APPROXIMATION OF BANDLIMITED FUNCTIONS BY EXPONENTIAL SUMS

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ABSTRACT. Let K be a compact set in $\mathbb{R}^n$. For $1 \leq p \leq \infty$, the Bernstein space B_K^p is the Banach space of all functions $f \in L^p(\mathbb{R}^n)$ such that its Fourier transform in a distributional sense is supported on K. If $f \in B_K^p$, then f is continuous on $\mathbb{R}^n$ and has an extension onto the complex space $\mathbb{C}^n$ to an entire function of exponential type K. We study the approximation of functions in B_K^p by finite τ-periodic exponential sums

$$\sum_m c_m e^{2\pi i(x,m)/\tau},$$

in $L^p(\tau\mathbb{T}^n)$ norm as $\tau \to +\infty$, where $\mathbb{T}^n = [-\tau/2, \tau/2]^n$.

KEYWORDS: Fourier transforms, bandlimited functions, entire functions of exponential type, Bernstein spaces, exponential sums, trigonometric polynomials, approximations by trigonometric polynomials.

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1 Introduction

The Fourier transform on $L^1(\mathbb{R}^n)$ is defined here by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t)e^{-i(x,t)}dt,$$

where $(x,t) = \sum_{k=1}^{\infty} x_k t_k$ denotes the inner product in $\mathbb{R}^n$. We normalize the inverse Fourier transform

$$\check{f}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(t)e^{i(\xi,t)}dt \tag{1.1}$$

so that the inversion formula $\widehat{(\check{f})} = f$ holds for suitable f. If $f \in L^p(\mathbb{R}^n)$

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and $p > 2$, then the Fourier transform $\hat{f}$ is understood as tempered distribution on $\mathbb{R}^n$.

Let K be a compact set in $\mathbb{R}^n$. For $1 \le p \le \infty$, the Bernstein space B_K^p is a closed subspace of $L^p(\mathbb{R}^n)$ consists of all $f \in L^p(\mathbb{R}^n)$ such that $\hat{f}$ is supported on K. In other terminology, any $f \in B_K^p$ is bandlimited to K. The Banach space B_K^p is equipped with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}, \quad \text{if } 1 \le p < \infty$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}_{\mathbb{R}^n} |f(x)|, \quad \text{if } p = \infty.$$

Throughout this article, we consider only the cases when B_K^p is infinite-dimensional Banach space. For this reason, we will assume that K is a body in $\mathbb{R}^n$, i.e., K coincides with the closure of its inner point set $\operatorname{int}(K)$. By the Paley-Wiener-Schwartz theorem (see [4, p. 181]), any $f \in B_K^p$ is continuous on $\mathbb{R}^n$ and has an extension onto the complex space $\mathbb{C}^n$ to an entire function of exponential type K; this means that

$$|f(z)| \le M e^{H_K(-y)}, \tag{1.2}$$

for all $z = x + iy$, x, $y \in \mathbb{R}^n$, where M is a positive constant and

$$H_K(\xi) = \max_{t \in K} (\xi, t),$$

$\xi \in \mathbb{R}^n$, is the supporting function of K.

Let $\mathbb{Z}^n$ denote the lattice of integer vectors in $\mathbb{R}^n$ and let $\mathbb{T}^n = (-1/2, 1/2]^n$. We write $\mathbb{N}$ for the set of natural numbers. Fix $\tau > 0$. Since each $f \in B_K^p$ is a continuous function on $\mathbb{R}^n$, we have that the Fourier coefficients of f on $\tau\mathbb{T}^n$

$$c_m(f, \tau) = \frac{1}{\tau^n} \int_{\tau\mathbb{T}^n} f(u) e^{-2\pi i (m,u)/\tau} du,$$

$m \in \mathbb{Z}^n$, are well-defined. The Fourier series of $f \in B_K^p$ is formally given by

$$\sum_{m \in \mathbb{Z}^n} c_m(f, \tau) e^{2\pi i (m,x)/\tau}. \tag{1.3}$$

We will take in (1.3) only those terms whose Fourier transforms are supported on K. More precisely, set

$$f_\tau(x) = \sum_{m \in \mathbb{Z}^n \cap n_\tau K} c_m(f, \tau) e^{2\pi i (m,x)/\tau}, \tag{1.4}$$

where $n_\tau = [\tau/2\pi]_{\mathbb{Z}}$ is the integer part of $\tau/2\pi$ and $n_\tau K = \{n_\tau u : u \in K\}$. Since K is a compact set, f_τ is a trigonometric polynomial on $\mathbb{R}^n$. We also consider the following $(\tau\mathbb{T}^n)$-truncated version of f_τ

$$\varphi_{f,\tau} = f_\tau \cdot \chi_{\tau\mathbb{T}^n}, \tag{1.5}$$

where $\chi_{\tau\mathbb{T}^n}$ is the characteristic (indicator) function of the set $\tau\mathbb{T}^n$.

