FRACTIONAL HARDY INEQUALITY WITH A REMAINDER TERM

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Abstract. We prove a Hardy inequality for the fractional Laplacian on the interval with the optimal constant and additional lower order term. As a consequence, we also obtain a fractional Hardy inequality with the best constant and an extra lower order term for general domains, following the method of M. Loss and C. Sloane [16].

1. Main result and discussion

Recently Loss and Sloane [16] have proven the following fractional Hardy inequality

$$\frac{1}{2}\int_{D\times D}\frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy \geq \kappa_{n,\alpha}\int_D \frac{u^2(x)}{\operatorname{dist}(x,D^c)^\alpha}\,dx, \quad u\in C_c(D), \tag{1.1}$$

for convex domains $D\subset\mathbb{R}^n$ and $1<\alpha<2$, where

$$\kappa_{n,\alpha} = \pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}\,\frac{B\left(\frac{1+\alpha}{2},\frac{2-\alpha}{2}\right)-2^\alpha}{\alpha 2^\alpha} \tag{1.2}$$

is the optimal constant. Here B is the Euler beta function, and $C_c(D)$ denotes the class of all continuous functions $u\colon\mathbb{R}^n\to\mathbb{R}$ with compact support in D. Inequality (1.1) with the optimal constant was earlier obtained for half-spaces and $\mathbb{R}^n\setminus\{0\}$, see [10, 11, 5, 9].

In this note we will prove the following strengthening of (1.1) for the interval.

Theorem 1.1. *Let $1<\alpha<2$, $-\infty<a<b<\infty$. For every $u\in C_c(a,b)$,*

$$\begin{aligned}\frac{1}{2}\int_a^b\int_a^b\frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy \;&\geq \kappa_{1,\alpha}\int_a^b u^2(x)\,\left(\frac{1}{x-a}+\frac{1}{b-x}\right)^\alpha dx\\ &+\frac{4-2^{3-\alpha}}{\alpha(b-a)}\int_a^b u^2(x)\,\left(\frac{1}{x-a}+\frac{1}{b-x}\right)^{\alpha-1} dx,\end{aligned} \tag{1.3}$$

and $\kappa_{1,\alpha}$ may not be replaced by a bigger constant in (1.3).

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For a domain $D \subset \mathbb{R}^n$ we consider a quadratic form

$$\mathcal{E}(u) = \frac{1}{2} \int_{D\times D} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy, \quad u \in C_c(D).$$

By the method developed by Loss and Sloane in [16], from Theorem 1.1 one gets the fractional Hardy inequality with a *remainder* term for general domains, namely Theorem 1.2. To state the result we recall the notation from [16, 7]. Let $D \subset \mathbb{R}^n$ be a bounded domain. For a direction $w \in S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ and $x \in D$ we define

$$d_{w,D}(x) = \min\{|t| \ : \ x + tw \notin D\}\ ,$$

$$\delta_{w,D}(x) = \sup\{|t| : x + tw \in D\}\ ,$$

(1.4)

$$\frac{1}{M_\alpha(x)^\alpha} = \frac{\int_{S^{n-1}} \left[\frac{1}{d_{w,D}(x)} + \frac{1}{\delta_{w,D}(x)}\right]^\alpha dw}{\int_{S^{n-1}} |w_n|^\alpha dw} = \frac{\int_{S^{n-1}} \left[\frac{1}{d_{w,D}(x)} + \frac{1}{\delta_{w,D}(x)}\right]^\alpha dw}{2\kappa_{n,\alpha}/\kappa_{1,\alpha}}\ .$$

Theorem 1.2. *Let $1 < \alpha < 2$ and we let $D \subset \mathbb{R}^n$ be a bounded domain. We have*

(1.5)

$$\mathcal{E}(u) \geq \kappa_{n,\alpha} \int_D \frac{u^2(x)}{M_\alpha(x)^\alpha}\,dx + \frac{\lambda_{n,\alpha}}{\operatorname{diam} D} \int_D \frac{u^2(x)}{M_{\alpha-1}(x)^{\alpha-1}}\,dx, \quad u \in C_c(D),$$

where

$$\lambda_{n,\alpha} = \frac{\pi^{(n-1)/2}\Gamma(\frac{\alpha}{2})(4-2^{3-\alpha})}{\alpha\Gamma(\frac{n+\alpha-1}{2})}.$$

