Generating semisimple groups by tori

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Abstract

Let G be a simple real Lie group with finite center. We prove that given a non–central element g in G there is an elliptic element h in G such that h and ghg^{-1} generate a dense subsemigroup of G.

1 Introduction

Every real or complex semisimple Lie algebra $\mathfrak{g}$ can be generated by two of its elements. This is an old result of Kuranishi [Ku1, Ku2]. If $\mathfrak{g}$ is simple then given a non–zero element $X \in \mathfrak{g}$ there is an element $Y \in \mathfrak{g}$ such that X and Y generate $\mathfrak{g}$, [I]. This property is called 1.5–generatedness. In control theory the question arises if a semisimple group G can be generated as a semigroup by two one–parameter subsemigroups. The main result of [AGK] gives a (generic) sufficient condition for a one–parameter subsemigroup and a one–parameter *subgroup* of G to generate G as a semigroup.

Another old result, also proved by Kuranishi [Ku3], is that any semisimple Lie group G has a dense subgroup with two generators. For the case of a compact semisimple Lie group this was already proved in 1935 by Auerbach [Au]. In this case the two group elements actually generate a dense subsemigroup, see remark 3.4. In a — not necessarily compact — semisimple Lie group elements close to the identity generate a dense subgroup if and only if their logarithms generate the Lie algebra, see [Ku3]. For group elements not near the identity and for subsemigroups this is a different question. Field and Rowley conjectured that 1.5–generatedness holds for compact simple Lie groups without center, see [Fi]. In that paper Field gives an elementary proof of the fact that for a compact semisimple Lie group G the pairs (x, y) in $G \times G$ which generate a dense subgroup form a non-empty Zariski open subset. This is not true for non–compact semisimple groups, see remark 3.9.

We now state the main results of our paper. We call an element g of a Lie group G *elliptic* if one of the following three equivalent conditions is satisfied (for the equivalence see remark 3.3):

a) g is contained in a compact subgroup of G;

b) the closure of the subsemigroup of G generated by g is compact;

c) the closure of the subsemigroup of G generated by g is a group.

If G is a closed subgroup of $GL(n, \mathbb{C})$ then $g \in G$ is elliptic if and only if g is semisimple and all the eigenvalues of g have absolute value 1.

Theorem 1.1. *Let G be a connected simple Lie group with finite center. Given a non–central element g in G there is an elliptic element $h \in G$ such that h and ghg^{-1} generate a dense subsemigroup of G.*

This is stronger than what is sometimes called 1.5–generatedness of a topological group: given one (non–central) element g there is an element h such that the subgroup generated by g and h is dense in G. See [GKS] and [B], where 1.5–generatedness is proved for certain groups resp. Lie algebras.

Theorem 1.1 is a consequence of the following result.

Theorem 1.2. *Let G be as above. Let g be a non–central element of G. Then there is a one–dimensional compact torus T in G such that $T \cup gTg^{-1}$ generates G (as a group or as a semigroup, there is no difference). If T is such a torus then for every $t \in T$, except for a finite set of torsion elements, the subsemigroup of G generated by $\{t, gtg^{-1}\}$ is dense in G.*

We also prove an infinitesimal version of our theorems which may be of independent interest, see proposition 2.8.

Theorem 1.2 implies in particular that there are two conjugate one–dimensional tori in G which generate G as a (semi–)group. The following example shows that there may be a one–dimensional torus in G no two conjugates of which generate G.

Example 1.3. Let G be the connected component of $SO(3,2)$ and let $K = SO_3 \times SO_2$ (a maximal compact subgroup of G). If we choose a one–dimensional torus T in either of the two factors of K then T has a 3–dimensional vector space of fixed points in the natural representation of G on $\mathbb{R}^5$, hence the union of two conjugates of T fixes a one-dimensional subspace and thus cannot generate G. This example also shows that one cannot interchange the roles of g and h: Given an elliptic non–central element h in G, it may happen that no two conjugates of h generate a dense subsemigroup of G.

Remark 1.4. Our theorems do not always hold if G has infinite center. E.g. the universal covering group of $SL_2(\mathbb{R})$ has no compact subgroup $\neq \{1\}$. But even if such G has a non–trivial compact subgroup, our theorem may fail, e.g. for the universal covering group of $SO(3,2)$, as follows from the argument in example 1.3.

In section 2 we give the essential part of the proof and in section 3 we draw conclusions, give generalizations and discuss the sets of relevant tori and group elements in our existence theorems.

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2 Proof

All our Lie groups will be connected and by a simple Lie group we mean one whose Lie algebra is simple.

In this section we will prove the following theorem.