Let $B^2_\sigma := B^2_{[-\sigma,\sigma]}$, where $\sigma > 0$. In a recent paper, Martin [7] studied the approximation of $f \in B^2_\sigma$ by the sums of one variable

$$\chi_{\tau\mathbb{T}}(x) \cdot \sum_{|m| \le [\tau\sigma/2\pi]_{\mathbb{Z}}} c_m(f,\tau) e^{2\pi i m x/\tau}. \tag{1.6}$$

Recall that a function f is called strictly bandlimited to $[-\sigma,\sigma]$, if it is bandlimited to $[-\varrho,\varrho]$ for some $0 \le \varrho < \sigma$. It was proved in [7] that if $f \in B^2_\sigma$ and f is strictly bandlimited to $[-\sigma,\sigma]$, then (1.6) converges to f in L^2 norm as $\tau \to \infty$ on $\mathbb{R}$ and on any horizontal lines in $\mathbb{C}$. Gröchening [2] studied similar to (1.6) trigonometric polynomials and obtained a quantitative estimate of approximation in $L^2(\tau\mathbb{T})$ norm. In [10], we investigated (1.5) for $f \in B^2_K$ in the case when K is an arbitrary Jordan-measurable compact subset of $\mathbb{R}^n$.

In the present paper, we give another proof of such approximation theorems. Furthermore, our approach extends to any B^p_K for $1 < p < \infty$.

To formulate the main result, recall that by an *n-polytope* in $\mathbb{R}^n$ we mean any intersection P of a finite number of half-spaces $L(a,\omega) = \{x \in \mathbb{R}^n : (x,a) \le \omega\}$, $a \in \mathbb{R}^n$, $\omega \in \mathbb{R}$, such that P has non-empty interior $\mathrm{int}(P)$.

Theorem 1.1. *Suppose $f \in B^p_K$ for $1 < p < \infty$. If K is an n-polytope, then*

$$\lim_{\tau\to\infty} \|f - \varphi_{f,\tau}\|_{L^p(\mathbb{R}^n)} = 0. \tag{1.7}$$

Until recently, the approximation of bandlimited functions by trigonometric polynomials was most completely investigated in the case $p = \infty$. Lewitan [6] was the first to define the trigonometric polynomials

$$\tilde{f}_\tau(x) := \sum_{k\in\mathbb{Z}} f(x + k\tau) \frac{\sin^2(\frac{x}{\tau} + k)}{(\frac{x}{\tau} + k)^2}, \ \tau > 0. \tag{1.8}$$

It was proved in [6] that if $f \in B^\infty_\sigma$, then $\tilde{f}_\tau \to f$ as $\tau \to \infty$ uniformly on compact subset of $\mathbb{R}$. Krein [5] given a new proof of this statement, obtained the degree of approximation, and considered several applications. Hörmander [3] considered trigonometric polynomials which are more general than (1.8) (for a more detailed history and references see [1], p.p. 146–152). For $f \in B^p_\sigma$

and $1 \le p < \infty$, Schmeisser [12] proved the convergence theorems for Lewitan (or Lewitan-Krein-Hörmander) polynomials (1.8) with respect to L^p norms on lines in $\mathbb{C}$ parallel to $\mathbb{R}$ and obtained the convergence of (1.8) with respect to l^p norms. The approximation by polynomials of several variables similar to (1.8) was also studied in the case $f \in B_K^\infty$ (for details see [9] and the literature cited there).

For our final theorem, we return to the polynomials (1.5). As in the case of Lewitan-Krein-Hörmander polynomials, it turns out that (1.5) also converges uniformly on compact subsets of $\mathbb{R}^n$.

Theorem 1.2. *Suppose $f \in B_K^p$, where $1 < p < \infty$. If K is an n-polytope, then $\lim_{\tau\to\infty}(f - \varphi_{f,\tau}) = 0$ uniformly on compact subsets of $\mathbb{R}^n$.*

2 Preliminaries

Let us start by introducing some terminology and notation. For $x \in \mathbb{R}^n$, we denote by $|x|_p$ the l_p norm on $\mathbb{R}^n$, i.e.,

$$|x|_p = \Big(\sum_k |x_k|^p\Big)^{1/p}, \quad \text{if } p \in [1, \infty)$$

and $|x|_\infty = \max_k |x_k|$. Set

$$\mathbb{U}^n = \{x \in \mathbb{R}^n : \ |x|_2 \le 1\} \quad \text{and} \quad \mathbb{Q}^n = \{x \in \mathbb{R}^n : \ |x|_\infty \le 1\}.$$