In particular, if D is a bounded and convex domain, then

(1.6)

$$\mathcal{E}(u) \geq \kappa_{n,\alpha} \int_D \frac{u^2(x)}{\operatorname{dist}(x,D^c)^\alpha}\,dx + \frac{\lambda_{n,\alpha}}{\operatorname{diam} D} \int_D \frac{u^2(x)}{\operatorname{dist}(x,D^c)^{\alpha-1}}\,dx, \quad u \in C_c(D).$$

Constant $\kappa_{n,\alpha}$ in (1.5) *and* (1.6) *may not be replaced by a bigger constant.*

The above theorem is a strengthening of [16, Theorem 1.1]. The main difference is the remainder term in (1.5) and (1.6), under the additional assumption that the domain is bounded. We like to note that for cones (e.g., $\mathbb{R}^n \setminus \{0\}$) there is no remainder term of the form (1.6). We consider the dilations of u and see that the homogeneity of $\mathcal{E}(u)$ and the term $\int_D \frac{u^2(x)}{\operatorname{dist}(x,D^c)^\alpha}\,dx$ is the same, but different from that of $\int_D \frac{u^2(x)}{\operatorname{dist}(x,D^c)^{\alpha-1}}\,dx$.

As a consequence of Theorem 1.2 we may obtain estimates for the first eigenvalue of regional fractional Laplacian, see e.g. [15], or [14] for other applications of Hardy inequalities.

We denote

$$Lu(x) = \lim_{\varepsilon\to 0^+} \int_{(-1,1)\cap\{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{1+\alpha}}\,dy,$$

L is, up to a multiplicative constant, the regional fractional Laplacian [13]. We calculate Lw for the function $w(x) = (1-x^2)^{(\alpha-1)/2}$, see Lemma 2.1, and the result follows from the ideas of [1, 8], see also [9, Proposition 2.3] or Lemma 2.2 below. We like to note that the calculation of Lw is done by using the Kelvin transform. For a discussion on the Kelvin transform and the fractional Laplacian we refer the reader to [6].

We note that the explicit formula for $Lu_p(x)$, where $u_p(x) = (1-x^2)^p$, may be deduced from [12] for $p = \alpha/2$, and for $p = \frac{\alpha-2}{2}$ from [2]. See the remarks after the proof of Lemma 2.1.

Finally, let us note that the symmetric bilinear form obtained from $\mathcal{E}$ by polarisation is, up to a multiplicative constant, the Dirichlet form of the censored stable process in $(-1,1)$, see [3]. The following result is a counterpart of Lemma 2.3 and Theorem 1.1 corresponding to a killed stable process [3] and it turns out to have a simple form.

Corollary 1.3. *Let $0 < \alpha < 2$ and $w(x) = (1-x^2)^{(\alpha-1)/2}$. For every $u \in C_c(-1,1)$*

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy =& \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{u(x)}{w(x)}-\frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{1+\alpha}}\,dx\,dy \\
&+ \frac{B(\frac{1+\alpha}{2},\frac{2-\alpha}{2})}{\alpha}\int_{-1}^{1} u^2(x)\,(1-x^2)^{-\alpha}dx \\
\geq& \frac{B(\frac{1+\alpha}{2},\frac{2-\alpha}{2})}{\alpha 2^\alpha}\int_{-1}^{1} u^2(x)\,\left(\frac{1}{x+1}+\frac{1}{1-x}\right)^\alpha dx. \qquad (1.7)
\end{aligned}$$

The constant $\frac{B(\frac{1+\alpha}{2},\frac{2-\alpha}{2})}{\alpha 2^\alpha}$ in the above inequality may not be replaced by a bigger one.

2. Proofs

We start by calculating the regional fractional Laplacian on power functions.