Theorem 2.1. *Let G be a semisimple real Lie group with finite center and let T be a maximal compact torus of G. Let $C \subset G$ be a conjugacy class, which is not contained in a proper normal subgroup of G. Then there is an element $g \in C$ for which $T \cup gTg^{-1}$ generates G. In fact, the set of such g is Zariski open (and hence topologically dense) in C.*

This theorem implies all the results claimed in the introduction and more technical versions of them, as will be explained in the next section. Clearly, if it is true for some Lie group with finite center locally isomorphic to G, then it is true for G.

Notation 2.2. For the proof we will pass to the complex case. We shall therefore use in the rest of this section a notation that differs from the one used so far. Let G be a simply connected semisimple complex Lie group and let $G(\mathbb{R})$ be a real form of G. Let σ be a Cartan involution of $G(\mathbb{R})$. We denote also by σ its extension to G. Let G_0 be the subgroup of fixed points of σ in G. Then $G_0(\mathbb{R})$ is a compact real form of G_0. If T_0 is a maximal algebraic torus of G_0 defined over $\mathbb{R}$, then $T_0(\mathbb{R})$ is a maximal compact torus of $G(\mathbb{R})$ (lying in $G_0(\mathbb{R})$). Finally, let $C \subset G$ be a conjugacy class defined over $\mathbb{R}$, and let $C(\mathbb{R})^\circ$ be a connected component of $C(\mathbb{R})$. Then $C(\mathbb{R})^\circ$ is a conjugacy class in $G(\mathbb{R})$. It suffices to prove theorem 2.1 for the group $G(\mathbb{R})$, the maximal compact torus $T_0(\mathbb{R})$, and the conjugacy class $C(\mathbb{R})^\circ$.

We will deduce theorem 2.1 from the following complex analog of it.

Theorem 2.3. *Let G be a simply connected semisimple complex Lie group, and let σ be an involutory automorphism of G. Let $G_0 \subset G$ be the subgroup of fixed points of σ, and let T_0 be a maximal torus of G_0. Let $C \subset G$ be a conjugacy class, which is not contained in a proper normal subgroup of G. Then there is an element $g \in C$ for which $T_0 \cup gT_0g^{-1}$ generates G. In fact, the set of such g is Zariski open in C.*

For a subset $M \subset G$ let $\langle M \rangle$ denote the subsemigroup of G generated by M. If $M = M^{-1}$ then $\langle M \rangle$ is in fact a subgroup. For any $x \in G$ set

$$G(x) = \langle T_0 \cup xT_0x^{-1} \rangle.$$

Clearly, $G(x)$ is an algebraic subgroup of G whose Lie algebra $\mathfrak{g}(x)$ is generated by $\mathfrak{t}_0 \cup \mathrm{Ad}(x)\mathfrak{t}_0$, where $\mathfrak{t}_0$ is the Lie algebra of T_0. It follows that $G(x) = G$ if and only if some (multiple) commutators of elements of $\mathfrak{t}_0$ and $\mathrm{Ad}(x)\mathfrak{t}_0$ form a basis of $\mathfrak{g}$. This shows that the set of $x \in G$, for which $G(x) = G$, is Zariski open in G, whence the last assertion of theorem 2.3 follows.

Let us now explain how to deduce theorem 2.1 from theorem 2.3. In the notation 2.2, theorem 2.3 asserts the existence of an element $g \in C$ such that $\mathfrak{t}_0 \cup \mathrm{Ad}(g)\mathfrak{t}_0$ generates the Lie algebra $\mathfrak{g}$. Since the set of such g is Zariski open in C, one may assume that $g \in C(\mathbb{R})^\circ$. Then $\mathfrak{t}_0(\mathbb{R}) \cup \mathrm{Ad}(g)\mathfrak{t}_0(\mathbb{R})$ generates the Lie algebra $\mathfrak{g}(\mathbb{R})$ and hence $T_0(\mathbb{R}) \cup gT_0(\mathbb{R})g^{-1}$ generates the group $G(\mathbb{R})$.

In the rest of this section, we shall use the notation of theorem 2.3.

Let us first reduce the proof of theorem 2.3 to the folllowing two cases:

Case 1. The group G is simple.

Case 2. The group G is a direct product of two simple factors permuted by σ and the conjugacy class C is σ-invariant.

If $G = G' \times G''$, then $C = C' \times C''$, where C' (resp. C'') is a conjugacy class in G' (resp. G''). If in addition G' and G'' are σ-invariant, then $G_0 = G'_0 \times G''_0$ and $T_0 = T'_0 \times T''_0$, where T'_0 (resp. T''_0) is a maximal torus in G'_0 (resp. G''_0). Hence, if the assertion of the theorem is true for G' and G'', it is true for G as well. Thus, it only remains to consider the case when G is a direct product of two simple factors permuted by σ.

