Infinite dimensional stochastic calculus via regularization.

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Abstract

This paper develops some aspects of stochastic calculus via regularization to Banach valued processes. An original concept of χ-quadratic variation is introduced, where χ is a subspace of the dual of a tensor product $B \otimes B$ where B is the values space of some process X process. Particular interest is devoted to the case when B is the space of real continuous functions defined on $[-\tau, 0]$, $\tau > 0$. Itô formulae and stability of finite χ-quadratic variation processes are established. Attention is deserved to a finite real quadratic variation (for instance Dirichlet, weak Dirichlet) process X. The $C([-\tau, 0])$-valued process $X(\cdot)$ defined by $X_t(y) = X_{t+y}$, where $y \in [-\tau, 0]$, is called *window* process. Let $T > 0$. If X is a finite quadratic variation process such that $[X]_t = t$ and $h = H(X_T(\cdot))$ where $H : C([-T, 0]) \longrightarrow \mathbb{R}$ is $L^2([-T, 0])$-smooth or H non smooth but finitely based it is possible to represent h as a sum of a real H_0 plus a forward integral of type $\int_0^T \xi d^- X$ where H_0 and ξ are explicitly given. This representation result will be strictly linked with a function $u : [0, T] \times C([-T, 0]) \longrightarrow \mathbb{R}$ which in general solves an infinite dimensional partial differential equation with the property $H_0 = u(0, X_0(\cdot))$, $\xi_t = D^{\delta_0} u(t, X_t(\cdot)) := Du(t, X_t(\cdot))(\{0\})$. This decomposition generalizes the Clark-Ocone formula which is true when X is the standard Brownian motion W. The financial perspective of this work is related to hedging theory of path dependent options without semimartingales.

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Contents

Chapter 1

Introduction

Classical stochastic calculus and integration come back at least to Itô [38] and it has been developed successfully by a huge number of authors. The most classical Itô's integrator is Brownian motion but the theory naturally extends to martingales and semimartingales. Stochastic integration with respect to semimartingales is now quite established and performing. For that topic, there are also many monographs, among them [41], [54] for continuous integrators and [40] and [53] for jump processes. In order to describe models coming especially from physics and biology, useful tools are infinite dimensional stochastic differential equation for which the classical stochastic integrals needed to be generalized. Those integrals involve Banach valued stochastic processes. At our knowledge the seminal book is [46], which generalizes stochastic integrals and Itô formulae, in a general framework, to a class of integrators called π-processes. Let B be a Banach space and X a B-valued continuous process. Let Y be an elementary B^*-valued process i.e. a finite sum of functions of the type $c\mathbb{1}_{]a,b]}$, where $a < b$ and c is a non-anticipating B^*-valued random variable. The integral $\int_0^T \langle c\mathbb{1}_{]a,b]}, dX\rangle$ can be obviously defined by $\langle c, X_b - X_a\rangle$. The integral $\int_0^T \langle Y, dX\rangle$ can be deduced by linearity. If X is a so-called π-process and Y is an elementary process then the following inequality holds $\mathbb{E}\left[\int_0^T \langle Y, dX\rangle\right]^2 \leq \mathbb{E}\left[\int_0^T \|Y\|^2 d\alpha\right]$, where α is a suitable measure on predictable sets. In other words for a π-process X it is possible, to write a generalization of the isometry property of real valued Itô integrals. If the Banach values space B is a Hilbert space then the concept of π-process generalizes the notion of square integrable martingale and bounded variation process. The infinite dimensional stochastic integration theory has known a big success in applications to different classes of stochastic partial differential equations. It concerns especially the case when B is a separable Hilbert space or Gelfand triples of Hilbert spaces. We mention at this level the significant early work by Pardoux, see [52], [50] and [51]. Successively the theory of stochastic partial differential equations was developed around the Da Prato-Zabczyk integral, see [13] (or for more recent issues [55]), and Walsh integral, e.g. [71] and [14]. A recent book completing the Metivier-Pellaumail approach is the [19]. Among the most successful application of stochastic calculus in an infinite dimensional separable Hilbert space are stochastic delay equations: a initial and fundamental paper is [9]. More recently an interesting and complete on this subject is [29]. A significant theory of infinite dimensional stochastic integration was developed when B is an M-type 2 Banach spaces, see [16, 15] and continued by several authors as e.g. [7], [1]. Interesting issues in this direction concern the case when B is a UMD space; one recent paper in this direction is [69]. A space which is neither a M-type 2 space nor a UMD space is $C([-\tau, 0])$ with $\tau > 0$, i.e. the Banach space of real continuous functions defined on $[-\tau, 0]$. This is the typical space in which stochastic integration is challenging. This context is natural when studying stochastic differential equations with functional dependence (as for instance delay equations). Due to the difficulty of stochastic integration and calculus in that space, most of the authors fit the problem in some ad hoc Hilbert space, see for instance [9]. A step in the investigation of stochastic integration for $C([-\tau, 0])$-valued and associated processes was done by [70].

The literature of stochastic integrals via regularizations and calculus concerns essentially real valued (and in some cases $\mathbb{R}^n$-valued) processes and it is very rich. This topic was studied first in [57] and [58, 59, 75]. Later significant developments appear for instance in [62, 26, 27] when the integrator is a real finite quadratic variation process and [24, 34, 33] when the integrator is not necessarily a finite quadratic variation process. Important investigations in the case of jump integrators were performed by [17] and [49]. Many applications were performed and it is impossible to list them all, in particular those to mathematical finance; in order to show the spirit we will quote [45], [42], [5]. A recent survey on the subject is [61]. Given an integrand process $Y = (Y_t)_{t \in [0,T]}$ and an integrator $X = (X_t)_{t \in [0,T]}$, a significant notion is the forward integral of Y with respect to X, denoted by $\int_0^T Y d^- X$. When X is a (continuous) semimartingale and Y is a cadlag adapted process, that integral coincides with Itô's integral $\int_0^T Y dX$. Stochastic calculus via regularization is a theory which allows, in many specific cases to manipulate those integrals when Y is anticipating or X is not a semimartingale. If $X = W$ is a Brownian motion and Y is a (possibly anticipating) process with some Malliavin differentiability, then $\int_0^T Y d^- W$ equals Skorohod integral $\int_0^T Y \delta W$ plus of a trace term. A version of this calculus when B has infinite dimension was not yet developed, even though interesting obervations in that direction were exploited in [35], in particular when the integrator is multi-parametric. The aim of the present work is to set up the basis of such a calculus with values on Banach spaces in the (simplified) case when integrals are real valued. The central object is a forward integral of the type $\int_0^T \langle Y, d^- X \rangle$, when Y (resp. X) is a B^*-valued (resp. B-valued) process. We show that when B is a separable Hilbert space, Y is a non-anticipating square integrable process and X is a Wiener process, $\int_0^T \langle Y, d^- X \rangle$ coincides with the Da Prato-Zabczyk integral, see Proposition 3.10.
One important object in calculus via regularization is the notion of the covariation $[X, Y]$ of two real processes X and Y. If $X = Y$, $[X, X]$ is called the so-called quadratic variation of X. If X is $\mathbb{R}^n$-valued process with components $X^1, \ldots, X^n$, the generalization of the notion of quadratic variation $[X, X]$ is provided by the matrix $([X^i, X^j])_{i,j=1,\ldots,n}$. If such a matrix indeed exists, one also says that X admits all

its mutual covariations or brackets.

In this paper we introduce a sophisticated notion of quadratic variation which generalizes the former one. This is called χ-quadratic variation in reference to a subspace χ of the dual of $B\hat{\otimes}_\pi B$. When B is the finite dimensional space $\mathbb{R}^n$, X admits all its mutual brackets if and only if X has a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$, see Proposition 6.2. A Banach valued *locally semi summable* process X in the sense of [19], has again a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$. We establish a general Itô's formula, see Theorem 8.1; we also show in Theorem 7.21 that if X has a χ-quadratic variation and $F : B \to \mathbb{R}$ is of class C^1 Fréchet with some supplementary properties on DF, then $F(X)$ is a real finite quadratic variation process. A specific attention is devoted to the case when $B = C([-\tau, 0])$ and $X(\cdot)$ is a *window* process associated to a real continuous process.

Definition 1.1. Given $0 < \tau \leq T$ and a real continuous process $X = (X_t)_{t\in[0,T]}$, we will call **window process**, and denoted by $X(\cdot)$, the $C([-\tau, 0])$-valued process

$$X(\cdot) = \big(X_t(\cdot)\big)_{t\in[0,T]} = \{X_t(x) := X_{t+x}; x \in [-\tau, 0], t \in [0, T]\} \ .$$

We emphasize that $C([-\tau, 0])$ is typical a non-reflexive Banach space. We obtain a generalized Doob-Meyer-Fukushima decomposition for $C^1\,(C([-T, 0]))$-functionals of window Dirichlet processes, see Theorems 7.34 and 7.33, or even $C^{0,1}\,([0, T] \times C([-T, 0]))$-functionals of window weak Dirichlet processes with finite quadratic variation, see Theorem 7.36.

Motivated by financial applications, we finally establish a Clark-Ocone type decomposition for a class of random variables h depending on the paths of a finite quadratic variation process X such that $[X]_t = t$. This chapter is motivated by the hedging problem of path-dependent options in mathematical finance. This generalizes some results included in [64, 3, 11] concerning the hedging of vanilla or Asiatic type options.

If the noise is modeled by (the derivative of) a Brownian motion W, the classical martingale representation theorem and classical Clark-Ocone formula is a useful tool for finding a portfolio hedging strategy. One of our results consists in expressing a random variable $h = H(X(\cdot))$, where $H : C([0, T]) \longrightarrow \mathbb{R}$, as

$$h = H_0 + \int_0^T \xi_s d^- X_s \tag{1.1}$$

under reasonable sufficient conditions on the functional H. H_0 is a real number and ξ is an non-anticipating process which are explicitly given. We will show that in most of the cases it is possible to exhibit a function $u : [0, T] \times C([-T, 0]) \longrightarrow \mathbb{R}$ which belongs to $C^{1,2}\,([0, T[\times C([-T, 0])) \cap C^0\,([0, T] \times C([-T, 0]))$ solving an infinite dimensional partial differential equation, such that the representation (1.1) of random variable h holds and $H_0 = u(0, X_0(\cdot))$, $\xi_t = D^{\delta_0}u\,(t, X_t(\cdot))$, where $D^{\delta_0}u\,(t, \eta)$ denotes the *projection* of the Fréchet derivative $Du\,(t, \eta)$ on the linear space generated by Dirac measure δ_0, i.e. such that $D^{\delta_0}u\,(t, \eta) := Du\,(t, \eta)(\{0\})$, see Proposition 9.27 and Corollary 9.28. Two types of general sufficient conditions on the functional H, for which such a function u exists, will be discussed. They concern cases when H is considered as defined on $L^2([-T,0])$, either when H has some Fréchet regularity, see Theorem 9.41 and Corollary 9.45, or when $\eta \mapsto H(\eta)$ is not smooth, but depends on a finite number of pathwise integrals, of the type $\int_0^T \varphi d\eta$, see Proposition 9.53 and Proposition 9.55; in that case, we will say that H is **finitely based**. Making use of some improper forward integral we also obtain some new representation results even when X is a Brownian motion W and H has no regularity, see Proposition 9.10 and Theorem 9.20. Expression (1.1) extends Clark-Ocone formula to the case when X is no longer a Brownian motion but it has the same quadratic variation. On the other hand, the mentioned decomposition reaches some r.v. h for which the classical Clark-Ocone formula is not true, even when X is the standard Brownian motion W.

The paper is organised as follows. After this introduction, Section 2 contains preliminary notations with some remarks on classical Dirichlet processes and Malliavin calculus and basic notions on tensor products analysis. In Section 3, we define the integral via regularization for infinite dimensional Banach valued processes and we establish a link with notion of Da Prato-Zabczyk's stochastic integral. Section 4 will be devoted to the definition of χ-quadratic variation and some related results and in Section 5, we will evaluate the χ-quadratic variation for different classes of processes. In Section 6 we will redefine some classical notions of quadratic variation in the spirit of χ-quadratic variation. In Section 7, we give the definition of χ-covariation and we establish C^1 stability properties and some basic facts about weak Dirichlet processes and Fukushima-Dirichlet decomposition of functions of the process $F(t, D_t(\cdot))$ with a sufficient condition to guarantee that the resulting process is a true Dirichlet process. In Section 8 we state and prove a C^2-Fréchet type Itô's formula. The final Section 9 is devoted to the Clark-Ocone type formula.

Chapter 2

Preliminaries

2.1 General notations

In this section we recall some definitions and notations concerning the whole paper. Let A and B be two general sets such that $A \subset B$; $\mathbb{1}_A : B \to \{0,1\}$ will denote the indicator function of the set A, so $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$. We also write $\mathbb{1}_A(x) = \mathbb{1}_{\{x \in A\}}$. If m, n are positive natural numbers, we will denote by $\mathbb{M}_{m\times n}(\mathbb{R})$ the space of real valued matrix of dimension $m \times n$. When $m = n$, this is the space of squared real valued matrix $n \times n$, denoted simply by $\mathbb{M}_n(\mathbb{R})$. If $m = 1$, $\mathbb{M}_{1\times n}(\mathbb{R})$ will be identified with $\mathbb{R}^n$.
Let $k \in \mathbb{N} \cup \{\infty\}$, we denote by $C^k(\mathbb{R}^n)$ the set of all function $g : \mathbb{R}^n \to \mathbb{R}$ which admits all partial derivatives of order $0 \leq p \leq k$. If I is a real interval and g is a function from $I \times \mathbb{R}^n$ to $\mathbb{R}$ which belongs to $C^{1,2}(I \times \mathbb{R}^n)$, the symbols $\partial_t g(t,x)$, $\partial_i g(t,x)$ and $\partial^2_{ij} g(t,x)$ will denote respectively the partial derivative with respect to variable I, the partial derivative with respect to the i-th component and the second order mixed derivative with respect to j-th and i-th component evaluated in $(t,x) \in I \times \mathbb{R}^n$.
We denote by $C^\infty_p(\mathbb{R}^n)$ (resp. $C^\infty_b(\mathbb{R}^n)$ and $C^\infty_0(\mathbb{R}^n)$) the set of all infinitely continuously differentiable functions $g : \mathbb{R}^n \to \mathbb{R}$ such that g and all its partial derivatives have polynomial growth (resp. g and all its partial derivatives are bounded and g has compact support).
Throughout this paper we will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a fixed probability space, equipped with a given filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ fulfilling the usual conditions. Let $a < b$ be two real numbers, $C([a,b])$ will denote the Banach linear space of real continuous functions equipped with the uniform norm denoted by $\|\cdot\|_\infty$ and $C_0([a,b])$ will denote the space of real continuous functions f on $[a,b]$ such that $f(a) = 0$. The letters B, E, F, G (respectively H) will denote Banach (respectively Hilbert) spaces over the scalar field $\mathbb{R}$. Given two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E, we say that $\|\cdot\|_1 \leq \|\cdot\|_2$ if for every $x \in E$ there is a positive constant c such that $\|x\|_1 \leq c\,\|x\|_2$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if they define the same topology, i.e. if there

exist positive real numbers c and C such that $c\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2$ for all $x \in E$.
The space of bounded linear mappings from B to E will be denoted by $L(B;E)$ and we will write $L(B)$ instead of $L(B;B)$. The topological dual space of B, i.e. when $L(B;\mathbb{R})$, will be denoted by B^*. If ϕ is a linear functional on B, we shall denote the value of ϕ at an element $b \in B$ either by $\phi(b)$ or $\langle \phi, b\rangle$ or even ${}_{B^*}\langle \phi, b\rangle_B$. Throughout the paper the symbols $\langle \cdot, \cdot\rangle$ will denote always some type of duality that will change depending on the context. Let K be a compact space, $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the so-called set of finite signed measures on K. We will say that two positive (or signed) measures μ and ν defined on a measurable space (Ω, Σ) are **singular** if there exist two disjoint sets A and B in Σ whose union is Ω such that μ is zero on all measurable subsets of B while ν is zero on all measurable subsets of A. This will be denoted by $\mu \perp \nu$. This definition generalizes to a family of measures. Let I be an index set and $(\mu_i)_{i\in I}$ a family of measures on a measurable space (Ω, Σ). $(\mu_i)_{i\in I}$ are called mutually singular if $\mu_i \perp \mu_j$ for any $i, j \in I$ such that $i \neq j$. In particular there exists a partition $(A_i)_{i\in I}$ of Σ such that $\forall\, i \in I$, $\mu_i(B) = 0$, $\forall\, B \subset A_i^c$, where A_i^c denotes the complementary set of A_i, i.e. $\Omega \setminus A_i$.
We recall the definition of the weak star topology: it is a topology defined on dual spaces as follows. Let B be a normed space; a sequence of B^*-valued elements $(\phi^n)_{n\in\mathbb{N}}$ converges **weak star** to $\phi \in B^*$, denoted by symbols $\phi^n \xrightarrow[n\longrightarrow+\infty]{w^*} \phi$, if ${}_{B^*}\langle \phi^n, b\rangle_B \xrightarrow[n\longrightarrow+\infty]{} {}_{B^*}\langle \phi, b\rangle_B$ for every $b \in B$. By definition, the weak star topology is weaker than the weak topology on B^*. An important fact about the weak star topology is the Banach-Alaoglu theorem: if B is normed, then the unit ball in B^* is weak star compact; more generally, the polar in B^* of a neighborhood of 0 in B is weak star compact. Given a Banach space B and its topological bidual space B^{**} the application $J : B \to B^{**}$ will denote the natural injection between a Banach space and its bidual. J is an injective linear mapping, though it is not surjective unless B is reflexive. J is an isometry with respect to the topology defined by the norm in B, the so-called *strong topology*, and $J(B)$ which is weak star dense in B^{**}. The weak star topology on B^* is the weak topology induced by the image of $J : J(B) \subset B^{**}$. For more informations about Banach spaces topologies, see [6, 74]. Let E, F, G be Banach spaces; we shall denote the space of G-valued bounded bilinear forms on the product $E \times F$ by $\mathcal{B}(E \times F; G)$ with the norm given by $\|\phi\|_{\mathcal{B}} = \sup\{\|\phi(e,f)\|_G : \|e\|_E \leq 1; \|f\|_F \leq 1\}$. Our principal references about functional analysis are [20, 21, 22, 6, 74].
The capital letters X, Y, Z will generally denote Banach valued continuous processes indexed by the time variable $t \in [0,T]$ with $T > 0$ (or $t \in \mathbb{R}_+$). A stochastic process X will be also denoted by $(X_t)_{t\in[0,T]}$, $\{X_t; t \in [0,T]\}$, or $(X_t)_{t\geq 0}$. A B-valued stochastic process X is a map $X : \Omega \times [0,T] \to B$ which will be always supposed to be measurable w.r.t. the product sigma-algebra. All the processes indexed by $[0,T]$ (respectively $\mathbb{R}^+$) will be naturally prolongated by continuity setting $X_t = X_0$ for $t \leq 0$ and $X_t = X_T$ for $t \geq T$ (respectively $X_t = X_0$ for $t \leq 0$). A sequence of continuous B-valued processes indexed by $[0,T]$, $(X^n)_{n\in\mathbb{N}}$ will be said to converge **ucp** (**uniformly convergence in probability**) to a process X if $\sup_{0\leq t\leq T}\|X^n - X\|_B$ converges to zero in probability when $n \to \infty$. The space $\mathcal{C}([0,T])$ will denote the linear space of continuous real processes equipped with the ucp topology and the metric

$d(X,Y) = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - Y_t| \wedge 1\right]$. The space $\mathcal{C}([0,T])$ is not a Banach space but equipped with this metric is a Fréchet space (or F-space shortly) see Definition II.1.10 in [20]. For more details about F-spaces and their properties see section II.1 in [20].
We recall Lemma 3.1 from [60]. The mentioned lemma states that a sequence of continuous increasing processes converging at each time in probability to a continuous process, converges ucp.

Lemma 2.1. Let $(Z^\epsilon)_{\epsilon>0}$ be a family of continuous processes. We suppose the following.

1) $\forall \epsilon > 0$, $t \to Z^\epsilon_t$ is increasing.

2) There is a continuous process $(Z_t)_{t>0}$ such that $Z^\epsilon_t \to Z_t$ in probability when ϵ goes to zero.

Then Z^ε converges to Z ucp.

We go on with other notations.
If X is a real continuous process indexed by $[0,T]$ and $0 < \tau \le T$, we will recall the fundamental definition of window process in Definition 1.1. Process $(X_t(\cdot))_{t\in[0,T]}$ will be also denoted by symbols $X(\cdot)$ or $\{X_t(\cdot); t \in [0,T]\}$. $X(\cdot)$ will be understood, sometimes without explicit mention, as $C([-\tau,0])$-valued. In view of some applications, sometimes, but it will be explicitly mentioned, $X(\cdot)$ will be considered as a $L^2([-T,0])$-valued process.

2.2 The forward integral for real valued processes

We will follow here a framework of calculus via regularizations started in [58]. In fact many authors have contributed to this and we suggest the reader consult the recent fairly survey paper [61] on it. We first recall basic concepts and some one dimensional results concerning calculus via regularization. For simplicity, all the considered integrator processes will be continuous processes. We recall now the notion of forward integral and covariation.

Definition 2.2. Let X (respectively Y) be a continuous (resp. locally integrable) process. If the random variables

$$\int_0^t Y_s d^- X_s := \lim_{\epsilon\to 0} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds \tag{2.1}$$

exist in probability for every $t \in [0,T]$ and the limiting process admits a continuous modification, then Y is said to be X-**forward integrable**. The limiting process is denoted by $\int_0^\cdot Y d^- X$ and it is called the **(proper) forward integral of Y with respect to X.**
Whenever the limit in (2.1) exists in the ucp topology the forward integral is of course a continuous process and $\int_0^\cdot Y d^- X$ is the **forward integral of Y with respect to X in the ucp sense**.

In fact, the definition in the ucp sense of the forward integral is the traditional one considered by F. Russo and P. Vallois, see for instance [61].

Definition 2.3. If $YI_{[0,t]}$ is X-forward integrable for every $0 \le t < T$, Y is said **locally X-forward integrable on** $[0,T[$. If moreover $\lim_{t\to T}\int_0^t Yd^-X$ exists in probability, the limiting process is called the **improper forward integral of Y with respect to X** and it is still denoted by $\int_0^\cdot Yd^-X$.

Definition 2.4.

1. The **covariation of X and Y** is defined by

$$[X,Y]_t = \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_0^t (X_{r+\epsilon}-X_r)(Y_{r+\epsilon}-Y_r)dr \tag{2.2}$$

   if the limit exists in the ucp sense with respect to t.

2. If $[X,X]$ exists then X is said to be a **finite quadratic variation** process. $[X,X]$ will also be denoted by $[X]$ and it will be called **quadratic variation of X**. According to the conventions of Section 2.1 we have

$$[X]_t = 0 \qquad\qquad \text{for } t<0. \tag{2.3}$$

3. If $[X]=0$, then X is said to be a **zero quadratic variation process**.

It follows by the definition that the covariation process defined in (2.2) is a continuous process. Obviously the covariation is a bilinear and symmetric operation.

Definition 2.5. If $X=(X^1,\dots,X^n)$ is a vector of continuous processes we say that it has all its **mutual covariations** (brackets) if $[X^i,X^j]$ exists for any $1\le i,j\le n$.

The definition of quadratic variation can be generalized for a $\mathbb{R}^n$ valued process. This generalization to multivalued processes will be studied in details in Section 6.1.

Definition 2.6. Let $X=(X^1,\cdots,X^n)$ be an $\mathbb{R}^n$-valued process having all its mutual covariations. The matrix in $\mathbb{M}_{n\times n}(\mathbb{R})$, denoted by $[X^*,X]$, and defined by $([X^*,X])_{1\le i,j\le n}=[X^i,X^j]$ is called the quadratic variation of X.

Remark 2.7. If $X=(X^1,\dots,X^n)$ has all its mutual covariations then by polarization (i.e. similarly to the case when a bilinear form is expressed as sum/difference of quadratic forms) we know that $[X^i,X^j]$ are locally bounded variation processes for $1\le i,j\le n$.

Lemma 2.8. Let $X=(X^1,\dots,X^n)$ be a vector of continuous processes such that

$$\frac{1}{\epsilon}\int_0^t (X^i_{s+\epsilon}-X^i_s)(X^j_{s+\epsilon}-X^j_s)ds \tag{2.4}$$

converges in probability for every $1\le i,j\le n$ to some continuous process for any $t\in[0,T]$. Then $[X^i,X^j]$ exists for every $1\le i,j\le n$.

Proof. Let i, j be fixed. The quantity

$$\frac{1}{\epsilon}\int_0^t \left[(X^i_{s+\epsilon}+X^j_{s+\epsilon})-(X^i_s+X^j_s)\right]^2 ds \tag{2.5}$$

is a linear combination of elements of type (2.4), therefore (2.5) converges in probability for any $t\in[0,T]$ to a continuous process. Since it is increasing, by Lemma 2.1, it converges ucp. By bilinearity (2.4) equals

$$\frac{1}{2\epsilon}\int_0^t (X^i_{s+\epsilon}-X^i_s)^2 ds+\frac{1}{2\epsilon}\int_0^t (X^i_{s+\epsilon}-X^i_s)^2 ds-\frac{1}{2\epsilon}\int_0^t \left[(X^i_{s+\epsilon}+X^j_{s+\epsilon})-(X^i_s+X^j_s)\right]^2 ds.$$

The first two integrals converges ucp because of Lemma 2.1 and the assumption. In conclusion also (2.4) converges ucp. □

Remark 2.9. 1. Let S be an $(\mathcal{F}_t)$-continuous semimartingale (resp. Brownian motion), (Y_t) be an adapted cadlag (resp. such that $\int_0^T Y_r^2 dr<\infty$). Then $\int_0^{\cdot} Y_r d^- S_r$ exists and equals the classical Itô integral $\int_0^{\cdot} Y_r dS_r$, see Proposition 6 in [61].

2. Let X (respectively Y) be a finite (respectively zero) quadratic variation process. Then (X,Y) has all its mutual covariations and $[X,Y]=0$, see Proposition 1, 6) in [61].

3. If S^1, S^2 are $(\mathcal{F}_t)$-semimartingale then $[S^1,S^2]$ coincides with the classical bracket $\langle S^1,S^2\rangle$ in the sense of [39, 53], see Corollary 2 in [61].

4. A bounded variation process is a zero quadratic variation process.

Definition 2.10. Let X and Y be two real continuous processes. We call **covariation structure of X and Y** the field $(x,y)\mapsto[X_{x+\cdot},Y_{y+\cdot}]$ whenever it exists for all $x\in\mathbb{R}_+$ and $y\in\mathbb{R}_+$. It will denoted by $([X_{x+\cdot},Y_{y+\cdot}],x,y\geq 0)$. Whenever $X=Y$, it will also be called **covariation structure of X**.

A not well known notion but however useful is the following. It was introduced in [12], Definition 3.5.

Definition 2.11. A real process R is called **strongly predictable** with respect to a filtration $(\mathcal{F}_t)$, if it exists $\delta>0$, such that $(R_{s+\epsilon})_{s\geq 0}$ is $(\mathcal{F}_t)$-adapted, for every $\epsilon\leq\delta$.

An important fact about the covariation structure of semimartingales is the following.

Proposition 2.12. Let S^1 and S^2 be two $(\mathcal{F}_t)$-continuous semimartingales. Then the covariation structure of S^1 and S^2 verifies $[S^1_{x+\cdot},S^2_{y+\cdot}]=0$ for all $x,y\in\mathbb{R}$ such that $x\neq y$.

Proof. Proposition 2.14.1) and the bilinearity of covariation helps us to reduce the problem to the case where $S^i=M^i, i=1,2$ are $(\mathcal{F}_t)$-local martingales. By definition of real covariation, we can just consider the case $y=0$ and $x<0$. Proposition 4.11 in [11] states that if M is a continuous $(\mathcal{F}_t)$-local martingale and Y is an $(\mathcal{F}_t)$-strongly predictable then then $[N,Y]=0$. Since process Y defined by $Y_t=Y_{t+x}, t\geq 0$ is $(\mathcal{F}_t)$-strongly predictable, the result follows. □

We recall the Itô formula for finite quadratic variation process.

Theorem 2.13. Let $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $F \in C^{1,2}([0,T[\times\mathbb{R})$ and X be a finite quadratic variation process. Then

$$\int_0^t \partial_x F(s, X_s) d^- X_s \tag{2.6}$$

exists in the ucp sense and equals

$$F(t, X_t) - F(0, X_0) - \int_0^t \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^t \partial_{x\,x} F(s, X_s) d[X]_s \; . \tag{2.7}$$

We recall also a useful result about integration by parts.

Proposition 2.14. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be continuous processes. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X d^- Y + \int_0^t Y d^- X + [X, Y]_t \tag{2.8}$$

where the forward integrals exist in the ucp sense.
If moreover Y is a bounded variation process, then

1) $[X, Y] = 0$.

2) $\int_0^t X d^- Y = \int_0^t X dY$ where $\int_0^t X dY$ is a Lebesgue-Stieltjes integral.

3) Consequently (2.8) simplifies in

$$\int_0^t Y d^- X = X_t Y_t - X_0 Y_0 - \int_0^t X dY \tag{2.9}$$

and previous forward integral $\int_0^t Y d^- X$ exists in the ucp sense.

2.3 About some classes of stochastic processes

We introduce now some peculiar continuous processes that will appear in the paper.

Definition 2.15. The **fractional Brownian motion** B^H of Hurst parameter $H \in (0, 1]$ is a centered Gaussian process with covariance

$$R^H(t, s) = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t - s|^{2H} \right)$$

If $H = 1/2$ it corresponds to a classical Brownian motion. The process is Hölder continuous of order γ for any $\gamma \in (0, H)$. This follows from the Kolmogorov criterion, see [41], Theorem 2.8, chapter 2.

Definition 2.16. The **bifractional Brownian motion** $B^{H,K}$ is a centered Gaussian process with covariance

$$R^{H,K}(t,s) = \frac{1}{2^K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right)$$

with $H \in (0,1)$ and $K \in (0,1]$.

Notice that if $K = 1$, then $B^{H,1}$ coincides with a fractional Brownian motion with Hurst parameter $H \in (0,1)$.
We recall some properties about quadratic variation in the particular case $HK = 1/2$ from Proposition 1 in [56]. If $K = 1$, then $H = 1/2$ and it is a Brownian motion. If $K \neq 1$, it provides an example of a Gaussian process, having non-zero finite quadratic variation which in particular equals $2^{1-K}t$, so, modulo a constant, the same as Brownian motion. The process is Hölder continuous of order γ for any $\gamma \in (0, HK)$. This follows again from Kolmogorov criterion.

The bifractional Brownian motion was introduced by Houdré and Villa in [36] and the related stochastic calculus via regularization was investigated in [56]. In particular, [56] shows that the bifractional Brownian motion behaves similarly to a fractional Brownian motion with Hurst parameter HK and developed a related stochastic calculus. Other properties were established by [43], [25] and [44].
In the whole paper W (respectively B^H and $B^{H,K}$) will denote a real $(\mathcal{F}_t)$-Brownian motion (resp. a fractional Brownian motion of Hurst parameter H and a bifractional Brownian motion of parameters H and K). We recall now definitions of some general classes of processes that we will frequently use in the paper. We start reminding the definition of an $(\mathcal{F}_t)$-semimartingale.

Definition 2.17. A real stochastic process S is an $(\mathcal{F}_t)$**-semimartingale** if S admits a decomposition $S = M + V$ where M is a $(\mathcal{F}_t)$-local square integrable martingale, V is a locally bounded variation process and $V_0 = 0$.

Definition 2.18. A real continuous process D is a called $(\mathcal{F}_t)$**-Dirichlet process** if D admits a decomposition $D = M + A$ where M is an $(\mathcal{F}_t)$-local martingale and A is a zero quadratic variation process. For convenience, we suppose $A_0 = 0$.

The decomposition is unique if for instance $A_0 = 0$, see Proposition 16 in [61]. An $(\mathcal{F}_t)$-Dirichlet process has in particular finite quadratic variation. An $(\mathcal{F}_t)$-semimartingale is also an $(\mathcal{F}_t)$-Dirichlet process, a locally bounded variation process is in fact a zero quadratic variation process.
The concept of $(\mathcal{F}_t)$-Dirichlet process can be weakened. An extension of such processes are the so-called $(\mathcal{F}_t)$-weak Dirichlet processes, which were first introduced and discussed in [23] and [32], but they appeared implicitly even in [24]. Recent developments concerning the subject appear in [10, 12, 65]. $(\mathcal{F}_t)$-weak Dirichlet processes are generally not a $(\mathcal{F}_t)$-Dirichlet processes but they preserve a decomposition property.

Definition 2.19. A real continuous process Y is called $(\mathcal{F}_t)$**-weak Dirichlet** if Y admits a decomposition $Y = M + A$ where M is an $(\mathcal{F}_t)$ local martingale and A is a process such that $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$ local martingale N. For convenience, we will always suppose $A_0 = 0$. A will be said to be an $(\mathcal{F}_t)$**-martingale orthogonal process**.

The decomposition is unique, see for instance Remark 3.5 in [32] or again Proposition 16 in [61]. Corollary 3.15 in [12] makes the following observation. If the underlying filtration $(\mathcal{F}_t)$ is the natural filtration associated with a Brownian motion W then an $(\mathcal{F}_t)$-adapted process A is an $(\mathcal{F}_t)$-martingale orthogonal process if and only if $[A, W] = 0$. An $(\mathcal{F}_t)$-Dirichlet process is also an $(\mathcal{F}_t)$-weak Dirichlet process, a zero quadratic variation process is in fact also an $(\mathcal{F}_t)$-martingale orthogonal process. An $(\mathcal{F}_t)$-weak Dirichlet process is not necessarily a finite quadratic variation process, but there are $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation that are not Dirichlet processes, see for instance [24]. In Theorem 7.34 we will provide another class of examples of $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation which are not $(\mathcal{F}_t)$-Dirichlet.

If W (resp. B^H, $B^{H,K}$, S, D,Y) is a Brownian motion (resp. a fractional Brownian motion of Hurst parameter H, a bifractional Brownian motion of parameters H and K, an $(\mathcal{F}_t)$-semimartingale, an $(\mathcal{F}_t)$-Dirichlet, an $(\mathcal{F}_t)$-weak Dirichlet), then $W(\cdot)$ (resp. $B^H(\cdot)$, $B^{H,K}(\cdot)$, $S(\cdot)$, $D(\cdot)$ and $Y(\cdot)$) will be called **window Brownian motion** (resp. **window fractional Brownian motion** of Hurst parameter H, **window bifractional Brownian motion** of parameters H and K, **window** $(\mathcal{F}_t)$**-semimartingale**, **window** $(\mathcal{F}_t)$**-Dirichlet** and **window** $(\mathcal{F}_t)$**-weak Dirichlet**). The window processes will constitute the main example of Banach valued process in the paper; in that case, as announced, the state space is $C([-\tau, 0])$. In the sequel the underlying filtration $(\mathcal{F}_t)$ will be often omitted.

2.4 Direct sum of Banach spaces

We recall the definition of *direct sum* of Banach spaces given in [20]. The vector space E is said to be the direct sum of vector spaces E_1 and E_2, symbolically $E = E_1 \oplus E_2$, if E_i are subspaces of E with property that every $e \in E$ has a unique decomposition $e = e_1 + e_2$, $e_i \in E_i$. The map $P_i : E \to E_i$ given by $P_i(e) = e_i$ is the projector of E onto E_i. This map will be denoted by P_{E_i} if necessary. If E_i are topological linear spaces, E is a topological linear space, equipped with the product topology. If E_i are Banach spaces, E is a Banach space under either of the p-norms, $1 \leq p \leq +\infty$:

$$\|e_1 + e_2\|_E := \|e_1 + e_2\|_p = \begin{cases} \max\{\|e_1\|_{E_1}, \|e_2\|_{E_2}\} & p = +\infty \\ \|e_1 + e_2\|_E = (\|e_1\|^p_{E_1} + \|e_2\|^p_{E_2})^{1/p} & 1 \leq p < +\infty \end{cases} \tag{2.10}$$

These norms are equivalent to the product topology and there is a real positive constant C such that $\|e_i\|_{E_i} \leq C\|e_1 + e_2\|_E$, for $i = 1, 2$ and all $e_1 \in E_1$ and $e_2 \in E_2$. If the p-norm is given by $p = 1, +\infty$ the

constant C is 1, if the p-norm is given by $1 < p < \infty$ the constant C will be $2^{1-1/p}$, it suffices to observe that the real function $f(x) = |x|^{1/p}$ is concave if $p > 1$.
Given $T \in (E_1 \oplus E_2)^*$, T admits a unique decomposition $T = T_1 \circ P_1 + T_2 \circ P_2$ with $T_1 \in E_1^*$ and $T_2 \in E_2^*$. In fact, we define T_1 by $T_1(e) = T(e)$ for all $e \in E_1$ and T_2 by $T_2(e) = T(e)$ for all $e \in E_2$. Clearly T_i, so defined, are linear and continuous. Whenever the direct sum of normed linear spaces is used as a normed space, the p-norm will be explicitly mentioned. If, however, each of the spaces E_i is a Hilbert space then it will be always understood, sometimes without explicit mention, that E is the uniquely determined Hilbert space with scalar product $\langle e, f\rangle_E = \langle e_1 + e_2, f_1 + f_2\rangle_E = \sum_{i=1}^{2}\langle e_i, f_i\rangle_i$, where $\langle \cdot, \cdot\rangle_i$ is the scalar product in E_i. Thus the norm in a direct sum of Hilbert spaces is always given by the p-norm considering $p = 2$ and, if necessary, will be called Hilbert direct sum and will be denoted by $E_1 \oplus_h E_2$. We remark that in a direct sum of Hilbert spaces it holds $\langle e, f\rangle_E = 0$ for all $e \in E_1$ and $f \in E_2$. The extension to any finite number of summands is immediate. If E_1 and E_2 are closed normed subspace of E, it holds $\overline{Span\{E_1, E_2\}} = E_1 \oplus E_2$.