Lemma 2.1. *Let $p > -1$ and $u_p(x) = (1-x^2)^p$. For $0 < \alpha < 2$ we have*

$$\begin{aligned}
Lu_p(x) = \frac{(1-x^2)^{p-\alpha}}{\alpha}\Bigg(&(1-x)^\alpha + (1+x)^\alpha \\
&- (2p+2-\alpha)B(p+1,1-\alpha/2) + \alpha I(p)\Bigg), \qquad (2.1)
\end{aligned}$$

where

$$I(p) = p.v.\int_{-1}^{1} \frac{(1-tx)^{\alpha-1-2p}-1}{|t|^{1+\alpha}}\,(1-t^2)^p\,dt.$$

We have $I(\frac{\alpha}{2}) = \frac{2}{\alpha}B(1+\frac{\alpha}{2}, 1-\frac{\alpha}{2})\big(1-(1-x^2)^{\alpha/2}\big)$, $I(\frac{\alpha-1}{2}) = I(\frac{\alpha-2}{2}) = 0$, *and if* $1<\alpha<2$, *then* $I(\frac{\alpha-3}{2}) = x^2 B(\frac{\alpha-1}{2}, 1-\alpha/2)$.

Proof. By changing variable $t = y^2$ and integrating by parts, we have

$$\begin{aligned} Lu_p(0) &= 2\lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{(1-y^2)^p - 1}{y^{1+\alpha}}\,dy \\ &= 2\lim_{\varepsilon\to 0^+}\left(\frac{1}{2}\int_{\varepsilon^2}^1 (1-t)^p t^{-1-\alpha/2}[(1-t)+t]\,dt - \int_\varepsilon^1 y^{-1-\alpha}\,dy\right) \\ &= 2\lim_{\varepsilon\to 0^+}\left(\frac{1}{\alpha}(1-\varepsilon^2)^{p+1}\varepsilon^{-\alpha} - \frac{p+1}{\alpha}\int_{\varepsilon^2}^1 (1-t)^p t^{-\alpha/2}\,dt \right.\\ &\qquad \left. + \frac{1}{2}\int_{\varepsilon^2}^1 (1-t)^p t^{-\alpha/2}\,dt + \frac{1}{\alpha} - \frac{\varepsilon^{-\alpha}}{\alpha}\right). \end{aligned}$$

It is easy to see that

$$\lim_{\varepsilon\to 0^+}\left(\frac{1}{\alpha}(1-\varepsilon^2)^{p+1}\varepsilon^{-\alpha} - \frac{\varepsilon^{-\alpha}}{\alpha}\right) = \lim_{\varepsilon\to 0^+}\frac{\varepsilon^{2-\alpha}}{\alpha}\,\frac{(1-\varepsilon^2)^{p+1}-1}{\varepsilon^2} = 0.$$

Hence

$$Lu_p(0) = \frac{2}{\alpha}\left[1 - (p+1-\alpha/2)B(p+1, 1-\alpha/2)\right].$$

For $x_0 \in (-1,1)$ we have

$$Lu_p(x_0) = p.v.\int_{-1}^1 \frac{(1-y^2)^p - (1-x_0^2)^p}{|y-x_0|^{1+\alpha}}\,dy.$$

We change the variable in the following way

$$t = \varphi(y) = \frac{x_0 - y}{1 - x_0 y}, \qquad y = \varphi(t), \qquad \text{then}$$

$$\varphi'(y) = \frac{x_0^2 - 1}{(1-x_0 y)^2}, \quad y - x_0 = \frac{t(1-x_0^2)}{t x_0 - 1}, \quad 1 - y^2 = \frac{(1-x_0^2)(1-t^2)}{(t x_0 - 1)^2}.$$

The principal value integral transforms as follows

$$\begin{aligned} Lu_p(x_0) &= (1-x_0^2)^{p-\alpha} p.v.\int_{-1}^1 \frac{(1-t^2)^p - (1-tx_0)^{2p}}{|t|^{1+\alpha}}\,(1-tx_0)^{\alpha-1-2p}\,dt \\ &= (1-x_0^2)^{p-\alpha}\left[Lu_p(0) - p.v.\int_{-1}^1 \frac{(1-tx_0)^{\alpha-1} - 1}{|t|^{1+\alpha}}\,dt \right. \\ &\quad \left. + p.v.\int_{-1}^1 \frac{(1-tx_0)^{\alpha-1-2p} - 1}{|t|^{1+\alpha}}\,(1-t^2)^p\,dt\right]. \end{aligned} \tag{2.2}$$

