ON THE ACTION OF THE GROUP OF ISOMETRIES ON A LOCALLY COMPACT METRIC SPACE

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Abstract. In this short note we give an answer to the following question. Let X be a locally compact metric space with group of isometries G. Let $\{g_i\}$ be a net in G for which $g_i x$ converges to y, for some $x, y \in X$. What can we say about the convergence of $\{g_i\}$? We show that there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : C_x \to X$ such that g_j converges to f pointwise on C_x and $f(C_x) = C_y$, where C_x and C_y denote the pseudo-components of x and y respectively. Applying this we give short proofs of the van Dantzig–van der Waerden theorem (1928) and Gao–Kechris theorem (2003).

1. The main result and some applications

A few words about the notation we shall be using. In what follows, X will denote a locally compact metric space with group of isometries G. If we endow G with the topology of pointwise convergence then G is a topological group [2, Ch. X, §3.5 Corollary]. On G there is also the topology of uniform convergence on compact subsets which is the same as the compact-open topology. In the case of a group of isometries these topologies coincide with the topology of pointwise convergence, and the natural action of G on X with $(g, x) \mapsto g(x)$, $g \in G$, $x \in X$, is continuous [2, Ch. X, §2.4 Theorem 1 and §3.4 Corollary 1]. For $F \subset G$, let $K(F) := \{x \in X \mid \text{the set } Fx \text{ has compact closure in } X\}$. The sets $K(F)$ are clopen [6, Lemma 3.1].

Lemma 1.1. *Let $\Gamma = \{g_i\}$ be a net in G and $x \in K(\Gamma)$ such that $g_i x$ converges to y for some $y \in X$. Then a subnet of Γ converges to an isometry $f : K(\Gamma) \to X$ on $K(\Gamma)$.*

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Proof. Let $g_i|_{K(\Gamma)}$ denote the restriction of g_i on $K(\Gamma)$. Arzela–Ascoli theorem implies that the set $\{g_i|_{K(\Gamma)} : K(\Gamma) \to X\}$ has compact closure in the set of all continuous maps from $K(\Gamma)$ to X. Thus, there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : K(\Gamma) \to X$ such that $g_j \to f$ on $K(\Gamma)$. □

In [4] S. Gao and A. S. Kechris introduced the concept of pseudo-components. These are the equivalence classes C_x of the following equivalence relation: $x \sim y$ if and only if x and y, as also y and x, can be connected by a finite sequence of intersecting open balls with compact closure. The pseudo-components are clopen [4, Proposition 5.3]. We call X pseudo-connected if it has only one pseudo-component. An immediate consequence of the definitions is that $gC_x = C_{gx}$ for every $g \in G$. Another notion, that will be used in the proofs, is the radius of compactness $\rho(x)$ of $x \in X$ [4]. Let $B_r(x)$ denote the open ball centered at x with radius $r > 0$. Then $\rho(x) := \sup\{ r > 0 \,|\, B_r(x) \text{ has compact closure}\}$. If $\rho(x) = +\infty$ for some $x \in X$ then every ball has compact closure (i.e., X has the Heine–Borel property), hence $\rho(x) = +\infty$ for every $x \in X$. If $\rho(x)$ is finite for some $x \in X$ then the radius of compactness is a Lipschitz map [4, Proposition 5.1]. Note that ρ is G-invariant.

Lemma 1.2. *Let $x, y \in X$ and $\{g_i\}_I$ be a net in G with $g_i x \to y$. Then there is an index $i_0 \in I$ such that $C_x \subset K(F)$, where $F := \{g_i \,|\, i \geq i_0\}$.*

Proof. Since X is locally compact there exists an index i_0 such that the set $F(x)$ has compact closure, where $F := \{g_i \,|\, i \geq i_0\}$. We claim that for every $z \in C_x$ the set $F(z)$ also has compact closure, hence $C_x \subset K(F)$. The strategy is to start with an open ball $B_r(x)$ with radius $r < \rho(x)$ and prove that $F(z)$ has compact closure for every $z \in B_r(x)$. Then our claim follows from the definition of C_x. To prove the claim take a sequence $\{g_n z\} \subset F$. Since the closure of $F(x)$ is compact we may assume, upon passing to a subsequence, that $g_n x \to w$ for some w in the closure of $F(x)$. Assume that $\rho(x)$ is finite and take a positive number ε such that $r + \varepsilon < \rho(x)$. Then for n big enough

$$d(g_n z, w) \leq d(g_n z, g_n x) + d(g_n x, w) = d(z, x) + d(g_n x, w) < r + \varepsilon < \rho(x).$$

Recall that the radius of convergence is a continuous map, and since $g_n x \to w$ then $\rho(x) = \rho(w)$. So, the sequence $\{g_n z\}$ is contained eventually in a ball of w with compact closure, hence it has a convergence subsequence. The same also holds in the case where $\rho(x) = +\infty$. □

Theorem 1.3. *Let X be a locally compact metric space with group of isometries G and let $\{g_i\}$ be a net in G for which $g_i x$ converges to y,* *for some $x, y \in X$. Then there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : C_x \to X$ such that g_j converges to f pointwise on C_x and $f(C_x) = C_{f(x)}$*