2.5 Tensor product of Banach spaces

In this section we recall some basic concepts and results about tensor products of two Banach spaces E and F. For details and a more complete description of these arguments, the reader may refer to [63, 18, 67], the case with E and F Hilbert spaces being particularly exhaustive in [47]. If E and F are Banach spaces, the vector space $E \otimes F$ will denote the **algebraic tensor product**. The typical description of an element $u \in E \otimes F$ is $u = \sum_{i=1}^{n} \lambda_i\, e_i \otimes f_i$ where n is a natural number, $\lambda_i \in \mathbb{R}$, $e_i \in E$ and $f_i \in F$. We observe that we can consider the mapping $(e, f) \mapsto e \otimes f$ as a sort of multiplication on $E \times F$ with values in the vector space $E \otimes F$. This product is itself bilinear, so in particular the representation of u is not unique. The general element u can always be rewritten in the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in E$, $y_i \in F$. We say that a norm, α, on $E \otimes F$ is a **reasonable crossnorm** if $\alpha(e \otimes f) \leq \|e\|_E\, \|f\|_F$ for every $e \in E$ and $f \in F$ and if for every $\phi \in E^*$ and $\psi \in F^*$, the linear functional $\phi \otimes \psi$ on $E \otimes F$ is bounded and $\|\phi \otimes \psi\| := \{\sup |\phi \otimes \psi(u)|;\, u \in E \otimes F; \alpha(u) \leq 1\} \leq \|\phi\|_{E^*}\, \|\psi\|_{F^*}$. We can define two different norms in the vector space $E \otimes F$: the so-called called **projective norm**, denoted by π and defined by

$$\pi(u) = \inf\left\{\sum_{i=1}^{n} \|x_i\|\, \|y_i\| \,:\, u = \sum_{i=1}^{n} x_i \otimes y_i\right\} \tag{2.11}$$

and the so-called **injective norm**, denoted by ϵ, defined by

$$\epsilon(u) = \sup\left\{\left|\sum_{i=1}^{n} \phi(x_i)\psi(y_i)\right| \,:\, \phi \in E^*, \|\phi\|_{E^*} \leq 1; \psi \in F^*, \|\psi\|_{F^*} \leq 1\right\}. \tag{2.12}$$

Those norms are reasonable and it holds that α is a reasonable crossnorm if and only if

$$\epsilon(u) \leq \alpha(u) \leq \pi(u) \tag{2.13}$$

for every $u \in E \otimes F$, i.e. the projective one is the largest one and ϵ is the smallest one. Moreover for every reasonable crossnorm in $E \otimes F$ we have $\alpha(e \otimes f) = \|e\| \, \|f\|$ and $\|\phi \otimes \psi\| = \|\phi\| \, \|\psi\|$. We will work principally with the projective norm π and a particular reasonable norm denoted by h, so-called **Hilbert tensor norm**. That reasonable norm h is called Hilbert norm because, whenever E and F are Hilbert spaces then h derives from a scalar product ${}_{E\otimes F}\langle \cdot, \cdot \rangle_{E\otimes F}$ on $E \otimes F$ verifying ${}_{E\otimes F}\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E\otimes F} = {}_E\langle e_1, e_2 \rangle_E \, {}_F\langle f_1, f_2 \rangle_F$. Given a reasonable crossnorm α, we denote by $E \otimes_\alpha F$ the tensor product vector space $E \otimes F$ endowed with the norm α. Unless the spaces E and F are finite dimensional, this space is not complete. We denote its completion by $E \hat{\otimes}_\alpha F$. The Banach space $\mathbf{E} \hat{\otimes}_\alpha \mathbf{F}$ will be referred to as the α **tensor product of the Banach spaces** E **and** F. If E and F are Hilbert spaces the Hilbert tensor product $E \hat{\otimes}_h F$ is a Hilbert space. We recall an important statement in the case of Hilbert spaces from chapter 6 in [47]. If $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ are two measure spaces, then $L^2(\Omega_1, \mathcal{F}_1, \mu_1) \hat{\otimes}_h L^2(\Omega_2, \mathcal{F}_2, \mu_2) \cong L^2(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. The symbols $E \hat{\otimes}_\alpha^2$, $e \otimes^2$ and $e \otimes_\alpha^2$ will denote respectively the Banach space $E \hat{\otimes}_\alpha E$, the elementary element $e \otimes e$ of the algebraic tensor product $E \otimes F$ and $e \otimes e$ in the Banach space $E \hat{\otimes}_\alpha E$. An important role in the paper will be played by topological duals of tensor product spaces denoted, as usual for a dual space of a Banach space, by $(E \hat{\otimes}_\alpha F)^*$ equipped with the operator norm, denoted by α^*; so, if $T \in (E \hat{\otimes}_\alpha F)^*$, $\alpha^*(T) = \sup_{\alpha(u) \le 1} |T(u)|$. By (2.13) we deduce following relation between the tensor dual norms:

$$\epsilon^*(u) \ge \alpha^*(u) \ge \pi^*(u). \tag{2.14}$$

We spend now some words on two special cases.

We have an isometric isomorphism between the Banach space of G-valued bounded bilinear operators (forms in the case $G = \mathbb{R}$) on the product $E \times F$, denoted by $\mathcal{B}(E \times F; G)$, and the Banach space of G-valued bounded linear operators on $E \hat{\otimes}_\pi F$.

If $\tilde{T} : E \times F \to G$ is a continuous bilinear mapping, it exists a unique bounded linear operator $T : E \hat{\otimes} F \to G$ satisfying ${}_{(E\hat{\otimes}_\pi F)^*}\langle T, e \otimes f \rangle_{E\hat{\otimes}_\pi F} = T(e \otimes f) = \tilde{T}(e, f)$ for every $e \in E$, $f \in F$. We observe moreover that it exists a canonical identification between $\mathcal{B}(E \times F; G)$ and $L(E; L(F; G))$ which identifies $\tilde{T}$ with $\bar{T} : E \to L(F; G)$ by $\tilde{T}(e, f) = \bar{T}(e)(f)$. Thus we have a chain of canonical identifications $L(E \hat{\otimes}_\pi F; G) \cong \mathcal{B}(E \times F; G) \cong L(E; L(F; G))$. If we take G to be the scalar field $\mathbb{R}$, we obtain an isometric isomorphism between the dual space of the projective tensor product equipped with the norm π^* with the space of bounded bilinear forms equipped with the usual norm:

$$(E \hat{\otimes}_\pi F)^* \cong \mathcal{B}(E \times F) \cong L(E; F^*) \tag{2.15}$$

With this identification, the action of a bounded bilinear form T as a bounded linear functional on $E \hat{\otimes}_\pi F$ is given by

$${}_{(E\hat{\otimes}_\pi F)^*}\langle T, \sum_{i=1}^n x_i \otimes y_i \rangle_{E\hat{\otimes}_\pi F} = T\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sum_{i=1}^n \tilde{T}(x_i, y_i) = \sum_{i=1}^n \bar{T}(x_i)(y_i). \tag{2.16}$$

It holds $\pi^*(T) = \|\tilde{T}\|_{\mathcal{B}}$ where $\|\cdot\|_{\mathcal{B}}$ was defined in Section 2.1. In the sequel that identification will be often used without explicit mention.
The importance of tensor product spaces and their duals is justified first of all from identification (2.15). In fact, as we will see in details in subsection 2.7, the second order derivative of a real function defined on a Banach space E belongs to $\mathcal{B}(E \times E)$.

We go on with properties of tensor products topologies. There is a of course a chain relation of inclusions between the following Banach tensor products. In particular we have

$$E\hat{\otimes}_\pi F \subset E\hat{\otimes}_\alpha F \subset E\hat{\otimes}_\epsilon F \qquad \text{densely and continuously.} \tag{2.17}$$

For their dual spaces it follows that

$$(E\hat{\otimes}_\epsilon F)^* \subset (E\hat{\otimes}_\alpha F)^* \subset (E\hat{\otimes}_\pi F)^* \qquad \text{continuously (but not necessarily densely).} \tag{2.18}$$

At this point, we would like to comment on a well-known functional analytical result, see Remark 1, after Theorem V.5 in [6].

Theorem 2.20. Let H be a Hilbert space equipped with its scalar product $\langle\ ,\ \rangle$ and associated norm $\|\cdot\|_H$. Let V be a reflexive Banach space equipped with its norm $\|\cdot\|_V$ such that $V \subset H$ continuously, i.e. $\|\cdot\|_H \leq \|\cdot\|_V$.
Then,

$$V \subset H \cong H^* \subset V^* \tag{2.19}$$

densely and continuously.

Remark 2.21.

1. It happens in some literature that previous statement appears without the assumption on V to be reflexive.
2. The statement of Theorem 2.20, is wrong without that reflexivity assumptions as the next item will confirm.
3. Let E and F be two Hilbert spaces. Then the statement of Theorem 2.20 cannot be true for instance with $H = E\hat{\otimes}_h F$ and $V = E\hat{\otimes}_\pi F$. In fact, if it were true Proposition 5.32 will induce a contradiction. As a consequence $E\hat{\otimes}_\pi F$ cannot be reflexive.
4. On the other hand, in general, the projective tensor product of (even reflexive) tensor products is not reflexive. Consider for instance $E = L^p([0,T])$ and $F = L^q([0,T])$, $p, q \in [1, +\infty]$ being conjugate. [63] at Section 4.2 proves that $E\hat{\otimes}_\pi F$ contains a complemented isomorphic copy of ℓ^1 and so it can not be reflexive.

Remark 2.22. Summarizing, if E and F are Hilbert spaces following triple continuously inclusion holds. The first-one holds even densely.

$$E\hat{\otimes}_\pi F \subset E\hat{\otimes}_h F \cong (E\hat{\otimes}_h F)^* \subset (E\hat{\otimes}_\pi F)^*$$

We state a useful result involving Hilbert tensor product and Hilbert direct sum norm.

Proposition 2.23. Let X and Y_1, Y_2 be Hilbert spaces such that $Y_1 \cap Y_2 = \{0\}$. We consider $Y = Y_1 \oplus Y_2$ equipped with the Hilbert direct norm. Then $X\hat{\otimes}_h Y = (X\hat{\otimes}_h Y_1) \oplus (X\hat{\otimes}_h Y_2)$.

Proof. Since $X \otimes Y_i \subset X \otimes Y$, $i = 1, 2$ we can write $X \otimes_h Y_i \subset X \otimes_h Y$ and so

$$(X\hat{\otimes}_h Y_1) \oplus (X\hat{\otimes}_h Y_2) \subset X\hat{\otimes}_h Y \tag{2.20}$$

Since we handle with Hilbert norms, it is easy to show that the norm topology of $X\hat{\otimes}_h Y_1$ and $X\hat{\otimes}_h Y_2$ is the same that the one induced by $X\hat{\otimes}_h Y$.
It remains to show the converse inclusion of (2.20). This follows because $X \otimes Y \subset X\hat{\otimes}_h Y_1 \oplus X\hat{\otimes}_h Y_1$. □

We state now some interesting results about tensor product topologies when $E = F = H$ and H is a separable Hilbert space. Those results involves Hilbert-Schmidt and Nuclear operators. The connection between those classes of operators and tensor product topologies will be deeply investigated in Section 6.2.1. We need a preliminary result.

Proposition 2.24. Let H be a separable Hilbert space and $\bar{T} \in L(H; H^*)$ defined by

$$\bar{T} : H \longrightarrow H^* \qquad g \mapsto \bar{T}(g) = \langle g, \cdot \rangle_H \tag{2.21}$$

Then $\bar{T} \notin L^2\big(H; H^*\big)$ where $L^2\big(H; H^*\big)$ is the space of Hilbert-Schmidt operators from H to H^*.

Proof. Let (e_i) be an orthonormal basis of H. By Riesz identification we have

$$\sum_{i=1}^{\infty} \big\|\bar{T}(e_i)\big\|^2_{H^*} = \sum_{i=1}^{\infty} \|e_i\|^2_H = +\infty$$

so $\bar{T}$ is not Hilbert-Schmidt. □

Corollary 2.25. Let H be a separable Hilbert space. Then $(H\hat{\otimes}_h H)^*$ is properly included in $(H\hat{\otimes}_\pi H)^*$.

Proof. As we have seen at (2.15), $(H\hat{\otimes}_\pi H)^*$ can be identified with $\mathcal{B}(H, H) \cong L(H; H^*)$. On the other hand $(H\hat{\otimes}_h H)^*$ can be identified with the Hilbert-Schmidt operators from H to H^*, denoted by $L^2(H; H^*)$, see Section 6.2.1. It is well known that $L^2(H; H^*)$ is properly included in $L(H; H^*)$.
A second, more direct argument, is the following. Let $T \in \mathcal{B}(H, H)$ defined by

$$T : H \times H \longrightarrow \mathbb{R} \qquad (g, h) \mapsto T(g, h) = \langle g, h \rangle_H. \tag{2.22}$$

The element $\bar{T} \in L(H; H^*)$ canonically associated with T in $\mathcal{B}(H, H)$ equals (2.21). By Proposition 2.24, $\bar{T} \notin L^2(H; H^*)$ and the result follows. □

We recall another important identification which helps to obtain a representation of a space of continuous functions of two variables as an injective tensor product of two spaces of continuous functions. More precisely if K_1, K_2 are compact spaces, by Section 3.2 in [63] we have

$$C(K_1)\hat{\otimes}_\epsilon C(K_2) = C(K_1; C(K_2)) = C(K_1 \times K_2)\ . \tag{2.23}$$

In particular, by (2.18), we have

$$\mathcal{M}(K_1 \times K_2) = (C(K_1)\hat{\otimes}_\epsilon C(K_2))^* \subset (C(K_1)\hat{\otimes}_\pi C(K_2))^* \cong \mathcal{B}(C(K_1), C(K_2)). \tag{2.24}$$

Let η_1, η_2 be two elements in $C([-\tau,0])$ (respectively $L^2([-\tau,0])$). The element $\eta_1 \otimes \eta_2$ in the algebraic tensor product $C([-\tau,0])\otimes^2$ (respectively $L^2([-\tau],0)\otimes^2$) will be identified with the element η in $C([-\tau,0]^2)$ (respectively $L^2([-\tau,0]^2)$) defined by $\eta(x,y) = \eta_1(x)\eta_2(y)$ for all x, y in $[-\tau,0]$. So if μ is a measure on $\mathcal{M}([-\tau,0]^2)$, the pairing duality ${}_{\mathcal{M}([-\tau,0]^2)}\langle \mu, \eta_1 \otimes \eta_2\rangle_{C([-\tau,0]^2)}$ has to be understood as the following pairing duality:

$${}_{\mathcal{M}([-\tau,0]^2)}\langle \mu, \eta\rangle_{C([-\tau,0]^2)} = \int_{[-\tau,0]^2} \eta(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2} \eta_1(x)\eta_2(y)\mu(dx,dy)\ . \tag{2.25}$$

2.6 Notations about subsets of measures

Spaces $\mathcal{M}([-\tau,0])$ and $\mathcal{M}([-\tau,0]^2)$ and their subsets will play a central role in the paper. We will introduce some other notations that will be used in the sequel. Let $-\tau = a_N < a_{N-1} < \ldots a_1 < a_0 = 0$ be $N+1$ fixed points in $[-\tau,0]$. Symbols a and A will refer respectively to the vector $(a_N, a_{N-1}, \ldots, a_1, 0)$ and to the matrix $(A_{i,j})_{0\le i,j\le N} = (\{a_i,a_j\})_{0\le i,j\le N}$. Vector a identifies $N+1$ points on $[-\tau,0]$ and matrix A identifies $(N+1)^2$ points on $[-\tau,0]^2$.

- Symbol $\mathcal{D}_i([-\tau,0])$ (shortly $\mathcal{D}_i$), will denote the one dimensional space of multiples of Dirac's measure concentrated at $a_i \in [-\tau,0]$, i.e.

  $$\mathcal{D}_i([-\tau,0]) := \{\mu \in \mathcal{M}([-\tau,0]);\ s.t.\mu(dx) = \lambda\,\delta_{a_i}(dx) \text{ with } \lambda \in \mathbb{R}\}\ ; \tag{2.26}$$

  we define the scalar product between $\mu^1 = \lambda^1\delta_{a_i}$ and $\mu^2 = \lambda^2\delta_{a_i}$ by $\langle\mu^1,\mu^2\rangle = \lambda^1\lambda^2$. $\mathcal{D}_i$ equipped with this scalar product is a Hilbert space. In particular for $a_0 = 0$, the space $\mathcal{D}_0$ will be the space of multiples of Dirac measure concentrated at 0.

- Symbol $\mathcal{D}_{i,j}([-\tau,0]^2)$ (shortly $\mathcal{D}_{i,j}$), will denote the one dimensional space of the multiples of Dirac measure concentrated at $(a_i,a_j) \in [-\tau,0]^2$, i.e.

  $$\mathcal{D}_{i,j}([-\tau,0]^2) := \{\mu \in \mathcal{M}([-\tau,0]^2);\ s.t.\mu(dx,dy) = \lambda\,\delta_{a_i}(dx)\delta_{a_j}(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_i\hat{\otimes}_h\mathcal{D}_j\ . \tag{2.27}$$

Let $\mu^1 = \lambda^1\,\delta_{a_i}(dx)\delta_{a_j}(dy)$ and $\mu^2 = \lambda^2\,\delta_{a_i}(dx)\delta_{a_j}(dy)$, $\mathcal{D}_{i\,j}$ is a Hilbert space equipped with the scalar product defined by $\langle \mu^1, \mu^2\rangle = \lambda^1\lambda^2$. The identification with $\mathcal{D}_i\hat{\otimes}_h\mathcal{D}_j$ is a trivial exercise. If $a_j = a_i = 0$, the space $\mathcal{D}_{0,0}$ will be the space of Dirac's measures concentrated at $(0,0)$.

- Symbol $\mathcal{D}_a([-\tau,0])$ (shortly $\mathcal{D}_a$), will denote the $N+1$ dimensional space of linear combination of Dirac measures concentrated at $(N+1)$ fixed points in $[-\tau,0]$ identified by a.

$$\mathcal{D}_a([-\tau,0]) := \{\mu \in \mathcal{M}([-\tau,0])\; s.t.\; \mu(dx) = \sum_{i=0}^{N} \lambda_i\delta_{a_i}(dx);\; \lambda_i \in \mathbb{R},\; i = 0,\ldots,N\} = \bigoplus_{i=0}^{N}\mathcal{D}_i\;. \quad (2.28)$$

Let $\mu^1 = \sum_{i=0}^N \lambda_i^1\delta_{a_i}(dx)$ and $\mu^2 = \sum_{i=0}^N \lambda_i^2\delta_{a_i}(dx)$. $\mathcal{D}_a$ is a Hilbert space with respect to the scalar product $\langle\mu^1,\mu^2\rangle = \sum_{i=0}^N \lambda_i^1\lambda_i^2$. By the second equality in 2.28, it is a direct sum is equipped with the corresponding Hilbert norm.

- Symbol $\mathcal{D}_A([-\tau,0]^2)$ (shortly $\mathcal{D}_A$), will denote the $(N+1)^2$ dimensional space of linear combination of Dirac measures concentrated at points $(a_i,a_j)_{0\le i,j\le N}$ in $[-\tau,0]^2$, i.e.

$$\mathcal{D}_A([-\tau,0]^2) := \{\mu \in \mathcal{M}([-\tau,0]^2);\; s.t.\mu(dx,dy) = \lambda_{i,j}\,\delta_{a_i}(dx)\delta_{a_j}(dy) \text{ with } \lambda_{i,j}\in\mathbb{R},\; i,j = 0,\ldots,N\}\;. \quad (2.29)$$

Let $\mu^1 = \lambda_{i,j}^1\,\delta_{a_i}(dx)\delta_{a_j}(dy)$ and $\mu^2 = \lambda_{i,j}^2\,\delta_{a_i}(dx)\delta_{a_j}(dy)$. $\mathcal{D}_A$ is a Hilbert space equipped with the scalar product defined by $\langle\mu^1,\mu^2\rangle = \sum_{0\le i,j\le N}\lambda_{i,j}^1\lambda_{i,j}^2$. Moreover we have the following useful identifications

$$\mathcal{D}_A \cong \mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a = \mathcal{D}_a\hat{\otimes}_h^2 \cong \left(\bigoplus_{i=0}^N \mathcal{D}_i\right)\hat{\otimes}_h^2 = \bigoplus_{i,j=0}^N \mathcal{D}_i\hat{\otimes}_h\mathcal{D}_j \cong \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}\;. \quad (2.30)$$

In particular there is an isometric isomorphism between $\mathcal{D}^A$ and $\mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a$. A generic element $\mu = \lambda_{i,j}\,\delta_{a_i}(dx)\delta_{a_j}(dy) \in \mathcal{D}_A$ is uniquely associated with the element $\tilde{\mu} \in \mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a$ identified by $\tilde{\mu} = \sum_{i,j=0}^N \lambda_{i,j}\delta_{a_i}\otimes\delta_{a_j}$. The isometry is trivial by equality between scalar products. Let $\mu^j = \sum_{i=0}^N \lambda_i^j\delta_{a_i}(dx)$ be elements which belongs to $\mathcal{D}_a$ for every $j = 1,2,3,4$. The Hilbert tensor product $\mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a$ is equipped with the scalar product $\langle\mu^1\otimes\mu^2,\mu^3\otimes\mu^4\rangle = \langle\mu^1,\mu^3\rangle\langle\mu^2,\mu^4\rangle = \left(\sum_{i=0}^N\lambda_i^1\lambda_i^3\right)\left(\sum_{i=0}^N\lambda_i^2\lambda_i^4\right) = \sum_{0\le i,j\le N}\lambda_i^1\lambda_i^3\lambda_j^2\lambda_j^4$. The other two identifications in (2.30) derive from (2.28), Proposition 2.23 and (2.27).

Dirac measures concentrated on points identified by vector a (by matrix A respectively) are of course mutually singular; this implies the direct sum representation for $\mathcal{D}_a$ and $\mathcal{D}_A$.

- Symbol $\mathcal{D}_d([-\tau,0]^2)$ (shortly $\mathcal{D}_d$), will denote the $N+1$ dimensional space of weighted Dirac measures concentrated at $(N+1)$ fixed points $(a_i,a_i)_{i=0,\ldots,N}$ on the diagonal of $[-\tau,0]^2$, i.e.

$$\mathcal{D}_d([-\tau,0]^2) := \{\mu \in \mathcal{M}([-\tau,0]^2)\, s.t.\, \mu(dx) = \sum_{i=0}^{N} \lambda_i \delta_{a_i}(dx)\delta_{a_i}(dy);\, \lambda_i \in \mathbb{R},\, i=0,\ldots,N\} \cong \bigoplus_{i=0}^{N} \mathcal{D}_{i,i}\ . \tag{2.31}$$

This a Hilbert space. It is a proper subspace of $\mathcal{D}_A([-\tau,0]^2)$.

Remark 2.26. There are natural identifications $\mathcal{D}_i \cong \mathcal{D}_{i,j} \cong \mathbb{R}$, $\mathcal{D}_a \cong \mathcal{D}_d \cong \mathbb{R}^{N+1}$ and $\mathcal{D}_A \cong \mathbb{M}_{(N+1)\times(N+1)}(\mathbb{R}) \cong \mathbb{R}^{N+1} \otimes \mathbb{R}^{N+1}$. All those spaces are finite dimensional separable Hilbert spaces which are subspace of the Banach space $\mathcal{M}([-\tau,0])$ or $\mathcal{M}([-\tau,0]^2)$.

We give some examples of infinite dimensional subsets of measures intervening in the sequel.

- $L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$, as well as $L^2([-\tau,0]^2) \cong L^2([-\tau,0])\hat{\otimes}_h^2$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$, both equipped with the norm derived from the usual scalar product. The Hilbert tensor product $L^2([-\tau,0])\hat{\otimes}_h^2$ will be always identified with $L^2([-\tau,0]^2)$, conformally to a quite canonical procedure, see [47], chapter 6.

- $\mathcal{D}_i([-\tau,0]) \oplus L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$. This is a direct sum in the space of measures $\mathcal{M}([-\tau,0])$. In fact given a measure $\mu \in \mathcal{M}([-\tau,0])$, it decomposes uniquely into $\mu^{ac}+\mu^s$ where μ^{ac} (respectively μ^s) is absolutely continuous (resp. singular) with respect to Lebesgue measure. If $\mu = \mu^1+\mu^2$, $\mu^1 \in \mathcal{D}_i([-\tau,0])$ and $\mu^2 \in L^2([-\tau,0])$, obviously $\mu^1 = \mu^s$ and $\mu^2 = \mu^{ac}$.
The particular case when $i=0$, the space $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$, shortly $\mathcal{D}_0 \oplus L^2$, will be often recalled in the paper. As generalization of previous space we have an ulterior subspace of measures.

- $\mathcal{D}_a([-\tau,0]) \oplus L^2([-\tau,0]) = \bigoplus_{i=0}^{N} \mathcal{D}_i([-\tau,0]) \oplus L^2([-\tau,0])$, this is a Hilbert separable subspace of $\mathcal{M}([-\tau,0])$.

- $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$.

- $Diag([-\tau,0]^2)$ (shortly $Diag$), will denote the subset of $\mathcal{M}([-\tau,0]^2)$ defined as follows:

$$Diag([-\tau,0]^2) := \left\{\mu \in \mathcal{M}([-\tau,0]^2)\, s.t.\, \mu(dx,dy) = g(x)\delta_y(dx)dy;\, g \in L^\infty([-\tau,0])\right\}\ . \tag{2.32}$$

$Diag([-\tau,0]^2)$, equipped with the norm $\|\mu\|_{Diag([-\tau,0]^2)} = \|g\|_\infty$, is a Banach space. Let f be a function in $C([-\tau,0]^2)$; the pairing duality between f and $\mu(dx,dy) = g(x)\delta_y(dx)dy \in Diag$ gives

$${}_{C([-\tau,0]^2)}\langle f,\mu\rangle_{Diag([-\tau,0]^2)} = \int_{[-\tau,0]^2} f(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2} f(x,y)g(x)\delta_y(dx)dy = \int_{-\tau}^{0} f(x,x)g(x)dx\ . \tag{2.33}$$

2.7 Fréchet derivative

We recall some notions about differential calculus in Banach spaces; for more details reader can refer to [8].

Let B and G be Banach spaces and $U \subset B$ be an open subspace of B. A function $F : U \longrightarrow G$ is called **Fréchet differentiable** at $x \in U$ if it exists a linear bounded application $A_x : B \longrightarrow G$ such that

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - A_x(h)\|_G}{\|h\|_B} = 0.$$

If this limit exists we denote $DF(x) = A_x$; $DF(x)$ the **derivative** of F at point x. If F is Fréchet differentiable for any $x \in U$, the application $x \mapsto DF(x)$ belongs to $L(B; G)$. If DF is continuous F is said to be $C^1(B; G)$ or once **continuously Fréchet differentiable**. Analogously this function DF may as well have a derivative, the second order derivative of F which will be a map $D^2F : U \longrightarrow L(B; L(B; G)) \cong \mathcal{B}(B \times B; G) \cong L(B\hat{\otimes}_\pi B; G)$. If D^2F is continuous F is said to be $C^2(B; G)$ or **twice continuously Fréchet differentiable**.

If I is an open interval, the function $F : I \times B \longrightarrow \mathbb{R}$, is said to belong to $C^{1,2}(I \times B)$, or $C^{1,2}$, if the following properties are fulfilled.

- F is once continuously differentiable;
- for any $t \in I$, $x \mapsto DF(t, x)$ is of class C^1 where DF denotes the derivative with respect to the second argument;
- the second order derivative with respect to the second argument $D^2F : I \times B \to (B\hat{\otimes}_\pi B)^*$ is continuous.

Previous considerations extend by the usual techniques to the case when I is a closed interval.

Remark 2.27. When $I = [0, T]$ and $B = C([-\tau, 0])$ we have the following.

$\partial_t F : [0, T] \times C([-\tau, 0]) \longrightarrow \mathbb{R}$

$DF : [0, T] \times C([-\tau, 0]) \longrightarrow C([-\tau, 0])^* \cong \mathcal{M}([-\tau, 0])$

$D^2F : [0, T] \times C([-\tau, 0]) \longrightarrow L\left(C([-\tau, 0]); C([-\tau, 0])^*\right) \cong \mathcal{B}(C([-\tau, 0]) \times C([-\tau, 0])) \cong \left(C([-\tau, 0])\hat{\otimes}_\pi C([-\tau, 0])\right)^*$

For all $(t, \eta) \in [0, T] \times C([-\tau, 0])$, we will denote by $D_{dx}F(t, \eta)$ the measure such that

$$_{\mathcal{M}([-\tau,0])}\langle DF(t, \eta), h\rangle_{C([-\tau,0])} = DF(t, \eta)(h) = \int_{[-\tau,0]} h(x) D_{dx}F(t, \eta) \qquad \forall\, h \in C([-\tau, 0]). \quad (2.34)$$

We recall that $\mathcal{M}([-\tau, 0]^2) \subset (C([-\tau, 0])\hat{\otimes}_\pi C([-\tau, 0]))^*$. If $D^2F\,(t, \eta) \in \mathcal{M}([-\tau, 0]^2)$ for all $(t, \eta) \in [0, T] \times C([-\tau, 0])$ (which will happen in most of the treated cases) we will denote with $D^2_{dx\,dy}F(t, \eta)$, or even $D_{dx}D_{dy}F(t,\eta)$, the measure on $[-\tau,0]^2$ such that

$$_{\mathcal{M}([-\tau,0]^2)}\langle D^2F(t,\eta),g\rangle_{C([-\tau,0]^2)} = D^2F(t,\eta)(g) = \int_{[-\tau,0]^2} g(x,y)\, D^2_{dx\,dy}F(t,\eta) \qquad \forall\, g\in C([-\tau,0]^2). \tag{2.35}$$

A useful notation that will be used along all the paper is the following.

Notation 2.28. Let $F:[0,T]\times C([-\tau,0])\longrightarrow\mathbb{R}$ be a Fréchet differentiable function, with Fréchet derivative $DF:[0,T]\times C([-\tau,0])\longrightarrow\mathcal{M}([-\tau,0])$. For any given $(t,\eta)\in[0,T]\times C([-T,0])$ and $a\in[-\tau,0]$, we denote by $D^{ac}F\,(\eta)$ the absolutely continuous part of measure $DF\,(t,\eta)$, and by $D^{\delta_a}F\,(t,\eta):=DF\,(t,\eta)(\{a\})$. We observe that $D^{\delta_a}F$ is a real valued function.

Example 2.29. If for example $DF(t,\eta)\in\mathcal{D}_0\oplus L^2$ for every $(t,\eta)\in[0,T]\times C([-\tau,0])$, then we will often write

$$D_{dx}F\,(t,\eta) = D^{ac}_xF\,(t,\eta)dx + D^{\delta_0}F\,(t,\eta)\delta_0(dx)\ . \tag{2.36}$$

2.8 Malliavin calculus

We recall some notions of stochastic calculus of variations, i.e. Malliavin calculus, that we need in the sequel. We refer the reader to [48] for a presentation of the subject. In this subsection, we will restrict to the case when the underlying process is a classical Brownian motion and H will denote $L^2([0,T])$. Let $(W_t)_{t\geq 0}$ be a standard Wiener process. If $h\in H$, $W(h)$ will denote the Wiener integral $\int_0^T hdW$. Let $\mathcal{S}$ denote the class of random variables F of the form

$$F = f(W(h_1),\dots,W(h_n)) \tag{2.37}$$

where $f\in C_p^\infty(\mathbb{R}^n)$, $h_1,\dots,h_n$ are in H and $n\geq 1$. Remark that $\mathcal{S}$ is dense in $L^2(\Omega)$. It is well known that we can identify $L^2(\Omega;H)$ with $L^2(\Omega\times[0,T])$.
The **Malliavin derivative** operator can be defined as in Definition 1.2.1 in [48], but it will denoted by D^m.

Definition 2.30. The derivative of a smooth random variable F of the form (2.37) is the H-valued random variable given by

$$D^mF = \sum_{i=1}^n \partial_i f(W(h_1),\dots,W(h_n))\, h_i \tag{2.38}$$

The operator D^m is closable from $L^p(\Omega)$ to $L^p(\Omega;H)$ for any $p\geq 1$, then for any $p\geq 1$ we will denote the domain of D^m in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ the closure of the class of smooth random variables

$\mathcal{S}$ with respect to the norm $\|F\|_{1,p} = (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_H^p])^{1/p}$. For $p = 2$, the space $\mathbb{D}^{1,2}$ is a Hilbert space with the scalar product $\langle F, G\rangle_{1,2} = \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG\rangle_H]$.
We recall Proposition 1.2.3 in [48] which will be useful for calculus.

Proposition 2.31. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded derivatives, and fix $p \geq 1$. Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Then $\varphi(F) \in \mathbb{D}^{1,p}$ and

$$D^m(\varphi(F)) = \sum_{i=1}^{m} \partial_i \varphi(F)\, D^m F^i \tag{2.39}$$

After extension, the derivative operator D^m is a closed and unbounded operator defined in the dense subset $\mathbb{D}^{1,2}$ of $L^2(\Omega)$ with values in $L^2(\Omega \times [0,T])$. We remind now the notion of it Skorohod integral or **adjoint operator** of D^m as defined in Definition 1.3.1 in [48]. This concept is narrowly related to the notion of integration by parts on Wiener space which will be often used in the sequel.

Definition 2.32. We denote by δ the adjoint of the operator D^m. δ is an unbounded operator on $L^2(\Omega \times [0,T])$ with values in $L^2(\Omega)$ such that:

1. The domain of δ, denoted by *Dom* δ is the set of processes $u \in L^2(\Omega \times [0,T])$ with the following properties.

   $$\left| \mathbb{E}\left[\int_0^T D_t^m F\, u_t\, dt \right] \right| \leq c\, \|F\|_{1,2}\ ,$$

   for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

2. If u belongs to *Dom* δ, then $\delta(u)$ is an element of $L^2(\Omega)$ characterized by

   $$\mathbb{E}\left[F\, \delta(u)\right] = \mathbb{E}\left[\int_0^T D_t^m F\, u_t\, dt \right] \tag{2.40}$$

   for any $F \in \mathbb{D}^{1,2}$.

The operator δ is sometimes called the divergence operator, and we will refer to it as the **Skorohod stochastic integral** of the process $u \in$ *Dom* δ. It transforms square integrable processes into random variables. We will often use the following notation:

$$\delta(u) := \int_0^T u_t \delta W_t\ .$$

Since adjoint operator are always closed, the operator δ is closed. Skorohod integral is an extension of the Itô stochastic integral allowing anticipating integrands.
We denote by $\mathbb{L}^{1,2}$ the class of processes $u \in L^2(\Omega \times [0,T])$ such that $u_t \in \mathbb{D}^{1,2}$ for almost all t, and there

exists a measurable version of the two-parameters process $D_s^m u_t$ verifying $\mathbb{E}\left[\int_0^T \int_0^T \left(D_s^m u_t\right)^2 \, ds\, dt\right] < +\infty$. $\mathbb{L}^{1,2}$ is a Hilbert space and $\mathbb{L}^{1,2} \subset Dom\ \delta$.

We recall now some useful rules of stochastic calculus of variations. By Propositions 1.3.8 and 1.3.18 in [48], if $(u_t)_{t\in[0,T]}$ is a square integrable adapted process with some Malliavin type regularity of type $\mathbb{L}^{1,2}$, we easily obtain the following identities. We omit here those details for simplicity.

1. $D_s^m(W_t) = \mathbb{1}_{[0,t]}(s) = \mathbb{1}_{\{s\le t\}}$.
2. $D_s^m\left(\int_0^t u_r\, dW_r\right) = u_s \mathbb{1}_{\{s\le t\}} + \int_s^t D_s^m(u_r)\, dW_r$.
3. $D_s^m\left(\int_0^t u_r\, dr\right) = \int_s^t D_s^m(u_r)\, dr$.

A well-known representation result is the celebrated **Clark-Ocone representation formula** and it is expressed in terms of Malliavin derivatives.
By martingale representation theorem we know that any square integrable random variable h, measurable with respect to $\mathcal{F}_T$, can be represented as

$$h = \mathbb{E}[h] + \int_0^T \xi_t dW_t \ , \tag{2.41}$$

where ξ_t is an adapted process such that $\mathbb{E}\left[\int_0^T \xi_t^2 dt\right] < \infty$.