We consider the integral in (2.2),

$$I = p.v.\int_{-1}^1 \frac{(1-tx_0)^{\alpha-1} - 1}{|t|^{1+\alpha}}\,dt = \lim_{\varepsilon\to 0^+}\left(J_\varepsilon(x_0) + J_\varepsilon(-x_0)\right),$$

where

$$\begin{aligned}
J_\varepsilon(x_0) &= \int_\varepsilon^1 \frac{(1-tx_0)^{\alpha-1}-1}{t^{1+\alpha}}\,dt = \int_\varepsilon^1 \Big(\frac{1}{t}-x_0\Big)^{\alpha-1}\frac{dt}{t^2} - \frac{\varepsilon^{-\alpha}-1}{\alpha}\\
&= \frac{1}{\alpha}\Big(\frac{1}{\varepsilon}-x_0\Big)^{\alpha} - \frac{1}{\alpha}\Big(1-x_0\Big)^{\alpha} - \frac{\varepsilon^{-\alpha}-1}{\alpha}\\
&= \frac{1}{\alpha} - \frac{1}{\alpha}\Big(1-x_0\Big)^{\alpha} + \frac{(1-\varepsilon x_0)^{\alpha}-1}{\alpha\varepsilon^{\alpha}}.
\end{aligned}$$

By l'Hôspital rule we find out that

$$I = \frac{2}{\alpha} - \frac{1}{\alpha}\Big(1-x_0\Big)^{\alpha} - \frac{1}{\alpha}\Big(1+x_0\Big)^{\alpha},$$

and the first part of the lemma is proved.

We have

$$\begin{aligned}
I(\alpha/2) &= p.v.\int_{-1}^1 \frac{(1-tx)^{-1}-1}{|t|^{1+\alpha}}\,(1-t^2)^{\alpha/2}\,dt\\
&= \int_{-1}^1 \frac{\sum_{k=2}^\infty (tx)^k}{|t|^{1+\alpha}}\,(1-t^2)^{\alpha/2}\,dt = 2\int_0^1 \frac{\sum_{k=1}^\infty (tx)^{2k}}{|t|^{1+\alpha}}\,(1-t^2)^{\alpha/2}\,dt\\
&= \sum_{k=1}^\infty B\Big(k-\frac{\alpha}{2}, 1+\frac{\alpha}{2}\Big)x^{2k}\\
&= \Gamma(1+\frac{\alpha}{2})\Gamma(\frac{-\alpha}{2})\left(\sum_{k=0}^\infty \frac{x^{2k}\Gamma(k-\frac{\alpha}{2})}{\Gamma(\frac{-\alpha}{2})k!} - 1\right)\\
&= \frac{2B(1+\frac{\alpha}{2}, 1-\frac{\alpha}{2})}{\alpha}\Big(1-(1-x^2)^{\alpha/2}\Big).
\end{aligned}$$

Calculating $I(p)$ for $p = \frac{\alpha-1}{2}$, $p = \frac{\alpha-2}{2}$ and $p = \frac{\alpha-3}{2}$ is easy and will be omitted. □

In the sequel we will need the above lemma only for $p = \frac{\alpha-1}{2}$. The fractional Laplacian on $u_{\alpha/2}$, extended to be zero on $\mathbb{R}\setminus(-1,1)$, was calculated by using Fourier transform and hypergeometric function in [12]. From those calculations we may confirm our formula for $Lu_{\alpha/2}$ and, consequently, for $I(\alpha/2)$. Also the value of $Lu_{(\alpha-2)/2}$ can be calculated from known results. Namely, $u_{(\alpha-2)/2}(x) = \frac{1}{2}(K(x,-1)+K(x,1))$ for $|x|<1$, where $K(x,Q) = \frac{(1-x^2)^{\alpha/2}}{|x-Q|}$ is the Martin kernel for the interval [2, (3.36)]. Hence $u_{(\alpha-2)/2}(x)$ extended to be zero on $\mathbb{R}\setminus(-1,1)$ annihilates on $(-1,1)$ the fractional Laplacian, see also [4, Chapter 3] and [3, (3.14)].