Let $G = H \times H$, where H is a simple group, and let $\sigma((x, y)) = (y, x)$ for $x, y \in H$. Then G_0 is the diagonal in $H \times H$ and one may assume that T_0 is the diagonal in $S \times S$, where S is a maximal torus in H. Furthermore, $C = C_1 \times C_2$, where C_1 and C_2 are some non-central conjugacy classes in H. Suppose that the assertion of the theorem is true for the group H and its identity automorphism. Then, for a generic $h_1 \in C_1$ (resp. $h_2 \in C_2$) $S \cup h_1Sh_1^{-1}$ (resp. $S \cup h_2Sh_2^{-1}$) generates H. It follows that, for a generic $g = (h_1, h_2) \in C_1 \times C_2$, the subgroup $G(g)$ projects onto each of the two simple factors of G. If nevertheless $G(g) \neq G$, then

$$G(g) = \{(h, \theta(h)) | h \in H\},$$

where θ is some automorphism of H.

Since $G(g) \supset T_0$, θ must fix S pointwise, which means that θ is the conjugation by some $s_1 \in S$. Furthermore, since $G(g) \supset gT_0g^{-1}$, one has

$$h_2sh_2^{-1} = s_1h_1sh_1^{-1}s_1^{-1} \text{ for all } s \in S,$$

which means that $s_1h_1 = h_2s_2$ for some $s_2 \in S$. Thus, the assertion of the theorem is false for $G = H \times H$ and $C = C_1 \times C_2$ if and only if $h_2 \in Sh_1S$ (or, equivalently, $h_1 \in Sh_2S$) for generic $h_1 \in C_1$ and $h_2 \in C_2$. In the latter case, C_1 lies in the closure of $S\,h_2S$ for a generic $h_2 \in C_2$ and C_2 lies in the closure of $S\,h_1S$ for a generic $h_1 \in C_1$, and hence C_1 lies in the closure of $S\,h_1 S$ for a generic $h_1 \in C_1$. Thus, if the assertion of the theorem does not hold for $C = C_1 \times C_2$, it also does not hold for $C = C_1 \times C_1$, that is, in the case when C is σ-invariant.

In the rest of this section we will suppose that one of the cases 1 and 2 occurs.

Let $x = x_s x_u$ be the Jordan decomposition of an element $x \in C$. Then the closure of C contains the conjugacy class C_s of x_s. Clearly, if $\sigma(C) = C$, then $\sigma(C_s) = C_s$. Assume that C_s does not lie in the center of G and that there is $g_s \in C_s$ such that $G(g_s) = G$. Then, since the set of $x \in G$, for which $G(x) = G$, is Zariski open in G, there is also $g \in C$ such that $G(g) = G$. If C_s does lie in the center of G, one can reduce the proof to the case when $C_s = \{e\}$, i.e. C consists of unipotent elements, by multiplying C by a suitable element of the center. Thus, it suffices to prove the theorem in the cases, when C consists of semisimple or of unipotent elements.

We will use the following lemma.

Lemma 2.4. *Let s be a semisimple automorphism of $\mathfrak{g}$. Then the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ generated by $(s-1)\mathfrak{g}$ is an ideal.*

Proof. Clearly, $\mathfrak{h}$ is normalized by $(s-1)\mathfrak{g}$ and also by the kernel $\ker(s-1)$ of $s-1$, which is the subalgebra of fixed points of s. Since $\mathfrak{g} = (s-1)\mathfrak{g} \oplus \ker(s-1)$, $\mathfrak{h}$ is normalized by $\mathfrak{g}$. □

The centralizer $\mathfrak{t} = \mathfrak{z}(\mathfrak{t}_0)$ of $\mathfrak{t}_0$ is a maximal torus in $\mathfrak{g}$, see e.g. [He, Ch. X, Lemma 5.3]. The connected component T of the centralizer of $\mathfrak{t}_0$ in G is then a maximal torus in G. (In fact the centralizer of $\mathfrak{t}_0$ is connected, see e.g. [GOV, Ch.3, Theorem 3.13], but we do not need this fact.)

Let

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \tag{2.5}$$

be the root space decomposition of $\mathfrak{g}$ with respect to T. For any $\alpha \in \Delta$, set $\bar{\alpha} := \alpha\,|_{T_0}$. Note that $\bar{\alpha} \neq 0$. Moreover, it is known that $\bar{\alpha} = \bar{\beta}$ for $\alpha, \beta \in \Delta$ only if $\beta = \alpha$ or $\beta = \sigma(\alpha)$, see e.g. [GOV, Ch. 3, Theorem 3.14(1)]. Let

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\bar{\alpha} \in \bar{\Delta}} \mathfrak{g}_{\bar{\alpha}} \tag{2.6}$$

be the root space decomposition of $\mathfrak{g}$ with respect to T_0. We have $\dim \mathfrak{g}_{\bar{\alpha}} = 1$ or 2 for any $\bar{\alpha} \in \bar{\Delta}$.