When the variable h belong to the space $\mathbb{D}^{1,2}$, it turns out that the process ξ_t can be identified as the predictable projection of the derivative of h.

Proposition 2.33. (Clark-Ocone representation formula)
Let $h \in \mathbb{D}^{1,2}$ and suppose that W is a one-dimensional Brownian motion equipped with its canonical filtration $(\mathcal{F}_t)$. Then

$$h = \mathbb{E}[h] + \int_0^T \mathbb{E}\left[D_t^m h | \mathcal{F}_t\right] dW_t \ . \tag{2.42}$$

Chapter 3

Calculus via regularization

In this section we will define a stochastic forward integral with respect to a Banach-valued stochastic process. We do not aim to consider a full generality: stochastic integrals will only be scalar valued. In this construction there are three difficulties.

- The integrator is generally not a semimartingale or the integrand may be anticipative.
- The integrator takes values in an infinite dimensional space B.
- B is a general Banach space, without reflexivity, or other classical properties related to classical stochastic integration.

As a special case, we will consider the $C([-\tau, 0])$-valued window Brownian motion $W(\cdot)$ as stochastic integrator. The general infinite dimensional integration theory with respect to martingales ([13, 46, 19]) does not apply, since $W(\cdot)$ is by no means a reasonable $C([-\tau, 0])$-valued semimartingale. In this section we also recall some properties of the Da Prato-Zabczyck integral and we will show that it coincides with ours when it exists.

3.1 Basic motivations

Definition 3.1. Let B be a Banach space and X be a B-valued stochastic process. We say that X is a **Pettis semimartingale** if, for every $\phi \in B^*$, $\langle \phi, X_t \rangle$ is a real semimartingale with respect to a filtration $(\mathcal{G}_t)$.

We remark the following.

- If X is a B-valued martingale in the sense of Section 1.17, [46], then it is also a Pettis semimartingale.

- If X is a B-valued semimartingale, in any reasonable sense, then X is expected to be a Pettis semimartingale.

Proposition 3.2. The $C([-\tau,0])$-valued window Brownian motion is not a Pettis semimartingale.

Proof. Let $(\mathcal{F}_t)$ be the natural filtration generated by the real Brownian motion W. It is enough to show that it exists an element μ in $B^* = \mathcal{M}([-\tau,0])$ such that $\langle \mu, W_t(\cdot)\rangle = \int_{[-\tau,0]} W_t(x)\mu(dx)$ is not a semimartingale with respect to any filtration. We will proceed by contradiction: we suppose that $W(\cdot)$ is a Pettis semimartingale, then in particular if we take $\mu = \delta_0 + \delta_{-\tau}$, the process $\langle \delta_0 + \delta_{-\tau}, W_t(\cdot)\rangle = W_t + W_{t-\tau} := X_t$ has to be a semimartingale with respect to some filtration $(\mathcal{G}_t)$. At the same time $W_t + W_{t-\tau}$ is $(\mathcal{F}_t)$-adapted, so by Stricker's theorem (see Theorem 4, pag. 53 in [53]), X is a semimartingale with respect to filtration $(\mathcal{F}_t)$. On the other hand $(W_{t-\tau})_{t\geq\tau}$ is a strongly predictable process with respect to $(\mathcal{F}_t)$, see Definition 2.11. By Proposition 4.11 in [11], it follows that $(W_{t-\tau})_{t\geq\tau}$ is an $(\mathcal{F}_t)$-martingale orthogonal process. Since W is an $(\mathcal{F}_t)$-martingale, the process $X_t = W_t + W_{t-\tau}$ is an $(\mathcal{F}_t)$-weak Dirichlet process. By uniqueness of the decomposition for $(\mathcal{F}_t)$-weak Dirichlet processes, $(W_{t-\tau})_{t\geq\tau}$ has to be a bounded variation process. This generates a contradiction because $(W_{t-\tau})_{t\geq\tau}$ is not a zero quadratic variation process. In conclusion $\langle \mu, W_t(\cdot)\rangle$ is not a semimartingale. □

Remark 3.3. Process X defined by $X_t = W_t + W_{t-\tau}$ is an example of $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation which is not an $(\mathcal{F}_t)$-Dirichlet process.

3.2 Definition of the integral for Banach valued processes

In subsection 2.2 we briefly recalled the definition of forward integral for real valued processes. We define now a forward stochastic integral for a Banach valued integrator and an integrand process with values in the dual of the Banach space.

Definition 3.4. Let $(X_t)_{t\in[0,T]}$ (respectively $(Y_t)_{t\in[0,T]}$) be a B-valued (respectively a B^*-valued) stochastic process, i.e. $X : \Omega \times [0,T] \longrightarrow B$ and $Y\colon \Omega \times [0,T] \longrightarrow B^*$. We suppose X to be continuous and $\int_0^T \|Y_s\|_{B^*} < +\infty$ a.s.
For every fixed $t \in [0,T]$ we define the **definite forward integral of Y with respect to X** denoted by $\int_0^t {}_{B^*}\langle Y_s, d^-X_s\rangle_B$ as the following limit in probability:

$$\int_0^t {}_{B^*}\langle Y_s, d^-X_s\rangle_B := \lim_{\epsilon\to 0}\int_0^t {}_{B^*}\langle Y(s), \frac{X(s+\epsilon)-X(s)}{\epsilon}\rangle_B ds\ .$$

We say that the **forward stochastic integral of Y with respect to X** exists if the process

$$\left(\int_0^t {}_{B^*}\langle Y_s, d^-X_s\rangle_B\right)_{t\in[0,T]}$$

admits a continuous version. In the sequel indices B and B^* will often be omitted.

Remark 3.5. 1. If B is a Hilbert space H, then, via the Riesz representation theorem, Definition 3.4 provides a natural definition in the case when X and Y are both H-valued.

2. Let B and H be respectively Banach and Hilbert spaces such that $B \subset H \cong H^* \subset B^*$. If X is a B-valued continuous process and Y is an H^*-valued process, then

$$\int_0^t {}_{B^*}\langle Y_s, d^- X_s \rangle_B = \int_0^t {}_{H^*}\langle Y_s, d^- X_s \rangle_H \ . \tag{3.1}$$

In the example below, we illustrate an elementary calculation of a forward integral related to window processes.

Example 3.6. Let X be a continuous finite quadratic variation process such that $[X]_t = t$ and $X_0 = 0$. We have the following equality

$$\int_0^t {}_{L^2([-T,0])}\langle X_s(\cdot), d^- X_s(\cdot) \rangle_{L^2([-T,0])} = \frac{1}{2}\int_0^t X_s^2 ds - \frac{t^2}{4}. \tag{3.2}$$

In fact, by the change of variables $v := u + s$, and usual conventions on the prolongation of processes, the left-hand side of (3.2) equals

$$\int_0^t \int_{-T}^0 X_{u+s} \frac{X_{u+s+\epsilon} - X_{u+s}}{\epsilon} du\, ds = \int_0^t \int_0^s X_v \frac{X_{v+\epsilon} - X_v}{\epsilon} dv\, ds. \tag{3.3}$$

According to Theorem 2.13, we have

$$\int_0^t X d^- X = \frac{1}{2}(X_t^2 - t),$$

in the ucp sense. Finally the right-hand side of (3.3) converges ucp to

$$\int_0^t \left(\int_0^s X d^- X \right) ds = \frac{1}{2}\int_0^t (X_s^2 - s) ds,$$

which is the right-hand side of (3.2).

3.3 Link with Da Prato-Zabczyk's integral

Let F and H be two separable Hilbert spaces. In the first part of this subsection we recall the definition of Hilbert valued Wiener processes (including the cylindrical case) and some properties of the Itô stochastic integral appearing in Da Prato-Zabczyk framework, see e.g. [13], when the integrator is a Wiener process. This integral will be denoted by

$$\int_0^t Y_s \cdot dW_s^{dz} \ t \in [0,T], \tag{3.4}$$

where W is a Wiener process on H and Y is a process with values in the space of linear but not necessarily bounded operators from H to F. In the second part we will illustrate the link with the forward integral defined in Definition 3.4. The central result will be Proposition 3.10. This states that whenever Y is a H^*-valued adapted process such that $\int_0^t \|Y_s\|^2_{H^*} ds < +\infty$ a.s. and W is a Q-Brownian motion W, Q being a nuclear operator on H, then the forward integral $\int_0^t \langle Y_s, d^- W_s\rangle$ exists as well as the Da Prato-Zabczyk integral $\int_0^t Y_s \cdot dW_s^{dz}$ and they are equal.

3.3.1 Notations

Notions of nuclear and Hilbert-Schimdt operator play a central role in the Da Prato-Zabczyk integral. We just recall that, let H and F be separable Hilbert spaces, $L^1(H;F)$ (resp. $L^2(H;F)$) denotes the separable Banach (resp. Hilbert) space of nuclear or trace class (resp. Hilbert-Schmidt) operators from H to F. If $H = F$ we simply denote $L^1(H)$ (resp. $L^2(H)$). We refer to Section 6.2.1 for a fairly survey about those classes of operators and their connection with tensor product of Banach spaces.

Let Q be a symmetric non negative operator in $L(H)$. We will consider first the case when Q is a trace class operator in H, i.e. $Q \in L^1(H)$. We assume that there exists a complete orthonormal system $\{e_i\}$ in H, and a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i$, for $i = 1, 2, \ldots$.

A random element $Z : \Omega \to H$ is said to be integrable (resp. square integrable) if $\mathbb{E}[\|Z\|_H] < \infty$. (resp. $\mathbb{E}\left[\|Z\|_H^2\right] < \infty$). If Z is integrable, it is possible to define its H-valued expectation $\mathbb{E}[Z]$ in the sense that $\mathbb{E}[{}_H\langle Z, h\rangle_H] = {}_H\langle \mathbb{E}[Z], h\rangle_H$. Given two square integrable (H-valued) random elements $Z_1, Z_2 : \Omega \to H$, we denote by $Cov(Z_1, Z_2)$ the map in $L^1(H)$ defined by

$$Cov(Z_1, Z_2)\,(h) = \mathbb{E}\left[(Z_1 - \mathbb{E}[Z_1])\; {}_H\langle Z_2 - \mathbb{E}[Z_2]\,,\, h\rangle_H\right] \qquad \forall\, h \in H\,.$$

Let $(\mathcal{F}_t)$ be a filtration fulfilling the usual conditions; it will be often implicit in this chapter. Symbol $\mathcal{M}^2_T(H)$ will denote the space of all H-valued continuous square integrable $(\mathcal{F}_t)$-martingales M. $\mathcal{M}^2_T(H)$ with the norm defined by $\|M\|^2_{\mathcal{M}^2_T(H)} = \mathbb{E}\left[\|M_T\|^2_H\right]$ is a Hilbert space. For a precise definition of H-valued martingale (resp. local) martingale, the reader may consult Section 3.4 of [13]. If M is a local martingale, we recall the notion of quadratic variation given in Proposition 3.2, [13]. That notion will be denoted by $[M]^{dz}$.

An $L^1(H)$-valued process V is said to be increasing if, for all $a \in H$, $\langle V_t\, a\,,\, a\rangle \geq 0$ and if $\langle V_t\, a\,,\, a\rangle \geq \langle V_s\, a\,,\, a\rangle$ if $0 \leq t \leq s \leq T$. The quadratic variation in the sense of Da Prato-Zabczyk of a local martingale M is a $L^1(H)$-valued continuous, adapted and increasing process V such that $V_0 = 0$ and for arbitrary $a, b \in H$ the process $\langle M_t, a\rangle\langle M_t, b\rangle - \langle V_t a, b\rangle$, $t \in [0, T]$, is an $(\mathcal{F}_t)$-martingale. This $L^1(H)$-valued process is uniquely determined and will be denoted by $[M]^{dz}$. It can be expressed also using the covariations by $v_t = \sum_{i,j=1}^{\infty} [\langle M_t, e_i\rangle, \langle M_t, e_j\rangle] e_i \otimes e_j$. The notion of quadratic variation will be more extensively investigated in Section 6.2.

Definition 3.7. An H-valued stochastic process $(W_t)_{t\geq 0}$ is called a Q**-Wiener process on H** (or Q-Brownian motion) if

(i) $W(0) = 0$.

(ii) W has continuous trajectories.

(iii) W has independent increments.

(iv) The random element $W(t) - W(s)$ is Gaussian for $t \geq s \geq 0$ with zero expectation and

$$Cov(W(t) - W(s), W(t) - W(s)) = (t - s)Q.$$

Proposition 3.8. Assume that W is a Q-Brownian motion with $Q \in L^1(H)$. Then for all $h_1, h_2, h_3, h_4 \in H$ and for all $t_1, t_2, t_3, t_4 \geq 0$ the following statements hold.

1. $\mathbb{E}\left[\langle W_{t_1}, h_1\rangle\right] = 0.$

2. $\mathbb{E}\left[\langle W_{t_1}, h_1\rangle\langle W_{t_2}, h_2\rangle\right] = t_1 \wedge t_2\langle Qh_1, h_2\rangle.$

3. $\mathbb{E}\left[\langle W_{t_1}, h_1\rangle\langle W_{t_2}, h_2\rangle\langle W_{t_3}, h_3\rangle\right] = 0.$

4. $$\begin{aligned}\mathbb{E}\left[\langle W_{t_1}, h_1\rangle\langle W_{t_2}, h_2\rangle\langle W_{t_3}, h_3\rangle\langle W_{t_4}, h_4\rangle\right] = (t_1 \wedge t_2 \wedge t_3 \wedge t_4)\Big(&\langle Qh_1, h_2\rangle\ \langle Qh_3, h_4\rangle + \\ &+ \langle Qh_1, h_4\rangle\ \langle Qh_2, h_3\rangle + \\ &+ \langle Qh_1, h_3\rangle\ \langle Qh_2, h_4\rangle\Big)\ .\end{aligned}$$

Proof. All the statements are easy to verify using the fact that $\langle W_t, h\rangle$ is a centered real Gaussian random variable and $E[\langle W_t, h_1\rangle\langle W_t, h_2\rangle] = t\langle Qh_1, h_2\rangle$. for all $t \geq 0$ and $h, h_1, h_2 \in H$. □

Note that the quadratic variation in the sense of Da Prato-Zabczyk of a Q-Brownian motion on H, with $Tr(Q) < +\infty$, is given by the deterministic process $[W]^{dz}_t = tQ$ where Q is a nuclear operator in $L^1(H)$. In fact for every $a, b \in H$ it holds $\langle W_t, a\rangle\langle W_t, b\rangle - \langle tQa, b\rangle$ is a real martingale. For the bilinearity of the scalar product we verify the result for $a = b$, i.e. that $\langle W_t, a\rangle^2 - \langle tQa, a\rangle$ is a martingale. It suffices to show that $\langle tQa, a\rangle$ is the bracket $[W](a \otimes a)$. In particular a Q-Brownian motion is a H-valued martingale which belong to $\mathcal{M}^2(H)$.

We summarize now the definition of stochastic integral with respect to a Q-Brownian motion W with values in H, where Q is a trace class operator.
Let F be a separable Hilbert space with complete orthonormal basis $\{f_j\}$ and let us fix a number $T > 0$. An $L(H; F)$-valued process $(\Phi_t)_{t\in[0,T]}$ is said to be **elementary** if there exists a sequence $0 = t_0 < t_1 < \ldots < t_M = T$ and sequence $\Phi_0, \Phi_1, \ldots, \Phi_{M-1}$ of $L(H; F)$-valued random variables taking only a finite

number of values such that Φ_m are $(\mathcal{F}_{t_m})$-measurable and $\Phi_t = \Phi_m$ for $t \in]t_m, t_{m+1}]$, $m = 0, \ldots, M-1$. For elementary processes Φ the **Da Prato-Zabczyk stochastic integral** is defined by the formula

$$\int_0^t \Phi_s \cdot dW_s^{dz} := \sum_{m=0}^{M-1} \Phi_m (W_{t_{m+1}\wedge t} - W_{t_m \wedge t}) \ .$$

To avoid complications, we suppose from now on that Q is strictly positive defined.

We introduce the subspace $H_0 = Q^{1/2}(H)$ of H, which, endowed with the inner product

$$\langle u, v \rangle_0 = \langle Q^{-1/2} u, Q^{-1/2} v \rangle = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \langle u, e_i \rangle \langle v, e_i \rangle$$

is a Hilbert space. The space of Hilbert-Schmidt operators from H_0 to F, denoted by $L^2(H_0; F)$, is also a separable Hilbert space, equipped with the norm

$$\begin{aligned} \|\Phi\|^2_{L^2(H_0;F)} &= \sum_{i=1}^{\infty} \|\Phi g_i\|_F^2 = \sum_{i,j=1}^{\infty} \lambda_i |\langle \Phi e_i, f_j \rangle|^2 = \|\Phi Q^{1/2}\|^2_{L^2(H;F)} = \\ &= \langle \Phi Q^{1/2}, \Phi Q^{1/2} \rangle_{L^2(H;F)} = Tr \left((\Phi Q^{1/2})(\Phi Q^{1/2})^* \right) = Tr(\Phi Q \Phi^*), \end{aligned}$$

where $g_i = \sqrt{\lambda_i} e_i$, $i = 1, 2, \ldots$, $\{e_i\}$ and $\{f_j\}$ are complete orthonormal bases in H_0, H and F. We remark here that the adjoint operator of $Q^{1/2}$ is $Q^{-1/2}$ from H_0 to H and that the operator $\Phi Q \Phi^*$ is of trace class being a composition of the Hilbert-Schmidt operator $(\Phi Q^{1/2})$ and its adjoint, which is also Hilbert-Schmidt, see properties in Section 2.2, [30]. Clearly $L(H; F) \subset L^2(H_0; F)$ but $L^2(H_0; F)$ also contains unbounded operators on H.

Let $(\Phi_t)_{t \in [0,T]}$ be a measurable $L^2(H_0; F)$-valued process; we define its norm by

$$|||\Phi|||_t^2 = \mathbb{E}\left[\int_0^t \|\Phi_s\|^2_{L^2(H_0;F)} ds \right] = \mathbb{E}\left[\int_0^t Tr(\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^* ds \right] \qquad t \in [0, T] \ .$$

We denote with $\mathcal{N}_W^2(0, T; L^2(H_0; F))$ the Hilbert space of all $L^2(H_0; F)$ predictable processes with $|||\Phi|||_T < +\infty$.

If a process Φ is elementary and $|||\Phi|||_T < +\infty$, then the stochastic integral $\int_0^{\cdot} \Phi_s \cdot dW_s^{dz}$ is a continuous square integrable F-valued martingale on $[0, T]$ and it holds the following identity:

$$\mathbb{E}\left[\left\| \int_0^t \Phi_s \cdot dW_s^{dz} \right\|_F^2 \right] = |||\Phi|||_t^2 \ , \qquad 0 \le t \le T \ . \tag{3.5}$$

The stochastic integral with respect to a Q-Brownian motion is an isometric transformation from the space of elementary processes equipped with the norm $||| \cdot |||$ into the space of F-valued square integrable martingale $\mathcal{M}_T^2(F)$. By the fact that elementary processes form a dense set in $\mathcal{N}_W^2(0, T; L^2(H_0; F))$, the definition of stochastic integral is extended to all elements in $\mathcal{N}_W^2(0, T; L^2(H_0; F))$ and (3.5) remains true.

Definition 3.9. For a general element $\Phi \in \mathcal{N}_W^2(0, T; L^2(H_0; F))$, we will denote **Brownian martingale** the martingale $M \in \mathcal{M}_T^2(F)$ given by the stochastic integral

$$M_\cdot = \int_0^\cdot \Phi_s \cdot dW_s^{dz} \ . \tag{3.6}$$

By the so called localization procedure, see Lemma 4.9 in [13], it is possible to extend the definition of the Da Prato-Zabczyk stochastic integral to $L^2(H_0; F)$-predictable processes satisfying even the weaker condition

$$\mathbb{P}\left[\int_0^T \|\Phi_s\|^2_{L^2(H_0;F)} ds < +\infty\right] = 1 \ . \tag{3.7}$$

All such processes are called **stochastically integrable** on $[0, T]$. They form a linear space denoted by $\mathcal{N}_W(0, T; L^2(H_0; F))$. In [13], Section 4.3 the definition of stochastic integral with respect to a Q-Brownian motion is extended to a a cylindrical Brownian motion. We suppose now that Q does not necessarily fulfill $Tr(Q) < +\infty$.

Let $H_0 = Q^{1/2}(H)$ with the induced norm and let H_1 be an arbitrary Hilbert space such that H is embedded continuously into H_1 and the embedding $\mathcal{J}$ of H_0 into H_1 is Hilbert-Schmidt. Let $\{g_j\}$ be an orthonormal and complete basis in H_0 and β_j a family of independent real valued standard Brownian motion then the the following series

$$W_t = \sum_{j=1}^{+\infty} g_j \beta j(t)$$

is convergent in $L^2(\Omega; H_1)$ and (W_t) is called a **cylindrical Brownian motion on** H. Let $Q_1 := \mathcal{J}\mathcal{J}^*$, we recall that W is a Q_1 Brownian motion on H_1 and $Tr(Q_1) < +\infty$. We remark that a Q Brownian motion with $Tr(Q) < +\infty$ is H-valued and has the same expansion of a cylindrical Brownian motion in $L^2(\Omega; H)$. The definition of stochastic integral is the same for a cylindrical Brownian motion because the class $\mathcal{N}_W^2(0; T; L^2(H_0; F))$ is independent of the space H_1 and the spaces $Q_1^{1/2}(H_1)$ are identical for all possible extensions H_1.

We recall some properties of Brownian stochastic integrals from Section 4.4 in [13].
If $\Phi \in \mathcal{N}_W^2(0, T; L^2(H_0; F))$, then the stochastic integral

$$M_t = \int_0^t \Phi(s) \cdot dW_s^{dz} \tag{3.8}$$

is a continuous square integrable martingale in $\mathcal{M}_T^2(F)$ and its quadratic variation is of the form

$$[M]_t^{dz} = \int_0^t \left(\Phi(s) Q^{1/2}\right) \left(\Phi(s) Q^{1/2}\right)^* ds \ . \tag{3.9}$$

Moreover if $\Phi_1, \Phi_2 \in \mathcal{N}^2_W(0,T;L^2(H_0;F))$ then

$$\mathbb{E}\left[\int_0^t \Phi_i(s)\cdot dW^{dz}_s\right] = 0\ , \qquad \mathbb{E}\left[\left\|\int_0^t \Phi_i(s)\cdot dW^{dz}_s\right\|^2\right] < +\infty \qquad s,t\in[0,T] \text{ and } i=1,2$$

and the covariance operator is given by the formula

$$V(t,s) = Cov\left[\int_0^t \Phi_1(r)\cdot dW^{dz}_r, \int_0^s \Phi_2(r)\cdot dW^{dz}_r\right] = \mathbb{E}\left[\int_0^{t\wedge s}\left(\Phi_1(r)Q^{1/2}\right)\left(\Phi_2(r)Q^{1/2}\right)^* dr\right]\ .$$

Under the same hypotheses we have

$$\mathbb{E}\left[{}_F\langle \int_0^t \Phi_1(r)\cdot dW^{dz}_r, \int_0^s \Phi_2(r)\cdot dW^{dz}_r\rangle_F\right] = \mathbb{E}\left[\int_0^{t\wedge s} Tr\left[\left(\Phi_1(r)Q^{1/2}\right)\left(\Phi_2(r)Q^{1/2}\right)^*\right] dr\right]\ .$$

We recall also that stochastic integration theory with respect to martingales $M\in\mathcal{M}^2_T(F)$, can be defined analogously to the one with respect to a Wiener process, see Chapter 6, Section 14 on [46]. The role of the process $t\,Q$ is played by the quadratic variation $[M]^{dz}_t$, $t\in[0,T]$. Even, if [13] defines the stochastic integral for a general martingale integrator, we will need this extension only in the case when the martingale M is itself a stochastic integral (3.8), otherwise denoted by $M=\Phi\cdot W$, with $\Phi\in\mathcal{N}^2_W(0,T;L^2(H_0;F))$. Let Ψ be a $L(F;\mathbb{R}) = F^*$-valued adapted process such that $\mathbb{E}\left[\int_0^T \|\Psi(s)\Phi_s\|^2_{F^*} ds\right] < +\infty$. Then the extension is straightforward, since we can define the stochastic integral

$$\Psi\cdot M^{dz}_t := \int_0^t \Psi(s)\cdot dM^{dz}_s := \int_0^t \Psi(s)\Phi(s)\cdot dW^{dz}_s,\ t\in[0,T]\ . \tag{3.10}$$

We remind that

$$[\Psi\cdot M]^{dz}_t = \int_0^t \left(\Psi(s)\Phi(s)Q^{1/2}\right)\left(\Psi(s)\Phi(s)Q^{1/2}\right)^* ds\ . \tag{3.11}$$

We recall that every operator in $L(H;F)$ is also in $L^2(H_0;F)$. In fact if $T\in L(H;F)$ then is well defined $L^2(H_0;F)$ because $H_0 = Q^{1/2}(H)$ is a subspace of H. Moreover if we suppose $T\in L(H;F)$, then, using the fact that $g_j = \sqrt{\lambda_j}e_j$ and $\|Te_j\|_F \le \|T\|_{L(H;F)}$ being $\{e_j\}$ a complete orthonormal system for H, we have

$$\|T\|^2_{L^2(H_0;F)} = \sum_{j=1}^{+\infty}\|Tg_j\|^2_F = \sum_{j=1}^{+\infty}\lambda_j\|Te_j\|^2_F \le \sum_{j=1}^{+\infty}\lambda_j\|T\|^2_{L(H;F)} = Tr(Q)\cdot\|T\|^2_{L(H;F)} < +\infty.$$

So for $L(H;F)$ predictable process Y such that $\mathbb{E}\left[\int_0^t \|Y_s\|^2_{L(H;F)} ds\right] < \infty$ it holds

$$\mathbb{E}\left[\int_0^t \|Y_s\|^2_{L^2(H_0;F)} ds\right] \le Tr(Q)\mathbb{E}\left[\int_0^t \|Y_s\|^2_{L(H;F)} ds\right] < \infty\ .$$

This implies that $Y\in\mathcal{N}^2_W(0,T;L^2(H_0;F))$, so the stochastic integral integral $\int Y\cdot dW^{dz}$ in the sense of [13] is a well defined F-valued process.

3.3.2 Connection with forward integral

We consider here the case $F = \mathbb{R}$.

Proposition 3.10. Let W a H-valued Q-Brownian motion with $Q \in L^1(H)$, i.e. $Tr(Q) = \sum_{j=1}^{+\infty} \lambda_j < +\infty$, and Y be a $L(H;\mathbb{R}) = H^*$ process such that $\int_0^t \|Y_s\|^2_{H^*} ds < \infty$ a.s. Then, for every $t \in [0,T]$,

$$\int_0^t \langle Y_s, d^- W_s\rangle = \int_0^t Y_s \cdot dW_s^{dz} \ . \tag{3.12}$$

Proof. 1) We first suppose that $\mathbb{E}\left[\int_0^T \|Y_s\|^2_{H^*} ds\right] < \infty$.
In this case Y in $\mathcal{N}_W^2(0,T;L^2(H_0;\mathbb{R})))$. The process on the right-hand side of 3.12 is an $\mathcal{M}_T^2(\mathbb{R})$ process because it is a stochastic integral for a process $Y \in \mathcal{N}_W^2(0,T;L^2(H_0;\mathbb{R}))$. We want to show that

$$\int_0^t \langle Y_s, \frac{W_{s+\epsilon} - W_s}{\epsilon}\rangle ds \xrightarrow[\epsilon \longrightarrow 0]{\mathbb{P}} \int_0^t Y_u \cdot dW_u^{dz}, \quad \forall t \in [0,T]. \tag{3.13}$$

We can represent $(W_{s+\epsilon} - W_s)$ as a H-valued Da Prato-Zabczyk stochastic integral whose integrand is the $L(H;H)$ elementary process identity on H. This integral will be denoted with the integration symbol dW^{dz^*}. Therefore, we write

$$W_{s+\epsilon} - W_s = \int_s^{s+\epsilon} dW_u^{dz^*}$$

and the left-hand side in (3.13) gives

$$\frac{1}{\epsilon}\int_0^t \langle Y_s, \int_s^{s+\epsilon} dW_u^{dz^*}\rangle ds = \frac{1}{\epsilon}\int_0^t \int_s^{s+\epsilon} Y_s \cdot dW_u^{dz}\, ds = \frac{1}{\epsilon}\int_0^t \int_{(u-\epsilon)^+}^u Y_s\, ds \cdot dW_u^{dz} = \int_0^t Y_u^\epsilon \cdot dW_u^{dz} \tag{3.14}$$

where

$$Y_u^\epsilon := \frac{1}{\epsilon}\int_{(u-\epsilon)^+}^u Y_s\, ds \ .$$

The first equality in (3.14) is true because, for a fixed $\epsilon > 0$ and $s \in [0,t]$, it holds

$$\langle Y_s, \int_s^{s+\epsilon} dW_u^{dz^*}\rangle = \int_s^{s+\epsilon} Y_s \cdot dW_u^{dz} \ .$$

In fact the constant random element Y_s is an elementary process so by definition the right-hand side stochastic integral gives

$$\int_s^{s+\epsilon} Y_s \cdot dW_u^{dz} = \langle Y_s, W_{s+\epsilon}\rangle - \langle Y_s, W_s\rangle = \langle Y_s, W_{s+\epsilon} - W_s\rangle = \langle Y_s, \int_s^{s+\epsilon} dW_u^{dz^*}\rangle \ .$$

The second equality in (3.14) is true by the Fubini stochastic theorem, see Theorem 4.18 in [13]. The term $\int_{u-\epsilon}^u Y_s\, ds$ has to be understood as a random Bochner type integral with values in H^*. We remark that $Y^\epsilon \in \mathcal{N}_W^2(0,T;L^2(H_0;\mathbb{R}))$ because

$$\int_0^T \|Y_u^\epsilon\|^2_{L^2(H_0;\mathbb{R})} du \le \int_0^T \|Y_u^\epsilon\|^2_{H^*} du \le \frac{1}{\epsilon}\int_0^T \int_{u-\epsilon}^u \|Y_s\|^2_{H^*}\, ds\, du \le \int_0^{T+1} \|Y_s\|^2_{H^*} ds \ .$$

We will in fact prove that convergence in (3.13) holds even in $L^2(\Omega)$. Because of the isometry property for the Da Prato-Zabczyk stochastic integral, we write

$$\mathbb{E}\left[\int_0^t (Y_u^\epsilon - Y_u) \cdot dW_u^{dz}\right]^2 = \mathbb{E}\left[\int_0^t \|Y_u^\epsilon - Y_u\|^2_{L^2(H_0;\mathbb{R})}\, du\right] \leq \mathbb{E}\left[\int_0^t \|Y_u^\epsilon - Y_u\|^2_{H^*}\, du\right] .$$

In fact previous expectation gives

$$\mathbb{E}\left[\int_0^t \|Y_u^\epsilon - Y_u\|^2_{H^*} du\right] = \mathbb{E}\left[\sum_{n=0}^{+\infty}\int_0^t \langle Y_u^\epsilon - Y_u\, ,\, e_n\rangle^2 du\right] \tag{3.15}$$

where (e_n) is an orthonormal basis of H^*. We have

$$\int_0^t \langle Y_u^\epsilon - Y_u\, ,\, e_n\rangle^2 du = \int_0^t \left(\frac{1}{\epsilon}\int_{(u-\epsilon)^+}^u \langle Y_s\, ,\, e_n\rangle ds - \langle Y_u\, ,\, e_n\rangle\right)^2 du\, .$$

We recall the maximal inequality, ([66], chapter I.1): there exists a universal constant C such that for any $\phi \in L^2([0,T])$,

$$\int_0^T \left(\sup_{0<\epsilon<1}\left\{\frac{1}{\epsilon}\int_{(v-\epsilon)^+}^v \phi_v\, dv\right\}\right)^2 du \leq C\int_0^T \phi_v^2\, dv\, . \tag{3.16}$$

By (3.16), we know then the existence of a constant $C > 0$ such that, for any n, ω a.s.

$$g_n(\omega, u) := \sup_{\epsilon>0}\left\{\frac{1}{\epsilon}\int_{(u-\epsilon)^+}^u \langle Y_s(\omega)\, ,\, e_n\rangle ds\right\}, \quad u \in [0,T],$$

is such that

$$\int_0^T g_n^2(\omega, u) du \leq C\int_0^T \langle Y_u(\omega)\, ,\, e_n\rangle^2 du\, . \tag{3.17}$$

We have of course

$$\mathbb{E}\left[\sum_{n=0}^{+\infty}\int_0^T \langle Y_u\, ,\, e_n\rangle^2 du\right] = \mathbb{E}\left[\int_0^T \|Y_u\|^2_{H^*} du\right] < \infty\, . \tag{3.18}$$

On the other hand for every n and ω a.s. we have

$$\int_0^t \left(\frac{1}{\epsilon}\int_{(u-\epsilon)^+}^u \langle Y_s\, ,\, e_n\rangle ds\right) du \longrightarrow \int_0^t \langle Y_u\, ,\, e_n\rangle du \tag{3.19}$$

when $\varepsilon \to 0$, by Lebesgue differentiation theorem. We recall that Lebesgue differentiation theorem says that if $\phi \in L^1_{loc}(ds)$, then a.e.

$$\frac{1}{\epsilon}\int_t^{t+\epsilon} \phi(s)\, ds \longrightarrow \phi(t)\, .$$

Finally Lebesgue dominated convergence theorem with respect to (ω, n) implies that right-hand side of (3.15) converges to zero.

2) It remains to treat the case when $\int_0^T \|Y_s\|^2_{H^*} ds < \infty$ a.s. In this case Y does not necessarily belong to $\mathcal{N}^2_W(0,T;L^2(H_0;\mathbb{R}))$. We proceed by localization. For $m > 0$ we define

$$\tau^m := \inf\left\{t > 0 \mid \int_0^t \|Y_s\|^2_{H^*} ds \geq m\right\}$$

and

$$Y_t^m = \begin{cases} Y_t & t < \tau^m \\ 0 & t > \tau^m \end{cases} .$$

Clearly $Y^m \in \mathcal{N}^2_W(0,T;L^2(H_0;\mathbb{R}))$. Let $\delta > 0$. We have to show that

$$I(\epsilon) := \mathbb{P}\left[\left|\frac{1}{\epsilon}\int_0^t \langle Y_s\,,\, W_{s+\epsilon} - W_s\rangle ds - \int_0^t Y_s \cdot dW_s^{dz}\right| > \delta\right] \xrightarrow[\epsilon \longrightarrow 0]{} 0 . \tag{3.20}$$

The left-hand side in (3.20) is bounded by $I_1(\epsilon) + I_2$ where

$$I_1(\epsilon) = \mathbb{P}\left[\left|\frac{1}{\epsilon}\int_0^t \langle Y_s\,,\, W_{s+\epsilon} - W_s\rangle ds - \int_0^t Y_s \cdot dW_s^{dz}\right| > \delta\ ;\ \tau^m > T\right] ,$$
$$I_2 = \mathbb{P}\left[\tau^m \leq T\right] .$$

Since Y^m belongs to $\mathcal{N}^2_W(0,T;L^2(H_0;\mathbb{R}))$ and by localization of Da Prato-Zabczyk integral, see Lemma 4.9 in [13], we obtain

$$I_1(\epsilon) = \mathbb{P}\left[\left|\frac{1}{\epsilon}\int_0^t \langle Y_s^m\,,\, W_{s+\epsilon} - W_s\rangle ds - \int_0^t Y_s^m \cdot dW_s^{dz}\right| > \delta\ ;\ \tau^m > T\right] .$$

Taking into account the first part of the proof already established we get $\lim_{\epsilon \to 0} I_1(\epsilon) = 0$. Consequently

$$\limsup_{\epsilon \to 0} I(\epsilon) \leq \mathbb{P}\left[\tau^m \leq T\right] .$$

Taking m large enough the right-hand side is arbitrarily small so the proof is finally concluded.

$\square$

In the special case $G = \mathbb{R}$, it is possible to establish a similar result with respect to Brownian martingale. We omit the details.

Proposition 3.11. Let M be the square integrable F-valued Brownian martingale defined by the Da Prato-Zabczyk stochastic integral $M = \Phi \cdot W$, where $\Phi \in \mathcal{N}^2_W(0,T;L^2(H_0;F))$. Let Y be a $L(F;\mathbb{R}) = F^*$-valued adapted process such that $\int_0^T \|Y(s)\Phi_s\|^2_{F^*} ds < +\infty$ a.s. Then for every $t \in [0,T]$

$$\int_0^t \langle Y_s, d^- M_s\rangle = \int_0^t Y_s \cdot dM_s^{dz} .$$

Chapter 4

Chi-quadratic variation

4.1 Comments

In this section we will define a new concept of quadratic variation which is suitable for Banach space valued processes. Let B be a Banach space.
We first try to explain why our concept is more general than other notions in the literature. The natural generalization notion coming from calculus related to semimartingales appear for instance in [46] (resp. [19]) for some classes of B-valued processes where B is a Hilbert (resp. Banach) space. One typical class is the family of π-processes which are not so far to Banach valued semimartingales, since their notion is constantly related to Itô type stochastic integrals. We remark that [13] introduces slight different notion of quadratic variation for B-valued martingales with B Hilbert separable space. For those processes [46] and [19] introduce two concepts of quadratic variation: the *real quadratic variation* and the *tensor quadratic variation.* The real one is characterized as a limit of discretization sums; the tensor quadratic variation is related to expressions of the type

$$X_t \otimes^2 - X_0 \otimes^2 - \int_{]0,t]} (X_{s^-} \otimes dX_s + dX_s \otimes X_{s^-}) \ .$$

In the language of regularizations we are also able to define a real and tensor quadratic variations processes, which are the true analogous of the mentioned concepts, but a priori for any process. However they will appear as particular cases of our theory as will be explained in details in Section 6.3.

