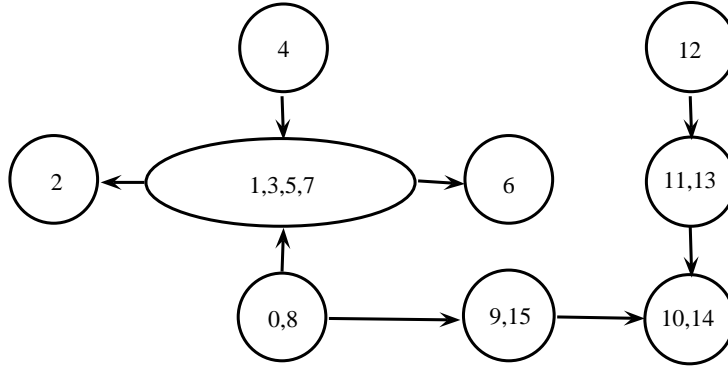


FIG 5. The resulting graph  $\mathcal{G}_{2,4}$

inverse map for  $\Delta$ . Denote

$$\begin{aligned}
 J_0 &:= \{j : i_j = i_0\}; \\
 J_k &:= \{j : j \neq 2mp - k, i_j = i_{2mp-k}\}, \quad k \in \{1, \dots, m-1\}; \\
 J_m &:= \{j : i_j = i_{2mp-m}\}; \\
 \overline{J}_k &:= \max(J_k), \quad \underline{J}_k := \min(J_k).
 \end{aligned}
 \tag{2.23}$$

We will prove that the sets  $J_k$  have some remarkable properties and after that it will be clear, how to obtain a collection of regular paths from one (big) regular path.

**Proposition 2.14.**  $J_k$  is nonempty.

*Proof.* Indeed, there is  $0 \in J_0$  and  $2mp - m \in J_m$ . If  $J_k$  is void with  $k \in \{1, \dots, m-1\}$ , then the index  $i_{2mp-k}$  has no equal indices in the path

i. But in this case the pair  $(i_{2mp-k-1}, i_{2mp-k})^-$  has no equivalent for the following reason. The index  $i_{2mp-k}$  appears in  $(2mp-k, 2mp-k+1)^-$  and in  $(2mp-k-1, 2mp-k)^-$  only and they are not equivalent. But each edge in a regular path has equivalent one, a contradiction. Therefore the initial assumption that  $J_k$  is void must be false.  $\square$

**Proposition 2.15.**  $J_k$  ( $0 \leq k \leq m$ ) are pairwise disjoint and if  $k < l$  then  $J_k < J_l$  (for all  $j \in J_k$  and for all  $i \in J_l$  the inequality  $j < i$  holds).

*Proof.* Indeed, if  $J_k \cap J_l \neq \emptyset$  then  $i_{2mp-k} = i_{2mp-l}$ . The edges of the path  $\mathbf{i}$  have the same orientation on the section  $(2mp-k, 2mp-k-1)^-, \dots, (2mp-l+1, 2mp-l)$  and therefore the graph  $G_{\mathbf{i}}$  has a cycle. But a regular graph is a tree, a contradiction. Thus  $J_k \cap J_l = \emptyset$ . We prove the second part of Proposition 2.15 for the case  $l = k+1$  only (which is sufficient). Consider the edge  $(2mp-(k+1), 2mp-k)^-$ . It must be equivalent to an edge  $(t, t+1)^+$  with some  $t \in J_k$  and  $t+1 \in J_{k+1}$ . If there exists  $s \in J_k$  such that  $s > t$  then  $s > t+1$  ( $J_k \cap J_{k+1} = \emptyset$ ). The edge  $(t, t+1)$  is not equivalent to any edge in the section  $(t+1, t+2), \dots, (s-1, s)$ , because it has only one equivalent edge  $(2mp-(k+1), 2mp-k)^-$ . It follows that there are two different paths in the graph  $G_{\mathbf{i}}$  which connect vertex  $\mathcal{U}$  (such that  $t+1 \in \mathcal{U}$ ) and vertex  $\mathcal{V}$  (such that  $s \in \mathcal{V}$  and  $t \in \mathcal{V}$ ), that is there is a cycle in the the graph  $G_{\mathbf{i}}$ , and hence there is a contradiction. Therefore  $t = \max J_k = \overline{J_k}$ . Similarly,  $t+1 = \underline{J_{k+1}}$ . It follows, that  $\overline{J_k} + 1 = \underline{J_{k+1}}$  and for all  $j \in J_k$  and for all  $i \in J_{k+1}$  the inequality  $j < i$  holds.  $\square$