Passing to the proof of the theorem in the case when C consists of semisimple elements, take $g_0 \in C \cap T$. By lemma 2.4 it suffices to prove that for a generic $x \in C$ the Lie algebra $\mathfrak{g}(x)$ generated by $\mathfrak{t}_0 \cup \mathrm{Ad}(x)\mathfrak{t}_0$ contains $(\mathrm{Ad}(g_0) - 1)\mathfrak{g}$. It thus suffices to prove that for every $\alpha \in \Delta$ with $\alpha(g_0) \neq 1$ we have $\mathfrak{g}(x) \supset \mathfrak{g}_\alpha$ for generic $x \in C$.

Suppose $\alpha(g_0) \neq 1$. Let $x(t) = \exp t\xi$, $t \in \mathbb{C}$, be the one–parameter subgroup of G with infinitesimal generator $\xi \in \mathfrak{g}$. For $\eta \in \mathfrak{t}_0$ we have

$$\frac{d}{dt}\mathrm{Ad}(x(t)g_0\, x(t)^{-1})\eta \mid_{t=0} = \frac{d}{dt}(\mathrm{Ad}(x(t))\mathrm{Ad}(g_0)\mathrm{Ad}(x(t)^{-1})\eta) \mid_{t=0}$$
$$= \mathrm{ad}(\xi)\mathrm{Ad}(g_0)\eta - \mathrm{Ad}(g_0)\mathrm{ad}(\xi)\eta = [(1-\mathrm{Ad}(g_0))\xi, \eta], \quad (2.7)$$

since $\mathrm{Ad}(g_0)\eta = \eta$. Choosing the $\mathfrak{g}_\alpha$–component of ξ to be non–zero we see that for a generic $x \in C$, $\mathrm{Ad}(x)\mathfrak{t}_0$ projects onto $\mathfrak{g}_\alpha$.

If $\sigma(\alpha) = \alpha$ then $\mathfrak{g}_{\bar\alpha} = \mathfrak{g}_\alpha$ and hence for a generic $x \in C$ we obtain $\mathfrak{g}(x) \supset \mathfrak{g}_\alpha$ since $\mathfrak{g}(x)$ is T_0-invariant.

If $\sigma(\alpha) \neq \alpha$ then $\mathfrak{g}_{\bar\alpha} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma(\alpha)}$. Take non-zero vectors $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Then $\sigma(e_\alpha) \in \mathfrak{g}_{\sigma(\alpha)}$ and $\sigma(e_{-\alpha}) \in \mathfrak{g}_{-\sigma(\alpha)}$. Put $h_a = [e_\alpha, e_{-\alpha}]$ and $h_{\sigma(\alpha)} = \sigma(h_\alpha) = [\sigma(e_\alpha), \sigma(e_{-\alpha})]$. We may assume that $\alpha(h_a) = 2$. Note that $\delta := \alpha - \sigma(\alpha)$ is not a root, since $\delta \mid_{\mathfrak{t}_0} = 0$. It follows that $[e_\alpha, \sigma(e_{-\alpha})] = [e_{-\alpha}, \sigma(e_\alpha)] = 0$ and $\sigma(\alpha)(h_a) = \alpha(\sigma(h_\alpha)) \leq 0$ (since $\sigma(e_\alpha)$ is a lowest weight vector for the adjoint representation of the subalgebra $\langle e_\alpha, h_\alpha, e_{-\alpha}\rangle$ in $\mathfrak{g}$).

Taking $\xi = e_\alpha + e_{-\alpha}$ in 2.7, we see that there exists $x \in C$ such that the projection of $\mathrm{Ad}(x)\mathfrak{t}_0$ to $\mathfrak{g}_{\bar\alpha}$ contains a one-dimensional subspace $\mathbb{C}\cdot e$ arbitrarily close to $\mathbb{C}\cdot e_\alpha$ and, at the same time, its projection to $\mathfrak{g}_{-\bar\alpha}$ contains a one-dimensional subspace $\mathbb{C}\cdot e_-$ arbitrarily close to $\mathbb{C}\cdot e_{-\alpha}$. Then $\mathfrak{g}(x) \supset \mathbb{C}\cdot e + \mathbb{C}\cdot e_-$ and hence $\mathfrak{g}(x) \ni [e, e_-] = \lambda\, h_\alpha + \mu\, \sigma(h_\alpha)$ with $\frac{\mu}{\lambda}$ small. In particular, we may assume that $\lambda \neq \mu$. Then $\mathrm{ad}(\lambda\, h_\alpha + \mu\, \sigma(h_\alpha)$ has different eigenvalues on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\sigma(\alpha)}$, namely $2\lambda + \mu\alpha(\sigma(h_\alpha))$ and $\lambda\sigma(\alpha)(h_\alpha) + 2\mu = \lambda\alpha(\sigma(h_\alpha)) + 2\mu$. Hence, $\mathfrak{g}(x) \supset \mathfrak{g}_\alpha$ for some $x \in C$.