Definition 4.1. Let X a B-valued stochastic process.

1. X is said to admit a **real quadratic variation** denoted by $[X]^{\mathbb{R}}$ if $[X]^{\mathbb{R}}$ is the real valued ucp limit for $\epsilon \downarrow 0$ of the sequence

$$[X]^{\mathbb{R},\epsilon}_{\cdot} = \int_0^{\cdot} \frac{\|X_{s+\epsilon} - X_s\|_B^2}{\epsilon} ds \ . \tag{4.1}$$

$[X]^{\mathbb{R}}$ will be indeed called real quadratic variation of X.

2. X admits a **tensor quadratic variation** if it admits a real quadratic variation and if it exists a $(B\hat{\otimes}_\pi B)$-valued process denoted by $[X]^{\otimes}$ such that the sequence of Bochner integrals

$$[X]^{\otimes,\epsilon}_{\cdot} = \int_0^{\cdot} \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon} ds \tag{4.2}$$

converges to $[X]^{\otimes}$ ucp for $\epsilon \downarrow 0$.

$[X]^{\otimes}$ will be indeed called tensor quadratic variation of X.

Remark 4.2. 1. Integrals in (4.2) are well defined in the Bochner sense as $B\hat{\otimes}_\pi B$-valued integral processes since the fact that X admits a real quadratic variation implies that

$$\frac{1}{\epsilon}\int_0^{\cdot} \left\|(X_{s+\epsilon} - X_s)\otimes^2\right\|_{B\hat{\otimes}_\pi B} ds = \int_0^{\cdot} \frac{\|X_{s+\epsilon} - X_s\|_B^2}{\epsilon} ds < +\infty \text{ a.s.}$$

2. In point 2. of the definition the condition of the existence of the real quadratic variation can be relaxed requiring that for all subsequences (ϵ_n) it exists a subsubsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^t \frac{\|X_{s+\epsilon_{n_k}} - X_s\|_B^2}{\epsilon_{n_k}} ds < +\infty \text{ a.s.}$$

similarly to the techniques developed in Section 4.3.

3. The tensor quadratic variation is the natural object intervening in the second order term of the Itô formula expanding $F(X)$ for some C^2-Fréchet function F.

4. Suppose that the limiting process in (4.2) exists. To insure that limit has bounded variation, the classical procedure consists in showing that the real quadratic variation exists, as required in the definition. In fact the variation of tensor quadratic variation is dominated by the variation of real quadratic variation, which is clearly of bounded variation being an increasing process.

Unfortunately, the existence of the real quadratic variation is a very requiring and rarely verified condition. For instance, the window Brownian motion $W(\cdot)$, which is our fundamental example, does not have, in principle, any real quadratic variation. In fact, even if for fixed ϵ the quantity

$$\int_0^t \frac{\|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|^2_{C([-\tau,0])}}{\epsilon} ds$$

exists, it is not possible to control its limit for ϵ going to zero as we will see in details in Remark 5.5 and Proposition 5.6.

We come back now to the convergence of (4.2): the projective norm π is may be too strong for its convergence even when $X = W(\cdot)$. One possible relaxation could consist in requiring a (strong) convergence with respect to a weaker tensor topology as the Hilbert or the injective ϵ-topology, however this route was not easily practicable for us. In fact our strategy is to introduce a convergence making use of a subspace χ of $(B\hat{\otimes}_\pi B)^*$; when χ coincides with the whole space $(B\hat{\otimes}_\pi B)^*$ our convergence coincides the classical weak star topology in $(B\hat{\otimes}_\pi B)^{**}$.
In such a case X will be said to have a χ-quadratic variation, see Defintion 4.19. Our χ-quadratic variation generalizes the concept of tensor quadratic variation at two levels.

- First we replace the (strong) convergence in (4.2) with a weak topology type convergence.
- Secondly the choice of a suitable subspace χ of $(B\hat{\otimes}_\pi B)^*$ gives a degree of freedom.

As we will see in Section 6, whenever X admits one of the classical quadratic variation (in the sense of [24, 13, 46, 19]), it admits a χ-quadratic variation with χ equal to the whole space. This corresponds to the *elementary* situation for us.
A window Brownian motion $X = W(\cdot)$ admits a χ- quadratic variation a priori only for strict subspaces χ. This will be particularly helpful in applications, in particular for obtaining at Section 9 some generalized Clark-Ocone formulae.

4.2 Notion and examples of Chi-subspaces

Let B be a Banach space.

Definition 4.3. A closed linear subspace χ of $(B\hat{\otimes}_\pi B)^*$, endowed with its own norm, such that

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B\hat{\otimes}_\pi B)^*} \tag{4.3}$$

will be called a **Chi-subspace of** $(B\hat{\otimes}_\pi B)^*$.

The result below follows immediately by the definition.

Proposition 4.4. Any closed subspace of a Chi-subspace is a Chi-subspace.

As the reader can see from Section 2.6, we are interested in expressing subsets of $(B\hat{\otimes}_\pi B)^*$ as direct sums of Chi-subspaces. This, together with Propositions 4.5 and 4.26 will help us to evaluate χ-quadratic variations of different processes.

Proposition 4.5. Let $\chi_1, \cdots, \chi_n$ be Chi-subspaces of $(B\hat{\otimes}_\pi B)^*$ such that $\chi_i \bigcap \chi_j = \{0\}$ for any $1 \leq i \neq j \leq n$. Then the normed space $\chi = \chi_1 \oplus \cdots \chi_n$ is a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$.

Proof. It is enough to prove the result for the case $n = 2$. If $\mu \in \chi$, then it admits decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \chi_1, \quad \mu_2 \in \chi_2$. It holds $\|\mu\|_{(B\hat{\otimes}_\pi B)^*} \leq \|\mu_1\|_{(B\hat{\otimes}_\pi B)^*} + \|\mu_2\|_{(B\hat{\otimes}_\pi B)^*}$ and Assumption (4.3) for χ_1 and χ_2 implies that $\|\mu_i\|_{(B\hat{\otimes}_\pi B)^*} \leq \|\mu_i\|_{\chi_i}$ for $i = 1, 2$. It follows then $\|\mu\|_{(B\hat{\otimes}_\pi B)^*} \leq \|\mu_1\|_{\chi_1} + \|\mu_2\|_{\chi_2}$, i.e. the norm (2.10) with $p = 1$ in the Banach space χ. Since all the norms defined in a direct sum of Banach spaces are equivalent to the product topology, then (4.3) is also verified for any norm and the result follows. □

Before providing the definition of the so-called χ-quadratic variation for a B-valued stochastic process, we will give some examples of Chi-subspaces that we will use frequently in the paper. For the notations we remind to Section 2.6.

Example 4.6. Let B be a general Banach space.

- $\chi = (B\hat{\otimes}_\pi B)^*$. This corresponds to our elementary situation anticipated at the end of Section 4.1. We anticipate that a process which admits a quadratic variation in the sense of [13, 46, 24], has a $(B\hat{\otimes}_\pi B)^*$-quadratic variation, see Section 6.

Example 4.7. Let $B = C([-\tau, 0])$.
This is the natural value space for all the window (continuous) processes. We list some examples of Chi-subspaces χ for which window processes have a χ-quadratic variation. Our basic reference Chi-subspace of $(C([-\tau, 0]\hat{\otimes}_\pi C([-\tau, 0]))^*$ will be $\mathcal{M}([-\tau, 0]^2)$ equipped with the usual total variation norm, denoted by $\|\cdot\|_{Var}$. This is in fact a proper subspace as it will be illustrated in the following lines. Condition (4.3) will be verified using properties of projective tensor products recalled at Section 2.5. All the other spaces considered in the sequel of the present example will be shown to be Chi-subspaces of $\mathcal{M}([-\tau, 0]^2)$; by Proposition 4.4 they will also be Chi-subspaces of $(B\hat{\otimes}_\pi B)^*$.
Here is the list.

- $\mathcal{M}([-\tau, 0]^2)$. This space, equipped with the total variation norm, is a Banach space. We can identify this space with the dual of the injective tensor product; in fact by (2.24)

$$\mathcal{M}([-\tau, 0]^2) = \big(C([-\tau, 0]^2)\big)^* = \big(C([-\tau, 0])\hat{\otimes}_\epsilon \mathcal{C}([-\tau, 0])\big)^* \subset \big(C([-\tau, 0])\hat{\otimes}_\pi C([-\tau, 0])\big)^*. \quad (4.4)$$

  In particular by properties of tensor product, (4.3) is verified because $\|\mu\|_{\epsilon^*} = \|\mu\|_{Var} \geq \|\mu\|_{(B\hat{\otimes}_\pi B)^*}$ for every $\mu \in \mathcal{M}([-\tau, 0]^2)$.

- $L^2([-\tau, 0]^2)$ identified with its dual. This is a Hilbert subspace of $\mathcal{M}([-\tau, 0]^2)$ and for $\mu \in L^2([-\tau, 0]^2)$ it holds obviously that $\|\mu\|_{Var} \leq \|\mu\|_{L^2([-\tau,0]^2)}$.

- $\mathcal{D}_{ij}([-\tau, 0]^2)$ for every $i, j = 0, \ldots, N$. If $\mu = \lambda\, \delta_{a_i}(dx)\delta_{a_j}(dy)$, $\|\mu\|_{Var} = |\lambda| = \|\mu\|_{\mathcal{D}_{i,j}}$.

- $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$. For a general element in this space $\mu = \lambda\delta_{a_i}(dx)\phi(y)dy$, $\phi \in L^2([-\tau,0])$, we have $\|\mu\|_{Var} \leq \|\mu\|_{L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])} = |\lambda|\cdot\|\phi\|_{L^2}$.
- $\chi^2([-\tau,0]^2) := (L^2([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0]))\hat{\otimes}_h^2$. This space will be denoted frequently shortly by χ^2. This is a well defined Hilbert space with the scalar product which derives from the scalar products in every Hilbert space and it is Chi-subspace of $\mathcal{M}([-\tau,0]^2)$ and consequently also of $(B\hat{\otimes}_\pi B)^*$.

Remark 4.8. 1. We could have shown that $\chi^2([-\tau,0]^2) \subset \mathcal{M}[-\tau,0]^2$ through an argument of tensor product theory. In fact if H is a Hilbert space such that $H \subset \mathcal{M}([-\tau,0])$ it holds $H\hat{\otimes}_h^2 \subset H\hat{\otimes}_\epsilon^2 \subset \mathcal{M}([-\tau,0])\hat{\otimes}_\epsilon^2 = C^*([-\tau,0])\hat{\otimes}_\epsilon^2 \subset (C([-\tau,0])\hat{\otimes}_\epsilon)^* = (C([-\tau,0]^2)^* = \mathcal{M}([-\tau,0]^2)$ because the ϵ-topology respects subspaces, see comment in relation to Proposition 3.2 on [63]. In our case setting $H = L^2([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])$, then H is a Hilbert subset of $\mathcal{M}([-\tau,0])$.

2. Using Proposition 2.23, we obtain:

$$\chi^2([-\tau,0]^2) = L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h^2 . \tag{4.5}$$

Using again Proposition 2.23 with (2.28) and (2.29) we can expand every addend in the right-hand side of (4.5), into a sum of elementary addends. For instance we have $L^2\hat{\otimes}_h\mathcal{D}_a = \bigoplus_{i=0}^N \left(L^2\hat{\otimes}_h\mathcal{D}_i\right)$ and $\mathcal{D}_a\hat{\otimes}_h^2 = \mathcal{D}_A = \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}$ so that (4.5) equals

$$L^2([-\tau,0]^2) \oplus \bigoplus_{i=0}^N \left(L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])\right) \oplus \bigoplus_{i=0}^N \left(\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])\right) \oplus \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}([-\tau,0]^2) . \tag{4.6}$$

Being χ^2 a finite direct sum of Chi-subspaces, Proposition 4.5 confirms that it is a Chi-subspace.

- As a particular case of $\chi^2([-\tau,0]^2)$ we will denote $\chi^0([-\tau,0]^2)$, χ^0 shortly, the subspace of measures defined as

$$\chi^0([-\tau,0]^2) := (\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0]))\hat{\otimes}_h^2 .$$

Again using Proposition 2.23, we obtain:

$$\chi^0([-\tau,0]^2) = L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_0([-\tau,0]) \oplus \mathcal{D}_0([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_{0,0}([-\tau,0]^2) . \tag{4.7}$$

Remark 4.9. 1. For every μ in $\chi^2([-\tau,0]^2)$ there exist $\mu_1 \in L^2([-\tau,0]^2)$, $\mu_2 \in L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a([-\tau,0])$, $\mu_3 \in \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ and $\mu_4 \in \mathcal{D}_a([-\tau,0])\hat{\otimes}_h^2$ such that

$$\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4, \tag{4.8}$$

with $\mu_1 = \phi_1$, $\mu_2 = \sum_{i=0,\ldots,N} \phi_2^i \otimes \delta_{a_i}$, $\mu_3 = \sum_{i=0,\ldots,N} \delta_{a_i} \otimes \phi_3^i$ and $\mu_4 = \sum_{i,j=0,\ldots,N} \lambda_{i,j} \delta_{a_i} \otimes \delta_{a_j}$, where $\phi_1 \in L^2([-\tau,0]^2)$, $\phi_2^i, \phi_3^i \in L^2([-\tau,0])$ and $\lambda_{i,j}$ are real numbers for every $i,j = 0,\ldots,N$. Components μ_1, μ_2 and μ_3 are singular with respect to the Dirac's measure on $\{(a_i,a_j)\}_{0\le i,j\le N}$, remarking that $\delta_{(a_i,a_j)} = \delta_{a_i} \otimes \delta_{a_j}$; in particular $\mu_k\{(a_i,a_j)\} = 0$ for $k = 1,2,3$. For a general μ it follows

$$\mu\{(a_i,a_j)\} = \mu_4\{(a_i,a_j)\} = \lambda_{i,j}\ . \tag{4.9}$$

2. Consequently an element $\mu \in \chi^0([-\tau,0]^2)$ can be uniquely decomposed as

$$\mu = \phi_1 + \phi_2 \otimes \delta_0 + \delta_0 \otimes \phi_3 + \lambda \delta_0 \otimes \delta_0, \tag{4.10}$$

where $\phi_1 \in L^2([-\tau,0]^2)$, ϕ_2, ϕ_3 are functions in $L^2([-\tau,0])$ and λ, α, β are real numbers and

$$\mu\left(\{0,0\}\right) = \mu_4\left(\{0,0\}\right) = \lambda\ . \tag{4.11}$$

We go on with other examples of Chi-subspaces.

- $Diag([-\tau,0]^2)$. Let $\mu \in Diag$, we have $\|\mu\|_{Var} \le \tau\, \|\mu\|_{Diag}$, so $Diag([-\tau,0]^2)$ is again a Chi-suspace of $\mathcal{M}([-\tau,0]^2)$.

- $\chi^3([-\tau,0]^2) := \chi^2([-\tau,0]^2) \oplus Diag([-\tau,0]^2)$. The sum is direct and obviously it is a subset of $\mathcal{M}([-\tau,0]^2)$. As a consequence of Proposition 4.5, χ^3 is a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$. This is a Banach space which fails to be Hilbert because $Diag$ is not Hilbert. We equip $\chi^3([-\tau,0]^2)$ with the norm (2.10), with $p = 2$, in the sense that, whenever μ is an element in $\chi^3([-\tau,0]^2)$ with decomposition $\mu = \mu_1 + \mu_2$, $\mu_1 \in \chi^2([-\tau,0]^2)$ and $\mu_2 \in Diag([-\tau,0]^2)$, we set

$$\|\mu\|^2_{\chi^3([-\tau,0]^2)} := \|\mu_1\|^2_{\chi^2([-\tau,0]^2)} + \|\mu_2\|^2_{Diag([-\tau,0]^2)}\ . \tag{4.12}$$

- $\chi^4([-\tau,0]^2)$ where

$$\chi^4([-\tau,0]^2) := \mathcal{D}_d([-\tau,0]^2) \oplus L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_a([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]). \tag{4.13}$$

  This is obviously a subspace of $\mathcal{M}([-\tau,0]^2)$ and it is a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$ because of Proposition 4.5.

The following examples are academic and they will not be used in the sequel in a relevant way. Some of them involve discrete measures with infinite (countable) support.

- $\chi^5([-\tau,0]^2) = \mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2)$ with

$$\mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2) := \left\{ \mu \in \mathcal{M}([-\tau,0]^2) : \mu = \sum_{i,j\in\mathbb{N}} \lambda_{i,j}\delta_{(\alpha_i,\alpha_j)}; \lambda_{i,j} \in \mathbb{R}, \sup_{i,j}\{|\lambda_{i,j}|i^2j^2\} < +\infty \right\} \quad (4.14)$$

where $(\alpha_i)_{i\in\mathbb{N}}$ and $(\alpha_j)_{j\in\mathbb{N}}$ are two sequences of given points in $[-\tau,0]$. An element of $\chi^5([-\tau,0]^2)$ is a discrete measure concentrated on a countable sequence of fixed points $(\alpha_i,\alpha_j)_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ on the square $[-\tau,0]^2$. The space $\mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2)$ equipped with the norm $\|\mu\|_{\mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2)} = \sup_{i,j}\{|\lambda_{i,j}|i^2j^2\}$, is a Banach subspace of $\mathcal{M}([-\tau,0]^2)$.
For $\chi^5([-\tau,0]^2)$, to be a Chi-subspace it remains to show $\|\mu\|_{Var} \le \|\mu\|_{\chi^5}$. For an element $\mu \in \chi^5$ the total variation norm is $\|\mu\|_{Var([-\tau,0]^2)} = \sum_{i,j\in\mathbb{N}} |\lambda_{i,j}|$ and it is finite. In particular $\|\mu\|_{Var([-\tau,0]^2)} = \sum_{i,j\in\mathbb{N}} |\lambda_{i,j}| = \sum_{i,j\in\mathbb{N}} |\lambda_{i,j}|i^2j^2\frac{1}{i^2j^2} \le \sup_{i,j}\{|\lambda_{i,j}|i^2j^2\}\sum_{i,j\in\mathbb{N}}\frac{1}{i^2j^2} = \|\mu\|_{\chi^5}\frac{\pi^4}{36}$.

- Let $\{\mu_i\}_{i=1,\dots,N}$ be N fixed mutually singular measures in $\mathcal{M}([-\tau,0]^2)$ with $\|\mu_i\|_{Var} = 1$. We define the space $\chi^6([-\tau,0]^2)$ as the space

$$\chi^6([-\tau,0]^2) := Span(\{\mu_i\}_{i=1,\dots,N}) = \left\{ \mu = \sum_{i=1,\dots,N} \lambda_i\mu_i;\ \mu_i \in \mathcal{M}([-\tau,0]^2),\ \lambda_i \in \mathbb{R} \right\}. \quad (4.15)$$

The space χ^6 equipped with the norm $\|\mu\|_{\chi^6} = \sqrt{\sum_{i=1}^N \lambda_i^2}$, is a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ of finite dimension N. The norm $\|\cdot\|_{\chi^6}$ is compatible with the induced topology defined by $\mathcal{M}([-\tau,0]^2)$. By Proposition 4.4, χ^6 is a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$. We observe that $\|\mu\|_{Var} = \sum_{i=1}^N |\lambda_i| \le \|\mu\|_{\chi^6} = \sqrt{\sum_{i=1}^N \lambda_i^2}$.

- Let μ be any fixed finite measure on $[-\tau,0]^2$.

$$\chi^\mu([-\tau,0]^2) = \{\nu \in \mathcal{M}([-\tau,0]^2); d\nu = g\,d\mu, g \in L^\infty(d\mu)\}\ . \quad (4.16)$$

Without restriction of generality we can consider μ being a positive measure. χ^μ is the space of absolutely continuous meausres with respect μ with Radon-Nikodym density in $L^\infty(d\mu)$. The space χ^μ, equipped with the norm $\|\nu\|_{\chi^\mu} := \|g\|_{L^\infty}$, is a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ and it is isomorphic to $L^\infty(d\mu)$. The norm $\|\nu\|_{\chi^\mu}$ of a general measure $\nu \in \chi^\mu$ will be denoted also by $\|\nu\|_{\infty,\mu}$. For a general measure $\nu \in \chi^\mu$ it holds $\|\nu\|_{Var} \le \|g\|_{L^\infty}\,\|\mu\|_{Var} = C\,\|\nu\|_{\chi^\mu}$ C being a constant, so χ^μ is a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$.
Next proposition shows that a χ^μ space can be constructed from a family of mutually singular measures.

Proposition 4.10. Let I be a countable set. Let $\{\mu_i\}_{i\in I}$, be mutually singular non-negative finite measures on $[-\tau,0]^2$ and set $\mu = \sum_{i\in I}\mu_i$ supposed to be finite.

Then

$$\chi^{\mu} = \bigoplus_{i\in I} \chi^{\mu_i} \tag{4.17}$$

and $\|\nu\|_{\infty,\mu} = \sup_{i\in I} \|\nu_i\|_{\infty,\mu_i}$.

Proof. Since μ_i, $i \in I$, are mutually singular, there is a partition $(A_i)_{i\in I}$ of $[-\tau, 0]^2$ such that $\mu_i(A_i^c) = 0$; we remark that if $A \subset A_i$ $\mu(A) = \mu_i(A)$, for all $i \in I$. Since $\mu_i(B) \le \mu(B)$, $\forall\ B \in \mathcal{B}([-\tau, 0]^2)$, $i \in I$, then $\mu_i << \mu$, $\forall\ i \in I$.
1) If $\nu = \sum_{i\in I} \nu_i$, $\nu_i \in \chi^{\mu_i}$ then $\nu \in \chi^{\mu}$.
We show first that $\nu << \mu$. In fact, for every $B \in \mathcal{B}([-\tau, 0]^2)$, if $\mu(B) = 0$ then $\nu(B) = \sum_{i\in I} \nu_i(B) = 0$ since $\nu_i << \mu_i << \mu$. On the other hand, it is possible to show that

$$\frac{d\nu}{d\mu}\mathbb{1}_{A_i} = \frac{d\nu_i}{d\mu_i}\mathbb{1}_{A_i} \qquad \mu\text{-a.e. (therefore}\mu_i - \text{ a.e.).} \tag{4.18}$$

In fact if $B \in \mathcal{B}([-\tau, 0]^2)$

$$\int_{B\cap A_i} \frac{d\nu}{d\mu} d\mu = \nu(B \cap A_i) = \nu_i(B \cap A_i) = \int_{B\cap A_i} \frac{d\nu_i}{d\mu_i} d\mu_i = \int_{B\cap A_i} \frac{d\nu_i}{d\mu_i} d\mu.$$

So (4.18) implies that

$$\left\|\frac{d\nu}{d\mu}\right\|_{\infty,\mu} \le \sup_{i\in I} \left\|\frac{d\nu_i}{d\mu_i}\right\|_{\infty,\mu_i}.$$

2) Viceversa, if $\nu \in \chi^{\mu}$, we set $\nu_i(B) = \nu(B \cap A_i)$ for $i \in I$ and B Borel set. Let B a Borel set such that $\mu_i(B) = 0$; then $\mu(B \cap A_i) = \mu_i(B \cap A_i) \le \mu_i(B) = 0$ and so $\nu_i(B) = \nu(B \cap A_i) = 0$; consequently $\nu_i << \mu_i$. Since again (4.18) holds, $\nu_i \in \chi^{\mu_i}$ and

$$\left\|\frac{d\nu_i}{d\mu_i}\right\|_{\infty,\mu_i} \le \left\|\frac{d\nu}{d\mu}\right\|_{\infty,\mu},$$

we conclude that $\nu \in \bigoplus_{i\in I} \chi^{\mu_i}$. $\square$

Remark 4.11. A particular case of the Proposition 4.10 is given when $\mu_i = \delta_{(a_i,b_i)}$ where $(a_i, b_i) \in [-\tau, 0]^2$, $i \in I = \{= 1, \dots, N\}$. Then $\nu \in \chi^{\mu}$ if and only if $\nu = \sum_{i=1}^{N} \lambda_i \delta_{(a_i,b_i)}$; in this case $\|\nu\|_{\infty,\mu} = \max_{1\le i\le N}\{|\lambda_i|\}$.

- A last example of Chi-subspace of $\mathcal{M}([-\tau, 0]^2)$ is $L^2([-\tau, 0]^2) \oplus \chi^{\mu}([-\tau, 0]^2)$, where μ is a given measure in $\mathcal{M}([-\tau, 0]^2)$, singular with respect to the Lebesgue measure. This is a Chi-subspace again because of Proposition 4.5.

Example 4.12. Let $B = H = L^2([-\tau, 0])$.

Before listing examples of Chi-subspaces of $(H\hat{\otimes}_\pi H)^*$ we need some preliminary results. We recall that

$$L^2([-\tau,0]^2) \cong (H\hat{\otimes}_h H) \cong (H\hat{\otimes}_h H)^* \subset (H\hat{\otimes}_\pi H)^*, \tag{4.19}$$

where $(H\hat{\otimes}_h H)$ and its dual are identified via the usual Riesz identification. On the other hand $(H\hat{\otimes}_\pi H)^*$ can be identified with $\mathcal{B}(H,H)$, see (2.15). Using this identification and (4.19) we inject $L^2([-\tau,0]^2)$ into $\mathcal{B}(H,H)$; in this way, the space $L^2([-\tau,0]^2)$ identifies a subspace of bilinear bounded (continuous) forms on $H \times H$. In other words, to every $f \in L^2([-\tau,0]^2)$ is associated the element $T^f \in \mathcal{B}(H,H)$ setting

$$T^f : L^2([-\tau,0]) \times L^2([-\tau,0]) \longrightarrow \mathbb{R}, \qquad (g,h) \mapsto T^f(g,h) = \int_{[-\tau,0]^2} g(x)h(y)f(x,y)\,dx\,dy\ . \tag{4.20}$$

Definition 4.13. We will denote by $L^2_{\mathcal{B}}([-\tau,0]^2)$ the set of all bilinear maps T^f. This space equipped with the norm $\|T^f\|_{L^2_{\mathcal{B}}([-\tau,0]^2)} := \|f\|_{L^2([-\tau,0]^2)}$, is a Hilbert space which indeed coincides with $L^2([-\tau,0]^2)^*$.

Remark 4.14.

1. By Proposition 2.24 we know that $L^2_{\mathcal{B}}([-\tau,0]^2)$ is properly included in $\mathcal{B}(H,H)$.
2. By definition, for every $T^f \in L^2_{\mathcal{B}}([-\tau,0]^2)$,
$$\left\|T^f\right\|_{\mathcal{B}} = \sup_{\|g\|\le 1, \|f\|\le 1} |T(g,h)| \le \|f\|_{L^2([-\tau,0]^2)} = \left\|T^f\right\|_{L^2_{\mathcal{B}}([-\tau,0]^2)}\ . \tag{4.21}$$
3. In Proposition 5.32 we will see that $L^2_{\mathcal{B}}([-\tau,0]^2)$ is not densely embedded into $\mathcal{B}(H,H)$.

The Banach space $(H\hat{\otimes}_\pi H)^*$ contains two significant Chi-subspaces; the first one is naturally associated with $L^2([-\tau,0]^2$ via $L^2_{\mathcal{B}}([-\tau,0]^2)$, the second one with $L^\infty([-\tau,0])$. Below we describe those announced Chi-subspaces.

- $\chi = L^2_{\mathcal{B}}([-\tau,0]^2)$ equipped with its norm. We recall the isometry between $(H\hat{\otimes}_\pi H)^*$ and $\mathcal{B}(H,H)$: the usual norm of the bilinear operator T^f, denoted by $\|\cdot\|_{\mathcal{B}}$, is equal to the norm of the corresponding element in $(H\hat{\otimes}_\pi H)^*$. By Remark 4.14.2. χ is clearly a Chi-subspace of $\mathcal{B}(H,H)$. Condition (4.3) could have been verified also using relations (2.17) and (2.18).

- $\chi = Diag_{\mathcal{B}}([-\tau,0]^2)$ where $Diag_{\mathcal{B}}([-\tau,0]^2)$ is the following set
$$\left\{ T^f \in \mathcal{B}(H,H),\ \text{s.t. } T^f(g,h) = \int_{[-\tau,0]} g(x)h(x)f(x)\,dx\ ; f \in L^\infty([-\tau,0]) \right\}\ . \tag{4.22}$$
By definition it is a subspace of $\mathcal{B}(H,H)$ and every operator T^f is determined by a function in $f \in L^\infty([-\tau,0])$. This space, equipped with the norm $\|T^f\|_{Diag_{\mathcal{B}}([-\tau,0]^2)} := \|f\|_{L^\infty([-\tau,0])} = \|f\|_\infty$ is a Banach space.
We verify condition (4.3). For $T^f \in Diag_{\mathcal{B}}([-\tau,0]^2)$, we have

$$\|T^f\|_{\mathcal{B}} = \sup_{\|g\|\le 1,\, \|h\|\le 1} |T(g,h)| = \sup_{\|g\|\le 1,\, \|h\|\le 1} \left| \int_{[-\tau,0]} g(x)h(x)f(x)dx \right| \le \|f\|_{L^\infty([-\tau,0])} = \|T\|_{Diag_{\mathcal{B}}([-\tau,0]^2)}\ .$$

Proposition 4.15. *$Diag_{\mathcal{B}}([-\tau,0]^2)$, equipped with the topology of $\mathcal{B}(H,H)$ is closed.*

Proof. Let $T^f \in Diag_{\mathcal{B}}([-\tau,0]^2)$ with $f \in L^\infty([-\tau,0])$. It is enough to show that

$$\|f\|_\infty = \|T^f\|_{\mathcal{B}} \ .$$

1) Obviously for every g, $h \in H$, $\left|T^f(g,h)\right| \leq \|f\|_\infty \|g\|_H \|h\|_H$, which implies that $\left\|T^f\right\|_{\mathcal{B}} \leq \|f\|_\infty$.
2) For proving the converse inequality of **1)** it is enough to find a sequence (g_N, h_N) of $H \times H$, such that $\|g_N\|_H = \|h_N\|_H = 1$, and $\left|T^f(g_N,h_N)\right| \xrightarrow[N\longrightarrow+\infty]{} \|f\|_\infty$.
Let $N > 0$ and define

$$\Lambda_N := \left\{ y \in [-\tau,0] \ ; \ |f(y)| \geq \|f\|_\infty - \frac{1}{N} \right\} \ .$$

By definition of essential supremum, it follows that $Leb(\Lambda_N) \xrightarrow[N\longrightarrow+\infty]{} 0$. We set

$$g_N(y) = 1\!\!1_{\Lambda_N}(y) \frac{1}{\sqrt{Leb(\Lambda_N)}}$$

$$h_N(y) = 1\!\!1_{\Lambda_N}(y) \frac{sign\,(f(y))}{\sqrt{Leb(\Lambda_N)}} \qquad \text{where} \qquad sign(x) = \begin{cases} +1 & \text{if } x \leq 0 \\ -1 & \text{if } x < 0 \end{cases} .$$

We have

$$\int_{-\tau}^0 g_N(y)h_N(y)f(y)dy = \int_{\Lambda_N} \frac{|f(y)|}{Leb(\Lambda_N)} \geq \|f\|_\infty - \frac{1}{N} \xrightarrow[N\longrightarrow+\infty]{} \|f\|_\infty \ .$$

This concludes the proof of the proposition. □

Remark 4.16. This space has been denoted with $Diag_{\mathcal{B}}$ because it has a strong relation with the space of measures $Diag$ defined in (2.32). In fact let φ be a function in $L^\infty([-\tau,0])$. φ can be either associated with a measure $\mu^\varphi \in Diag([-\tau,0]^2)$ or with an operator $T^\varphi \in Diag_{\mathcal{B}}([-\tau,0]^2)$. The measure is identified by $\mu^\varphi(dx,dy) = \varphi(x)\delta_y(dx)dy$. The bilinear operator is identified by $T^\varphi(g,h) = \int_{[-\tau,0]} g(x)h(x)\varphi(x)\,dx$. Let η_1, η_2 be two elements in $C([\tau,0]) \subset H$,

$$_{\mathcal{M}([-\tau,0]^2)}\langle \mu^\varphi, \eta_1 \otimes \eta_2 \rangle_{C([-\tau,0]^2)} = T^\varphi(\eta_1,\eta_2) \ . \tag{4.23}$$

In fact the left-hand side in (4.23) equals

$$\langle \mu^\varphi(dx,dy), \eta_1(x)\cdot\eta_2(y)\rangle = \int_{[-\tau,0]^2} \eta_1(x)\eta_2(y)\varphi(x)\delta_y(dx)dy = \int_{[-\tau,0]} \eta_1(x)\eta_2(x)\varphi(x)dx \ .$$

For instance if φ is the constant function equal to 1, then diagonal measure μ^1 corresponds to the inner product in $L^2([-\tau,0])$ in the sense that

$$_{\mathcal{M}([-\tau,0]^2)}\langle \mu^1, \eta_1 \otimes \eta_2 \rangle_{C([-\tau,0]^2)} = T^1(\eta_1,\eta_2) = {}_{L^2([-\tau,0])}\langle \eta_1, \eta_2 \rangle_{L^2([-\tau,0])}.$$

Remark 4.17. We recall that the bilinear functions in $L^2_{\mathcal{B}}([-\tau,0]^2)$ identified with $L^2([-\tau,0]^2)$, can also be observed as a subspace of $\mathcal{M}([-\tau,0]^2)$.

4.3 Definition of χ-quadratic variation and some related results

In this subsection, we introduce the definition of the χ-quadratic variation of a B-valued stochastic process X. We remind that $\mathcal{C}([0,T])$ denotes the space of continuous processes equipped with the ucp topology.

Let χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$, X be a B-valued stochastic process and $\epsilon > 0$. We denote by $[X,X]^\epsilon$, or simply by $[X]^\epsilon$, the following application

$$[X]^\epsilon : \chi \longrightarrow \mathcal{C}([0,T]) \qquad \text{defined by} \qquad \phi \mapsto \left(\int_0^t {}_\chi\langle \phi, \frac{J\left((X_{s+\epsilon} - X_s)\otimes^2\right)}{\epsilon} \rangle_{\chi^*} \, ds \right)_{t\in[0,T]} \tag{4.24}$$

where

$$J : B\hat{\otimes}_\pi B \longrightarrow (B\hat{\otimes}_\pi B)^{**} \tag{4.25}$$

denotes the canonical injection between a space and its bidual as introduced in Section 2.1.

With application $[X]^\epsilon$ it is possible to associate another one, denoted by $\widetilde{[X,X]}^\epsilon$, or simply by $\widetilde{[X]}^\epsilon$, defined by

$$\widetilde{[X]}^\epsilon(\omega,\cdot) : [0,T] \longrightarrow \chi^* \qquad \text{given by} \qquad t \mapsto \left(\phi \mapsto \int_0^t {}_\chi\langle \phi, \frac{J\left((X_{s+\epsilon} - X_s)\otimes^2\right)}{\epsilon} \rangle_{\chi^*} \, ds \right) . \tag{4.26}$$

Remark 4.18.

1. We recall that $\chi \subset (B\hat{\otimes}_\pi B)^*$ implies $(B\hat{\otimes}_\pi B)^{**} \subset \chi^*$.

2. As indicated ${}_\chi\langle \cdot,\cdot\rangle_{\chi^*}$ denotes the duality between the space χ and its dual χ^* in fact by assumption, ϕ is an element of χ and element $J\left((X_{s+\epsilon} - X_s)\otimes^2\right)$ naturally belongs to $(B\hat{\otimes}_\pi B)^{**} \subset \chi^*$.

3. With a slight abuse of notation, in the sequel application J will be omitted. The tensor product $(X_{s+\epsilon} - X_s)\otimes^2$ has to be considered as the element $J\left((X_{s+\epsilon} - X_s)\otimes^2\right)$ which belongs to χ^*.

4. Suppose $B = C([-\tau,0])$ and χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$.
   An element of the type $\eta = \eta_1 \otimes \eta_2$, η_1, $\eta_2 \in B$, can be either considered as an element of the type $B\hat{\otimes}_\pi B \subset (B\hat{\otimes}_\pi B)^{**} \subset \chi^*$ or as an element of $C([-\tau,0]^2)$ defined by $\eta(x,y) = \eta_1(x)\eta_2(y)$. When χ is indeed a Chi-subspace of $\mathcal{M}([\tau,0]^2)$, then the pairing between χ and χ^* will be compatible with the pairing duality between $\mathcal{M}([\tau,0]^2)$ and $C([-\tau,0]^2)$ given in (2.25).