If A and B are two sets in $\mathbb{R}^n$, then we write $A + B$ for the set of all sums $a + b$, $a \in A$, $b \in B$. In particular, if $A = \{a\}$, $a \in \mathbb{R}^n$, then we write simply $a + B$. For $\delta \in \mathbb{R} \setminus \{0\}$ we denote by δA the set $\{\delta a : \ a \in A\}$. Let K be a compact subset of $\mathbb{R}^n$. The Lebesgue measure (in $\mathbb{R}^n$) of K will be denoted by $\Omega_n(K)$. Set

$$r_K = \inf\Big\{|x|_2 : \ x \in \mathbb{R}^n \setminus K\Big\} \tag{2.1}$$

and

$$R_K = \max\Big\{|x|_2 : \ x \in K\Big\}. \tag{2.2}$$

For $\theta > 0$, we denote by K_θ the θ-envelope of K, i.e.

$$K_\theta = \cup_{x\in K}(x + \theta\mathbb{U}^n).$$

We define the following "θ-kernel of K"

$$K_{(-\theta)} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus K)_\theta. \tag{2.3}$$

In the sequel, we assume that in $K_{(-\theta)}$ the number θ is less than the diameter of K. This implies that $K_{(-\theta)}$ is nonempty set.

Let $f \in B_K^p$. For $\theta > 0$, we say that f is θ-strictly bandlimited to K if $\operatorname{supp} \widehat{f} \subset K_{(-\theta)}$. Throughout, the term "$f$ is strictly bandlimited to K" means that f is θ-strictly bandlimited to K for some $\theta > 0$.

We need a few preliminary properties of B_K^p. Let us start with some Nikol'skii's inequalities (see [8, Sec. 3]). Let $f \in B_{\mathbb{Q}_\sigma^n}^p$, where $\mathbb{Q}_\sigma^n = \{x \in \mathbb{R}^n : |x_k| \le \sigma_k < \infty, k = 1, \dots, n\}$. If $1 \le p \le r \le \infty$, then

$$\|f\|_{L^r(\mathbb{R}^n)} \le 2^n \Big(\prod_{k=1}^n \sigma_k\Big)^{1/p-1/r} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.4}$$

In the case $1 \le p < \infty$

$$\lim_{x \to \infty} f(x) = 0. \tag{2.5}$$

Of course, (2.4) and (2.5) still hold for all $f \in B_K^p$ in the case if K is a compact body in $\mathbb{R}^n$. More precisely, given a compact body $K \subset \mathbb{R}^n$ and $1 \le p \le r \le \infty$, there exists a positive constant $A = A(n, p, r, K) < \infty$ depending on n, p, r, and K so that

$$\|f\|_{L^r(\mathbb{R}^n)} \le A \, \|f\|_{L^p(\mathbb{R}^n)} \tag{2.6}$$

for all $f \in B_K^p$. In other words, the embedding mapping

$$B_K^p \subset B_K^r \tag{2.7}$$

is continuous.

Let p be a 2π-periodic in each variable trigonometric polynomial on $2\pi\mathbb{T}^n$. If $1 \le p \le r \le \infty$, then there exists a constant $c = c(n)$ such that

$$\|p\|_{L^r(2\pi\mathbb{T}^n)} \le c\Big(\prod_{k=1}^n m_k\Big)^{1/p-1/r} \|p\|_{L^p(2\pi\mathbb{T}^n)}, \tag{2.8}$$

where m_k is the degree of p in the variable x_k (see [8, Sec. 3], [13]).

Recall that the Schwartz class of test functions $S(\mathbb{R}^n)$ consists of the complex-valued infinitely differentiable functions ξ on $\mathbb{R}^n$ satisfying

$$\sup_{x \in \mathbb{R}^n, |u| \le k} |(1 + |x|_2)^s D^u \xi(x)| < \infty \tag{2.9}$$

for all $k, s \in \mathbb{N}$, where u is a nonnegative multi-index and $|u| = \sum_{j=1}^n u_j$. The dual space $S'(\mathbb{R}^n)$ is the space of tempered distributions. If $\xi \in S(\mathbb{R}^n)$ and $\tau > 0$, then the Poisson summation formula gives

$$\sum_{l \in \mathbb{Z}^n} \xi(x + \tau l) = \frac{1}{\tau^n} \sum_{m \in \mathbb{Z}^n} \widehat{\xi}\Big(\frac{2\pi}{\tau} m\Big) e^{2\pi i (m,x)/\tau}, \tag{2.10}$$

$x \in \mathbb{R}^n$ (see [4, p. 177, (7.2.1)]).

Given a compact set K in $\mathbb{R}^n$, we denote by $S_K^\circ(\mathbb{R}^n)$ the subspace of $S(\mathbb{R}^n)$ consisting of strictly bandlimited to K functions. We will need the following easily implicit auxiliary statement. For completeness, we give also its proof.