It remains to consider the case when C consists of unipotent elements.

We first deal with the infinitesimal case which may be of independent interest.

Proposition 2.8. *With notation and hypotheses as in theorem 2.3, let $\mathfrak{c} \subset \mathfrak{g}$ be an adjoint orbit, which is not contained in a proper ideal of $\mathfrak{g}$. Then there is an element $\gamma \in \mathfrak{c}$ such that $\mathfrak{t}_0 \cup [\gamma, \mathfrak{t}_0]$ generates the algebra $\mathfrak{g}$. In fact, the set of such γ is Zariski open in $\mathfrak{c}$.*

(Note that we do not assume here that γ is nilpotent.)

Proof. For any $\gamma \in \mathfrak{g}$, let $\mathfrak{g}(\gamma)$ denote the subalgebra generated by $\mathfrak{t}_0 \cup [\gamma, \mathfrak{t}_0]$. For the same reason as in the proof of the second assertion of 2.2, the set of elements $\gamma \in \mathfrak{g}$ such that $\mathfrak{g}(\gamma) = \mathfrak{g}$, is Zariski open in $\mathfrak{g}$, whence the second assertion of the proposition follows.

For any $\xi \in \mathfrak{g}$ and $\alpha \in \Delta$ we will denote by ξ_α the projection of ξ to $\mathfrak{g}_\alpha$ in the decomposition 2.5. Similarly, for any $\bar\alpha \in \bar\Delta$ we will denote by $\xi_{\bar\alpha}$ the projection of ξ to $\mathfrak{g}_{\bar\alpha}$ in the decomposition 2.6.

Since the linear span of $\mathfrak{c}$ is an ideal, it coincides with $\mathfrak{g}$. It follows that for a generic $\gamma \in \mathfrak{c}$ and any $\alpha \in \Delta$ we have $\gamma_\alpha \neq 0$ and hence $\mathfrak{g}(\gamma)$ contains a one-dimensional subspace of $\mathfrak{g}_{\bar\alpha}$. If moreover $\sigma(\alpha) = \alpha$ then $\mathfrak{g}(\gamma) \supset \mathfrak{g}_\alpha = \mathfrak{g}_{\bar\alpha}$.

Let now $\sigma(\alpha) \neq \alpha$. Since the roots α and $\sigma(\alpha)$ are of equal length and their difference is not a root, they generate a σ-invariant root subsystem of type $A_1 + A_1$ or A_2, in which they are simple roots.

Take $\gamma \in \mathfrak{c}$ with non-zero components $\gamma_\alpha, \gamma_{-\alpha}, \gamma_{\sigma(\alpha)}, \gamma_{-\sigma(\alpha)}$ and set

$$\gamma(t) = \mathrm{Ad}(\exp te_\alpha)\gamma = (\exp t\mathrm{ad}(e_\alpha))\gamma.$$

Note that

$$\frac{d^k}{dt^k}\gamma(t)\mid_{t=0} = (\mathrm{ad}(e_\alpha))^k\gamma.$$

Since 2α, $2\alpha - \sigma(\alpha)$, $\alpha - \sigma(\alpha)$ are not roots, we obtain

$$\frac{d^2}{dt^2}[\gamma(t)_{\bar{\alpha}}, \gamma(t)_{-\bar{\alpha}}]\mid_{t=0} = \frac{d^2}{dt^2}[\gamma(t)_\alpha, \gamma(t)_{-\alpha}]\mid_{t=0} + \frac{d^2}{dt^2}[\gamma(t)_{\sigma(\alpha)}, \gamma(t)_{-\sigma(\alpha)}]\mid_{t=0} = [(\mathrm{ad}(e_\alpha))^2\gamma_{-\alpha}, \gamma_{-\alpha}],$$

which is a non–zero multiple of h_α. It follows that for some t the subalgebra $\mathfrak{g}(\gamma(t))$ contains an element $\lambda h_\alpha + \mu\sigma(h_\alpha)$ with $\lambda \neq \mu$, and hence $\mathfrak{g}(\gamma(t))$ contains $\mathfrak{g}_{\bar{\alpha}}$ and $\mathfrak{g}_{-\bar{\alpha}}$.