The next lemma may be considered a special case of Proposition 2.3 from [9], see also [8]. For reader's convenience we give an elementary proof following [5].

Lemma 2.2. *Let $D \subset \mathbb{R}^n$ be an open set. For every $u \in C_c(D)$ and any strictly positive function $w \in C^2(D) \cap L^1(D, (1+|x|)^{-n-\alpha}\,dx)$, we have*

$$\mathcal{E}(u) = \int_D u^2(x)\frac{-Lw(x)}{w(x)}\,dx + \frac{1}{2}\int_D\int_D \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{n+\alpha}}\,dx\,dy.$$

Proof. We have

$$(u(x)-u(y))^2 + u^2(x)\frac{w(y)-w(x)}{w(x)} + u^2(y)\frac{w(x)-w(y)}{w(y)}$$
$$= \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 w(x)w(y).$$

We integrate against $1_{\{|x-y|>\varepsilon\}}|x-y|^{-n-\alpha}\,dx\,dy$, and let $\varepsilon \to 0$. We can use Taylor's expansion for w and the compactness of the support of u to justify the application of Lebesgue dominated convergence theorem. □

We next state a result analogous to the ground state representation obtained for half-spaces and $\mathbb{R}^n \setminus \{0\}$ by Frank and Seiringer [9, 11]. In the following result we return to $D = (-1,1)$ and $n = 1$.

Lemma 2.3. *Let $0 < \alpha < 2$. Let $w(x) = (1-x^2)^{(\alpha-1)/2}$. For every $u \in C_c(-1,1)$,*

$$\begin{aligned}\mathcal{E}(u) &= \frac{1}{2}\int_{-1}^1\int_{-1}^1 \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{1+\alpha}}\,dx\,dy \\ &+ 2^\alpha \kappa_{1,\alpha}\int_{-1}^1 u^2(x)\,(1-x^2)^{-\alpha}dx \\ &+ \frac{1}{\alpha}\int_{-1}^1 u^2(x)\,\left[2^\alpha - (1+x)^\alpha - (1-x)^\alpha\right](1-x^2)^{-\alpha}\,dx.\end{aligned}$$

Proof of Lemma 2.3. The lemma follows immediately from Lemma 2.2 applied to $w(x) = (x^2-1)^{(\alpha-1)/2}$ and Lemma 2.1 with $p = (\alpha-1)/2$. □

Proof of Theorem 1.1. By scaling we may and do assume that $a = -1$ and $b = 1$. By Lemma 2.3 it is enough to verify that

$$2^\alpha - (1+x)^\alpha - (1-x)^\alpha \ge (2^\alpha - 2)(1-x^2), \quad 1 \le \alpha \le 2, \ \ 0 \le x \le 1. \tag{2.3}$$

Substituting $u = x^2$, it is enough to verify that

$$g(u) = (2^\alpha - 2)u - (1-\sqrt{u})^\alpha - (1+\sqrt{u})^\alpha + 2$$

is concave, or

$$g'(u) = 2^\alpha - 2 + \frac{\alpha}{2\sqrt{u}}\left((1-\sqrt{u})^{\alpha-1} - (1+\sqrt{u})^{\alpha-1}\right)$$

is decreasing. We substitute $u = t^2$ and observe that

$$\frac{(1-t)^{\alpha-1} - (1+t)^{\alpha-1}}{t} = \frac{h(t)-h(0)}{t},$$

where $h(t) = (1-t)^{\alpha-1} - (1+t)^{\alpha-1}$. Since h is concave, the function $t \mapsto \frac{h(t)-h(0)}{t}$ is decreasing, and so is g'. This proves (2.3), and (1.3).