Definition 4.19. Let χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X a B-valued stochastic process. We say that X **admits a χ-quadratic variation** if the following assumptions are fulfilled.

H1 For all $(\epsilon_n) \downarrow 0$ it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, \frac{(X_{s+\epsilon_{n_k}} - X_s)\otimes^2}{\epsilon_{n_k}} \rangle_{\chi^*} \right| ds = \sup_k \frac{1}{\epsilon_{n_k}} \int_0^T \left\| (X_{s+\epsilon_{n_k}} - X_s)\otimes^2 \right\|_{\chi^*} ds \quad < +\infty \quad a.s.$$

(4.27)

H2 (i) It exists an application $\chi \longrightarrow \mathcal{C}([0,T])$, denoted by $[X,X]$ or simply by $[X]$, such that

$$[X]^\epsilon(\phi) \xrightarrow[\epsilon\longrightarrow 0_+]{ucp} [X](\phi) \qquad \text{for every } \phi \in \chi \,. \tag{4.28}$$

(ii) There is a measurable process $\widetilde{[X,X]} : \Omega \times [0,T] \longrightarrow \chi^*$, also denoted by $\widetilde{[X]}$, such that

- for almost all $\omega \in \Omega$, $\widetilde{[X]}(\omega,\cdot)$ is a (cadlag) bounded variation process.
- $\widetilde{[X]}(\cdot,t)(\phi) = [X](\phi)(\cdot,t)$ a.s. for all $\phi \in \chi$.

When X admits a χ-quadratic variation, we will call **χ-quadratic variation** of X the χ^*-valued process $(\widetilde{[\mathbf{X}]})_{\mathbf{0\le t\le T}}$ defined for every $\omega \in \Omega$ and $t \in [0,T]$ by $\phi \mapsto \widetilde{[X]}(\omega,t)(\phi) = [X](\phi)(\omega,t)$. Sometimes, with a slight abuse of notation, even $[\mathbf{X}]$ will be called χ-quadratic variation and it will be confused with $\widetilde{[X]}$.

Remark 4.20.

1. For every fixed $\phi \in \chi$, the processes $\widetilde{[X]}(\cdot,t)(\phi)$ and $[X](\phi)(\cdot,t)$ are indistinguishable. In particular the χ^*-valued process $\widetilde{[X]}$ is weakly star continuous, i.e. $\widetilde{[X]}(\phi)$ is continuous for every fixed ϕ.

2. In fact the existence of $\widetilde{[X]}$ guarantees that $[X]$ admits a proper version which allows to consider it as pathwise integral.

3. The quadratic variation $\widetilde{[X]}$ will be the object intervening in the second order term of the Itô formula expanding $F(X)$ for some C^2-Fréchet function F.

4. We will show in Corollaries 4.38 and 4.39 that, when χ is separable (the most of cases) Condition **H2** can be relaxed in a significant way. For instance convergence (4.28) can be verified only in probability on a dense subspace of χ and **H2**(ii) is automatically verified.

Remark 4.21.

1. A practical criterion to verify Condition **H1** is

$$\frac{1}{\epsilon}\int_0^T \left\|(X_{s+\epsilon}-X_s)\otimes^2\right\|_{\chi^*} ds \le B(\epsilon) \tag{4.29}$$

where $B(\epsilon)$ converges in probability. In fact the convergence in probability implies the a.s. convergence of a subsequence.

2. A consequence of Condition **H1** is that for all $(\epsilon_n) \downarrow 0$ there exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \|\widetilde{[X]}^{\epsilon_{n_k}}\|_{Var[0,T]} < \infty \quad a.s. \tag{4.30}$$

In fact $\|\widetilde{[X]}^{\epsilon}\|_{Var[0,T]} \leq \frac{1}{\epsilon}\int_0^T \|(X_{s+\epsilon}-X_s)\otimes^2\|_{\chi^*} ds$, this implies that $\widetilde{[X]}^{\epsilon}$ is a χ^*-valued process of bounded variation on $[0,T]$. As a consequence, for a χ-valued continuous stochastic process Y the integral $\int_0^t {}_{\chi}\langle Y_s, d\widetilde{[X]}_s^{\epsilon_{n_k}}\rangle_{\chi^*}$ is a well-defined Lebesgue-Stieltjes type integral for almost all $\omega \in \Omega$, $t \in [0,T]$.

Remark 4.22.

1. Given $G : \chi \longrightarrow C([0,T])$ we can associate $\tilde{G} : [0,T] \longrightarrow \chi^*$ setting $\tilde{G}(t)(\phi) = G(\phi)(t)$. $\tilde{G} : [0,T] \longrightarrow \chi^*$ has bounded variation if

$$\|\tilde{G}\|_{Var[0,T]} = \sup_{\sigma\in\Sigma_{[0,T]}} \sum_{i|(t_i)_i=\sigma} \left\|\tilde{G}(t_{i+1}) - \tilde{G}(t_i)\right\|_{\chi^*} = \sup_{\sigma\in\Sigma_{[0,T]}} \sum_{i|(t_i)_i=\sigma} \sup_{\|\phi\|_{\chi}\leq 1} |G(\phi)(t_{i+1}) - G(\phi)(t_i)| < +\infty$$

   where $\Sigma_{[0,T]}$ is the set of all possible partitions of the interval $[0,T]$ and $\sigma = (t_i)_i$ is an element of $\Sigma_{[0,T]}$. This quantity is called **total variation** of $\tilde{G}$.
   For example if $G(\phi) = \int_0^t \dot{G}_s(\phi)\, ds$ then $\|G\|_{Var[0,T]} \leq \int_0^T \sup_{\|\phi\|_{\chi}\leq 1} |\dot{G}_s(\phi)|\, ds$.

2. If $G(\phi)$, $\phi \in \chi$ is a family of stochastic processes, it is not obvious to find a good version $\tilde{G} : [0,T] \longrightarrow \chi^*$ of G. This will be the object of Theorem 4.35.

Definition 4.23. We say that a continuous B-valued process X admits **global quadratic variation** if it admits a χ-quadratic variation with $\chi = (B\hat{\otimes}_{\pi}B)^*$.

Remark 4.24. We observe some interesting features in the case $\chi = (B\hat{\otimes}_{\pi}B)^*$.

1. The natural convergence topology is the weak star convergence in the space $(B\hat{\otimes}_{\pi}B)^{**}$ for elements $\widetilde{[X]}^{\epsilon}$. In fact, at least when χ is separable, for any $t \in [0,T]$, it exists a null subset N of Ω such and a sequence (ε_n) such that

$$\widetilde{[X]}^{\epsilon_n}(\omega,t) \xrightarrow[\epsilon\longrightarrow 0]{w^*} \widetilde{[X]}(\omega,t)$$

   weak star, see Lemma 4.32. We recall that $J(B\hat{\otimes}_{\pi}B)$ is weak star dense in $(B\hat{\otimes}_{\pi}B)^{**}$, so $\widetilde{[X]}$ takes values "a priori" in $(B\hat{\otimes}_{\pi}B)^{**}$.

2. The weak star convergence is weaker then the strong convergence in $B\hat{\otimes}_{\pi}B$, i.e. the convergence with respect to the topology defined by the norm. A strong convergence is required for example in the definition of a tensor quadratic variation, see Definition 4.1.2, or in the definition of quadratic variation for an $\mathbb{R}^n$-valued process, see Definition 2.6. In a finite dimensional spaces all topologies are equivalent. If the Banach space $B\hat{\otimes}_{\pi}B$ is not reflexive, then $(B\hat{\otimes}_{\pi}B)^{**}$ strictly contains $B\hat{\otimes}_{\pi}B$.

3. In general $B\hat{\otimes}_{\pi}B$ is not reflexive even if B is an Hilbert space, see Remark 2.21.3.

Proposition 4.25. Let X be a B-valued process admitting a tensor quadratic variation then X admits a global quadratic variation. In particular the global quadratic variation takes valued in $B\hat{\otimes}_\pi B$ and $\widetilde{[X]} = [X]^{\otimes}$ a.s.

Proof. We set $\chi = (B\hat{\otimes}_\pi B)^*$. We observe that the existence of $[X]^{\mathbb{R}}$ implies the validity of Condition **H1**. Recalling definition of $[X]^\epsilon$ at (4.24) and the definition of injection J we observe that

$$[X]^\epsilon(\phi)(\cdot,t) = \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, \frac{J\left((X_{s+\epsilon}-X_s)\otimes^2\right)}{\epsilon}\rangle_{(B\hat{\otimes}_\pi B)^{**}}\, ds = \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle_{B\hat{\otimes}_\pi B}\, ds\ . \tag{4.31}$$

Since Bochner inegrability implies Pettis integrability, for details see Appendix A, in particular Proposition A.1, we also have that for every $\phi \in (B\hat{\otimes}_\pi B)^*$,

$${}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, [X]^{\otimes,\epsilon}_t\rangle_{B\hat{\otimes}_\pi B} = \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle_{B\hat{\otimes}_\pi B}\, ds\ . \tag{4.32}$$

(4.31) and (4.32) imply that

$$[X]^\epsilon(\phi)(\cdot,t) = {}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, [X]^{\otimes,\epsilon}_t\rangle_{B\hat{\otimes}_\pi B} \qquad \text{a.s.} \tag{4.33}$$

We go on now with the proof of Condition **H2**. We will show that

$$\sup_{t\le T}\left| [X]^\epsilon(\phi)(\cdot,t) - {}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, [X]^{\otimes}_t\rangle_{B\hat{\otimes}_\pi B}\right| \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0\ . \tag{4.34}$$

Developing the left-hand side of (4.34) and using (4.33), we obtain

$$\begin{aligned}\sup_{t\le T}\left| [X]^\epsilon(\phi)(\cdot,t) - {}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, [X]^{\otimes}_t\rangle_{B\hat{\otimes}_\pi B}\right| &= \sup_{t\le T}\left| {}_{(B\hat{\otimes}_\pi B)^*}\langle \phi, [X]^{\otimes,\epsilon}_t - [X]^{\otimes}_t\rangle_{B\hat{\otimes}_\pi B}\right| \\ &\le \|\phi\|_{(B\hat{\otimes}_\pi B)^*}\ \sup_{t\le T}\left\| [X]^{\otimes,\epsilon}_t - [X]^{\otimes}_t\right\|_{B\hat{\otimes}_\pi B}\ ,\end{aligned}$$

where the last quantity converges to zero in probability by Definition 4.1.2 of tensor quadratic variation. This implies (4.34). The tensor quadratic variation has always bounded variation because of existence of real quadratic variation, see Remark 4.2.4. In particular **H2**(ii) is also verified. □

We go on with some related results about χ-quadratic variation.

Proposition 4.26. Let X be a B-valued process and χ_1, χ_2 be two Chi-subspaces of $(B\hat{\otimes}_\pi B)^*$ with $\chi_1 \cap \chi_2 = \{0\}$. Let $\chi = \chi_1 \oplus \chi_2$. If X admits a χ_i-quadratic variation $[X]_i$ for $i = 1, 2$ then it admits a χ-quadratic variation $[X]$ and it holds $[X](\phi) = [X]_1(\phi_1) + [X]_2(\phi_2)$ for all $\phi \in \chi$ with unique decomposition $\phi = \phi_1 + \phi_2$.

Proof. χ is a Chi-subspace because of Proposition 4.5. will be enough to show the result for a fixed norm in the space χ. We choose $\|\phi\|_\chi = \|\phi_1\|_{\chi_1} + \|\chi_2\|_2$.
We remark that for all possible norms in $\chi_1 \oplus \chi_2$ we have $\|\phi\|_\chi \geq \|\phi_i\|_{\chi_i}$. Then condition **H1** follows immediately by inequality

$$\int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| {}_\chi\langle \phi, (X_{s+\epsilon} - X_s)\otimes^2\rangle_{\chi^*} \right| ds \leq \int_0^T \sup_{\|\phi_1\|_{\chi_1} \leq 1} \left| {}_{\chi_1}\langle \phi_1, (X_{s+\epsilon} - X_s)\otimes^2\rangle_{\chi_1^*} \right| ds +$$
$$+ \int_0^T \sup_{\|\phi_2\|_{\chi_2} \leq 1} \left| {}_{\chi_2}\langle \phi_2, (X_{s+\epsilon} - X_s)\otimes^2\rangle_{\chi_2^*} \right| ds \ .$$

Condition **H2**(i) follows by linearity; in fact

$$[X]^\epsilon(\phi) = \int_0^t {}_\chi\langle \phi_1 + \phi_2, (X_{s+\epsilon} - X_s)\otimes^2\rangle_{\chi^*} ds =$$
$$= \int_0^t {}_{\chi_1}\langle \phi_1, (X_{s+\epsilon} - X_s)\otimes^2\rangle_{\chi^*} ds + \int_0^t {}_{\chi_2}\langle \phi_2, (X_{s+\epsilon} - X_s)\otimes^2\rangle_{\chi^*} ds \xrightarrow[\epsilon\to 0]{ucp} [X]_1(\phi_1) + [X]_2(\phi_2) \ .$$

Concerning Condition **H2**(ii), for $\omega \in \Omega$, $t \in [0,T]$ we can obviously set $\widetilde{[X]}(\omega,t)(\phi) = \widetilde{[X]}_1(\omega,t)(\phi_1) + \widetilde{[X]}_2(\omega,t)(\phi_2)$. □

Proposition 4.27. Let X be a B-valued stochastic process. Let χ_1 χ_2 be two subspaces $\chi_1 \subset \chi_2 \subset (B\hat{\otimes}_\pi B)^*$ such that χ_1 is a Chi-subspace of χ_2 and χ_2 is a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. If X admits a χ_2-quadratic variation $[X]_2$, then it also admits a χ_1-quadratic variation $[X]_1$ and it holds $[X]_1(\phi) = [X]_2(\phi)$ for all $\phi \in \chi_1$.

Remark 4.28. If Condition **H1** is valid for χ_2 then it is also verified for χ_1. In fact we remark that $(X_{s+\epsilon} - X_s)\otimes^2$ is an element in $(B\hat{\otimes}_\pi B) \subset (B\hat{\otimes}_\pi B)^{**} \subset \chi_2^* \subset \chi_1^*$. If $A := \{\phi \in \chi_1 \, ; \|\phi\|_{\chi_1 \leq 1}\}$ and $B := \{\phi \in \chi_2 \, ; \|\phi\|_{\chi_2 \leq 1}\}$, then $A \subset B$ and clearly $\int_0^t \sup_A |\langle \phi, (X_{s+\epsilon} - X_s)\otimes^2\rangle| ds \leq \int_0^t \sup_B |\langle \phi, (X_{s+\epsilon} - X_s)\otimes^2\rangle| ds$. This implies the inequality $\left\|(X_{s+\epsilon} - X_s)\otimes^2\right\|_{\chi_1^*} \leq \left\|(X_{s+\epsilon} - X_s)\otimes^2\right\|_{\chi_2^*}$ and Assumption **H1** follows immediately.

Proof of Proposition 4.27. The validity of Assumption **H1** with respect to χ_1 was the object of Remark 4.28. Assumption **H2**(i) is trivially verified because for all $\phi \in \chi_1$, by hypothesis, we have $[X]^\epsilon(\phi) \xrightarrow[\epsilon\to 0]{ucp} [X]_2(\phi)$. In particular $[X]_1(\phi) = [X]_2(\phi)$, $\forall\, \phi \in \chi_1$. We set $\widetilde{[X]}_1(\omega,t)(\phi) = \widetilde{[X]}_2(\omega,t)(\phi)$, for all $\omega \in \Omega$, $t \in [0,T]$, $\phi \in \chi_1$. Condition **H2**(ii) follows because given $G : [0,T] \longrightarrow \chi_1$ we have $\|G(t) - G(s)\|_{\chi_1^*} \leq \|G(t) - G(s)\|_{\chi_2^*}$, $\forall\, 0 \leq s \leq t \leq T$. □

Remark 4.29. On the contrary, let χ_1, χ_2 be two Chi-subspaces as in Proposition 4.27. It may happens that a B-valued process X does not admit a $(B\hat{\otimes}_\pi B)^*$-quadratic variation or not even a χ_2-quadratic variation but it admits a χ_1-quadratic variation. For this reason the fact to introduce a subspace of $(B\hat{\otimes}_\pi B)^*$ gives much more opportunities of calculus. For example that the $C([-\tau,0])$-valued window

Brownian motion admits a χ^2-quadratic variation but it does not have a $\mathcal{M}([-\tau,0]^2)$-quadratic variation. This will be seen in details in Section 5.

We continue with some general properties of χ-quadratic variation.

Lemma 4.30. Let X be a B-valued stochastic process. Suppose that $\frac{1}{\epsilon}\int_0^T \|(X_{s+\epsilon}-X_s)\otimes^2\|_{\chi^*}\, ds$ converges to 0 in probability when $\epsilon \to 0$.

1. Then X admits a zero χ-quadratic variation.
2. If $\chi = (B\hat{\otimes}_\pi B)^*$ then X admits a zero real and tensor quadratic variation.

Proof.

1. Condition **H1** is verified because of Remark 4.21.1. We verify **H2**(i) directly. For every fixed $\phi \in \chi$ we have

$$\begin{aligned}
|[X]^\epsilon(\phi)(t)| = & \left|\int_0^t {}_\chi\langle \phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle_{\chi^*}\, ds\right| \le \\
\le & \int_0^t \left|{}_\chi\langle \phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle_{\chi^*}\right|\, ds \le \\
\le & \int_0^T \left|{}_\chi\langle \phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle_{\chi^*}\right|\, ds\ .
\end{aligned}$$

So we obtain

$$\sup_{t\in[0,T]} |[X]^\epsilon(\phi)(t)| \le \int_0^T \left|{}_\chi\langle \phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle_{\chi^*}\right|\, ds \le \|\phi\|_\chi \frac{1}{\epsilon}\int_0^T \|(X_{s+\epsilon}-X_s)\otimes^2\|_{\chi^*}\, ds \xrightarrow{\mathbb{P}} 0$$

in probability by the hypothesis. Since condition **H2 ii)** holds trivially, this allows to conclude.

2. By definition the real quadratic variation is zero and this forces the tensor quadratic variation also to be zero.

$\square$

An important proposition used later to prove different fundamental results, as Itô's formula, is the following.

Proposition 4.31. Let χ be a separable Banach space, a sequence $F^n : \chi \longrightarrow \mathcal{C}([0,T])$ of linear continuous maps and measurable random fields $\widetilde{F}^n : \Omega\times[0,T] \longrightarrow \chi^*$ such that $\widetilde{F}^n(\cdot,t)(\phi) = F^n(\phi)(\cdot,t)$ a.s. $\forall\, t\in[0,T]$, $\phi\in\chi$. We suppose the following.

i) For all (n_k) it exists (n_{k_j}) such that $\sup_j \|\widetilde{F}^{n_{k_j}}\|_{Var[0,T]} < \infty$.

ii) There is a linear continuous map $F : \chi \longrightarrow \mathcal{C}([0,T])$ such that for all $t \in [0,T]$ and for every $\phi \in \chi$ $F^n(\phi)(\cdot,t) \longrightarrow F(\phi)(\cdot,t)$ in probability.

iii) There is $\widetilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ of such that for ω a.s. $\widetilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ has bounded variation and $\widetilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t)$a.s. $\forall\, t \in [0,T]$ and $\phi \in \chi$.

iv) $F^n(\phi)(0) = 0$ for every $\phi \in \chi$.

Then for every $t \in [0,T]$ and every continuous process $H : \Omega \times [0,T] \longrightarrow \chi$

$$\int_0^t {}_{\chi}\langle H(\cdot,s), d\widetilde{F}^n(\cdot,s)\rangle_{\chi^*} \longrightarrow \int_0^t {}_{\chi}\langle H(\cdot,s), d\widetilde{F}(\cdot,s)\rangle_{\chi^*} \qquad \text{in probability.} \tag{4.35}$$

Before writing the proof we need a technical lemma. In the sequel indices χ and χ^* in the duality, will often be omitted.

Lemma 4.32. Let $t \in [0,T]$. There is a subsequence of (n_k) still denoted by the same symbol and a null subset N of Ω such that

$$\widetilde{F}^{n_k}(\omega,t)(\phi) \longrightarrow_{k\to\infty} \widetilde{F}(\omega,t)(\phi) \tag{4.36}$$

for every $\phi \in \chi$ and $\omega \notin N$.

Proof of Lemma 4.32 . Let $\mathcal{S}$ be a dense countable subset of χ. By a diagonalization principle for extracting subsequences, there is a subsequence (n_k), a null subset N of Ω such that for all $\omega \notin \Omega$,

$$\widetilde{F}_\infty(\omega,t)(\phi) := \lim_{k\to+\infty} \widetilde{F}^{n_k}(\omega,t)(\phi) \tag{4.37}$$

exists for any $\phi \in \mathcal{S}$, $\omega \notin N$ and $\forall\, t \in [0,T]$.
By construction, for every $t \in [0,T]$, $\phi \in \mathcal{S}$

$$\widetilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t) = \widetilde{F}_\infty(\cdot,t)(\phi) \qquad a.s.$$

Let $t \in [0,T]$ be fixed. A slight modification of the null set N, yields that for every $\omega \notin N$,

$$\widetilde{F}(\omega,t)(\phi) = \widetilde{F}_\infty(\omega,t)(\phi) \qquad \forall\, \phi \in \mathcal{S}\ .$$

At this point (4.37) becomes

$$\widetilde{F}(\omega,t)(\phi) = \lim_{k\to+\infty} \widetilde{F}^{n_k}(\omega,t)(\phi) \tag{4.38}$$

for every $\omega \notin N$, $\phi \in \mathcal{S}$.
It remains to show that (4.38) still holds for $\phi \in \chi$. Therefore we fix $\phi \in \chi$, $\omega \notin N$. Let $\epsilon > 0$ and $\phi_\epsilon \in \mathcal{S}$

such that $\|\phi - \phi_\epsilon\|_\chi \le \epsilon$. We can write

$$\begin{aligned}\left|\widetilde{F}(\omega,t)(\phi) - \widetilde{F}^{n_k}(\omega,t)(\phi)\right| &\le \left|\widetilde{F}(\omega,t)(\phi-\phi_\epsilon)\right| + \left|\widetilde{F}(\omega,t)(\phi_\epsilon) - \widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon)\right| + \left|\widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon-\phi)\right| \le \\ &\le \left\|\widetilde{F}(\omega,t)\right\|_{\chi^*}\|\phi-\phi_\epsilon\|_\chi + \sup_k\left\|\widetilde{F}^{n_k}(\omega,t)\right\|_{\chi^*}\|\phi-\phi_\epsilon\|_\chi + \\ &\quad + \left|\widetilde{F}(\omega,t)(\phi_\epsilon) - \widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon)\right| .\end{aligned}$$

Taking the $\limsup_{k\to+\infty}$ in previous expression and using (4.38) yields

$$\limsup_{k\to+\infty}\left|\widetilde{F}(\omega,t)(\phi) - \widetilde{F}^{n_k}(\omega,t)(\phi)\right| \le \left\|\widetilde{F}(\omega,t)\right\|_{\chi^*}\epsilon + \sup_k\left\|\widetilde{F}^{n_k}(\omega,\cdot)\right\|_{Var[0,T]}\epsilon .$$

Since $\epsilon > 0$, the result follows. □

Proof of the Proposition 4.31 . Let $t \in [0,T]$ be fixed. We denote

$$I(n)(\omega) := \int_0^t \langle H(\omega,s), d\widetilde{F}^n(\omega,s)\rangle - \int_0^t \langle H(\omega,s), d\widetilde{F}(\omega,s)\rangle .$$

Let $\delta > 0$ and a subdivision of $[0,t]$ given by $0 = t_0 < t_1 < \cdots < t_m = t$ with mesh smaller than δ. Let (n_k) be a sequence diverging to infinity. We need to exhibit a subsequence (n_{k_j}) such that

$$I(n_{k_j})(\omega) \longrightarrow 0 \qquad \text{a.s.} \tag{4.39}$$

Lemma 4.32 implies the existence of a null set N, a subsequence (n_{k_j}) such that

$$\left|\widetilde{F}^{n_{k_j}}(\omega,t_l)(\phi) - \widetilde{F}(\omega,t_l)(\phi)\right| \xrightarrow[j\longrightarrow+\infty]{} 0 \qquad \forall\,\phi\in\chi \quad \text{and for every} \quad l\in\{0,\dots,m\} . \tag{4.40}$$

Let $\omega \notin N$. We have

$$\begin{aligned}\left|I(n_{k_j})(\omega)\right| &= \left|\sum_{i=1}^m\left(\int_{t_{i-1}}^{t_i}\langle H(\omega,s), d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle - \langle H(\omega,s), d\widetilde{F}(\omega,s)\rangle\right)\right| \le \\ &\le \sum_{i=1}^m\left|\int_{t_{i-1}}^{t_i}\langle H(\omega,s) - H(\omega,t_{i-1}) + H(\omega,t_{i-1}), d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle + \right. \\ &\qquad\left. - \int_{t_{i-1}}^{t_i}\langle H(\omega,s) - H(\omega,t_{i-1}) + H(\omega,t_{i-1}), d\widetilde{F}(\omega,s)\rangle\right| \le \\ &\le I_1(n_{k_j})(\omega) + I_2(n_{k_j})(\omega) + I_3(n_{k_j})(\omega) ,\end{aligned}$$

where

$$
\begin{aligned}
I_1(n_{k_j})(\omega) &= \sum_{i=1}^{m} \left| \int_{t_{i-1}}^{t_i} \langle H(\omega,s)-H(\omega,t_{i-1}), d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle \right| \leq \varpi_{H(\omega,\cdot)}(\delta)\, \sup_j \|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]} \\
I_2(n_{k_j})(\omega) &= \sum_{i=1}^{m} \left| \int_{t_{i-1}}^{t_i} \langle H(\omega,s)-H(\omega,t_{i-1}), d\widetilde{F}(\omega,s)\rangle \right| \leq \varpi_{H(\omega,\cdot)}(\delta)\, \|\widetilde{F}(\omega)\|_{Var[0,T]} \\
I_3(n_{k_j})(\omega) &= \sum_{i=1}^{m} \left| \int_{t_{i-1}}^{t_i} \langle H(\omega,t_{i-1}), d(\widetilde{F}^{n_{k_j}}(\omega,s)-\widetilde{F}(\omega,s))\rangle \right| = \\
&= \sum_{i=1}^{m} \left| \langle H(\omega,t_{i-1}), \widetilde{F}^{n_{k_j}}(\omega,t_i)-\widetilde{F}(\omega,t_i)-\widetilde{F}^{n_{k_j}}(\omega,t_{i-1})+\widetilde{F}(\omega,t_{i-1})\rangle \right| \leq \\
&\leq \sum_{i=1}^{m} \left| F^{n_{k_j}}(H(\omega,t_{i-1}))(\omega,t_i) - F(H(\omega,t_{i-1}))(\omega,t_i) \right| + \\
&\qquad \sum_{i=1}^{m} \left| F^{n_{k_j}}(H(\omega,t_{i-1}))(\omega,t_{i-1}) - F(H(\omega,t_{i-1}))(\omega,t_{i-1}) \right| .
\end{aligned}
$$

The notation $\varpi_{H(\omega,\cdot)}$ indicates the modulus of continuity for H and it is a random variable; in fact it depends on ω in the sense that

$$
\varpi_{H(\omega,\cdot)}(\delta) = \sup_{|s-t|\leq\delta} \|H(\omega,s)-H(\omega,t)\|_{\chi} \ .
$$

By (4.40) applied to $\phi = H(\omega,t_{i-1})$ we obtain

$$
\limsup_{j\to\infty} |I(n_{k_j})(\omega)| \leq \left(\sup_j \|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]} + \|\widetilde{F}(\omega)\|_{Var[0,T]} \right) \varpi_{H(\omega,\cdot)}(\delta) \tag{4.41}
$$

Since $\delta>0$ is arbitrary and H is uniformly continuous on $[0,t]$ so that $\varpi_{H(\omega,\cdot)}(\delta)\to 0$ a.s. for $\delta\to 0$, then $\limsup_{j\to\infty}|I(n_{k_j})(\cdot)|=0$ a.s..
This concludes (4.39) and the proof of the Proposition. □

Corollary 4.33. Let B be a Banach space and χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. Let X be a B-valued stochastic process with χ-quadratic variation and H a continuous measurable process $H:\Omega\times[0,T]\longrightarrow\mathcal{V}$ where $\mathcal{V}$ is a closed separable subspace of χ. Then for every $t\in[0,T]$

$$
\int_0^t {}_{\chi}\langle H(\cdot,s), d\widetilde{[X]}^{\epsilon}(\cdot,s)\rangle_{\chi^*} \longrightarrow \int_0^t {}_{\chi}\langle H(\cdot,s), d\widetilde{[X]}(\cdot,s)\rangle_{\chi^*} \tag{4.42}
$$

in probability.

Proof. By Proposition 4.4, $\mathcal{V}$ is a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. By Proposition 4.27 X admits a $\mathcal{V}$-quadratic variation $[X]_{\mathcal{V}}$ and $[X]_{\mathcal{V}}(\phi)=[X](\phi)$ for all $\phi\in\mathcal{V}$; in the sequel of the proof, $[X]_{\mathcal{V}}$ will be still denoted by

$[X]$. Since the ucp convergence implies the convergence in probability for every $t \in [0, T]$, by Proposition 4.31 and definition of $\mathcal{V}$-quadratic variation, it follows

$$\int_0^t {}_{\mathcal{V}}\langle H(\cdot, s), d\widetilde{[X]}^{\epsilon}(\cdot, s)\rangle_{\mathcal{V}^*} \xrightarrow[\epsilon \longrightarrow 0]{\mathbb{P}} \int_0^t {}_{\mathcal{V}}\langle H(\cdot, s), d\widetilde{[X]}(\cdot, s)\rangle_{\mathcal{V}^*} . \tag{4.43}$$

Since the pairing duality between χ and χ^* is compatible with the one between $\mathcal{V}$ and $\mathcal{V}^*$, result (4.42) is now established. □

An important and useful theorem to find sufficient conditions for the existence of the χ-quadratic variation of a Banach valued process is given below. It will be a consequence of a Banach-Steinhaus type result for Fréchet spaces, see Theorem II.1.18, pag. 55 in [20]. We start with a remark.

Remark 4.34.

1. In the mentioned Banach-Steinhaus theorem intervenes the following notion. Let E be a Fréchet spaces, F-space shortly. A subset B of E is called **bounded** if for all $\epsilon > 0$ it exists δ_ϵ such that for all $0 < \alpha \leq \delta_\epsilon$, αB is included in the open ball $\mathcal{B}(0, \epsilon) := \{e \in E; d(0, e) < \epsilon\}$.

2. Let (Y^n) be a sequence of random elements with values in a Banach space $(B, \| \cdot \|_B)$ such that $\sup_n \|Y^n\|_B \leq Z$ a.s. for some positive random variable Z. Then (Y^n) is bounded in the F-space of random elements equipped with the convergence in probability which is governed by the metric

$$d(X, Y) = \mathbb{E}\left[\|X - Y\|_B \wedge 1\right].$$

In fact by Lebesgue dominated convergence theorem it follows $\lim_{\gamma \to 0} \mathbb{E}[\gamma Z \wedge 1] = 0$.

3. In particular taking $B = C([0, T])$ a sequence of continuous processes (Y^n) such that $\sup_n \|Y^n\|_\infty \leq Z$ a.s. is bounded for the usual metric in $\mathcal{C}([0, T])$ equipped with the topology related to the ucp convergence.

Theorem 4.35. Let $F^n : \chi \longrightarrow \mathcal{C}([0, T])$ be a sequence of linear continuous maps such that $F^n(\phi)(0) = 0$ a.s. and there is $\tilde{F}^n : \Omega \times [0, T] \longrightarrow \chi^*$ a.s. for which we have the following.

i) $F^n(\phi)(\cdot, t) = \tilde{F}^n(\cdot, t)(\phi)$ a.s. $\forall\, t \in [0, T]$, $\phi \in \chi$.

ii) $\forall\, \phi \in \chi$, $t \mapsto \tilde{F}^n(\cdot, t)(\phi)$ is cadlag.

iii) $\sup_n \|\tilde{F}^n\|_{Var} < \infty$ a.s.

iv) There is a subset $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$ and a linear application $F : \mathcal{S} \longrightarrow \mathcal{C}([0, T])$ such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$.

1) Suppose that χ is separable. Then there is a linear and continuous extension $F : \chi \longrightarrow \mathcal{C}([0,T])$ and there is $\tilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ such that $\tilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t)$ a.s. for every $t \in [0,T]$. Moreover the following properties hold.

- **a)** For every $\phi \in \chi$, $F^n(\phi) \xrightarrow{ucp} F(\phi)$.
  In particular for every $t \in [0,T]$, $\phi \in \chi$, $F^n(\phi)(\cdot,t) \xrightarrow{\mathbb{P}} F(\phi)(\omega,t)$.
- **b)** $\tilde{F}$ has bounded variation a.s. and $t \mapsto \tilde{F}(\omega,t)$ is ω-a.s. weakly star continuous.

2) Suppose the existence of $\tilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ such that $t \mapsto \tilde{F}(\omega,t)$ has bounded variation and weakly star cadlag such that

$$\tilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t) \qquad \text{a.s.} \qquad \forall\, t \in [0,T],\ \forall\, \phi \in \mathcal{S}\ .$$

Then point **a)** still follows.

Remark 4.36. In point **2)** we do not necessarily suppose χ to be separable.

Proof of the Theorem 4.35.

a) We recall that $\mathcal{C}([0,T])$ is an F-space. Let $\phi \in \chi$. Clearly $(F^n(\phi)(\cdot,t))_t$ and $\left(\tilde{F}^n(\cdot,t)(\phi)\right)_t$ are indistinguishable processes and so $\left(\tilde{F}^n(\phi)(\cdot,t)\right)_t$ is a continuous process. So it follows

$$\begin{aligned}\|F^n(\phi)\|_\infty &= \sup_{t\in[0,T]} |F^n(\phi)(t)| = \sup_{t\in[0,T]} |\tilde{F}^n(\cdot,t)(\phi)| \leq \\ &\leq \sup_{t\in[0,T]} \left\|\tilde{F}^n(\cdot,t)\right\|_{\chi^*} \|\phi\|_\chi \leq \sup_n \|\tilde{F}^n\|_{Var} \|\phi\|_\chi < +\infty\end{aligned}$$

a.s. by the hypothesis. By Remark 4.34.2. and 3. it follows that the set $\{F^n(\phi)\}$ is a bounded subset of the F-space $\mathcal{C}([0,T])$ for every fixed $\phi \in \chi$.
We can apply the Banach-Steinhaus Theorem II.1.18, pag. 55 in [20] and point **iv)**, which imply the existence of $F : \chi \longrightarrow \mathcal{C}([0,T])$ linear and continuous such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \chi$. So **a)** is established in both situations **1)** and **2)**.

b) It remains to show the rest in situation **1)**, i.e. when χ is separable.

b.1) We first prove the existence of a suitable version $\tilde{F}$ of F such that $\tilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ is weakly star continuous ω a.s.
Since χ is separable, we consider a dense countable subset $\mathcal{D} \subset \chi$. Point **a)** implies that for a fixed $\phi \in \mathcal{D}$ there is a subsequence (n_k) such that $F^{n_k}(\phi)(\omega,\cdot) \xrightarrow{C([0,T])} F(\phi)(\omega,\cdot)$ a.s. Since $\mathcal{D}$ is countable there is a null set N and a further subsequence still denoted by (n_k) such that

$$\tilde{F}^{n_k}(\omega,\cdot)(\phi) \xrightarrow{C([0,T])} F(\phi)(\omega,\cdot) \qquad \forall\, \phi \in \mathcal{D},\ \forall \omega \notin N\ . \tag{4.44}$$

For $\omega \notin N$, we set $\tilde{F}(\omega,t)(\phi) = F(\phi)(\omega,t)\ \forall\ \phi \in \mathcal{S},\ t \in [0,T]$. By a slight abuse of notation the sequence $\tilde{F}^{n_k}$ can be seen as applications

$$\tilde{F}^{n_k}(\omega,\cdot) : \chi \longrightarrow C([0,T])$$

which are linear continuous maps verifying the following

- $\tilde{F}^{n_k}(\omega,\cdot)(\phi) \longrightarrow \tilde{F}(\omega,\cdot)(\phi)$ in $C([0,T])$ for all $\phi \in \mathcal{D}$, because of (4.44).
- For every $\phi \in \chi$, we have

$$\begin{aligned}\sup_k \sup_t |\tilde{F}^{n_k}(\omega,t)(\phi)| &\leq \sup_k \sup_t \sup_{\|\phi\|_\chi \leq 1} |\tilde{F}^{n_k}(\omega,t)(\phi)|\ \|\phi\|_\chi \leq \sup_k \sup_t \|\tilde{F}^{n_k}(\omega,t)\|\ \|\phi\|_\chi \\ &\leq \sup_k \|\tilde{F}^{n_k}(\omega,t)\|_{Var} \|\phi\|_\chi < +\infty.\end{aligned}$$

Banach-Steinhaus thereom for Banach spaces implies the existence of a linear continuous map

$$\tilde{F}(\omega,\cdot) : \chi \longrightarrow C([0,T])$$

extending previous map $\tilde{F}(\omega,\cdot)$ from $\mathcal{D}$ to χ with values on $C([0,T])$. Moreover

$$\tilde{F}^{n_k}(\omega,\cdot)(\phi) \xrightarrow{C([0,T])} \tilde{F}(\omega,\cdot)(\phi) \qquad \forall\phi \in \chi,\ \forall\omega \notin N$$

and for every $\omega \notin N$ the application

$$\tilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^* \qquad t \mapsto \tilde{F}(\omega,t)$$

is weakly star continuous. $\tilde{F}$ is measurable from $\Omega \times [0,T]$ to χ^* being limit of measurable processes.

b.2) We prove now that the χ^*-valued process $\tilde{F}$ has bounded variation.
Let $\omega \notin N$ fixed again. Let $(t_i)_{i=0}^M$ be a subdivision of $[0,T]$ and let $\phi \in \chi$. Since the functions

$$F^{t_i,t_{i+1}} : \phi \longrightarrow \Big(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\Big)(\phi) \qquad F^{n_k,t_i,t_{i+1}} : \phi \longrightarrow \Big(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\Big)(\phi)$$

belong to χ^*, Banach-Steinhaus theorem says

$$\begin{aligned}\sup_{\|\phi\|\leq 1} \left|\Big(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\Big)(\phi)\right| &= \|F^{t_i,t_{i+1}}\|_{\chi^*} \leq \liminf_{k\to\infty} \|F^{n_k,t_i,t_{i+1}}\|_{\chi^*} = \\ &= \liminf_{k\to\infty} \sup_{\|\phi\|\leq 1} \left|\Big(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\Big)(\phi)\right| .\end{aligned}$$

Taking the sum over $i = 0, \ldots, (M-1)$ we get

$$\begin{aligned}\sum_{i=0}^{M-1} \sup_{\|\phi\|\leq 1} \left|\Big(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\Big)(\phi)\right| &\leq \sum_{i=0}^{M-1} \liminf_{k\to\infty} \sup_{\|\phi\|\leq 1} \left|\Big(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\Big)(\phi)\right| \leq \\ &\leq \sup_k \sum_{i=0}^{M-1} \sup_{\|\phi\|\leq 1} \left|\Big(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\Big)(\phi)\right| \leq \sup_k \|\tilde{F}^{n_k}\|_{Var} ,\end{aligned}$$

where the second inequality is justified by the relation $\liminf a_i^n + \liminf b_i^n \le \sup(a_i^n + b_i^n)$. Taking the sup over all subdivision $(t_i)_{i=0}^M$ we obtain

$$\|\tilde{F}\|_{Var} \le \sup_k \|\tilde{F}^{n_k}\|_{Var} < +\infty \ .$$

This shows finally the fact that $\tilde{F}(\omega, \cdot) : [0,T] \longrightarrow \chi^*$ has bounded variation.