Thus, for a generic $\gamma \in \mathfrak{c}$ the subalgebra $\mathfrak{g}(\gamma)$ contains all the subspaces $\mathfrak{g}_{\bar{\alpha}}$, $\bar{\alpha} \in \bar{\Delta}$, and hence coincides with $\mathfrak{g}$. □

We now pass to the proof of theorem 2.3 in the case, when C consists of unipotent elements. In this case $C = \exp \mathfrak{c}$, where $\mathfrak{c} \subset \mathfrak{g}$ is an adjoint orbit consisting of nilpotent elements. By proposition 2.5 there exists an element $\gamma \in \mathfrak{c}$ such that $\mathfrak{t}_0 \cup [\gamma, \mathfrak{t}_0]$ generates the algebra $\mathfrak{g}$. This means that some (multiple) commutators of elements of $\mathfrak{t}_0$ and $[\gamma, \mathfrak{t}_0]$ constitute a basis of $\mathfrak{g}$. Replacing each element $[\gamma, \eta]$, $\eta \in \mathfrak{t}_0$ by $(\mathrm{Ad}(\exp t\gamma) - 1)\eta$ with sufficiently small $|t|$, we shall still obtain a basis of $\mathfrak{g}$. Since (due to the Morozov–Jacobson theorem) $t\gamma \in \mathfrak{c}$ for all $t \neq 0$, the assertion of the theorem follows.

3 Conclusions

For the whole section let G be a semisimple real Lie group with finite center, and let g_0 be a fixed element of G which is not contained in a proper normal subgroup of G. Equivalently, $\mathrm{Ad}(g_0)$ fixes no non–zero ideal in the Lie algebra $\mathfrak{g}$ of G, or, again equivalently, for every non-trivial factorgroup of G the image of g is not central.

Recall that we denote by $\langle M \rangle$ the subsemigroup of G generated by M.

Theorem 3.1. *There is a one–dimensional compact torus T^1 in G such that $\langle T^1 \cup gT^1g^{-1}\rangle = G$. For such T^1 the subgroup $\langle t, gtg^{-1}\rangle$ is dense in G for every $t \in T^1$ except for a finite set of torsion elements.*

Proof. By theorem 2.1, there is a (maximal) compact torus T in G such that $\langle T \cup gTg^{-1}\rangle = G$. Take a dense one–parameter subgroup $H_1 = \{\exp t\xi, t \in \mathbb{R}\}$ in T. Then the group $H = \langle H_1 \cup gH_1g^{-1}\rangle$ is dense in G. Clearly, H is a (not necessarily closed) Lie subgroup of G whose Lie algebra $\mathfrak{h}$ is generated by $\{\xi, \mathrm{Ad}(g)\xi\}$. But the only dense Lie subgroup in a semisimple Lie group is the group itself. Thus $\mathfrak{h} = \mathfrak{g}$ is generated by $\{\xi, \mathrm{Ad}(g)\xi\}$. The set of pairs of vectors in $\mathfrak{g}$ which generate the Lie algebra $\mathfrak{g}$ is Zariski open in $\mathfrak{g}\times\mathfrak{g}$. So our claim holds for all one–dimensional compact tori $\{\exp t\xi', t \in \mathbb{R}\}$ contained in T with ξ' near ξ. There are such ξ' near ξ since the set of infinitesimal generators of one–dimensional compact tori is topologically dense in the Lie algebra of any compact torus. This yields the first assertion of the theorem.

To continue the proof of theorem 3.1, let us recall the following theorem.

Theorem 3.2. *[BG1]. There is a neighborhood U of the unit e in G such that the exponential map has an analytic inverse* $\log : U \to G$ *and a tuple $(g_1, \dots, g_d) \in U^d$ generates a dense subgroup of G if and only if $\{\log g_1, \dots, \log g_d\}$ generates the Lie algebra $\mathfrak{g}$.*

It follows that there is a neighborhood V of e in G such that log is defined on V and $\{h, ghg^{-1}\}$ generates a dense subgroup of G if and only if $\{\log h, \mathrm{Ad}(g)\log h\}$ generates the Lie algebra $\mathfrak{g}$. Thus if $h \in V$ is elliptic then the subsemigroup $\langle h, ghg^{-1}\rangle$ is dense in G if and only if $\{\log h, \mathrm{Ad}(g)\log h\}$ generates the Lie algebra $\mathfrak{g}$.