The fact that $\kappa_{1,\alpha}$ in (1.3) is optimal follows from [5]. □

The constant $2^\alpha - 2$ in (2.3) is the largest possible, consider $x = 0$. However it is not clear if the constant $\frac{4-2^{3-\alpha}}{\alpha(b-a)}$ is optimal in (1.3).

Proof of Theorem 1.2. The proof is analogous to the proof of Theorem 1.1 in [16], but instead of applying [16, Corollary 2.3] we use Theorem 1.1. For reader's convenience we repeat the beginning of the argument of Loss and Sloane. By [16, Lemma 2.4] and [16, Corollary 2.3] we find that

$$
\begin{aligned}
\mathcal{E}(u) &= \frac{1}{4}\int_{S^{n-1}} dw \int_{\{x:x\cdot w=0\}} d\mathcal{L}_w(x) \int_{x+sw\in D} ds \int_{x+tw\in D} dt \frac{|u(x+sw)-u(x+tw)|^2}{|s-t|^{1+\alpha}}\\
&\geq \kappa_{1,\alpha}\frac{1}{2}\int_{S^{n-1}} dw \int_{\{x:x\cdot w=0\}} d\mathcal{L}_w(x) \int_{x+sw\in D} ds |u(x+sw)|^2\\
&\qquad \times \left[\frac{1}{d_w(x+sw)} + \frac{1}{\delta_w(x+sw)}\right]^\alpha\\
&+ \frac{4-2^{3-\alpha}}{2\alpha}\int_{S^{n-1}} dw \int_{\{x:x\cdot w=0\}} d\mathcal{L}_w(x) \int_{x+sw\in D} ds |u(x+sw)|^2\\
&\qquad \times \left[\frac{1}{d_w(x+sw)} + \frac{1}{\delta_w(x+sw)}\right]^{\alpha-1} \frac{1}{d_w(x+sw)+\delta_w(x+sw)}\\
&= \kappa_{1,\alpha}\frac{1}{2}\int_{S^{n-1}} dw \int_D |u(x)|^2 \left[\frac{1}{d_{w,D}(x)} + \frac{1}{\delta_{w,D}(x)}\right]^\alpha dx\\
&+ \frac{4-2^{3-\alpha}}{2\alpha}\int_{S^{n-1}} dw \int_D |u(x)|^2 \left[\frac{1}{d_{w,D}(x)} + \frac{1}{\delta_{w,D}(x)}\right]^{\alpha-1} \frac{1}{d_{w,D}(x)+\delta_{w,D}(x)} dx\\
&\geq \kappa_{n,\alpha}\int_D \frac{|u(x)|^2}{M_\alpha(x)^\alpha} dx + \frac{\lambda_{n,\alpha}}{\operatorname{diam} D}\int_D \frac{|u(x)|^2}{M_{\alpha-1}(x)^{\alpha-1}} dx.
\end{aligned}
$$

In the last line we have used [16, (7)], which is valid for any $\alpha > 0$, hence also for $\alpha - 1$ in place of α. This proves (1.5).

Inequality (1.6) follows from [16, (9)], which is also valid for any $\alpha > 0$. □

Proof of Corollary 1.3. The equality follows from Lemma 2.3 and the following formula, where we take $D = (-1,1)$ in the definition of $\mathcal{E}(u)$

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\, dx\, dy &= \mathcal{E}(u) + \int_{-1}^1 u^2(x)\frac{(1+x)^{-\alpha}+(1-x)^{-\alpha}}{\alpha}\, dx\\
&= \frac{1}{2}\int_{-1}^1\int_{-1}^1 \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 \frac{w(x)w(y)}{|x-y|^{1+\alpha}}\, dx\, dy\\
&\quad + \frac{2^\alpha(\kappa_{1,\alpha}+\alpha)}{\alpha}\int_{-1}^1 u^2(x)\,(1-x^2)^{-\alpha} dx.
\end{aligned}
\tag{2.4}
$$

□

The sharpness of the constant $\frac{B(\frac{1+\alpha}{2},\frac{2-\alpha}{2})}{\alpha 2^\alpha}$ in (1.7) follows from [5].

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