Lemma 2.1. *Let K be a convex compact body in $\mathbb{R}^n$. If $1 \le p < \infty$, then $S_K^\circ(\mathbb{R}^n)$ is an $L^p(\mathbb{R}^n)$-dense subspace of B_K^p.*

Proof. Suppose first that $0 \in \operatorname{int} K$. Take any $\zeta \in S(\mathbb{R}^n)$ such that ζ is non-negative on $\mathbb{R}^n$, $\operatorname{supp}\zeta \subset U^n$, and

$$\int_{\mathbb{R}^n} \zeta(x)\,dx = 1. \tag{2.11}$$

By the Bochner theorem, the function $\check{\zeta}$ is continuous and positive definite on $\mathbb{R}^n$. This, conjugate with (2.11), implies that

$$\sup_{\mathbb{R}^n} |\check{\zeta}(t)| \le \check{\zeta}(0) = (2\pi)^{-n}. \tag{2.12}$$

Let $f \in B_K^p$. Given $\delta \in (0,1)$, we set $f_\delta(x) = f(\delta x)$. Let

$$F_\delta(x) = (2\pi)^n f_\delta(x) \cdot \check{\zeta}\Big(\frac{(1-\delta)r_K}{2}x\Big), \tag{2.13}$$

where r_K is defined in (2.1). We consider $\widehat{f}$ as element of $S'(\mathbb{R}^n)$ with compact support. Then $\widehat{f_\delta} * \zeta$ is well defined in $S'(\mathbb{R}^n)$ and

$$\operatorname{supp}\widehat{F_\delta} = \operatorname{supp}(\widehat{f_\delta} * \zeta) \subset \delta K + \frac{(1-\delta)r_K}{2}\mathbb{U}^n \subset K_{-(\theta)}$$

for all $\theta \in (0, (1-\delta)r_K/2)$. By (2.5), each $f \in B_K^p$, $p \in [1,\infty]$, is a bounded function on $\mathbb{R}^n$. Hence $F_\delta \in S_K^\circ(\mathbb{R}^n)$.

Let us prove that

$$\lim_{\delta\to 1} \|f - F_\delta\|_{L^p(\mathbb{R}^n)} = 0. \tag{2.14}$$

To this end, we estimate $\|f - f_\delta\|_{L^p(\mathbb{R}^n)}$ and $\|f_\delta - F_\delta\|_{L^p(\mathbb{R}^n)}$. Set $\varepsilon > 0$. Then there exists $a_\varepsilon > 0$ such that

$$\|f\|_{L^p(\mathbb{R}^n\setminus a_\varepsilon \mathbb{U}^n)} < \varepsilon. \tag{2.15}$$

Fix any such a_ε. If $1/2 < \delta < 1$, then

$$\|F_\delta\|_{L^p(\mathbb{R}^n\setminus a_\varepsilon \mathbb{U}^n)} \le \|f_\delta\|_{L^p(\mathbb{R}^n\setminus a_\varepsilon \mathbb{U}^n)} < 2^{n/p}\varepsilon \tag{2.16}$$

for all $\delta \in (1/2, 1)$.

Now we consider the functions f, f_δ, and F_δ on $a_\varepsilon\mathbb{U}^n$. It is known (see, for example, [10]), that if $g \in B_K^p$, $1 \le p \le \infty$, $x, y \in \mathbb{R}^n$, and $\max_{t\in K} |(x-y,t)| \le \pi$, then

$$|g(x) - g(y)| \le 2\sin\Big(\frac{1}{2}\max_{t\in K}|(t, x-y)|\Big) \cdot \|g\|_{L^\infty(\mathbb{R}^n)}. \tag{2.17}$$

Therefore, if $x \in a_\varepsilon \mathbb{U}^n$ and $1-\delta$ is sufficiently small, then by (2.17), we obtain that

$$\begin{aligned}\|f - f_\delta\|_{L^p(a_\varepsilon \mathbb{U}^n)} &\le 2\big(a_\varepsilon^n \Omega_n(\mathbb{U}^n)\big)^{1/p}\Big|\sin\Big(\frac{1-\delta}{2}\max_{t\in K, x\in a_\varepsilon \mathbb{U}^n}|(x,t)|\Big)\Big| \\ &\le \big(a_\varepsilon^{n+p}\Omega_n(\mathbb{U}^n)\big)^{1/p}(1-\delta)n^{1/2}R_K < \varepsilon, \qquad (2.18)\end{aligned}$$

where R_K is defined in (2.2). In view of (2.12), we have for $x \in a_\varepsilon \mathbb{U}^n$