Let now $\mathfrak{t}^1$ be the Lie algebra of our one–dimensional torus T^1 for which $\langle T^1 \cup gT^1g^{-1}\rangle = G$. Then $\mathfrak{t}^1 \cup \mathrm{Ad}(g)\mathfrak{t}^1$ generates γ, and by theorem 3.2 $\langle t, gtg^{-1}\rangle = G$ for every $t \in T^1 \cap V$, $t \neq e$. Let A be the set of $t \in T^1$ for which $\langle t, gtg^{-1}\rangle \neq G$. Clearly, if $t \in A$ then $t^m \in A$ for every $m \in \mathbb{Z}$. Thus, if $t \in T$ has infinite order, then $t \notin A$. Moreover, the orders of elements of A are bounded. It follows that the set A is finite. □

Remark 3.3. The equivalence of the different properties characterizing elliptic elements stated before theorem 1.1 is implied by the following facts.

1) Every non–empty compact subsemigroup of a topological group is a group. (See e.g. Theorem 30.6 in [St].)

2) [Po, Section 39, Lemma 1] Every element g of a locally compact topological group G has exactly one of the following two properties:

   α) g is contained in a compact subgroup of G;

   β) the subgroup of G generated by g is infinite cyclic and discrete.

Remark 3.4. Note that 1) implies that the closure of a subsemigroup generated by elliptic elements is actually a group and hence is also the closure of the group generated by these elements.

Remark 3.5. Theorem 3.2 implies that the set of pairs $(x, y) \in G\times G$ for which the group generated by $\{x, y\}$ is dense in G form a topologically open subset of $G \times G$, see [BG1].

Let $\mathfrak{E}(G)$ be the set of elliptic elements of G. Remark 3.5 together with theorem 3.1 yields

Corollary 3.6. *The set of $h \in \mathfrak{E}(G)$ such that $\langle h, ghg^{-1}\rangle$ is dense in G is topologically dense in $\mathfrak{E}(G)$.*

Let $\mathfrak{a}$ be the (algebraic) subset of elements $X \in \mathfrak{g}$ such that $\{X, \mathrm{Ad}(g)X\}$ does not generate the Lie algebra $\mathfrak{g}$. Let A be the subset of elements $h \in G$ such that the subgroup of G generated by $\{h, g\,hg^{-1}\}$ is not dense in G.

Proposition 3.7. *Let T be a compact torus in G and let $\mathfrak{t}$ be its Lie algebra. Then $\mathfrak{a}\cap\mathfrak{t}$ is a finite union of Lie algebras of subtori of T. Moreover, if A does not contain T, then $A \cap T$ is contained in a finite union of proper closed subgroups of T.*

Proof. There is a neighborhood V of 0 in $\mathfrak{g}$ and a neighborhood U of e in G such that the exponential map $\exp : \mathfrak{g} \to G$ induces a diffeomorphism $V \to U$ and $V \cap \mathfrak{a}$ maps onto $U\cap A$, by theorem 3.2. Since $\mathbb{R}\,\mathfrak{a} \subset \mathfrak{a}$ and $x^n \in A$ if $n \in \mathbb{Z}$ and $x \in A$, for any $X \in \mathfrak{a}$ the one–parameter subgroup $\{\exp tX, t \in \mathbb{R}\}$ is contained in A. Since A is closed by remark 3.5, the closure of this one–parameter subgroup is also contained in A, and its Lie algebra is contained in $\mathfrak{a}$. Thus, $\mathfrak{a}$ is a union of the Lie algebras of subtori of T. Since there are only countably many of such subtori and an irreducible algebraic variety cannot be covered by countably many of proper closed subvarieties, every irreducible component of $\mathfrak{a} \cap \mathfrak{t}$ is the Lie algebra of a subtorus of T, whence the first assertion of the proposition follows.

Now let $T_1, \ldots, T_s$ be subtori of T such that $\mathfrak{a}\cap\mathfrak{t}$ is the union of their Lie algebras. Then $A \cap T \cap U = \bigcup_{i=1}^{s}(T_i \cap U)$. For every $x \in T$ there is a positive integer $m = m_x$ and a neighborhood $W = W_x$ of x in T such that $W^m \subset U$. Hence if $y \in A\cap W$ then $y^m \in T_i$ for some i. Since a finite number of such neighborhoods W cover T, a finite union of proper closed subgroups of T of the form $Y_{m,i} = \{y \in T; y^m \in T_i\}$ contains $A\cap T$. □

Proposition 3.8. *Let $X \in \mathfrak{g}$ be such that the corresponding one–parameter subgroup has compact closure. Then we have the following dichotomy. The set of $t \in \mathbb{R}$ such that $\exp tX \in A$ is either $\mathbb{R}$ or a discrete subset of $\mathbb{R}$.*

Proof. Let T be the closure of our one parameter group, a compact torus, and let $\mathfrak{t}$ be its Lie algebra. We have to show that the set $B = \{t \in \mathbb{R}; \exp tX \in A\}$ equals $\mathbb{R}$ if B is not discrete. Let b be a cluster point of B. Since $\mathbb{Z}B \subset B$ we may assume that $b \in U$, with notations as in the proof of proposition 3.7. The image under $\log\circ\exp$ of some interval W around b is contained in an affine line of the form