Proof. By Lemma 1.2 there is an index $i_0 \in I$ such that $C_x \subset K(F)$, where $F := \{g_i \,|\, i \geq i_0\}$. Hence, by Lemma 1.1, there exists a subnet $\{g_j\}$ of $\{g_i\}$ which converges to an isometry $f : K(F) \to X$ on $K(F)$. Therefore, $g_j \to f$ on C_x. Let us show that $f(C_x) = C_{f(x)}$. Since $d(x, g_j^{-1}f(x)) = d(g_jx, f(x)) \to 0$ it follows that $g_j^{-1}f(x) \to x$. Hence, by repeating the previous procedure, there exist a subnet $\{g_k\}$ of $\{g_j\}$ and an isometry $h : C_{f(x)} \to X$ such that $g_k^{-1} \to h$ pointwise on $C_{f(x)}$ and $h(f(x)) = x$. Note that $g_kx \in C_{f(x)}$ eventually for every k, since $g_kx \to f(x)$ and $C_{f(x)}$ is clopen. Therefore, $g_kC_x = C_{g_kx} = C_{f(x)}$. Take a point $z \in C_x$. Then, $g_kz \to f(z)$ and since $C_{f(x)}$ is clopen then $f(z) \in C_{f(x)}$, so $f(C_x) \subset C_{f(x)}$. By repeating the same arguments as before, it follows that $hC_{f(x)} \subset C_x$. Take now a point $w \in C_{f(x)}$. Then $h(w) \in C_x$, hence $g_k^{-1}(w) \in C_x$ eventually for every k. So, $w = g_kg_k^{-1}(w) \to f(h(w)) \in f(C_x)$ from which follows that $C_{f(x)} \subset f(C_x)$. □

A few words about properness. A continuous action of a topological group H on a topological space Y is called proper (or Bourbaki proper) if the map $H \times Y \to Y \times Y$ with $(g, x) \mapsto (x, gx)$, for $g \in H$ and $x \in Y$, is proper, i.e., it is continuous, closed and the inverse image of a singleton is a compact set [1, Ch. III, §4.1 Definition 1]. In terms of nets, a continuous action is proper if and only if whenever we have two nets $\{g_i\}$ in H and $\{x_i\}$ in Y, for which both $\{x_i\}$ and $\{g_ix_i\}$ converge, then $\{g_i\}$ has a convergent subnet. For isometric actions, it is easy to see that a continuous action is proper if and only if whenever we have a net $\{g_i\}$ in H for which $\{g_ix\}$ converges for some $x \in Y$, then $\{g_i\}$ has a convergent subnet. If H is locally compact and Y is Hausdorff, then H acts properly on Y if and only if for every $x, y \in Y$ there exist neighborhoods U and V of x and y, respectively, such that the set $\{g \in H \,|\, gU \cap V \neq \emptyset\}$ has compact closure in H [1, Ch. III, §4.4 Proposition 7]. Observe that if H acts properly on a locally compact space Y then H is also locally compact.

A direct implication of Theorem 1.3 is the van Dantzig–van der Waerden theorem [3]. The advantage of our proof, comparing to the proofs given in the original work of van Dantzig–van der Waerden or in [5, Theorem 4.7, pp. 46–49], is that it is considerably shorter.

Corollary 1.4. *(van Dantzig–van der Waerden theorem 1928) Let X be a connected locally compact metric space with group of isometries G. Then G acts properly on X and is locally compact.*

Another application of Theorem 1.3 is that we can rederive the results of Gao and Kechris in [4, Theorem 5.4 and Corollary 6.2].

Corollary 1.5. *(Gao–Kechris theorem 2003) Let X be a locally compact metric space with finitely many pseudo-components. Then the group of isometries G of X is locally compact. If X is pseudo-connected, then G acts properly on X.*

Proof. Let $C_1, C_2, \ldots, C_n$ denote the pseudo-components of X and take points $x_1 \in C_1, x_2 \in C_2, \ldots, x_n \in C_n$ and open balls $B_r(x_m) \subset C_m$, $m = 1, 2, \ldots, n$, $r > 0$ such that all $B_r(x_m)$ have compact closures. We will show that the set $V := \bigcap_{m=1}^{n} \{g \in G \,|\, gx_m \in B_r(x_m)\}$ is an open neighborhood of the identity in G with compact closure. Indeed, take a net $\{g_i\}$ in V. Since each $B_r(x_m)$ has compact closure there exist a subnet $\{g_j\}$ of $\{g_i\}$ and points $y_1 \in C_1, y_2 \in C_2, \ldots, y_n \in C_n$ such that $g_j x_m \to y_m$ for every $m = 1, 2, \ldots, n$. Theorem 1.3 implies that there exist a subnet $\{g_l\}$ of $\{g_j\}$ and isometries $f_m : C_m \to X$ such that $g_l \to f_m$ on C_m and $f_m(C_m) = C_m$ for all m. The last implies that $\{g_l\}$ converges to an isometry on X, hence V has compact closure.

If X is pseudo-connected the proof of the statement follows directly from Theorem 1.3. □

Remark 1.6. Note that in Corollary 1.5 we do not require that X is separable as in [4, Theorem 5.4 and Corollary 6.2]. This is not a real improvement since if X has countably many pseudo-components then it is separable. Indeed, we define a relation on X by $x\mathcal{S}y$ if and only if there exist separable balls $B_r(x)$ and $B_l(y)$ with $y \in B_r(x)$ and $x \in B_l(y)$. Let $U(x)$ be the equivalence class of x in the transitive closure of the relation $\mathcal{S}$. Then, each $U(x)$ is a separable clopen subset of X [5, Lemma 3 in Appendix 2]. By construction $C_x \subset U(x)$, therefore X is separable.

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