$$\begin{aligned}|f_\delta(x) - F_\delta(x)| &= (2\pi)^n |f_\delta(x)|\Big|\check{\zeta}(0) - \check{\zeta}\Big(\frac{(1-\delta)r_K}{2}x\Big)\Big| \\ &\le \|f\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n}\Big|1 - e^{i(1-\delta)r_K(x,t)/2}\Big|\zeta(t)\,dt \\ &\le \max_{t\in\mathbb{U}^n, x\in a_\varepsilon\mathbb{U}^n}\Big|1 - e^{i(1-\delta)r_K(x,t)/2}\Big|\,\|f\|_{L^\infty(\mathbb{R}^n)} \\ \le \max_{t\in\mathbb{U}^n, x\in a_\varepsilon\mathbb{U}^n}|(x,t)|\frac{(1-\delta)r_K\,\|f\|_{L^\infty(\mathbb{R}^n)}}{\sqrt{2}} &\le (n/2)^{1/2}(1-\delta)r_K a_\varepsilon \|f\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Hence

$$\|f_\delta(x) - F_\delta(x)\|_{L^p(a_\varepsilon\mathbb{U}^n)} \le \big(a_\varepsilon^n\Omega_n(\mathbb{U}^n)\big)^{1/p}(n/2)^{1/2}(1-\delta)r_K a_\varepsilon\|f\|_{L^\infty(\mathbb{R}^n)} < \varepsilon$$

for sufficiently small $1-\delta$. Since ε is an arbitrary positive number, combining (2.15), (2.16), (2.18), and (2.19), we have (2.14).

In the general case, we take an arbitrary $x_0 \in \operatorname{int} K$. Let $g \in B_K^p$. If

$$f(x) = g(x)e^{-i(x,x_0)},$$

then $f \in B^p_{-x_0+K}$, where the body $\operatorname{int}(-x_0+K)$ contains origin. For $\delta \in (0,1)$, we set $G_\delta(x) = F_\delta(x)e^{i(x,x_0)}$, where F_δ is defined in (2.13). Then (2.14) becomes $\lim_{\delta\to 1}\|G_\delta - g\| = 0$. Finally, each G_δ belongs to $S_K^\circ(\mathbb{R}^n)$, and this complete the proof of the lemma.

3 Proofs

For $f \in S(\mathbb{R}^n)$, let

$$E_{f,\tau}(x) := \frac{1}{\tau^n}\sum_{m\in\mathbb{Z}^n\cap\frac{\tau}{2\pi}K}\widehat{f}\Big(\frac{2\pi}{\tau}m\Big)e^{2\pi i(m,x)/\tau}, \qquad (3.1)$$

$x \in \mathbb{R}^n$. We will compare this polynomial with our main trigonometric polynomial (1.4).

Lemma 3.1. *Let K be a convex compact body in $\mathbb{R}^n$. If $f \in S_K^\circ(\mathbb{R}^n)$ and $p \in [1,\infty)$, then*

$$\lim_{\tau\to\infty} \|f_\tau - E_{f,\tau}\|_{L^p(\tau\mathbb{T}^n)} = 0. \tag{3.2}$$

Proof. First we claim that there exists an $\tau_0 > 0$ such that if $\tau \geq \tau_0$, then $E_{f,\tau}$ can be written as

$$E_{f,\tau}(x) = \frac{1}{\tau^n} \sum_{m\in\mathbb{Z}^n\cap n_\tau K} \widehat{f}\Big(\frac{2\pi}{\tau}m\Big) e^{2\pi i(m,x)/\tau}, \tag{3.3}$$

where $n_\tau = [\tau/2\pi]_{\mathbb{Z}}$. To that end, it suffices to show that if $\widehat{f}((2\pi/\tau)m_0) \neq 0$ for $m_0 \in \mathbb{Z}^n$, then $m_0 \in n_\tau K$. Since f is θ-strictly bandlimited to K for some $\theta > 0$ and K is convex, we have $(2\pi/\tau)m_0 + \theta\mathbb{U}^n \subset K$. Hence there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0]$, then

$$\frac{2\pi}{\tau}(1+\delta)m_0 \in K. \tag{3.4}$$

Suppose τ_0 is defined be means of the equality $(1+\delta_0)(1-2\pi/\tau_0) = 1$. Since $1-2\pi/\tau < 2\pi n_\tau/\tau \leq 1$, we have that for any $\tau \in [\tau_0, \infty)$ there exists $\delta \in (0, \delta_0]$ such that $2\pi n_\tau(1+\delta)/\tau = 1$. Using (3.4), we obtain that $m_0 \in n_\tau K$. This proofs our claim (3.3).