**Proposition 2.16.** For all  $k$  the difference  $(\overline{J_k} - \underline{J_k})$  is divisible by  $2m$ .

*Proof.* Denote  $(\overline{J_k} - \underline{J_k}) \pmod{2m}$  by  $d_k$ . Notice, that  $(\overline{J_k} - \underline{J_k})$  is the number of edges in the path's section  $(\underline{J_k}, \underline{J_k} + 1), \dots, (\overline{J_k} - 1, \overline{J_k})$ . Notice, that the orientation of edges changes after every  $m$  steps. Edges of the form  $(i_{\overline{J_k}}, i_{\underline{J_{k+1}}})^+$  have the same orientation. It follows, that  $d_0 \leq m-1$ ,  $d_1 \leq m-1$ ,  $(d_0 + 1 + d_1) \leq m-1 \pmod{2m}$  (and so  $(d_0 + 1 + d_1) \leq m-1$ , because  $0 \leq d_0 + 1 + d_1 \leq 2m-1$ ),  $\dots$ ,  $0 \leq d_0 + 1 + d_1 + 1 + \dots + d_{m-2} + 1 + d_{m-1} \leq m-1$  (similarly), i.e.  $0 \leq \sum_{k=0}^{m-1} d_k + m - 1 \leq m-1$ . Therefore,  $d_k = 0$  for all  $k = 0, 1, \dots, m-1$ . Consider all edges of the path  $\mathbf{i}$ . There are  $m$  edges of the form  $(\overline{J_k}, \underline{J_{k+1}})$ ,  $m$  edges of the form  $(2mp-k, 2mp-k+1)^-$  with some  $k = 1, 2, \dots, m$  and all the remaining ones are in sections of the form  $(\underline{J_k}, \underline{J_k} + 1), \dots, (\overline{J_k} - 1, \overline{J_k})$ . There are  $2mp$  edges in total. Therefore,  $\sum_{k=0}^m (\overline{J_k} - \underline{J_k}) + m + m = 2mp$  and hence  $\sum_{k=0}^m d_k = 0 \pmod{2m}$ . It follows, that  $d_m = 0$  too.  $\square$

**Proposition 2.17.** *If  $\underline{J}_k < t < \overline{J}_k$  and  $\underline{J}_l < s < \overline{J}_l$ , then  $i_t \neq i_s$ . In other words, sections of the path  $\mathbf{i}$  of the form  $(\underline{J}_k, \underline{J}_k + 1), \dots, (\overline{J}_k - 1, \overline{J}_k)$  with  $k = 0, 1, \dots, m$  are disjoint.*

*Proof.* Without loss of generality we consider  $l > k$ . Assume that  $i_t = i_s$ . In this case the section  $(\underline{J}_k, \underline{J}_k + 1), \dots, (s - 1, s)$  contains the edge  $(\overline{J}_k, \underline{J}_{k+1})$ , and the section  $(t, t + 1), \dots, (\overline{J}_k - 1, \overline{J}_k)$  does not contain it or its equivalent  $(2mp - k - 1, 2mp - k)^-$ . Thus, there are two non-equal paths in the regular graph  $G_{\mathbf{i}}$  which connected vertex  $\mathcal{U}$  (such that  $\overline{J}_k \in \mathcal{U}$ ) and vertex  $\mathcal{V}$  (such that  $s \in \mathcal{V}$  and  $t \in \mathcal{V}$ ), that is there is a cycle in the the graph  $G_{\mathbf{i}}$ . Therefore, the initial assumption must be false.  $\square$

Now we can describe the inverse map for  $\Delta$ . Let  $p_k := (\overline{J}_k - \underline{J}_k)/2m$  ( $p_k$  is a nonnegative integer by Proposition 2.16). Furthermore, we have for sum  $\sum_{k=0}^{m-1} p_k = p - 1$  (see the proof of Proposition 2.16). Denote by  $\mathbf{j}^{(k)}$  the  $k$ -th resulting path (it has a length  $2mp_k$  and if  $p_k = 0$  then  $\mathbf{j}_k$  is empty). Let

$$j_t^{(k)} := i_{\underline{J}_k + ((t-k) \bmod 2mp_k)}, t \in \{0, \dots, 2mp_k - 1\}, k \in \{0, \dots, m\}. \quad (2.24)$$

Now one obtains the collection  $(\mathcal{G}_{2,2}, \mathcal{G}_{2,0}, G_{2,1})$  (see fig. 4) from the graph  $\mathcal{G}_{2,4}$  (see fig.5) in the way described in (2.24).

**Proposition 2.18.** *The collection of paths  $(\mathbf{j}^{(0)}, \mathbf{j}^{(1)}, \dots, \mathbf{j}^{(m)})$  (defined by (2.24)) is the collection of regular paths.*

*Proof.* In fact, the path  $\mathbf{j}^{(k)}$  is almost the same as the section  $(\underline{J}_k, \underline{J}_k + 1), \dots, (\overline{J}_k - 1, \overline{J}_k)$  of the regular path  $\mathbf{i}$ . This section contains  $2mp_k$  edges. Each of these edges has an equivalent one in the same section by Proposition 2.17. Therefore this section contains exactly  $mp_k + 1$  distinct indices because of connectivity and non-cyclicity. Hence the path  $\mathbf{j}^{(k)}$  is a regular path.  $\square$

Thus,  $\tilde{\Delta}$  is the bijection between  $T_{m,p}^{reg}$  and  $\bigcup T_{m,p_0}^{reg} \times T_{m,p_1}^{reg} \times \dots \times T_{m,p_m}^{reg}$ , where the union is taken over all  $p_0 + p_1 + \dots + p_m = p - 1$ . Hence  $\#T_{m,p}^{reg} = M_p^{(m)}$  and Lemma 2.10 is proved.  $\square$

Lemmas 2.9 and 2.10 show that the moments of the spectral distribution converge to  $M_p^{(m)}$ . Thus Theorem 1.1 is proved.

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