□

Proposition 4.37. The statement of Theorem 4.35 holds replacing condition **iv)** with the one below.

iv') There is a subset $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$ and a linear application $F : \mathcal{S} \longrightarrow \mathcal{C}([0,T])$ such that for every $\phi \in \mathcal{S}$.

- $F^n(\phi)(t) \longrightarrow F(\phi)(t)$ for every $t \in [0,T]$ in probability.
- $F^n(\phi)$ is an increasing process.

Proof. Since for every $\phi \in \mathcal{S}$, $F(\phi)$ is an increasing process, Lemma 2.1 implies that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$, so **iv)** is established. □

Important implications of Theorem 4.35 and Proposition 4.37 are Corollaries 4.38 and 4.39, which give us easier conditions for the existence of the χ-quadratic variation as anticipated in Remark 4.20.4.

Corollary 4.38. Let B be a Banach space, X be a B-valued stochastic process and χ be a separable Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. We suppose the following.

H0' There is $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, \frac{(X_{s+\epsilon_{n_k}} - X_s)^{\otimes 2}}{\epsilon_{n_k}} \rangle_{\chi^*} \right| ds \quad < +\infty \ .$$

H2' There is $\mathcal{T} : \chi \longrightarrow \mathcal{C}([0,T])$ such that $[X]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ ucp for all $\phi \in \mathcal{S}$.

Then X admits a χ-quadratic variation and application $[X]$ is equal to $\mathcal{T}$.

Proof. Condition **H1** is verified by assumption. Conditions **H2**(i) and (ii) follow by Theorem 4.35 setting $F^n(\phi)(\cdot, t) = [X]^{\varepsilon_n}(\phi)(t)$ and $\tilde{F}^n = [\tilde{X}]^{\varepsilon_n}$ for a suitable sequence (ε_n). □

Corollary 4.39. Let B be a Banach space, X be a B-valued stochastic process and χ be a separable Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. We suppose the following.

H0" There are subsets $\mathcal{S}$, $\mathcal{S}^p$ of χ such that $\overline{Span(\mathcal{S})} = \chi$, $Span(\mathcal{S}) = Span(\mathcal{S}^p)$ and $\mathcal{S}^p$ is constituted by **positive definite** elements ϕ in the sense that $\langle \phi, b \otimes b \rangle \ge 0$ for all $b \in B$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| {}_\chi\langle \phi, \frac{(X_{s+\epsilon_{n_k}} - X_s)^{\otimes 2}}{\epsilon_{n_k}} \rangle_{\chi^*} \right| ds \quad < +\infty \; .$$

H2” There is $\mathcal{T} : \chi \longrightarrow \mathcal{C}([0,T])$ such that $[X]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ in probability for every $\phi \in \mathcal{S}$ and for every $t \in [0,T]$.

Then X admits a χ-quadratic variation and application $[X]$ is equal to $\mathcal{T}$.

Proof. We verify the conditions of Corollary 4.38. Conditions **H0’** and **H1** are verified by assumption. We observe that for every $\phi \in \mathcal{S}^p$, $[X]^\epsilon(\phi)$ is an increasing process. By linearity, it follows that for any $\phi \in \mathcal{S}^p$, $[X]^\epsilon(\phi)(t)$ converges in probability to $\mathcal{T}(\phi)(t)$ for any $t \in [0,T]$. Lemma 2.1 implies that $[X]^\epsilon(\phi)$ converges ucp for every $\phi \in \mathcal{S}^p$ and therefore in $\mathcal{S}$. Conditions **H2’** of Corollary 4.38 is now verified. $\square$

Chapter 5

Evaluations of χ-quadratic variations of window processes

In this section $(X_t)_{0\leq t\leq T}$ will be a real continuous process as usual prolongated by continuity and $(X_t(\cdot))_{0\leq t\leq T}$ its associated window process. We are interested in evaluations of some χ-quadratic variations for process $X(\cdot)$. In Section 5.1, $X(\cdot)$ will be considered with values in $B = C([-\tau, 0])$; in Section 5.2, $X(\cdot)$ will be considered with values in $H = L^2([-\tau, 0])$. For simplicity of exposition, we will consider in most of the cases $\tau = T$. Only when it is really necessary in view of further applications we develop computations in the general case $0 < \tau \leq T$.

5.1 Window processes with values in $C([-\tau, 0])$

In this section we set $B = C([-\tau, 0])$, $X(\cdot)$ has to be considered as a B-valued process and χ has to be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$, as listed in Example 4.7.
We start with some examples of χ-quadratic variation calculated directly through the definition.

Proposition 5.1. Let X be a real valued process with Hölder continuous paths of parameter $\gamma > 1/2$. Then $X(\cdot)$ admits a zero real and tensor quadratic variation. In particular X admits a zero global quadratic variation.

Proof. By Lemma 4.30, point 2. and by Proposition 4.25 we only need to verify the zero real quadratic variation. By Lemma 2.1 we only need to show the convergence to zero in probability of following quantity.

$$\frac{1}{\epsilon}\int_0^T \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|_B^2\, ds = \frac{1}{\epsilon}\int_0^T \sup_{u\in[-T,0]} |X_{s+u+\epsilon} - X_{s+u}|^2\, ds\ . \tag{5.1}$$

Since X is a.s. γ-Hölder continuous, then (5.1) is bounded by a sequence of random variables $Z(\epsilon)$ defined by $Z(\epsilon) := \epsilon^{2\gamma-1} Z\,T$ where Z is a non-negative finite random variable. This implies that (5.1) converges to zero a.s. for $\gamma > \frac{1}{2}$. □

Remark 5.2. By Proposition 4.27 every window process $X(\cdot)$ associated to a continuous process with Hölder continuous paths of parameter $\gamma > 1/2$ admits zero χ-quadratic variation for every Chi-subspace of $(B\hat{\otimes}_\pi B)^*$, for instance $\chi = \mathcal{M}([-T,0]^2)$.

Remark 5.3. As immediate applications of Proposition 5.1 and properties stated in Section 2.3, we obtain the following results.

1. The fractional window Brownian motion $B^H(\cdot)$ with $H > 1/2$ admits a zero real, tensor and global quadratic variation.

2. The bifractional window Brownian motion $B^{H,K}(\cdot)$ with $KH > 1/2$ admits a zero real, tensor and global quadratic variation.

Remark 5.4. We recall that a Brownian motion W has Hölder continuous paths of parameter $\gamma < 1/2$ so that we can not use Proposition 5.1.

Remark 5.5. In principle the window Brownian motion $W(\cdot)$ does not even admit an $\mathcal{M}([-T,0]^2)$-quadratic variation because the first condition is not verified. However we do not have a quite formal proof of this. Presumably the window Brownian motion $W(\cdot)$ does not admit a global quadratic variation, even though it is possible to show that the expectation

$$\lim_{\epsilon\to 0} \mathbb{E}\left[\int_0^T \frac{1}{\epsilon}\left\|W_{u+\epsilon}(\cdot) - W_u(\cdot)\right\|_B^2 du\right] = +\infty. \tag{5.2}$$

This is a consequence of the following result.

Proposition 5.6. Let W be a classical Brownian motion. Let $0 < \tau_1 < \tau_2$, then there are positive constants C_1, C_2 such that

$$C_1 \le \mathbb{E}\left[\sup_{u\in[\tau_1,\tau_2]} \frac{|W_{u+\epsilon} - W_u|^2}{\epsilon\ln(1/\epsilon)}\right] \le C_2$$

Proof. See [37]. □

The following proposition constitutes an existence result of a χ-quadratic variation calculated with the help of Corollaries 4.38 and 4.39. We remind that $\mathcal{D}_i([-\tau,0])$ and $\mathcal{D}_{i,j}([-\tau,0]^2)$ were defined at (2.26) and (2.27).

Proposition 5.7. Let X be a real continuous process with finite quadratic variation and $0 < \tau \le T$. The following properties hold true.

1) $X(\cdot)$ admits zero χ-quadratic variation, where $\chi = L^2([-\tau,0]^2)$.

2) $X(\cdot)$ admits zero χ-quadratic variation for every $i \in \{0,\ldots,N\}$, where $\chi = L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_i([-\tau,0])$.

If moreover the covariation $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ exists for a given $i,j \in \{0,\ldots,N\}$, we have the validity of the following statement.

3) $X(\cdot)$ admits χ-quadratic variation, where $\chi = \mathcal{D}_{i,j}([-\tau,0]^2)$ and it equals

$$[X(\cdot)](\mu) = \mu(\{a_i,a_j\})[X_{\cdot+a_i}, X_{\cdot+a_j}], \quad \forall \mu \in \chi. \tag{5.3}$$

Proof. The proof will be the same. Example 4.7 says that the three involved sets χ are separable Chi-subspaces of $(B\hat{\otimes}_\pi B)^*$.
Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for $L^2([-\tau,0])$; $\{f_i = \delta_{a_i}\}$ is clearly a basis for $\mathcal{D}_i([-\tau,0])$. Then $\{e_i \otimes e_j\}_{i,j\in\mathbb{N}}$ is a basis of $L^2([-\tau,0]^2)$, $\{e_j \otimes f_i\}_{j\in\mathbb{N}}$ is a basis of $L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])$ and $\{f_i\otimes f_j\}$ is a basis of $\mathcal{D}_{i,j}([-\tau,0]^2)$. We will show the results using Corollary 4.39. To verify Condition **H1** we consider

$$A(\epsilon) := \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle_{\chi^*}\right| ds$$

for the three Chi-subspaces. In all the three situations we will show the existence of a family of random variables $\{B(\epsilon)\}$ converging in probability to some random variable B, such that $A(\epsilon) \le B(\epsilon)$ a.s. By Remark 4.21.1 this will imply Assumption **H1**.

1) Suppose $\chi = L^2([-\tau,0]^2)$. By Cauchy-Schwarz inequality we have

$$\begin{aligned} A(\epsilon) &\le \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{L^2([-\tau,0]^2)}\le 1} \|\phi\|_{L^2([-\tau,0]^2)} \cdot \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|^2_{L^2([-\tau,0])} \le \\ &\le \frac{1}{\epsilon}\int_0^T\int_0^s (X_{u+\epsilon} - X_u)^2\, du\, ds \le B(\epsilon) \end{aligned}$$

where

$$B(\epsilon) = T\int_0^T \frac{(X_{u+\epsilon}-X_u)^2}{\epsilon}\, du$$

which converges in probability to $T\,[X]_T$.

2) We proceed now similarly for $\chi = L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])$.

We consider ϕ of the form $\phi = \tilde{\phi}\otimes\delta_{\{a_i\}}$, where $\tilde{\phi}$ is an element of $L^2([-\tau,0])$. We first observe

$$\|\phi\|_{L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i} = \left\|\tilde{\phi}\right\|_{L^2([-\tau,0])}\cdot\left\|\delta_{\{a_i\}}\right\|_{\mathcal{D}_i} = \sqrt{\int_{[-\tau,0]}\tilde{\phi}(s)^2\, ds}\,.$$

Then

$$
\begin{aligned}
A(\epsilon) &= \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_i}\leq 1} \left| (X_{s+\epsilon}(a_i)-X_s(a_i)) \int_{[-\tau,0]} (X_{s+\epsilon}(x)-X_s(x))\,\tilde{\phi}(x)\,dx \right| ds \leq \\
&\leq \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|\leq 1} \left(\sqrt{(X_{s+\epsilon}(a_i)-X_s(a_i))^2}\right)\cdot \\
&\qquad\qquad \cdot \left(\left\|\tilde{\phi}\right\|_{L^2([-\tau,0])} \sqrt{\int_{[-\tau,0]} (X_{s+\epsilon}(x)-X_s(x))^2\,dx} \right) ds \leq \\
&\leq \int_0^T \sqrt{\frac{(X_{s+\epsilon}(a_i)-X_s(a_i))^2}{\epsilon}} \sqrt{\int_{[-T,0]} \frac{(X_{s+\epsilon}(x)-X_s(x))^2}{\epsilon}\,dx}\,ds \leq B(\epsilon)
\end{aligned}
$$

where

$$
B(\epsilon) = \sqrt{T}\int_0^T \frac{(X_{y+\epsilon}-X_y)^2}{\epsilon}\,dy\;,
$$

sequence that converges in probability to $\sqrt{T}\,[X]_T$.

3) For the last case $\chi = \mathcal{D}_{i,j}([-\tau,0]^2)$. A general element ϕ which belongs to χ admits a representation $\phi = \lambda\,\delta_{\{(a_i,a_j)\}}$, with norm equals to $\|\phi\|_{\mathcal{D}_{i,j}} = |\lambda|$. We have

$$
\begin{aligned}
A(\epsilon) &= \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{\mathcal{D}_{i,j}}\leq 1} \left|\lambda\,(X_{s+a_i+\epsilon}-X_{s+a_i})\left(X_{s+a_j+\epsilon}-X_{s+a_j}\right)\right| ds \leq \\
&\leq \frac{1}{\epsilon}\int_0^T \left|(X_{s+a_i+\epsilon}-X_{s+a_i})\left(X_{s+a_j+\epsilon}-X_{s+a_j}\right)\right| ds
\end{aligned}
\tag{5.4}
$$

and using again Cauchy-Schwarz inequality, previous quantity is bounded by

$$
\sqrt{\int_0^T \frac{(X_{s+a_i+\epsilon}-X_{s+a_i})^2}{\epsilon}\,ds}\sqrt{\int_0^T \frac{\left(X_{s+a_j+\epsilon}-X_{s+a_j}\right)^2}{\epsilon}\,ds} \leq B(\epsilon)
\tag{5.5}
$$

where

$$
B(\epsilon) = \int_0^T \frac{(X_{s+\epsilon}-X_s)^2}{\epsilon}\,ds
$$

which converges in probability to $[X]_T$.

We verify now Conditions **H0"** and **H2"**.

1) A general element in $\{e_i\otimes e_j\}_{i,j\in\mathbb{N}}$ is difference of two positive definite elements in $\{e_i\otimes^2, (e_i+e_j)\otimes^2\}_{i,j\in\mathbb{N}}$. Therefore we set $\mathcal{S} = \{e_i\otimes e_j\}_{i,j\in\mathbb{N}}$ and $\mathcal{S}^p = \{e_i\otimes^2, (e_i+e_j)\otimes^2\}_{i,j\in\mathbb{N}}$. This implies **H0"**. It remains to verify

$$
[X(\cdot)]^\epsilon(e_i\otimes e_j)(t) \xrightarrow[\epsilon\longrightarrow 0]{} 0
\tag{5.6}
$$

in probability for any $i,j \in \mathbb{N}$ in order to conclude to the validity of Condition **H2"**. Clearly we can suppose $\{e_i\}_{i\in\mathbb{N}} \in C^1([-\tau,0])$. We fix $\omega \in \Omega$, outside some null set, fixed but omitted. We have

$$[X(\cdot)]^\epsilon(e_i \otimes e_j)(t) = \int_0^t \frac{\gamma_j(s,\epsilon)\,\gamma_i(s,\epsilon)}{\epsilon} ds \tag{5.7}$$

where

$$\gamma_j(s,\epsilon) = \int_{(-\tau)\vee(-s)}^0 e_j(y)\,(X_{s+y+\epsilon} - X_{s+y})\,dy$$

and

$$\gamma_i(s,\epsilon) = \int_{(-\tau)\vee(-s)}^0 e_i(x)\,(X_{s+x+\epsilon} - X_{s+x})\,dx\ .$$

Without restriction of generality, in the purpose not to overcharge notations, we can suppose from now on that $\tau = T$.

For every $s \in [0,T]$, we have

$$\begin{aligned} |\gamma_j(s,\epsilon)| &= \left| \int_{-s}^0 (e_j(y-\epsilon) - e_j(y))\,X_{s+y}dy + \int_0^\epsilon e_j(y-\epsilon)X_{s+y}dy - \int_{-s}^{-s+\epsilon} e_j(y-\epsilon)X_{s+y}dy \right| \le \\ &\le \epsilon \left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty \right) \sup_{s\in[0,T]} |X_s|\ . \end{aligned} \tag{5.8}$$

For $t \in [0,T]$, this implies that

$$\begin{aligned} \int_0^t \left| \frac{\gamma_j(s,\epsilon)\,\gamma_i(s,\epsilon)}{\epsilon} \right| ds &\le \int_0^T \left| \frac{\gamma_j(s,\epsilon)\,\gamma_i(s,\epsilon)}{\epsilon} \right| ds \\ &\le T\,\epsilon\, \left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty \right) \left(\int_{-T}^0 |\dot{e_i}(y)|dy + 2\|e_i\|_\infty \right) \left(\sup_{s\in[0,T]} |X_s| \right)^2 \end{aligned}$$

which trivially converges a.s. to zero when ϵ goes to zero and therefore (5.6) is established.

2) A general element in $\{e_j \otimes f_i\}_{j\in\mathbb{N}}$ is difference of two positive definite elements of type $\{e_j\otimes^2, f_i\otimes^2, (e_j + f_i)\otimes^2\}_{j\in\mathbb{N}}$. This shows **H0"**. It remains to show that

$$[X(\cdot)]^\epsilon\,(e_j \otimes f_i)\,(t) \longrightarrow 0 \tag{5.9}$$

in probability for every $j \in \mathbb{N}$. In fact the left-hand side equals

$$\int_0^t \frac{\gamma_j(s,\epsilon)}{\epsilon}\,(X_{s+a_i+\epsilon} - X_{s+a_i})\ ds\ .$$

Using estimate (5.8), we obtain

$$\int_0^t \left| \frac{\gamma_j(s,\epsilon)}{\epsilon}\,(X_{s+a_i+\epsilon} - X_{s+a_i}) \right| ds \le T \left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty \right) \varpi_X(\epsilon) \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0$$

where $\varpi_X(\epsilon)$ is the usual (random in this case) continuity modulus, so the result follows.

3) A general element $f_i \otimes f_j$ is difference of two positive definite elements $(f_i + f_j)\otimes^2$ and $f_i \otimes^2 + f_j \otimes^2$. So that Condition **H0”** is fulfilled. Concerning Condition **H2”** we have, for $0 \le i, j \le N$,

$$[X(\cdot)]^\epsilon (f_i \otimes f_j)(t) = \frac{1}{\epsilon} \int_0^t (X_{s+a_i+\epsilon} - X_{s+a_i}) \left(X_{s+a_j+\epsilon} - X_{s+a_j}\right) ds \, .$$

This converges to $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ which exists by hypothesis.

This finally concludes the proof of Proposition 5.7. □

We recall that $\mathcal{D}_d$, $\mathcal{D}_A$, χ^2, χ^0 and χ^4 were defined respectively at (2.31), (2.29), (4.5), (4.7) and (4.13).

Corollary 5.8. Let X be a real continuous process with finite quadratic variation $[X]$. Then for every $i \in \{0, \dots, N\}$, it yields

4) $X(\cdot)$ admits zero χ-quadratic variation, where $\chi = \mathcal{D}_i([-\tau, 0])\hat{\otimes}_h L^2([-\tau, 0])$.

5) $X(\cdot)$ admits a $\mathcal{D}_{i,i}([-\tau, 0]^2)$-quadratic variation which equals

$$[X(\cdot)](\mu) = \mu(\{a_i, a_i\})[X_{\cdot+a_i}, X_{\cdot+a_i}], \ \forall \mu \in \mathcal{D}_{i,i}([-\tau, 0]^2). \tag{5.10}$$

6) $X(\cdot)$ admits a $\mathcal{D}_d([-\tau, 0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[X]_{t+a_i}, \ \forall \mu \in \mathcal{D}_d([-\tau, 0]^2), t \in [0, T]. \tag{5.11}$$

7) $X(\cdot)$ admits a $\chi^0([-\tau, 0]^2)$-quadratic variation which equals

$$[X(\cdot)](\mu) = \mu(\{0, 0\})[X], \ \forall \mu \in \chi^0. \tag{5.12}$$

8) $X(\cdot)$ admits a $\chi^6([-\tau, 0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[X]_{t+a_i}, \ \forall \mu \in \chi^6, t \in [0, T]. \tag{5.13}$$

Corollary 5.9. Let X be a real continuous process such that $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ exists for all $i, j = 0, \dots, N$, in particular it is has finite quadratic variation. Then

9) $X(\cdot)$ admits a $\mathcal{D}_A([-\tau, 0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i,j=0}^{N} \mu(\{a_i, a_j\})[X_{\cdot+a_i}, X_{\cdot+a_j}]_t, \forall \mu \in \mathcal{D}_A([-\tau, 0]^2), t \in [0, T]. \tag{5.14}$$

10) $X(\cdot)$ admits a $\chi^2([-\tau,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu)=\sum_{i,j=0}^{N}\mu(\{a_i,a_j\})[X_{\cdot+a_i},X_{\cdot+a_j}]_t,\forall\mu\in\chi^2([-\tau,0]^2),t\in[0,T]. \tag{5.15}$$

Proof of Corollaries 5.8 and 5.9. The considered χ^2 admits a finite direct sum decomposition given by (4.6). Also χ^6, χ^0, $\mathcal{D}_d$ and $\mathcal{D}_A$ admit a finite direct sum decomposition by definition. Results follow immediately applying Propositions 4.26 and 5.7 $\square$

Remark 5.10. We mention a particular case of Corollary 5.9 that we will frequently meet in the sequel. Let X be a real continuous process with covariation structure such that $[X_{\cdot+a_i},X_{\cdot+a_j}]=0$ for $i\neq j$. In this case the $\chi^2([-\tau,0]^2)$-quadratic variation of $X(\cdot)$ simplifies in

$$[X(\cdot)]_t(\mu)=\sum_{i=0}^{N}\mu(\{a_i,a_i\})[X_{\cdot+a_i}]_t=\sum_{i=0}^{N}\mu(\{a_i,a_i\})[X_\cdot]_{t+a_i}\ . \tag{5.16}$$

Remark 5.11. We remark that in Corollary 5.8 the quadratic variation $[X]$ of the real finite quadratic variation process X not only insures the existence of χ-quadratic variation but completely determines the χ-quadratic variation. For example if X is a real finite quadratic variation process such that $[X]_t=t$, then $X(\cdot)$ has the same χ^0-quadratic variation as the window Brownian motion.

Now we list two corollaries of Propositions 5.7 and 4.26 that will be useful in the application to Dirichlet processes in Section 7.3.

Corollary 5.12. Let V be a real continuous zero quadratic variation process. Then the associated window process $V(\cdot)$ has zero $\mathcal{D}_A([-\tau,0]^2)$-quadratic variation. In particular the associated window process of a real bounded variation process has zero $\mathcal{D}_A([-\tau,0]^2)$-quadratic variation.

Corollary 5.13. Let V be a real continuous zero quadratic variation process. Then $V(\cdot)$ has zero $\chi^2([-\tau,0]^2)$-quadratic variation.

Proposition 5.14. Let V a real absolutely continuous process such that $V'\in L^2([0,T])$ ω-a.s. Then the associated B-valued window process $V(\cdot)$ has zero real and tensor quadratic varaition. In particular it has zero global quadratic variation.

Proof. Using Lemma 4.30 point 2. and Proposition 4.25, we only need to show the real zero quadratic variation. By Lemma 2.1 we only need to show the convergence to zero in probability of the quantity

$$\int_0^T\frac{1}{\epsilon}\left\|V_{s+\epsilon}(\cdot)-V_s(\cdot)\right\|_B^2ds=\int_0^T\frac{1}{\epsilon}\sup_{x\in[-\tau,0]}\left|V_{s+\epsilon}(x)-V_s(x)\right|^2ds; \tag{5.17}$$

in fact we will even show the a.s. convergence. Quantity (5.17) equals

$$\int_0^T\frac{1}{\epsilon}\sup_{x\in[-\tau,0]}\left|\int_{s+x}^{s+x+\epsilon}V'(y)dy\right|^2ds\leq\int_0^T\frac{1}{\epsilon}\max_{x\in[-\tau,0]}\int_{s+x}^{s+x+\epsilon}V'(y)^2dyds\leq T\,\varpi_{\int_0^\cdot(V'^2)(y)dy}(\epsilon)\xrightarrow[\epsilon\longrightarrow0]{a.s.}0,$$

since $\varpi_{\int_0^\cdot g^2(y)dy}$ denotes the modulus of continuity of the a.s. continuous function $t \mapsto \int_0^t (V'^2)(y)dy$. $\square$

Example 5.15. We list some examples of processes X fulfilling the assumptions of Corollary 5.9 or only those of Corollary 5.8. If we only know the quadratic variation but we do not know the mutual covariations $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ for $i, j \in \{0, \ldots, N\}$ we use Corollary 5.8.

1) All continuous real semimartingale S. In fact S has finite quadratic variation and it holds $[S_{\cdot+a_i}, S_{\cdot+a_j}] = 0$ for $i \neq j$, see Proposition 2.12.

2) In particular if X is the Brownian motion W. In fact $[W]_t = t$ and $[W_{\cdot+a_i}, W_{\cdot+a_j}] = 0$ for $i \neq j$ because W is a semimartingale.

3) Consider a bifractional Brownian motion $B^{H,K}$ with parameters H and K.

   Proposition 5.16. Let $B^{H,K}$ be a Bifractional Brownian motion with $HK = 1/2$. Then $[B^{H,K}] = 2^{1-K}t$ and $[B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] = 0$ for $i \neq j$.

   Remark 5.17.

   - If $K = 1$, then $H = 1/2$ and $B^{H,K}$ is a Brownian motion, case already treated.
   - In the case $K \neq 1$ we recall that the bifractional Brownian motion $B^{H,K}$ is not a semimartingale, see Proposition 6 from [56].

   Proof of Proposition 5.16. Proposition 1 in [56] says that $B^{H,K}$ has finite quadratic variation which is equal to $[B^{H,K}] = 2^{1-K}t$. By Proposition 1 and Theorem 2 in [44] there are two constants α and β depending on K, a centered Gaussian process $X^{H,K}$ with absolutely continuous trajectories on $[0, +\infty[$ and a standard Brownian motion W such that $\alpha X^{H,K} + B^{H,K} = \beta W$. Then

$$[\alpha X^{H,K}_{\cdot+a_i} + B^{H,K}_{\cdot+a_i}, \alpha X^{H,K}_{\cdot+a_j} + B^{H,K}_{\cdot+a_j}] \quad = \quad \beta^2 [W_{\cdot+a_i}, W_{\cdot+a_j}]. \tag{5.18}$$

   Using the bilinearity of the covariation, we expand the left-hand side in (5.18) into a sum of four terms

$$\alpha^2 [X^{H,K}_{\cdot+a_i}, X^{H,K}_{\cdot+a_j}] + \alpha [B^{H,K}_{\cdot+a_i}, X^{H,K}_{\cdot+a_j}] + \alpha [X^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] + [B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] \tag{5.19}$$

   Since $X^{H,K}$ has bounded variation then first three terms on (5.19) vanish because of point 1) of Proposition 2.14. On the other hand term the right-hand side in (5.18) is equal to zero for $i \neq j$ since W is a semimartingale, see point 1). We conclude that $[B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] = 0$ for $i \neq j$. $\square$

4) Let D be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $D = M + A$, M local martingale and A zero quadratic variation process. Then D satisfies the hypotheses of the Corollary 5.9, in

particular of Remark 5.10. In fact $[D]_t = [M]_t$ and $[D_{\cdot+a_i}, D_{\cdot+a_j}] = 0$ for $i \neq j$. Consequently the associated window Dirichlet process admits a χ^2-quadratic variation.
More details about Dirichlet processes and their properties will be given in section 7.3. Examples of finite quadratic variation weak Dirichlet processes are provided in Section 2 of [24].

5) Let X be a $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $X = W + A$, W being a $(\mathcal{F}_t)$-Brownian motion and the process A which is $(\mathcal{F}_t)$-adapted with $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale N. Moreover we suppose that A is a finite quadratic variation process. Then X is an example of finite quadratic variation process in fact $[X] = [W] + [A]$. However the covariations $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ are not determined. This is an example where we only can use Corollary 5.8 but not Corollary 5.9.

We will show now that, under the same assumptions of Corollary 5.9, a finite quadratic variation process X admits a $Diag([-\tau,0]^2)$-quadratic variation. This $Diag([-\tau,0]^2)$-quadratic variation will be used in Example 8.3 about the application of Itô's formula to the window process associated to a finite quadratic variation process.

Proposition 5.18. Let $0 < \tau \leq T$. Let X be a real continuous process with finite quadratic variation $[X]$. Then $X(\cdot)$ admits a $Diag([-\tau,0]^2)$-quadratic variation, where $Diag([-\tau,0]^2)$ was defined in (2.32). Moreover we have

$$[X(\cdot)] : Diag([-\tau,0]^2) \longrightarrow \mathcal{C}([0,T])$$

given by

$$\mu \mapsto [X(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} g(-x)[X]_{t-x}dx \qquad\qquad t \in [0,T] \,, \tag{5.20}$$

where μ is a generic element in $Diag([-\tau,0]^2)$ of type $\mu(dx,dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau,0])$.

Remark 5.19. Taking into account the usual convention $[X]_t = 0$ for $t < 0$, the process $\left(\int_0^{t\wedge\tau} g(-x)[X]_{t-x}dx\right)_{t\geq 0}$ can also be written as $\left(\int_0^{\tau} g(-x)[X]_{t-x}dx\right)_{t\geq 0}$.

Proof. We recall that for a generic element μ we have $\|\mu\|_{Diag} = \|g\|_\infty$.
First we verify Condition **H1**. We can write

$$\frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{Diag}\leq 1} \left|\langle \mu, (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes^2 \rangle\right| ds \leq \frac{1}{\epsilon}\int_0^T \sup_{\|g\|_\infty\leq 1} \left|\int_{-T}^0 g(x)\left(X_{s+\epsilon}(x) - X_s(x)\right)^2 dx\right| ds =$$
$$= \int_0^T \sup_{\|g\|_\infty\leq 1} \left|\int_0^s \frac{\left(X_{x+\epsilon} - X_x\right)^2}{\epsilon} g(x-s)\, dx\right| ds \leq T[X]_T,$$

and Condition **H1** is verified by Remark 4.21.1.
It remains to prove Condition **H2**. Using Fubini's theorem, we write

$$
\begin{aligned}
[X(\cdot)]^{\epsilon}_{t}(\mu) &= \frac{1}{\epsilon}\int_0^t \langle \mu(dx,dy), (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes^2 \rangle \, ds = \\
&= \frac{1}{\epsilon}\int_0^t \int_{[-\tau,0]^2} \left(X_{s+\epsilon}(x) - X_s(x)\right)\left(X_{s+\epsilon}(y) - X_s(y)\right) g(x)\delta_x(dy)dx\, ds = \\
&= \frac{1}{\epsilon}\int_0^t \int_{[-\tau,0]} \left(X_{s+\epsilon}(x) - X_s(x)\right)^2 g(x)dx\, ds = \\
&= \int_{(-t)\vee(-\tau)}^{0} g(x) \int_{-x}^{t} \frac{\left(X_{s+x+\epsilon} - X_{s+x}\right)^2}{\epsilon} ds\, dx = \\
&= \int_{(-t)\vee(-\tau)}^{0} g(x) \int_{0}^{t+x} \frac{\left(X_{s+\epsilon} - X_{s}\right)^2}{\epsilon} ds\, dx = \\
&= \int_{0}^{t\wedge\tau} g(-x) \int_{0}^{t-x} \frac{\left(X_{s+\epsilon} - X_{s}\right)^2}{\epsilon} ds\, dx\, .
\end{aligned}
\tag{5.21}
$$

It remains to show the ucp convergence,

$$
\left(\int_0^{t\wedge\tau} g(-x) \int_0^{t-x} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon} ds\, dx \right)_{t\in[0,T]} \xrightarrow[\epsilon\longrightarrow 0]{ucp} \left(\int_0^{t\wedge\tau} g(-x)[X]_{t-x}\, dx \right)_{t\in[0,T]}
$$

i.e.

$$
\sup_{t\leq T} \left| \int_0^{t\wedge\tau} g(-x) \int_0^{t-x} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon} ds - [X]_{t-x}\, dx \right| \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0
\tag{5.22}
$$

The left-hand side of (5.22) is bounded by

$$
\begin{aligned}
\int_0^T |g(-x)| \sup_{t\in[0,T]} \left| \int_0^{t-x} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon} ds - [X]_{t-x} \right| dx &\leq \int_0^T |g(-x)| \sup_{t\in[0,T]} \left| \int_0^{t} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon} ds - [X]_{t} \right| dx \leq \\
&\leq T\, \|g\|_\infty \sup_{t\in[0,T]} \left| \int_0^{t} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon} ds - [X]_{t} \right| .
\end{aligned}
$$

Since X is a finite quadratic variation process, previous expression converges to zero.
More explicitly we obtain

$$
[X(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} g(-x)[X]_{t-x}\, dx = \begin{cases} \displaystyle\int_0^t g(-x)[X]_{t-x}dx & 0 \leq t \leq \tau \\ \displaystyle\int_0^\tau g(-x)[X]_{t-x}dx & \tau < t \leq T \end{cases} .
$$

Previous expression has an obvious modification $\widetilde{[X(\cdot)]}$ which has finite variation with value in χ^*. The total variation is in fact easily dominated by $\tau[X]_T$. $\square$

Example 5.20. For the sake of further calculations we illustrate a direct application of Proposition 5.18.

1. Suppose that $[X]$ is absolutely continuous with $A_t = \frac{d[X]_t}{dt}$. For $\mu \in Diag([-\tau,0]^2)$, $\mu(dx,dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau,0])$, we have

$$\int_0^T \langle \mu, d\widetilde{[X(\cdot)]}\rangle_t = \int_0^T \int_0^{t\wedge\tau} g(-x)A_{t-x}dx\, dt\ .$$

2. In particular if $A \equiv 1$, as for standard Brownian motion,

$$\int_0^T \langle \mu, d\widetilde{[X(\cdot)]}\rangle_t = \int_0^T \int_0^{t\wedge\tau} g(-x)dx\, dt\ . \tag{5.23}$$

Direct consequences of Propositions 5.9, 5.8, 5.18 and 4.26 are the two corollaries below.