Let $\tau \geq \tau_0$. Using (3.3) we can estimate $\|f_\tau - E_{f,\tau}\|_{L^p(\tau\mathbb{T}^n)}$ as follows:

$$\begin{aligned}
&\|f_\tau - E_{f,\tau}\|_{L^p(\tau\mathbb{T}^n)} \\
&= \frac{1}{\tau^n}\bigg(\int_{\tau\mathbb{T}^n}\Big|\sum_{m\in\mathbb{Z}^n\cap n_\tau K} \tau^n c_m e^{2\pi i(m,x)/\tau} - \sum_{k\in\mathbb{Z}^n\cap\frac{\tau}{2\pi}K} \widehat{f}\Big(\frac{2\pi}{\tau}k\Big) e^{2\pi i(k,x)/\tau}\Big|^p dx\bigg)^{1/p} \\
&= \frac{1}{\tau^n}\bigg(\int_{\tau\mathbb{T}^n}\Big|\sum_{m\in\mathbb{Z}^n\cap n_\tau K}\Big(\tau^n c_m - \widehat{f}\Big(\frac{2\pi}{\tau}m\Big)\Big)\Big|^p dx\bigg)^{1/p} \\
&\leq \tau^{n/p-n}\sum_{m\in\mathbb{Z}^n\cap\frac{\tau}{2\pi}K}\Big|\tau^n c_m - \widehat{f}\Big(\frac{2\pi}{\tau}m\Big)\Big|.
\end{aligned} \tag{3.5}$$

Let $s \in \mathbb{N}$. By (2.9) (with $u = 0$), there exists a constant $c(f,s) < \infty$ such that

$$|f(x)| \leq c(f,s)|x|_2^{-s}$$

for $x \in \mathbb{R}^n \setminus \frac{\tau_0}{2}\mathbb{U}^n$. Since $\tau \geq \tau_0$, it follows that

$$\begin{aligned}
|\tau^n c_m - \widehat{f}\Big(\frac{2\pi}{\tau}m\Big)\Big| &= \Big|\int_{\mathbb{R}^n\setminus\tau\mathbb{T}^n} f(x)e^{-2\pi i(m,x)/\tau}\,dx\Big| \leq \int_{\mathbb{R}^n\setminus\tau\mathbb{T}^n}|f(x)|\,dt \\
&\leq \int_{\mathbb{R}^n\setminus\frac{\tau}{2}\mathbb{U}^n}|f(x)|\,dx \leq c(f,s)\int_{\mathbb{R}^n\setminus\frac{\tau}{2}\mathbb{U}^n}|x|_2^{-s}\,dx.
\end{aligned}$$

If $s > n$, then via polar coordinates in $\mathbb{R}^n$

$$\Big|\tau^n c_m - \widehat{f}\Big(\frac{2\pi}{\tau}m\Big)\Big| \le c(f,s)\frac{2\pi^{n/2}}{\Gamma\big(\frac{n}{2}\big)}\int_{\frac{\tau}{2}}^{\infty} \varrho^{n-s-1}\, d\varrho = c(f,s)\frac{2\pi^{n/2}}{(s-n)\Gamma\big(\frac{n}{2}\big)}\tau^{n-s},$$

where Γ is the usual Gamma function. Finally, if we choose in (2.9) $u = 0$ and $s > 0$ so that $s > n(1+1/p)$, then (3.5) together with (3.6) proves (3.2) and the lemma.

Let K be a compact subset of $\mathbb{R}^n$. If $k \in \mathbb{N}$, then

$$D_{k,K}(x) = \sum_{m\in\mathbb{Z}^n\cap kK} e^{i(x,m)}$$

is called the Dirichlet kernel of order k related to K. Now the polynomial (1.4) can be presented as the value of the following integral operator

$$f_\tau(x) = \frac{1}{\tau^n}\int_{\tau\mathbb{T}^n} f(t) D_{n_\tau,K}\Big(\frac{2\pi}{\tau}(x-t)\Big)\, dt. \tag{3.6}$$

Therefore, setting

$$\Phi_\tau(f) = \varphi_{f,\tau},$$

$\tau \in (0,\infty)$, we may assume that (1.4) defines on B^p_K the one-parametric family of bounded linear operators $\Phi_\tau : B^p_K \to L^p(\mathbb{R}^n)$. In the next lemma we give some sufficient conditions under which the family $(\Phi_\tau)_{\tau>0}$ has uniformly bounded norms.

Lemma 3.2. *Let K be an n-polytope in $\mathbb{R}^n$. If $1 < p < \infty$, then there exists a constant $\varpi(K,p) < \infty$ such that*

$$\|\Phi_\tau(f)\|_{L^p(\mathbb{R}^n)} \le \varpi(K,p)\|f\|_{B^p_K} \tag{3.7}$$

for every $f \in B^p_K$ and all $\tau \in (0,\infty)$.