$\mathbb{R}X + Y$ with $Y \in \ker \exp$. The interval $WX + Y$ in this line is contained in V and intersects the algebraic set $\mathfrak{a}$ in a non–discrete set, hence $WX + Y \subset \mathfrak{a}$ and hence $\mathbb{R}X + Y \subset \mathfrak{a}$. It follows that $X \in \mathfrak{a}$ since $\mathfrak{a}$ is homogeneous and thus $\exp tX \in A$ for every $t \in \mathbb{R}$. □

Remark 3.9. The pairs $(x, y) \in G \times G$ generating a Zariski dense subgroup form a topologically dense subset, see [V, Proposition 2]. But if G is non–compact, there is a topologically open non–empty subset of pairs $(x, y) \in G \times G$ for which the group generated by $\{x, y\}$ is free and discrete (but Zariski dense). This can be seen by a Schottky type construction using proximality.

References

[Au] Auerbach, H., *Sur les groupes linéaires bornés (III)*, Studia Math. 5 (1935), 43–49

[AGK] El Assoudi, R., J.P. Gauthier and I.A.K. Kupka, *On subsemigroups of semisimple Lie groups*, Annales de l' I.H.P., Section C, **13** (1996), 117–133

[B] Bois, J.-M., *Generators of simple Lie algebras in arbitrary characteristics*, Math. Z. 2008

[BG1] Breuillard E. and T. Gelander, *On dense free subgroups of Lie groups*, J. of Algebra **261** (2003), 448–467

[BG2] Breuillard E. and T. Gelander, *A topological Tits alternative*, Ann. of Math. (2) **166** (2007), 427–474

[Fi] Field, M., *Generating sets for compact semisimple Lie groups*, Proc. Amer. Math. Soc. **127** (1999), 3361–3365

[GOV] Gorbatsevich, V. V., Onishchik, A. L. and Vinberg, E. B., *Lie groups and Lie algebras, III., Structure of Lie groups and Lie algebras*, A translation of Current problems in mathematics. Fundamental directions. Vol. 41 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990, translation by V. Minachin [V. V. Minakhin], Encyclopaedia of Mathematical Sciences, 41. Springer-Verlag, Berlin, 1994. iv+248 pp

[GKS] Guralnick, R.M., Kantor, W.M. and Saxl, J., *The probability of generating a classical group*, Communications in Algebra **22** (1994), 1395–1402, see also the review MR 1261266 of this paper by A. Lubotzky

[He] Helgason, S., *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics **34**, corrected reprint of the 1978 original, American Mathematical Society, Providence, RI, (2001), xxvi+641

[Ho] Hochschild, G., *The structure of Lie groups*, Holden-Day Inc., San Francisco (1965), ix+230

[HM] Hofmann, K.H. and S.A. Morris, *Finitely generated connected locally compact groups*, Seminar Sophus Lie 2 (1992), 123–134

[I] Ionescu, T., *On the generators of semisimple Lie algebras*, Linear Algebra and Appl. **15** (1976), 271–292

[KoNo1] Kobayashi, S. and Nomizu, K., *Foundations of differential geometry,* Vol. I, Interscience Publishers, a division of John Wiley & Sons, New York-London, (1963), xi+329

[KoNo2] Kobayashi, S. and Nomizu, K., *Foundations of differential geometry*, Vol. II, Interscience Tracts in Pure and Applied Mathematics, No. 15, Vol. II Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, (1969), xv+470

[Ku1] Kuranishi, M., *Two elements generations on semi-simple Lie groups*, {Volume numbers not printed on issues until Vol. **7**}, (1955), Kodai Math. Sem. Rep., Kodai Mathematical Seminar Reports, **1**, (1949), No. 5-6, 9–10

[Ku2] Kuranishi, M., *On everywhere dense imbedding of free groups in Lie groups*, Nagoya Math. J., Nagoya Mathematical Journal **2** (1951), 63–71

[Ku3] Kuranishi, M., *On everywhere dense imbedding of free groups in Lie groups*, Nagoya Mathematical Journal **2** (1951), 63–71

[P] Poguntke, D., *The coproduct of two circles*, General Topology and its Applications **6** (1976), 127–144

[Po] Pontryagin, L.S., *Topological groups*, Translated from the second Russian edition by Arlen Brown, Gordon and Breach Science Publisher, Inc., New York, (1966), xv+543

[St] Stroppel, M., *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, (2006), x+302

[V] Vinberg, E.B., *On invariants of sets of matrices*, Journal of Lie Theory **6** (1996), 249–269

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