Corollary 5.21. Let $0 < \tau \leq T$ and X be a real continuous process such that $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ exists for all $i,j \in \{0,\dots,N\}$. Then $X(\cdot)$ admits a $\chi^3([-\tau,0]^2)$-quadratic variation where $\chi^3([-\tau,0]^2) = \chi^2([-\tau,0]^2) \oplus Diag([-\tau,0]^2)$. The $\chi^3([-\tau,0]^2)$-quadratic variation is

$$[X(\cdot)]_t(\mu) = \sum_{i,j=0}^N \mu_2(\{a_i,a_j\})[X_{\cdot+a_i}, X_{\cdot+a_j}]_t + \int_0^{t\wedge\tau} g(-x)[X]_{t-x}dx$$

where $\mu = \mu_1 + \mu_2$ is a generic element of χ^3 with $\mu_1 \in L^2([-\tau,0]^2), \mu_2 \in Diag([-\tau,0]^2)$ of type $\mu_2(dx,dy) = g_2(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau,0])$.

Corollary 5.22. Let $0 < \tau \leq T$ and X be a real continuous process with finite quadratic variation $[X]$. Then $X(\cdot)$ admits a $\mathcal{D}_d([-\tau,0]^2) \oplus Diag([-\tau,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i=0}^N \mu_1(\{a_i,a_i\})[X_{\cdot+a_i}]_t + \int_0^{t\wedge\tau} g_2(-x)[X]_{t-x}dx, t \in [0,T],$$

where $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \mathcal{D}_d([-\tau,0]^2)$, μ_2 is a generic element in $Diag([-\tau,0]^2)$ with associated g_2 in $L^\infty([-\tau,0])$.

We go on with other evaluations of χ-quadratic variations. We first recall that $\chi^5([-T,0]^2)$ was defined at (4.14).

Proposition 5.23. Let X be a real continuous process admitting $[X_{\cdot+\alpha_i}, X_{\cdot+\alpha_j}]$ for every $i,j \in \mathbb{N}$. Then $X(\cdot)$ admits a $\chi^5([-T,0]^2)$-quadratic variation equals to

$$[X(\cdot)]_t(\mu) = \sum_{i,j\in\mathbb{N}} \mu(\{\alpha_i,\alpha_j\})[X_{\cdot+\alpha_i}, X_{\cdot+\alpha_j}]_t\ , \tag{5.24}$$

where μ is a general element in χ^5 which can be written $\mu = \sum_{i,j\in\mathbb{N}} \lambda_{i,j}\delta_{(\alpha_i,\alpha_j)}$.

Proof. Obviously χ^5 is a separable Chi-subspace of $(B\hat{\otimes}_\pi B)^*$, so we make use of Corollary 4.39. We recall that for a general $\mu \in \chi^5$ the norm is $\|\mu\|_{\chi^5} = \sup_{i,j}\{|\lambda_{i,j}|i^2j^2\}$, so

$$\frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{\chi^5}\leq 1} \left|\langle \mu, (X_{s+\epsilon}(\cdot)-X_s(\cdot))\otimes^2\rangle\right| ds =$$

$$\begin{aligned}
&= \frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{\chi^5}\leq 1}\left|\sum_{i,j\in\mathbb{N}} \lambda_{i,j}(X_{s+\alpha_i+\epsilon}-X_{s+\alpha_i})(X_{s+\alpha_j+\epsilon}-X_{s+\alpha_j})\right| ds = \\
&= \int_0^T \sup_{\|\mu\|\leq 1}\left|\sum_{i,j\in\mathbb{N}} \lambda_{i,j} i^2 j^2 \frac{(X_{s+\alpha_i+\epsilon}-X_{s+\alpha_i})(X_{s+\alpha_j+\epsilon}-X_{s+\alpha_j})}{\epsilon i^2 j^2}\right| ds \leq \\
&\leq \sum_{i,j\in\mathbb{N}} \frac{1}{i^2j^2}\sqrt{\int_0^T \frac{(X_{s+\alpha_i+\epsilon}-X_{s+\alpha_i})^2}{\epsilon}ds}\sqrt{\int_0^T \frac{(X_{s+\alpha_j+\epsilon}-X_{s+\alpha_j})^2}{\epsilon}ds} \leq \\
&\leq \sum_{i,j\in\mathbb{N}} \frac{1}{i^2j^2}\int_0^T \frac{(X_{s+\epsilon}-X_s)^2}{\epsilon}ds = \\
&= \left(\frac{\pi^2}{6}\right)^2 \int_0^T \frac{(X_{s+\epsilon}-X_s)^2}{\epsilon}ds \xrightarrow{\mathbb{P}} \frac{\pi^4}{36}[X]_T\ .
\end{aligned}$$

Condition **H1** follows by using Remark 4.21.1.
We set $\mathcal{S} = \{\delta_{(\alpha_i,\alpha_i)},\}_{i,j\in\mathbb{N}}$ and $\mathcal{S}^p = \{\delta_{\alpha_i}\otimes^2, \left(\delta_{\alpha_i}+\delta_{\alpha_j}\right)\otimes^2\}_{i,j\in\mathbb{N}}$ and **H0"** is verified. Also Condition **H2"** can be proved; in fact for every element in $\mathcal{S}$ we have

$$\int_0^t \frac{(X_{s+\alpha_i+\epsilon}-X_{s+\alpha_i})(X_{s+\alpha_j+\epsilon}-X_{s+\alpha_j})}{\epsilon}ds \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} [X_{\cdot+\alpha_i}, X_{\cdot+\alpha_j}]_t\ .$$

As announced the result follows by Corollary 4.39. □

In the next examples, the knowledge of the whole covariation structure of the process is needed. We remind to (4.15) for the definition of $\chi^6([-\tau,0]^2)$.

Proposition 5.24. Let X be a real continuous process with given covariation structure $([X_{\cdot+x}, X_{\cdot+y}],\ x,y\in[-\tau,0])$, in particular X has finite quadratic variation. Then $X(\cdot)$ admits a $\chi^6([-\tau,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \int_{[-\tau,0]^2}[X_{\cdot+x}, X_{\cdot+y}]_t\mu(dx,dy) = \sum_{i=1}^N \lambda_i\int_{[-\tau,0]^2}[X_{\cdot+x}, X_{\cdot+y}]_t\mu_i(dx,dy)\ , \tag{5.25}$$

where μ is a general element in $\chi^6([-\tau,0]^2)$ which can be written as $\mu = \sum_{i=1}^N \lambda_i\mu_i$, i.e. μ is a linear composition of N fixed measures $(\mu_i)_{i=1,\ldots,N}$ with total variation 1.

Proof. χ^6 is obviously a separable Chi-subspace of $(B\hat{\otimes}_\pi B)^*$, and we make again use of Corollary 4.39.

We verify **H1**. Recalling that $\|\mu\|^2_{\chi^6} = \sum_{i=1}^N \lambda_i^2$, it yields

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{\chi^6}\le 1} \left|\langle \mu, (X_{s+\epsilon}(\cdot)-X_s(\cdot))\otimes^2\rangle\right| ds = \\
&= \int_0^T \sup_{\|\mu\|_{\chi^6}\le 1} \left|\int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon}-X_{s+x})(X_{s+y+\epsilon}-X_{s+y})}{\epsilon}\mu(dx,dy)\right| ds = \\
&= \int_0^T \sup_{\|\mu\|^2_{\chi^6}\le 1} \left|\sum_{i=1}^N \lambda_i \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon}-X_{s+x})(X_{s+y+\epsilon}-X_{s+y})}{\epsilon}\mu_i(dx,dy)\right| ds \le \\
&\le \int_0^T \sum_{i=1}^N \sup_{\sum \lambda_i^2\le 1} \left\{|\lambda_i| \left|\int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon}-X_{s+x})(X_{s+y+\epsilon}-X_{s+y})}{\epsilon}\mu_i(dx,dy)\right|\right\} ds\ .
\end{aligned}$$

Since $|\lambda_i|\le 1$ for every i, Fubini's theorem and Cauchy-Schwarz inequality imply that previous quantity is less or equal than

$$\begin{aligned}
&\sum_{i=1}^N \int_0^T \int_{[-\tau,0]^2} \frac{|(X_{s+x+\epsilon}-X_{s+x})(X_{s+y+\epsilon}-X_{s+y})|}{\epsilon}|\mu_i|(dx,dy)ds = \\
&= \sum_{i=1}^N \int_{[-\tau,0]^2}\int_0^T \frac{|(X_{s+x+\epsilon}-X_{s+x})(X_{s+y+\epsilon}-X_{s+y})|}{\epsilon} ds\, |\mu_i|(dx,dy) \le \\
&\le \sum_{i=1}^N \int_{[-\tau,0]^2} \sqrt{\int_0^T \frac{(X_{s+x+\epsilon}-X_{s+x})^2}{\epsilon}ds}\sqrt{\int_0^T \frac{(X_{s+y+\epsilon}-X_{s+y})^2}{\epsilon}ds}\, |\mu_i|(dx,dy) \le \\
&\le \sum_{i=1}^N \int_{[-\tau,0]^2}\int_0^T \frac{(X_{s+\epsilon}-X_s)^2}{\epsilon}ds\, |\mu_i|(dx,dy) \xrightarrow{\mathbb{P}} [X]_T \sum_{i=1}^N \int_{[-\tau,0]^2} |\mu_i|(dx,dy) = \\
&= [X]_T \sum_{i=1}^N |\mu_i|([-\tau,0]^2) < +\infty\ a.s..
\end{aligned}$$

By Remark 4.21.1, Condition **H1** is verified.
Since the signed measure μ_i can be decomposed into differences of positive and negative components μ_i^+ and μ_i^-, setting $\mathcal{S}=\{\mu_i\}_{i\in\{1,\dots,N\}}$ and $\mathcal{S}^p=\{\mu_i^+,\mu_i^-\}_{i\in\{1,\dots,N\}}$ then **H0"** is verified. To verify Condition **H2"** we consider a fixed positive measure μ with unitary total variation. For any $t\in[0,T]$ we need to prove:

$$\int_0^t \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon}-X_{s+x})(X_{s+y+\epsilon}-X_{s+y})}{\epsilon}\mu(dx,dy)\, ds \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \mu(dx,dy)\ . \quad (5.26)$$

Let ϵ_n be a sequence converging to zero. It will be enough to show that (5.26) holds for $\epsilon=\epsilon_{n_k}$, when $k\to+\infty$ and (n_k) is a subsequence.

Let $\delta > 0$. Using Fubini's theorem, convergence (5.26) we have to verify that

$$\mathbb{P}\left[\int_{[-\tau,0]^2} \left|\gamma_t^{\epsilon_{n_k}}(x,y) - [X_{\cdot+x}, X_{\cdot+y}]_t\right| |\mu|(dx,dy) \; > \delta\right] \xrightarrow[k\longrightarrow+\infty]{} 0\,, \tag{5.27}$$

where

$$\gamma_t^\epsilon(x,y) = \int_0^t \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} ds\;.$$

Since

$$|\gamma_t^\epsilon(x,y)| \le \int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon} ds\;,$$

we consider (n_k) such that

$$\int_0^T \frac{(X_{s+\epsilon_{n_k}} - X_s)^2}{\epsilon_{n_k}} ds$$

converges a.s. We set

$$Z := \sup_k \int_0^T \frac{(X_{s+\epsilon_{n_k}} - X_s)^2}{\epsilon_{n_k}} ds\;.$$

Clearly

$$[X_{\cdot+x}, X_{\cdot+y}]_t \le Z \qquad a.s. \qquad \forall\, x,y \in [-\tau, 0]\;.$$

Let M be a positive number; the left-hand side of (5.27) is bounded by

$$\begin{aligned}
&\mathbb{P}\left[\int_{[-\tau,0]^2} \left|\gamma_t^{\epsilon_{n_k}}(x,y) - [X_{\cdot+x}, X_{\cdot+y}]_t\right| |\mu|(dx,dy) \; > \delta \quad ; \quad Z \le M\right] + \mathbb{P}\left[Z \ge M\right] \le \\
&\le \frac{1}{\delta}\mathbb{E}\left[\int_{[-\tau,0]^2} \left|\gamma_t^{\epsilon_{n_k}}(x,y) - [X_{\cdot+x}, X_{\cdot+y}]_t\right| \cdot 1\!\!1_{\{Z\le M\}} \cdot |\mu|(dx,dy)\right] + \mathbb{P}\left[Z \ge M\right] = \\
&= \frac{1}{\delta}\int_{[-\tau,0]^2} \mathbb{E}\left[\left|\gamma_t^{\epsilon_{n_k}}(x,y) - [X_{\cdot+x}, X_{\cdot+y}]_t\right| \cdot 1\!\!1_{\{Z\le M\}}\right] |\mu|(dx,dy) + +\mathbb{P}\left[Z \ge M\right]\;.
\end{aligned} \tag{5.28}$$

Now

$$\left|\gamma_t^{\epsilon_{n_k}}(x,y) - [X_{\cdot+x}, X_{\cdot+y}]_t\right| \cdot 1\!\!1_{\{Z\le M\}} \le 2M$$

and

$$\gamma_t^{\epsilon_{n_k}}(x,y) \xrightarrow[k\longrightarrow+\infty]{\mathbb{P}} [X_{\cdot+x}, X_{\cdot+y}]_t \qquad \text{for every } x,y \in [-\tau,0]\;.$$

Consequently by uniform integrability the same sequence converges in $L^1(\Omega)$ i.e.

$$\mathbb{E}\left[\left|\gamma_t^{\epsilon_{n_k}}(x,y) - [X_{\cdot+x}, X_{\cdot+y}]_t\right| \cdot 1\!\!1_{Z\le M}\right] \xrightarrow[k\longrightarrow+\infty]{} 0\;.$$

By Lebesgue dominated convergence theorem (5.28) converges to $\mathbb{P}[Z \geq M]$. This show that the

$$\limsup_{k\longrightarrow+\infty} \mathbb{P}\left[\int_{[-\tau,0]^2} \left|\gamma_t^{\epsilon_{n_k}}(x,y) - [X_{\cdot+x}, X_{\cdot+y}]_t\right| |\mu|(dx,dy) \ > \delta\right] \leq \mathbb{P}[Z \geq M] \ .$$

Setting $M \longrightarrow +\infty$ previous lim sup vanishes. Convergence (5.27) follows and therefore also (5.26). As announced the result is established by Corollary 4.39. □

Remark 5.25. As a particular case of Proposition 5.24 we consider the case when X is a real continuous process with finite quadratic variation $[X]$ and covariation structure such that $[X_{\cdot+x}, X_{\cdot+y}] = 0$ for $x \neq y$. Then the $\chi^6([-\tau,0]^2)$-quadratic variation of $X(\cdot)$ equals

$$[X(\cdot)]_t(\mu) = \int_{[-\tau,0]^2} [X]_{t+x} \mathbb{1}_D(x,y)\mu(dx,dy) \tag{5.29}$$

where μ is a general element $\chi^6([-\tau,0]^2)$ and $D = \{(x,y) \in [-\tau,0]^2; x = y\}$ is the diagonal of the square $[-\tau,0]^2$.

Another significant example is the following. Let μ be a fixed positive, finite measure on $[-\tau,0]^2$; μ could be for instance singular with respect to Lebesgue measure. We recall that notation χ^μ has been introduced at (4.16).

Proposition 5.26. Let μ be a given positive, finite measure on $[-\tau,0]^2$ and X be a real process admitting a covariation structure $([X_{\cdot+x}, X_{\cdot+y}], x, y \in [-\tau,0])$. Then $X(\cdot)$ admits a χ^μ-quadratic variation which equals, for a measure $d\nu = g\, d\mu$ with $g \in L^\infty(d\mu)$ and

$$[X(\cdot)]_t(\nu) = \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \nu(dx,dy) \ . \tag{5.30}$$

Proof. Concerning **H1**, we write

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^T \sup_{\|\nu\|_{\chi^\mu}\leq 1} \left|\langle \nu, (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle\right| ds = \\
&\qquad = \int_0^T \sup_{\|g\|_{L^\infty(d\mu)}\leq 1} \left| \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} g(x,y)\mu(dx,dy)\right| ds \leq \\
&\qquad = \int_0^T \int_{[-\tau,0]^2} \left|\frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon}\right| |\mu|(dx,dy) ds \leq \\
&\qquad \leq \int_{[-\tau,0]^2} \int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon} ds\, |\mu|(dx,dy) \xrightarrow{\mathbb{P}} [X]_T\, |\mu|([-\tau,0]^2),
\end{aligned}$$

which is an a.s. finite random variable. So **H1** is established via Remark 4.21.1. Concerning **H2**, writing $g = g^+ - g^-$ it will be enough to show that (4.28) converges in probability for any $t \in [0,T]$. That

convergence follows similarly to the proof of Proposition 5.24.
For every $d\nu = g\,d\mu$, i.e. $\nu(dx,dy) = g(x,y)\mu(dx,dy)$ we are able to show

$$\int_0^t \int_{[\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} \nu(dx,dy)\,ds \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} \cdot \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \nu(dx,dy)$$

Point (ii) in condition **H2** can be easily verified showing that

$$t \mapsto \left(\nu \mapsto \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \nu(dx,dy) \right)$$

has bounded variation as a $\chi^{\mu *}$-valued function. In particular it is easy to show that its total variation is bounded by $[X]_T^2 \mu(([-T,0]^2)$. □

5.2 Window processes with values in $L^2([-\tau,0])$

Let $(X_t)_{0\le t\le T}$ be again a real continuous process. In this section we consider its window processes $(X_t(\cdot))_{0\le t\le T}$ as process with values in the Hilbert space $H = L^2([-\tau,0])$. Below, we will compute some χ-quadratic variations, χ belonging to a class of Chi-subspaces of $(H\hat{\otimes}_\pi H)^*$, as listed in Example 4.12. We start with a preliminary result.

Proposition 5.27. Let X be a real continuous process with finite quadratic variation $[X]_t = t$. Then $X(\cdot)$ admits a real quadratic variation in the sense of Definition 4.1.1 and $[X(\cdot)]^{\mathbb{R}}_t = \int_0^{t\wedge\tau} (t-x)\,dx,\ t\in[0,T]$.

Proof. We have to show that

$$\frac{1}{\epsilon}\int_0^t \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|_H^2\,ds \xrightarrow[\epsilon\to 0]{ucp} \begin{cases} \dfrac{t^2}{2} & 0 \le t \le \tau \\ \tau\left(t - \dfrac{\tau}{2}\right) & \tau < t \le T \end{cases} . \tag{5.31}$$

Since the real processes appearing in the left-hand side of (5.31) are increasing, by Lemma 2.1 it will be enough to show convergence in (5.31) in probability for every fixed $t\in[0,T]$. We have in fact

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^t \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|_H^2\,ds = \frac{1}{\epsilon}\int_0^t \int_{-\tau}^0 (X_{s+r+\epsilon} - X_{s+r})^2\,dr\,ds = \\
&= \int_0^{t\wedge\tau}\int_{-s}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\,ds + \int_{t\wedge\tau}^t \int_{-\tau}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\,ds = \\
&= \begin{cases} \displaystyle\int_0^t\int_{-s}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\,ds \xrightarrow[\epsilon\to 0]{\mathbb{P}} \frac{t^2}{2} & 0\le t\le\tau \\ \displaystyle\int_0^\tau\int_{-s}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\,ds + \int_\tau^t\int_{-\tau}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\,ds \xrightarrow[\epsilon\to 0]{\mathbb{P}} \frac{\tau^2}{2} + \tau(t-\tau) & \tau < t\le T \end{cases} .
\end{aligned}$$

□

Remark 5.28. Let X be a real continuous process with finite quadratic variation $[X]_t = t$. As consequences of Proposition 5.27 we have the following.

1. Condition **H1** for existence of global quadratic variation of $X(\cdot)$ is verified. By Remark 4.28, it follows that Condition **H1** for existence of χ-quadratic variation of $X(\cdot)$ is even verified for any Chi-subspace χ of $(H\hat{\otimes}_\pi H)^*$.

2. If we could show that $X(\cdot)$ has a tensor quadratic variation, then by Proposition 4.25, we would know that $X(\cdot)$ admits a global quadratic variation.

3. However, we are not able to prove the existence of a global quadratic variation because we can not prove Condition **H2**, i.e. that there exists an application $[X(\cdot)]$, such that $[X(\cdot)]^\epsilon(T) \xrightarrow[\epsilon\longrightarrow 0]{ucp} [X(\cdot)](T)$ for every $T \in (H\hat{\otimes}_\pi H)^* \cong \mathcal{B}(H,H)$. Nevertheless we have an expression of this limit for some particular $T \in (H\hat{\otimes}_\pi H)^*$. For instance if we fix the bilinear bounded operator $T : H \times H \to \mathbb{R}$, defined by $(h,g) \mapsto T(h,g) = \langle h,g\rangle_H$ we can show that $[X(\cdot)]^\epsilon(T) \xrightarrow[\epsilon\longrightarrow 0]{ucp} [X(\cdot)](T)$ where $[X(\cdot)](T)$ is exactly the real quadratic variation calculated at Proposition 5.27.

If X is a zero quadratic variation process, then the situation for $X(\cot)$ is clearer and simpler.

Corollary 5.29. Let X be a real continuous process with zero quadratic variation $[X] = 0$. Then $X(\cdot)$ admits zero real, tensor and global quadratic variation.

Proof. The result follows immediately by Lemma 4.30 point 2. and Proposition 5.27. ☐

We keep in mind the definitions of $L^2_{\mathcal{B}}([-\tau,0]^2)$ and $Diag_{\mathcal{B}}([-\tau,0]^2)$ given respectively in Definition 4.13 and in (4.22) and we recall that they are Chi-subspaces of $(H\hat{\otimes}_\pi H)^*$.

Proposition 5.30. Let X be a real continuous process with finite quadratic variation. We have the following.

1. $X(\cdot)$ admits zero $L^2_{\mathcal{B}}([-\tau,0]^2)$-quadratic variation.

2. $X(\cdot)$ admits a $Diag_{\mathcal{B}}([-\tau,0]^2)$-quadratic variation which equals, for every $T^f \in Diag_{\mathcal{B}}([-\tau,0]^2)$,

$$[X(\cdot)]_t(T^f) = \int_0^{t\wedge\tau} f(x)[X]_{t-x}dx \qquad t \in [0,T] \tag{5.32}$$

remembering that $[X]_u = 0$ for $u < 0$. In particular that quadratic variation is non zero.

Proof.

1. The proof follows the same lines as the one of Proposition 5.7 where we have evaluated the $L^2([-\tau,0]^2)$-quadratic variation of $X(\cdot)$ considered as $C([-\tau,0])$-valued process.

2. The proof is again vary similar to the one of Proposition 5.18 where we have evaluated the $Diag([-\tau,0]^2)$-quadratic variation of $X(\cdot)$ considered as $C([-\tau,0])$-valued process.

$\square$

Remark 5.31. We recall that $H = L^2([-\tau,0])$, so $H\hat{\otimes}_\pi H$ is densely embedded into $H\hat{\otimes}_h H$ because of (2.17). $H\hat{\otimes}_h H$ is the Hilbert space identified canonically with $L^2([-\tau,0]^2)$ or $L^2_{\mathcal{B}}([-\tau,0]^2)$ and $(H\hat{\otimes}_\pi H)^*$ is the Banach space identified canonically with $\mathcal{B}(H,H)$.
Let $(e_i)_{i\in\mathbb{N}}$ an orthonormal basis of H. We consider $T_n = \sum_{i=1}^n e_i \otimes e_i$ as en element of $(H\hat{\otimes}_h H)^* \subset (H\hat{\otimes}_\pi H)^*$. We also define $T \in (H\hat{\otimes}_\pi H)^*$ through the relation $T(h,f) = \langle h, f\rangle_H$.

1. By (2.13) we have the norm inequality $\|\cdot\|_{(H\hat{\otimes}_h H)^*} \geq \|\cdot\|_{(H\hat{\otimes}_\pi H)^*}$. However those norms are not equivalent.
   In fact, it holds $\|T_n\|^2_{(H\hat{\otimes}_h H)^*} = n$. On the other hand, let h and f in H; $T_n(h,f) = \sum_{i=1}^n \langle h, e_i\rangle\langle f, e_i\rangle = \sum_{i=0}^n \langle h\otimes f, e_i \otimes e_i\rangle$. So $|T_n(h,f)| \leq \sqrt{\sum_{i=1}^n \langle h, e_i\rangle^2 \sum_{j=1}^n \langle f, e_j\rangle^2} = \|h\|\,\|f\|$, where the last equality comes by Parseval's identity. Then $\|T_n\|_{(H\hat{\otimes}_\pi H)^*} = \|T_n\|_{\mathcal{B}} = \sup_{\|h\|,\|f\|\leq 1} |T_n(h,f)| \leq 1$.

2. The sequence T_n weakly converges to T as element of $(H\hat{\otimes}_\pi H)^*$ for the following reasons.
   - For h, f in H, we have $T_n(h,f) \xrightarrow[n\longrightarrow+\infty]{} T(h,f)$. In fact
   $$T(h,f) = \langle h, f\rangle_H = \sum_{i=0}^{\infty} \langle h, e_i\rangle\langle f, e_i\rangle = \lim_{n\to+\infty} \sum_{i=0}^{n} \langle h\otimes f, e_i\otimes e_i\rangle = \lim_{n\to+\infty} T_n(h,f)\ .$$
   - Since $\|T_n\|_{(H\hat{\otimes}_\pi H)^*} \leq 1$, for any $\phi \in H\hat{\otimes}_\pi H$, the sequence $(T_n(\phi))_n$ is obviously bounded by $\|\phi\|_{H\hat{\otimes}_\pi H}$.

   By Banach-Steinhaus theorem is follows that $T_n(\phi) \xrightarrow[n\longrightarrow+\infty]{} T(\phi)$ for any $\phi \in H\hat{\otimes}_\pi H$.

3. The sequence T_n does not converge strongly to T as element of $(H\hat{\otimes}_\pi H)^*$.
   In fact the sequence (T_n) is not Cauchy. For $m, n \in \mathbb{N}$, $m > n$, for h, f in H we have
   $$(T_n - T_m)(h,f) = \sum_{i=m+1}^{n} \langle e_i \otimes e_i, h\otimes f\rangle_{(H\hat{\otimes}_h H)}\ .$$
   Taking $h = f = e_n$, previous quantity equals 1 so that $\|T_n - T_m\|_{(H\hat{\otimes}_\pi H)^*} = 1$

Proposition 5.32. With previous conventions $(H\hat{\otimes}_h H)^*$ is not densely embedded in $(H\hat{\otimes}_\pi H)^*$.

Proof. We two arguments: a first one probabilistic and the second one analytical.

1. Let $W(\cdot)$ be a window Brownian motion considered with values in H. Point 1 of Proposition 5.30 says that $W(\cdot)$ has zero $(H\hat{\otimes}_h H)^*$-quadratic variation. We suppose ab absurdo that $(H\hat{\otimes}_h H)^*$ is

densely embedded in $(H\hat{\otimes}_\pi H)^*$. We recall by Remark 5.28 1. that Condition **H1** for the existence of global quadratic variation for $W(\cdot)$ is always verified. Setting $\mathcal{S} = (H\hat{\otimes}_h H)^*$, Conditions **H0'** and **H2'** of Corollary 4.38 are verified. Consequently $W(\cdot)$ has a global quadratic variation $[W(\cdot)]$. Since the quadratic variation $[W(\cdot)] : (H\hat{\otimes}_\pi H)^* \longrightarrow \mathcal{C}([0,T])$ is continuous, it must be identically zero. This contradicts Point 2 of the same Proposition 5.30.

2. Proposition 4.15 says that $Diag_{\mathcal{B}}([-\tau, 0]^2)$ is a closed subspace of $\mathcal{B}(H,H)$ then $L^2_{\mathcal{B}}([-\tau, 0]^2)$ can not be densely embedded in $\mathcal{B}(H,H)$.

$\square$

Chapter 6

Link with quadratic variation concepts in the literature

In this section we will investigate the link with other definitions of quadratic variation for a B-valued process X. Our approach extends at least three classical notions of quadratic variation.
The first treated case will be the quadratic variation defined by [24] for a $\mathbb{R}^n$-valued process, which generalizes the notion of quadratic variation of multi-dimensional semimartingales. The quadratic variation defined there, is a $\mathbb{M}_{n\times n}(\mathbb{R})$-valued process denoted by $[X^*, X]$, see Definition 2.6; that matrice is constituted by all the mutual covariations of vector X. The second case will be the quadratic variation, denoted by $[X]^{dz}$, of a martingale X with values in a separable Hilbert space H defined in [13]. In this definition $[X]^{dz}$ is a $L^1(H)$-valued process, i.e. a nuclear operator on the space H. The third one is the tensor quadratic variation, see Definition 4.1.2, denoted by $[X]^{\otimes}$ which is very closed to the concept defined by Pellaumail and Metivier in [46] and similarly by Dinculeanu in [19] . Those authors consider Banach valued processes, which are practically semimartingales. We recall that $[X]^{\otimes}$ is a bounded variation process with values in $(B\hat{\otimes}_\pi B)$.

For each one of those cases we will show that if the B-valued process X admits a quadratic variation $[X^*, X]$ (respectively $[X]^{dz}$ or $[X]^{\otimes}$) then X admits a global quadratic variation, (i.e. a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$), with $B = \mathbb{R}^n$ (respectively $B = H$, separable Hilbert space and B general Banach space). Moreover, the global quadratic variation and each one of classical quadratic variation concept will be essentially identified.
For the first case, we will establish an equivalence between $\mathbb{M}_{n\times n}(\mathbb{R})$ and $(\mathbb{R}^n \otimes \mathbb{R}^n)$ which allows us to identify $[X^*, X]$ and $\widetilde{[X]}$. For the second case, we establish a correspondence between the set of nuclear operators $L^1(H)$ and the projective tensor product $(H\hat{\otimes}_\pi H)$ and $[X]^{dz}$ will be identified to $\widetilde{[X]}$, but this will be delicate. For the last case we refer essentially to Proposition 4.25 which identifies $[X]^{\otimes}$ and $[X]$.

Indeed, when B is a Hilbert space, the pairing duality between $B\hat{\otimes}_\pi B$ and its dual $(B\hat{\otimes}_\pi B)^*$ coincides with the trace pairing duality, see Proposition 6.2, point 3. when B is finite dimensional and more generally Proposition 6.12 when $B = H$ is a separable Hilbert space.

We emphasize that with respect to the classical quadratic variation concepts, the χ-quadratic variation introduces two levels of generalization. First, in the classical cases χ always equals the full space $(B\hat{\otimes}_\pi B)^*$; second, our quadratic variation $\widetilde{[X]}$ takes values in χ^* therefore in the bidual space $(B\hat{\otimes}_\pi B)^{**}$ instead of $(B\hat{\otimes}_\pi B)$ as it happens in [46, 19].

6.1 The finite dimensional case

We begin this section recalling some notions about the finite dimensional case $B = \mathbb{R}^n$ and the quadratic variation in the sense of [24]. The duality between tensor product $\mathbb{R}^n \otimes \mathbb{R}^m$ and its dual will be associated with the trace of an operator (matrix). The second order term in Itô's formula involving quadratic variation will be linked, as for Itô's calculus, to the integral of a trace.

We first remind some notions about tensor products of finite dimensional spaces and integrals as well as covariations in a multidimensional setting. The algebraic tensor product $\mathbb{R}^n \otimes \mathbb{R}^m$ is complete with respect to every possible norm α, in particular with respect to any reasonnable one; so it coincides with $\mathbb{R}^n \otimes_\alpha \mathbb{R}^m$. Therefore it is a Hilbert space (therefore reflexive) with finite dimension $n \times m$. It exists a canonical identification between $\mathbb{R}^n \otimes \mathbb{R}^m$ (respectively its dual space $(\mathbb{R}^n \otimes \mathbb{R}^m)^*$) and the space of real matrix of dimension $n \times m$, $\mathbb{M}_{n\times m}(\mathbb{R})$ (respectively the space $\mathbb{M}_{m\times n}(\mathbb{R})$).

Let $(e_i)_{1\leq i\leq n}$, $(f_j)_{1\leq j\leq m}$ be the canonical basis for $\mathbb{R}^n$ and $\mathbb{R}^m$. Every element $u \in \mathbb{R}^n \otimes \mathbb{R}^m$ of the form $u = \sum_{1\leq i\leq n;1\leq j\leq m} u_{i,j}\ e_i \otimes e_j$ is associated to a unique matrix $U = (u_{i,j})_{1\leq i\leq n,1\leq j\leq m}$, $U \in \mathbb{M}_{n\times m}(\mathbb{R})$. Conversely given a matrix $U \in \mathbb{M}_{n\times m}(\mathbb{R})$ of the form $U = (U_{i,j})_{1\leq i\leq n,1\leq j\leq m}$, it is associated to a unique element $u \in \mathbb{R}^n \otimes \mathbb{R}^m$ of the form $u = \sum_{1\leq i\leq n;1\leq j\leq m} U_{i,j}\ e_i \otimes e_j$. Concerning the dual space, we recall, from the preliminaries, that $(\mathbb{R}^n \otimes \mathbb{R}^m)^* \cong L(\mathbb{R}^n; L(\mathbb{R}^m))$ which is naturally identified with $\mathbb{M}_{m\times n}(\mathbb{R})$. So a matrix $T \in \mathbb{M}_{m\times n}(\mathbb{R})$ of the form $T = (T_{i,j})_{1\leq i\leq m,1\leq j\leq n}$ is associated with the linear form $t : \mathbb{R}^n \otimes \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $t(x \otimes y) = {}_{\mathbb{R}^m}\langle Tx\ , y\rangle_{\mathbb{R}^m}$.

For a general matrix $A = (A_{i,j})_{i\in I,j\in J}$, $A_{\cdot,j}$ ($A_{i,\cdot}$ respectively) will denote the j-th column of the matrix A (the i-th row of the matrix A respectively).

In this section we will show that the quadratic variation in the sense of [24, 58, 32], whenever it exists, i.e. when X has all its mutual covariations, coincides with the global quadratic variation. Moreover we will show that the duality pairing between an element $t \in (\mathbb{R}^n \otimes_\pi \mathbb{R}^m)^*$ (or simply $(\mathbb{R}^n \otimes \mathbb{R}^m)^*$) and an element $u \in (\mathbb{R}^n \otimes_\pi \mathbb{R}^m)$ (or simply $(\mathbb{R}^n \otimes \mathbb{R}^m)$, denoted by $t(u)$ or even by $\langle t, u\rangle$, coincides with the trace $Tr(TU)$ of the matrix TU, whenever U is the $\mathbb{M}_{n\times m}(\mathbb{R})$ matrix associated with u and T is the $\mathbb{M}_{m\times n}(\mathbb{R})$ matrix associated with t. For this task we express integrals and covariations in a multidimensional setting. If Y is a $m\times n$ matrix of continuous processes $(Y^{i,j})_{1\leq i\leq n,1\leq j\leq m}$, and A is a $m\times d$ matrix $(A^{j,k})_{1\leq j\leq m,1\leq k\leq d}$

then $[Y, A]_t$ is the $n \times d$ matrix constituted by the following ucp limit (if it exists)

$$\frac{1}{\epsilon}\int_0^t (Y_{s+\epsilon} - Y_s)(A_{s+\epsilon} - A_s)ds \; . \tag{6.1}$$

If $X = (X^1, \cdots, X^n)$, $Y = (Y^1, \cdots, Y^m)$ such that (X, Y) (resp. X) has all its mutual covariations, we denote by $[X^*, Y]$ the $n \times m$ matrix defined by $([X^*, Y])_{1\le i\le n; 1\le j\le m} = [X^i, Y^j]$ and $[X^*, X]$ is a $n \times n$ matrix defined by $([X^*, X])_{1\le i,j\le n} = [X^i, X^j]$.

Proposition 6.1. Let u (resp. t) be element of $(\mathbb{R}^n \otimes \mathbb{R}^m)$ (resp. in $(\mathbb{R}^n \otimes \mathbb{R}^m)^*$) and U (resp. T) be the corresponding matrix in $\mathbb{M}_{n\times m}(\mathbb{R})$ (resp. in $\mathbb{M}_{m\times n}(\mathbb{R})$). Then

$$Tr(TU) = \langle t, u\rangle = t(u) = \sum_{1\le i\le n, 1\le j\le m} U_{i,j}T_{j,i} \tag{6.2}$$

Proof. Let $(f_j)_{1\le j\le m}$ be the canonical basis for $\mathbb{R}^m$. The left-hand side in (6.2) equals

$$\begin{aligned} Tr(TU) &= \sum_{j=1}^m \langle TU(f_j), f_j\rangle_{\mathbb{R}^m} = \sum_{j=1}^m \langle T(U_{\cdot,j}), f_j\rangle_{\mathbb{R}^m} = \sum_{j=1}^m \langle U_{\cdot,j}, T^*(f_j)\rangle_{\mathbb{R}^n} = \\ &= \sum_{j=1}^m \langle U_{\cdot,j}, T_{j,\cdot}\rangle_{\mathbb{R}^n} = \sum_{j=1}^m \sum_{i=1}^n U_{i,j}T_{j,i} \end{aligned} \tag{6.3}$$

because it is well-known that the adjoint of a matrix coincides with its transposed so $T^*(f_j) = T_{j,\cdot}$. Concerning the right-hand side of (6.2) we have

$$\begin{aligned} \langle t, u\rangle = t(u) &= t\left(\sum_{1\le i\le n, 1\le j\le m} U_{i,j} e_i \otimes f_j\right) = \sum_{1\le i\le n, 1\le j\le m} U_{i,j} t\left(e_i \otimes f_j\right) = \\ &= \sum_{1\le i\le n, 1\le j\le m} U_{i,j}\langle T(e_i), f_j\rangle_{\mathbb{R}^m} = \sum_{1\le i\le n, 1\le j\le m} U_{i,j}\langle T_{\cdot,i}, f_j\rangle_{\mathbb{R}^m} = \\ &= \sum_{j=1}^m \sum_{i=1}^n U_{i,j}T_{j,i} \; . \end{aligned}$$

The proof is now concluded. □

Proposition 6.2. Let $X = (X^1, \cdots, X^n)$ be an $\mathbb{R}^n$-valued process.