Proof Given $f \in B^p_K$, we define on $\mathbb{R}^n$ the $2\pi\mathbb{T}^n$-periodic function g_f such that $g_f(v) = f((\tau/2\pi)v)$ for $v \in 2\pi\mathbb{T}^n$. Now, (3.7) implies

$$\begin{aligned}\|\Phi_\tau(f)\|^p_{L^p(\mathbb{R}^n)} &= \|\Phi_\tau(f)\|^p_{L^p(\tau\mathbb{T}^n)}\\ &= \frac{1}{\tau^{np}}\int_{\tau\mathbb{T}^n}\Big|\int_{\tau\mathbb{T}^n} f(t)D_{n_\tau,K}\Big(\frac{2\pi}{\tau}(x-t)\Big)\,dt\Big|^p\,dx\\ &= \frac{\tau^n}{(2\pi)^{n(p+1)}}\int_{2\pi\mathbb{T}^n}\Big|\int_{2\pi\mathbb{T}^n} g_f(v)D_{n_\tau,K}(y-v)\,dv\Big|^p\,dy.\end{aligned} \tag{3.8}$$

In [13] (theorem 2.4.5, p. 56), it is shown that under the conditions of this lemma, there exists a constant $c(K) > 0$, depending only on K, such that

$$\frac{1}{(2\pi)^n}\Big(\int_{2\pi\mathbb{T}^n}\Big|\int_{2\pi\mathbb{T}^n} h(v)D_{s,K}(y-v)\,dv\Big|^p\,dy\Big)^{1/p} \le c(K)\Big(1+\frac{4p^2}{p-1}\Big)^n\|h\|_{L^p(2\pi\mathbb{T}^n)}$$

for all $s \in \mathbb{N}$ and $h \in L^p(2\pi\mathbb{T}^n)$. Combining this with (3.9), we get

$$\|\Phi_\tau(f)\|_{L^p(\mathbb{R}^n)} \le c(K)\Big(1 + \frac{4p^2}{p-1}\Big)^n \|f\|_{L^p(\tau\mathbb{T}^n)}.$$

Thus we have (3.8) with $\varpi(K,p) = c(K)(1 + 4p^2/(p-1))^n$, and the lemma is proved.

Proof of theorem 1.1. The Banach–Steinhaus theorem together with Lemma 2.1 and Lemma 3.2 imply that it suffices to check (1.7) only on $S_K^\circ(\mathbb{R}^n)$. Let $f \in S_K^\circ(\mathbb{R}^n)$. The Poisson summation formula (2.10) gives

$$\sum_{l\in\mathbb{Z}^n} f(x+\tau l) = E_{f,\tau}(x), \tag{3.9}$$

where $E_{f,\tau}$ is defined in (3.1). Here the series converges absolutely and uniformly on compact subset of $\mathbb{R}^n$. In particular, the series

$$\sum_{l\in\mathbb{Z}^n\setminus\{0\}} f(x+\tau l)$$

defines on $\tau\mathbb{T}^n$ an $L^p(\tau\mathbb{T}^n)$ function. Since $\lim_{\tau\to\infty} \|f\|_{L^p(\mathbb{R}^n\setminus\tau\mathbb{T}^n)} = 0$, we have from (3.2) and (3.10) that it remains to show that

$$\lim_{\tau\to\infty}\Big\|\sum_{l\in\mathbb{Z}^n\setminus\{0\}} f(x+\tau l)\Big\|_{L^p(\tau\mathbb{T}^n)} = 0. \tag{3.10}$$

Let $s \in \mathbb{N}$. If $x \in \tau\mathbb{T}^n$ and $l \in \mathbb{Z}^n \setminus \{0\}$, then (2.9) implies that there exists $M(s) > 0$ such that

$$|f(x+\tau l)| \le \frac{M(s)}{(1+|x+\tau l|_2)^s} \le \frac{M(s)}{(1+\frac{\tau}{2}|l|_2)^s} \le \frac{2^s M(s)}{\tau^s |l|_2^s} \le \frac{2^s M(s)}{\tau^s |l|_\infty^s}.$$

We may take $s \in \mathbb{N}$ so that $s > n$. Then

$$\Big|\sum_{l\in\mathbb{Z}^n\setminus\{0\}} f(x+\tau l)\Big| \le \frac{2^s M(s)}{\tau^s} \sum_{l\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|l|_\infty} \le \frac{2^s M(s)}{\tau^s} \sum_{m=1}^{\infty} \frac{2^n(m-1)^{n-1}}{m^s}.$$

Therefore exists $M_1(s) > 0$ such that

$$\Big\|\sum_{l\in\mathbb{Z}^n\setminus\{0\}} f(x+\tau l)\Big\|_{L^p(\tau\mathbb{T}^n)} \le M_1(s)\tau^{n/p-s}.$$

This proves (3.11) and the proof of theorem 1.1 is complete.

Proof of theorem 1.2. Suppose $f \in B_K^p$. We claim that

$$M(f) := \sup_{\tau>0} \|f - f_\tau\|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{3.11}$$

In fact, by (2.4), it would suffice to show that

$$\sup_{\tau>0} \|f_\tau\|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{3.12}$$

To see this, let $p_\tau(t) := f_\tau(\tau t/2\pi)$, $t \in 2\pi\mathbb{T}^n$. Since $K \subset R_K\mathbb{U}^N \subset R_K\mathbb{Q}^n$, where R_K is defined in (2.2), the polynomial p_τ is 2π-periodic and has degree at most $n_\tau R_K$ in each variable. Now, by (2.8), there exists an $c < \infty$ such that

$$\|p_\tau\|_{L^\infty(2\pi\mathbb{T}^n)} \le c\ (n_\tau R_K)^{n/p}\|p_\tau\|_{L^p(2\pi\mathbb{T}^n)}.$$

Then

$$\|f_\tau\|_{L^\infty(\mathbb{R}^n)} = \|f_\tau\|_{L^\infty(\tau\mathbb{T}^n)} = \|p_\tau\|_{L^\infty(2\pi\mathbb{T}^n)} \le c\ (n_\tau R_K)^{n/p}\|p_\tau\|_{L^p(2\pi\mathbb{T}^n)} =$$
$$c\left(\frac{2\pi n_\tau}{\tau}R_K\right)^{n/p}\|f_\tau\|_{L^p(\tau\mathbb{T}^n)} \le c\ R_K^{n/p}\|f_\tau\|_{L^p(\tau\mathbb{T}^n)}.$$

Thus, using lemma 3.2, we get

$$\|f_\tau\|_{L^\infty(\mathbb{R}^n)} \le c\ R_K^{n/p}\|\Phi_\tau(f)\|_{L^p(\mathbb{R}^n)} \le c\ R_K^{n/p}\varpi(K,p)\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.13}$$

and this proves (3.13).

In view of (2.5), to prove the theorem, it is enough to show that

$$\lim_{\tau\to\infty} \|f - \varphi_{f,\tau}\|_{L^\infty(\tau\mathbb{T}^n)} = 0.$$

Assume, to the contrary, that there exist an $a > 0$ and a sequence $\tau_m \to \infty$ such that $\|f - \varphi_{f,\tau_m}\|_{L^\infty(\tau_m\mathbb{T}^n)} \ge a$ for all $m \in \mathbb{N}$. Then we may choose a sequence $(x_m)_1^\infty$, where $x_m \in \tau_m\mathbb{T}^n$, so that

$$|(f - f_{\tau_m})(x_m)| = |(f - \varphi_{f,\tau_m})(x_m)| \ge a. \tag{3.14}$$

We fix $\delta \in (0, 1]$ and define

$$\Delta_m = x_m + \frac{\pi\delta}{R_K}\ \mathbb{U}^n,$$

$m = 1, 2, \ldots$. Suppose $y \in \Delta_m$. Since $y = x_m + (\pi\delta/R_K)w$, for some $w \in \mathbb{U}^n$, we get

$$\max_{t\in K} |(x_m - y, t)| = \frac{\pi\delta}{R_K}\max_{t\in K}|(w, t)| \le \frac{\pi\delta}{R_K}\max_{t\in K}|t|_2 \le \pi\delta \le \pi.$$

Now using (2.17) and (3.15), we obtain

$$|(f - f_{\tau_m})(y)|$$
$$\ge |(f - f_{\tau_m})(x_m)| - 2\|f - f_{\tau_m}\|_{L^\infty(\mathbb{R}^n)}\sin\Big(\frac{1}{2}\max_{t\in K}|(x_m - y, t)|\Big)$$
$$\ge a - \pi\delta\|f - f_{\tau_m}\|_{L^\infty(\mathbb{R}^n)}. \tag{3.15}$$

Since $a \le M(f) < \infty$, we may take

$$\delta = \frac{a}{2\pi M(f)}.$$

Then (3.16) implies

$$\int_{\tau_m \mathbb{T}^n} |(f - f_{\tau_m})(x)|^p \, dx \ge \int_{\tau_m \mathbb{T}^n \cap \Delta_m} |(f - f_{\tau_m})(y)|^p \, dy \ge \left(\frac{a}{2}\right)^p \Omega_n(\tau_m \mathbb{T}^n \cap \Delta_m). \tag{3.16}$$

We may assume without of generality that

$$\tau_m \ge \frac{2\pi}{\sqrt{n} R_K}$$

for all $m \in \mathbb{N}$. Then

$$\Omega_n(\Delta_m \cap \tau_m \mathbb{T}^n) \ge \frac{\pi \delta \, \Omega_n(\mathbb{U}^n)}{2^n R_K} = \frac{a \, \Omega_n(\mathbb{U}^n)}{2^{n+1} M(f) R_K}. \tag{3.17}$$

Then combining (3.17) and (3.18) we obtain

$$\int_{\tau_m \mathbb{T}^n} |(f - f_{\tau_m})(x)|^p \, dx \ge \frac{a^{p+1} \, \Omega_n(\mathbb{U}^n)}{2^{n+p+1} M(f) R_K}$$

for $m = 1, 2, \ldots$. This contradicts (1.7) and theorem 1.2 is proved.

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