1. The following properties are equivalent.
   (a) X has all its mutual covariations.
   (b) X admits a real and tensor quadratic variation in the sense of Definition 4.1.
   (c) X admits a global quadratic variations.
2. If one of the three previous properties holds, the following statements are valid.

(a) The tensor quadratic variation $[X]^{\otimes}$ coincides with the element in the tensor product associated to the matrix $[X^*, X]$.

(b) The real quadratic variation $[X]^{\mathbb{R}}$ coincides with $[X, X^*]$.

(c) $\widetilde{[X]}(\cdot, t) = [X]^{\otimes}_t$.

(d) Let H be an $M_{n\times n}(\mathbb{R})$-valued continuous process and $H^{\otimes}$ be the element in the tensor product associated to H. Then, setting $B = \mathbb{R}^n$,

$$\int_0^t Tr\left(H(\cdot, s) \cdot d[X^*, X]_s\right) = \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle H^{\otimes}(\cdot, s), d\widetilde{[X]}_s\rangle_{B\hat{\otimes}_\pi B} \ . \tag{6.4}$$

Proof. We observe that point 2. (a) is a consequence of the natural identification between a matrix and tensor product. The proof of points 2.(b) and (c) will naturally appear as side-effect of point 1 proof. So we go on with the proof of the equivalences in point 1.

1. (a)$\Rightarrow (b)$. In order to show the existence of the real quadratic variation we need to show the ucp convergence of following sequence

$$\int_0^{\cdot} \frac{\|X_{s+\epsilon} - X_s\|^2_{\mathbb{R}^n}}{\epsilon} ds = \sum_{i=1}^n \int_0^{\cdot} \frac{\left(X^i_{s+\epsilon} - X^i_s\right)^2}{\epsilon} ds \ . \tag{6.5}$$

By hypothesis X has all its mutual covariations, so in particular every term in the sum converges ucp to $[X^i, X^i]$. Consequently (6.5) converges to

$$\sum_{i=1}^n [X^i] = [X, X^*] = [X]^{\mathbb{R}}$$

which gives the real quadratic variation. This also establishes point 2. (b).
By identification between $\mathbb{R}^n \otimes \mathbb{R}^n$ and $M_{n\times n}(\mathbb{R})$ we have the following

$$\int_0^{\cdot} \frac{(X_{s+\epsilon} - X_s) \otimes^2}{\epsilon} ds = \int_0^{\cdot} \frac{(X_{s+\epsilon} - X_s)^* \, (X_{s+\epsilon} - X_s)}{\epsilon} ds \ . \tag{6.6}$$

In order to show now the existence of the tensor quadratic variation we need only to show the ucp convergence of the right-hand side of (6.6) which is a matrix valued sequence with component $1 \le i, j, \le n$ equals to

$$\int_0^{\cdot} \frac{(X^i_{s+\epsilon} - X^i_s)(X^j_{s+\epsilon} - X^j_s)}{\epsilon} ds. \tag{6.7}$$

(6.7) converges by hypothesis and this forces the convergence of (6.6) because the convergence in $M_{n\times n}(\mathbb{R})$ is equivalent to the convergence of every component.

1. (b)$\Rightarrow (c)$ This is a consequence of Proposition 4.25. In particular we also get $\widetilde{[X]} = [X]^{\otimes}$ a.s. which also shows point 2. (c).

1. (c)⇒(a) Let $(e_i^* \otimes e_j^*)$ be the canonical basis of $(\mathbb{R}^n \otimes \mathbb{R}^n)^*$. By Condition **H2** i), (4.28) holds true for every fixed ϕ in $(\mathbb{R}^n \otimes \mathbb{R}^n)^*$, in particular setting $\phi = e_i^* \otimes e_j^*$ for all $1 \leq i, j \leq n$. Consequently

$$\int_0^{\cdot} \frac{(X_{s+\epsilon}^i - X_s^i)(X_{s+\epsilon}^j - X_s^j)}{\epsilon} ds = [X]^{\epsilon}(e_i^* \otimes e_j^*)$$

converges ucp and X has all its mutual covariations.

It remains to show the identity in point 2. (d) for fixed $\omega \in \Omega$. By (6.3) and the classical characterization of traces for matrices, the left-hand side of (6.4) can be developed as a finite sum of well defined Lebesgue-Stieltjes integrals; therefore

$$\int_0^t Tr\left(H(\cdot, s) \cdot d[X^*, X]_s\right) = \sum_{i,j=1}^n \int_0^t H_{i,j}(\cdot, s) d[X^j, X^i]_s. \tag{6.8}$$

The last equality in Proposition 6.1, the duality in a finite tensor product and the corresponding canonical identification say that (6.8) equals

$$\int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle H^{\otimes}(\cdot, s), d\widetilde{[X]}_s\rangle_{B\hat{\otimes}_\pi B}.$$

□

Corollary 6.3. Let S be a an $(\mathcal{F}_t)$-semimartingale with values in $\mathbb{R}^n$. Then S admits a global quadratic variation.

Proof. According to Remark 2.9, point 3. S admits all its mutual covariations $[S^*, S]$. The result follows using Proposition 6.2. □

6.2 The quadratic variation in the sense of Da Prato and Zabczyk

In this section, we will adopt the same notations as in Section 3.3.1, where we gave a short presentation of the Da Prato-Zabczyk stochastic integral.

Let H and F be two separable Hilbert spaces with complete orthonormal basis $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$, I and J countable sets. Let also H^* be the topological dual space of H with canonical complete orthonormal basis $\{e_i^*\}_{i \in \mathbb{I}}$ defined by $(e_i^*)(e_j) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker's delta, i.e. $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.
X will denote a H-valued continuous stochastic process. The principal goal will be to recover the quadratic variation given by G. Da Prato and J. Zabczyk, denoted by $[X]^{dz}$, in our framework. In general $[X]^{dz}$ is a stochastic process with values in the space of nuclear operators $L^1(H)$. In Section 6.2.1 we will establish a link between the language of some classes of operators (as Hilbert-Schmidt operators, nuclear operators and trace class operators) and tensor products. In particular Propositions 6.6 and 6.7 will identify the

space of nuclear operator $L^1(H)$ with the space $H\hat{\otimes}_\pi H$. We will also recall the so called *approximation property* in a general Banach space and some important consequences in tensor products theory. In the following sections we will show that, if X admits a quadratic variation $[X]^{dz} \in L^1(H)$, then it admits a global quadratic variation denoted by $[X]$. Moreover $\widetilde{[X]}$ will be exactly the element in $H\hat{\otimes}_\pi H$ associated to the nuclear operator valued quadratic variation $[X]^{dz}$. This identification will be made step by step following the construction of a stochastic integral made in [13]. In this section capital letters will denote operators and small letters will denote tensor products.

6.2.1 Nuclear and Hilbert-Schmidt operators, approximation property

For more details about this part the reader may refer to Appendix C in [13], Chapter 6 in [47] and Chapter 4 in [63]. We recall the definitions of nuclear and Hilbert-Schmidt operators.

If E and G are Banach spaces, E^* and G^* will denote their duals and $L(E;G)$ will be the Banach space of all linear bounded operators from E into G endowed with the usual operator norm, simply denoted by $\|\cdot\|$. An element $T \in L(E;G)$ is said to be a **nuclear operator** if there exist two sequences $(a_j) \in G$ and $(\phi_j) \in E^*$ such that $\sum_{j=1}^{\infty} \|a_j\|\,\|\phi_j\| < \infty$ and T has the representation $Tx = \sum_{j=1}^{\infty} a_j\,\phi_j(x)$ for every $x \in E$. The space of all nuclear operators from E to G, endowed with the norm

$$\|T\|_1 = \inf\left\{\sum_{j=1}^{\infty} \|a_j\|\,\|\phi_j\| \;:\; Tx = \sum_{j=1}^{\infty} a_j\,\phi_j(x)\right\}$$

is a Banach space and will be denoted by $L^1(E;G)$.
It is well-known that if $T \in L^1(E;G)$ then its adjoint $T^* \in L^1(G^*;E^*)$, furthermore $\|T^*\|_1 \le \|T\|_1$ and if $S \in L(F;E)$ and $T \in L^1(E;G)$ then $TS \in L^1(F;G)$ and $\|TS\|_1 \le \|T\|_1\,\|S\|$.
If H is a separable Hilbert space, $T \in L^1(H;H)$ then the trace of T, defined by $Tr(T) = \sum_{\in I}\langle T(e_i), e_i\rangle$ is a well-defined number independent of the choice of the orthonormal basis $\{e_i\}_{i\in I}$ and $|Tr(T)| \le \|T\|_1$. Moreover a nonnegative operator $T \in L(H;H)$ (non-negativity means $\langle T(f), f\rangle \ge 0$ for any $f \in H$) is nuclear if and only if for an orthonormal basis $\{e_i; i \in I\}$ on H we have $\sum_{i=1}^{\infty}\langle T(e_i), e_i\rangle < \infty$; in this case, we have $Tr(T) = \|T\|_1$. $L^1(H)$ will also be a shortened symbol for $L^1(H;H)$.

An element $T \in L(H;F)$ is said to be a **Hilbert-Schmidt operator** if $\sum_{i\in I} \|T(e_i)\|_F^2 < \infty$. This definition is independent of the choice of the basis. The space of all Hilbert-Schmidt operators from H to F equipped with the norm

$$\|T\|_2 = \left(\sum_{i\in I} \|T(e_i)\|_F^2\right)^{1/2}$$

is a separable Hilbert space with scalar product $\langle S, T\rangle = \sum_{i\in I}\langle S(e_j), T(e_j)\rangle$. It is easy to show that if

$T \in L^2(H;F)$ then its adjoint operator $T^* \in L^2(F^*;H^*)$ and $\|T^*\|_2 = \|T\|_2$. In fact $\sum_{i\in I} \|T(e_i)\|_F^2 = \sum_{i\in I}\sum_{j\in J}\langle T(e_i), f_j\rangle^2 = \sum_{i\in I}\sum_{j\in J}\langle e_i, T^*(f_j)\rangle^2 = \sum_{j\in J}\|T^*(f_j)\|_{H^*}^2$. Similarly as for $L^1(H)$, $L^2(H)$ will stand for $L^2(H;H)$.

Given two separable Hilbert spaces H, F, another important property is the following. A linear operator T is nuclear, i.e. belongs to $L^1(H;F)$, if and only if it exists a third Hilbert space G and a factorization $T = UV$ such that $U \in L^2(G;F)$ and $V \in L^2(H;G)$; in this case, $\|T\|_1 = \inf\{\|U\|_2\,\|V\|_2\}$ over all the possible factorizations of the operator T.

Let $T \in L^1(H;F)$ be a nuclear operator among separable Hilbert spaces. According to the definition of nuclear operator there exists two sequences $(h_j^*) \in H^*$ and $(f_j) \in F$ such that $\sum_{j=1}^{\infty}\|h_j^*\|\,\|f_j\| < \infty$ and T has the representation $Tx = \sum_{j=1}^{\infty} h_j^*\, f_j(x)$ for every $x \in H$. We denote by t the element $t \in H^*\hat{\otimes}_\pi F$ defined by $t = \sum_{j=1}^{\infty} h_j^* \otimes f_j$. The tensor element t will be called the **nuclear representation** of the nuclear operator T.

We aim at characterizing tensor products of two Hilbert spaces in terms of classes of operators. For a complete presentation of tensor products of two Hilbert spaces the reader may refer to chapter six of [47]. To this extent, the main point concerns the identification of the Hilbert tensor product $H\hat{\otimes}_h F$ with the space of Hilbert-Schmidt operators $L^2(H;F^*)$ and the identification of the projective tensor product Banach space $H\hat{\otimes}_\pi F$, which is a subspace of $H\hat{\otimes}_h F$, with the space of nuclear operator $L^1(H,F^*)$, which is a subspace of $L^2(H;F^*)$. In particular when $H = F$ using the identification of the Riesz's representation theorem we will have $H\hat{\otimes}_h H \cong L^2(H;H^*) \cong L^2(H)$ and $H\hat{\otimes}_\pi H \cong L^1(H;H^*) \cong L^1(H)$. (We will see that the identification above is true in a more general case, i.e. every time that H has approximation property).

If $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are respectively orthonormal basis of H and F, then $\{e_i \otimes f_j; (i,j) \in I \times J\}$ is an orthonormal basis for $H\hat{\otimes}_h F$. Since $I \times J$ is countable then also $H\hat{\otimes}_h F$ is a separable Hilbert space equipped with the Hilbert tensor norm $h(\cdot)$, see Section 2.5. This means that a general $u \in H\hat{\otimes}_h F$ has a representation $u = \sum_{i\in I, J\in J} u_{i,j}\, e_i \otimes f_j$ with $\sum_{i\in I, j\in J}|u_{i,j}|^2 < \infty$ and in particular we have $h(u)^2 = \sum_{i\in I, j\in J}|u_{i,j}|^2$.

The isomorphism between $H\hat{\otimes}_h F$ and $L^2(H;F^*)$ is identified as follows. To every $u \in H\hat{\otimes}_h F$ corresponds a unique Hilbert-Schmidt operator $U \in L^2(H;F^*)$ such that ${}_{F^*}\langle U(h), f\rangle_F = \langle h \otimes f, u\rangle_{H\hat{\otimes}_h F}$ for all $h \in H$ and $f \in F$. Moreover it holds $\|U\|_2^2 = \|u\|_{H\hat{\otimes}_h F}^2 = h(u)^2 < \infty$ for every orthonormal basis $\{e_i\}_{i\in I}$ of H. Conversely every Hilbert-Schmidt operator $V \in L^2(H;F^*)$ is associated with an element $v \in H\hat{\otimes}_h F$. If $H = F$ the element $u \in H\hat{\otimes}_h H$ is symmetric, i.e. $\langle h \otimes g, u\rangle = \langle g \otimes h, u\rangle$, if and only if the associated operator U is selfadjoint. We remark that this characterization coincides with the definition of Hilbert-Schmidt operators given by Neveu in [47]. $H\hat{\otimes}_h F$ can also be identified with $L^2(F;H^*)$ via the association $u \mapsto U^*$, where U^* is the adjoint of U. We remind that $\|U^*\|_2 = \|U\|_2$.

The range of previous identifications $u \mapsto U$ restricted to $H\hat{\otimes}_\pi F$ coincides with $L^1(H; F^*)$. This provides an isomorphism of Banach spaces, i.e. in particular $\pi(u) = \|u\|_{H\hat{\otimes}_\pi F} = \|U\|_1$. $H\hat{\otimes}_\pi F$ is in fact the subset of $H\hat{\otimes}_h F$ such that if $u = \sum_{I\times J} u_{i,j} e_i \otimes f_j$ it holds $\pi(u) = \sum_{i\in I, j\in J} |u_{i,j}| < +\infty$.

In the particular case when $H = F$, the element *Trace* is by definition the unique element of $(H\hat{\otimes}_\pi H)^*$ verifying $h \otimes g \longrightarrow \langle h, g\rangle$. By continuity and bilinearity, *Trace* is extended to all $H\hat{\otimes}_\pi H$ and is still denoted by the same symbol. For an element $u \in H\hat{\otimes}_\pi H$ with representation $u = \sum_m h_m \otimes f_m$ it holds $\mathit{Trace}(u) = \sum_m \langle h_m, f_m\rangle_H$ and $\mathit{Trace}(u) \le \pi(u)$. We observe that $\mathit{Trace}(u) = Tr(U)$.

Let $u \in H\hat{\otimes}_h F$ of the form $u = \sum_{i,j} u_{ij} e_i \otimes f_j$. We summarize previous considerations through the following table. The equivalents between the second and third column is given through Riesz's isometry between F and F^*.

$$\begin{array}{ccccc}
H\hat{\otimes}_h F & \cong & L^2(H;F^*) & \cong & L^2(H;F) \\
\|u\|^2_{H\hat{\otimes}_h F} = \sum_{i,j} u_{ij}^2 = h(u)^2 & = & \|U\|_2^2 & = & \|\tilde{U}\|_2 \\
\cup & & \cup & & \cup \\
H\hat{\otimes}_\pi F & \cong & L^1(H;F^*) & \cong & L^1(H;F) \\
\|u\|_{H\hat{\otimes}_\pi F} = \sum_{i,j} |u_{i,j}| = \pi(u) & = & \|U\|_N = \|U\|_1 & = & \|\tilde{U}\|_1
\end{array}$$

If $H = F$ we can define the trace

$$\begin{array}{ccccc}
H\hat{\otimes}_\pi H & \cong & L^1(H;H^*) & \cong & L^1(H) \\
Tr(u) = \sum_{i,i} u_{i,i} & & & & Tr(\tilde{U}) = \sum_i \langle \tilde{U}(e_i), e_i\rangle_H
\end{array}$$

We introduce now the concept of approximation property of the theory of tensor product of Banach spaces For more details the reader can refer to chapter 4 in [63]. By the definition 4.1 on [63]

Definition 6.4. A Banach space is said to have the **approximation property** if, for every Banach space Y, every bounded linear operator $T : X \longrightarrow Y$, every compact subset K of X and every $\epsilon > 0$, there exists a finite rank operator $S : X \longrightarrow Y$ such that $\|Tx - Sx\| \le \epsilon$ for every $x \in K$.

If X is an (infinite dimensional) Banach space with the approximation property it is possible to approximate any bounded linear mapping $T : X \to Y$ on each compact subset K of X with a finite rank operator for every Banach space Y.

Remark 6.5. The following statements hold.

1. Every Banach space having a Schauder basis (for example a separable Hilbert space) has the approximation property.

2. If X^* has approximation property so does X.

3. Some examples of spaces with this property are $L^p(\mu)$ with $p \in [1,\infty)$, c_0, l^p, $C(K)^* = \mathcal{M}(K)$, $C(K)$, $(L^\infty(\mu))^*$, $L^\infty(\mu)$ for a given σ-finite measure.

4. If X^* or Y has approximation property then $X^*\hat{\otimes}_\pi Y = L^1(X;Y)$, see Proposition 4.9 in [63]. If $t \in X^*\hat{\otimes}_\pi Y$ with a nuclear representation $t = \sum_{m=1}^\infty \phi_m \otimes y_m$ we can associate a nuclear operator $T : X \to Y$ such that by $T(x) = \sum_{m=1}^\infty \phi_m(x) y_m$ for every $x \in X$. If $X = Y = H$ with H separable Hilbert space, $T \in H^*\hat{\otimes}_\pi H$, with the trace operator characterized by $Tr(T) := \sum_{m=1}^\infty \phi_m(y_m)$ and $H^{**}\hat{\otimes}_\pi H = H\hat{\otimes}_\pi H = L^1(H^*;H)$.

In the proposition below we show that if we consider an element $a \in \left(H\hat{\otimes}_\pi H\right)^*$ with its correspondent operator $A \in L(H;H^*)$ and $b \in H\hat{\otimes}_\pi H$ with its correspondent operator $B \in L^1(H^*;H)$ then the duality in the projective tensor product space corresponds to the trace of the product AB. This fact can be useful at the moment of establishing Itô's formula developing $F(X)$ where X is a H valued process and $F \in C^2(H)$. In our language in the second order term will appear the second order derivative D^2F as an element of $\left(H\hat{\otimes}_\pi H\right)^*$ and the quadratic variation $\widetilde{[X]}$ as an element of $H\hat{\otimes}_\pi H$. In the language of Da Prato-Zabczyk, the duality pairing between $(H\hat{\otimes}_\pi H)^*$ and $H\hat{\otimes}_\pi H$ corresponds to the trace of the corresponding operators.

Proposition 6.6. Let $a \in \left(H\hat{\otimes}_\pi H\right)^*$ and the associated $A \in L(H;H^*)$ such that and $\langle a, x\otimes y\rangle = {}_{H^*}\langle A(x), y\rangle_H$. Let $b \in H\hat{\otimes}_\pi H$ of the form $b = \sum_{j=1}^\infty c_j \otimes d_j$ and the associated $B \in L^1(H^*;H)$ such that $B(h^*) = \sum_{j=1}^\infty \langle c_j, h^*\rangle d_j$ for every $h^* \in H^*$.
Then $BA \in L^1(H;H)$ and $Tr(BA) = {}_{(H\hat{\otimes}_\pi H)^*}\langle a, b\rangle_{H\hat{\otimes}_\pi H}$.

Proof. $BA \in L^1(H;H)$ because of considerations at the beginning of Section 6.2.1. We will show the result for $b = x\otimes y$ and the associated $B \in L^1(H^*;H)$ such that $B(h^*) ={}_H \langle x, h^*\rangle_{H^*} y$ for every $h^* \in H^*$. We have

$$Tr(BA) = \sum_n \langle BAe_n, e_n\rangle = \sum_n \langle {}_H\langle x, Ae_n\rangle_{H^*} y, e_n\rangle$$

because $BA(e_n) = B(Ae_n)$ and $A(e_n) \in H^*$. The equality ${}_H\langle x, A(e_n)\rangle_{H^*} = \langle a, e_n \otimes x\rangle$ implies

$$Tr(BA) = \sum_n \langle A(e_n), x\rangle\langle y, e_n\rangle = \sum_n \langle a, e_n\otimes x\rangle\langle y, e_n\rangle = \sum_{n,m} \langle x, e_m\rangle\langle a, e_n\otimes x\rangle\langle y, e_n\rangle$$

because $x = \sum_m \langle x, e_m\rangle e_m$. On the other hand

$$\langle a, b\rangle = \langle a, x\otimes y\rangle = \langle a, \sum_n \langle x, e_n\rangle e_n \otimes \sum_m \langle y, e_m\rangle e_m\rangle = \sum_{n,m} \langle x, e_m\rangle\langle a, e_n\otimes x\rangle\langle y, e_n\rangle\ .$$

So the result follows for $b = x\otimes y$. Since a and A are linear and continuous, the result is extended to all $b \in H\hat{\otimes}_\pi H$ by density. $\square$

In order to make the link with Da Prato-Zabczyk framework, previous proposition has to be read identifying H and H^* through Riesz's representation theorem.

Proposition 6.7. Let $\phi : H \to H^*$ be that Riesz's isomorphism

1. We identify $L(H; H^*)$ with $L(H)$ in the following way: to $A \in L(H; H^*)$ we associate $\tilde{A} \in L(H)$ by $\tilde{A}(h) := \phi^{-1}(A(h))$ for every $h \in H$.

2. We identify $L^1(H^*; H)$ with $L^1(H)$ in the following way: to $B \in L^1(H^*; H)$ we associate $\tilde{B} \in L^1(H)$ by $\tilde{B}(h) := B(\phi(h))$ for every $h \in H$.

3. If a is the element in $\left(H\hat{\otimes}_\pi H\right)^*$ associated to A and b is the element in $H\hat{\otimes}_\pi H$ associated to B, we also have $Tr(\tilde{B}\tilde{A}) = Tr(BA) = {}_{(H\hat{\otimes}_\pi H)^*}\langle a, b\rangle_{H\hat{\otimes}_\pi H}$.

Proof. We only have to prove first equality in 3. In fact, using the definitions of $\tilde{A}$ and $\tilde{B}$ given in 1. and 2. we obtain

$$Tr(\tilde{B}\tilde{A}) = \sum_n \langle \tilde{B}\tilde{A}e_n, e_n\rangle = \sum_n \langle \tilde{B}\phi^{-1}Ae_n, e_n\rangle = \sum_n \langle B\phi\phi^{-1}Ae_n, e_n\rangle = \sum_n \langle BAe_n, e_n\rangle = Tr(BA) \ .$$

□

In the sequel of this chapter we will definitely identify $H\hat{\otimes}_\pi H \cong L^1(H)$. If $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of a Hilbert space H we denote by $(e_i^*)_{i\in\mathbb{N}}$ the orthonormal basis of H^* such that $e_j^*(e_i) = \delta_{i,j}$ We remind that $(H\hat{\otimes}_h H)$ is an Hilbert separable space with basis $(e_i \otimes e_j)_{i,j\in\mathbb{N}}$. Again $\{(e_i \otimes e_j)^*\}_{i,j\in\mathbb{N}}$ denotes the canonical basis for $(H\hat{\otimes}_h H)^*$.

Corollary 6.8. For every $i, j \in \mathbb{N}$, let $(e_i \otimes e_j)^*$ be an element of the basis $(H\hat{\otimes}_h H)^*$. Then $(e_i \otimes e_j)^* = e_i^* \otimes e_j^*$.

Proof. By definition it holds $\langle (e_i \otimes e_j)^*, e_m \otimes e_n\rangle = \delta_{i,m}\,\delta_{j,n}$. On the other hand the element $e_i^* \otimes e_j^*$ belongs to the dual space $(H\hat{\otimes}_h H)^*$ and by properties of Hilbert tensor product it holds $\langle e_i^* \otimes e_j^*, e_m \otimes e_n\rangle = \langle e_i^*, e_m\rangle\,\langle e_j^*, e_n\rangle = \delta_{i,m}\,\delta_{j,n}$. Then $\{e_i^* \otimes e_j^*\}_{i,j\in\mathbb{N}}$ is the canonical basis for $(H\hat{\otimes}_h H)^*$. □

6.2.2 The case of a Q-Brownian motion, Q being a trace class operator

Let W be a Q Brownian motion, where Q is a trace class operator in H. We will show that W admits a global quadratic variation $[W]$ that we can identify with $[W]^{dz}$. We recall that from Section 3.3.1 that the Da Prato-Zabczyk quadratic variation of a Q-Wiener process on H with $Tr(Q) < +\infty$ is given by the formula $[W]_t^{dz} = t\,Q$.

Proposition 6.9. Let $Q \in L^1(H)$ (respectively $\Phi \in L(H)$) and let q be the element in $H\hat{\otimes}_\pi H$ associated to Q (respectively ϕ be the element in $(H\hat{\otimes}_\pi H)^*$ associated to Φ).
Let W be a Q-Brownian motion. The following statements hold.

1. W admits a global quadratic quadratic variation $\widetilde{[W]}(\cdot,t) = tq$ a.s.
   Moreover for every $t \in [0,T]$, $\widetilde{[W]}(\cdot,t)(\phi) = [W](\phi)(t) = t\, {}_{H\hat{\otimes}_\pi H}\langle q, \phi\rangle_{(H\hat{\otimes}_\pi H)^*} = t\,Tr(Q\Phi)$.

2. W admits a real quadratic variation $[W]^{\mathbb{R}}_t = t\,Tr(Q)$.

Proof. We first prove point 2., taking into account Lemma 2.1 and showing that

$$\int_0^t \frac{\|W_{s+\epsilon} - W_s\|_H^2}{\epsilon} ds \xrightarrow{L^2(\Omega)} t\,Tr(Q)\ . \tag{6.9}$$

Taking the expectation of the left-hand side of (6.9) gives

$$\begin{aligned}
\mathbb{E}\left(\int_0^t \frac{\|W_{s+\epsilon} - W_s\|_H^2}{\epsilon} ds\right) &= \int_0^t \frac{\mathbb{E}\left(\sum_{i=1}^\infty \langle W_{s+\epsilon} - W_s, e_i\rangle^2\right)}{\epsilon} ds = \\
&\int_0^t \frac{\sum_{i=1}^\infty \mathbb{E}\left(\langle W_{s+\epsilon} - W_s, e_i\rangle^2\right)}{\epsilon} ds = \\
&= \int_0^t \frac{\sum_{i=1}^\infty \epsilon\langle Qe_i, e_i\rangle}{\epsilon} ds = t\,Tr(Q)
\end{aligned}$$

We show now that the the variance of the left-hand side of (6.9) converges to 0. In fact

$$\begin{aligned}
Var\left[\int_0^t \frac{\|W_{s+\epsilon} - W_s\|_H^2}{\epsilon} ds\right] &= \mathbb{E}\left[\int_0^t \left(\frac{\|W_{s+\epsilon} - W_s\|_H^2}{\epsilon} - Tr(Q)\right) ds\right]^2 = \\
&= \frac{2}{\epsilon^2}\int_0^t \int_{(s_1-\epsilon)^+}^{s_1} Cov\left[\|W_{s_1+\epsilon} - W_{s_1}\|^2, \|W_{s_2+\epsilon} - W_{s_2}\|^2\right] ds_2\, ds_1 = \\
&= \frac{2}{\epsilon^2}\int_0^t \int_{(s_1-\epsilon)^+}^{s_1} \sum_{i,j=1}^\infty Cov\left[\langle W_{s_1+\epsilon} - W_{s_1}, e_i\rangle^2, \langle W_{s_2+\epsilon} - W_{s_2}, e_j\rangle^2\right] ds_2\, ds_1 = \\
&= 4\|Q\|_2^2 \epsilon t \xrightarrow[\epsilon\to 0]{} 0
\end{aligned}$$

because

$$Cov\left[\langle W_{s_1+\epsilon} - W_{s_1}, e_i\rangle^2, \langle W_{s_2+\epsilon} - W_{s_2}, e_j\rangle^2\right] = 2(Cov\left[\langle W_{s_1+\epsilon} - W_{s_1}, e_i\rangle, \langle W_{s_2+\epsilon} - W_{s_2}, e_j\rangle\right])^2$$

and this equals $2(s_2 + \epsilon - s_1)^2\langle Qe_i, e_j\rangle^2$ by Proposition 3.8. We recall that $Q \in L^1(H)$ implies $Q \in L^2(H)$ and in particular the Hilbert-Schmidt norm equals to $\sum_{i,j=1}^\infty \langle Qe_i, e_j\rangle^2 = \|Q\|_2^2$.
This allows to conclude that W admits a real quadratic variation and $[W]^{\mathbb{R}}_t = t\,Tr(Q)$.
This shows point 2. and Condition **H1** related to point 1. at the same time.

Concerning Condition **H2**, we only prove convergence in probability for every fixed $t \in [0,T]$, of (4.28). For this we even show $L^2(\Omega)$-convergence for every fixed $\phi \in (H\hat{\otimes}_\pi H)^*$, i.e.

$$\frac{1}{\epsilon}\int_0^t \langle \phi, (W_{s+\epsilon}-W_s)\otimes^2\rangle ds \xrightarrow{L^2(\Omega)} t\,Tr(Q\Phi) = t\langle q, \phi\rangle \ . \tag{6.10}$$

which also implies that the global quadratic variation is the $H\hat{\otimes}_\pi H$-valued deterministic process $t\, q$.
In order to establish the convergence in (6.10), we first evaluate the expectation of the left-hand side of (6.10):

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{\epsilon}\int_0^t \langle \phi, (W_{s+\epsilon}-W_s)\otimes^2\rangle ds\right] &= \frac{1}{\epsilon}\int_0^t \mathbb{E}\left[\phi(W_{s+\epsilon}-W_s, W_{s+\epsilon}-W_s)ds\right] = \\
&= \frac{1}{\epsilon}\int_0^t \mathbb{E}\left[\sum_{i,j=1}^\infty \phi(e_i,e_j)\langle W_{s+\epsilon}-W_s, e_i\rangle\,\langle W_{s+\epsilon}-W_s, e_j\rangle ds\right] = \\
&= \frac{1}{\epsilon}\int_0^t \sum_{i,j=1}^\infty \phi(e_i,e_j)\mathbb{E}\left[\langle W_{s+\epsilon}-W_s, e_i\rangle\,\langle W_{s+\epsilon}-W_s, e_j\rangle\right] ds \ .
\end{aligned} \tag{6.11}$$

Again by Proposition 3.8 we obtain that $\mathbb{E}\left[\langle W_{s+\epsilon}-W_s, e_i\rangle\,\langle W_{s+\epsilon}-W_s, e_j\rangle\right] = \epsilon\langle Qe_i, e_j\rangle$ and by Proposition 6.6 and usual properties of nuclear operators it is easy to verify that

$$\sum_{i,j=1}^\infty \langle \phi, e_i\otimes e_j\rangle\langle Qe_i, e_j\rangle = \langle \phi, q\rangle = Tr(Q\Phi) \ . \tag{6.12}$$

Therefore (6.11) equals $t\langle \phi, q\rangle = t\,Tr(Q\Phi)$.
In order to conclude to the validity of the $L^2(\Omega)$-convergence in (6.10), we show that the variance of its left-hand side converges to zero. In fact

$$\begin{aligned}
Var\left[\frac{1}{\epsilon}\int_0^t \langle \phi, (W_{s+\epsilon}-W_s)\otimes^2\rangle ds\right] &= \frac{1}{\epsilon^2}\int_0^t\int_0^t Cov\left[\langle \phi, (W_{s_1+\epsilon}-W_{s_1})\otimes^2\rangle, \langle \phi, (W_{s_2+\epsilon}-W_{s_2})\otimes^2\rangle\right] ds_1\, ds_2 = \\
&= I_1(\epsilon) + I_2(\epsilon)
\end{aligned} \tag{6.13}$$

where

$$\begin{aligned}
I_1(\epsilon) &= \frac{2}{\epsilon^2}\int_0^t \int_{(s_1-\epsilon)^+}^{s_1} Cov\left[\langle \phi, (W_{s_1+\epsilon}-W_{s_1})\otimes^2\rangle, \langle \phi, (W_{s_2+\epsilon}-W_{s_2})\otimes^2\rangle\right] ds_2\, ds_1 \\
I_2(\epsilon) &= \frac{2}{\epsilon^2}\int_0^t \int_0^{s_1-\epsilon} Cov\left[\langle \phi, (W_{s_1+\epsilon}-W_{s_1})\otimes^2\rangle, \langle \phi, (W_{s_2+\epsilon}-W_{s_2})\otimes^2\rangle\right] ds_2\, ds_1 = 0 \ .
\end{aligned}$$

Consequently Cauchy-Schwarz implies that (6.13) is bounded by

$$\frac{2}{\epsilon^2}\int_0^t \sqrt{Var\left[\langle \phi, (W_{s_1+\epsilon}-W_{s_1})\otimes^2\rangle\right]}\int_{(s_1-\epsilon)^+}^{s_1} \sqrt{Var\left[\langle \phi, (W_{s_2+\epsilon}-W_{s_2})\otimes^2\rangle\right]} ds_2\, ds_1 \ . \tag{6.14}$$

On the other hand, for any $s \in [0,T]$, $Var\left[\langle \phi, (W_{s+\epsilon}-W_s)\otimes^2\rangle\right]$ equals

$$\sum_{i,j,n,m=1}^{\infty} \phi(e_i,e_j)\phi(e_n,e_m) Cov\left[\langle W_{s+\epsilon}-W_s,e_i\rangle\langle W_{s+\epsilon}-W_s,e_j\rangle\,,\,\langle W_{s+\epsilon}-W_s,e_n\rangle\langle W_{s+\epsilon}-W_s,e_m\rangle\right]. \tag{6.15}$$

Using Proposition 3.8 we obtain that

$$Var\left[\langle W_{s+\epsilon}-W_s,e_i\rangle\langle W_{s+\epsilon}-W_s,e_j\rangle\right] = 2\epsilon^2\langle Qe_i,e_j\rangle^2$$
$$Var\left[\langle W_{s+\epsilon}-W_s,e_n\rangle\langle W_{s+\epsilon}-W_s,e_m\rangle\right] = 2\epsilon^2\langle Qe_n,e_m\rangle^2 \ .$$

Again by Cauchy-Schwarz inequality with respect to the covariance in (6.15) and by (6.12) we deduce that

$$Var\left[\langle\phi,(W_{s+\epsilon}-W_s)\otimes^2\rangle\right] \le 2\epsilon^2\left(\sum_{i,j=1}^{\infty}\phi(e_i,e_j)\langle Qe_i,e_j\rangle\right)^2 = 2\,\epsilon^2\left[Tr(Q\Phi)\right]^2 \ .$$

This implies that

$$(6.14) \le 4\,\epsilon\, t\ \left[Tr(Q\Phi)\right]^2 \xrightarrow[\epsilon\longrightarrow 0]{} 0\ .$$

This concludes the proof. □

Remark 6.10.

- The question wether the Q-Wiener process W admits a tensor quadratic variation is beyond our capabilities. We do not know how to prove (or disprove) that for fixed $t\in[0,T]$,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t (W_{s+\varepsilon}-W_s)\otimes^2\, ds \tag{6.16}$$

  exists in the (strong) norm of $H\hat{\otimes}_\pi H$. If it exists, by point 1. of Proposition 6.9 that limit has to be equal to tq.

- We are able however to show that (6.16) converges according to the (Hilbert) norm $H\hat{\otimes}_h H$. In fact, using the bilinearity of the inner product and Proposition 3.8 like in the proof of Proposition 6.9, we can indeed show that

$$\lim_{\varepsilon\to 0} E\left(\|\int_0^t \frac{1}{\varepsilon}(W_{s+\varepsilon}-W_s)\otimes^2\, ds\|^2_{H\hat{\otimes}_h H}\right) = 0.$$

Proposition 6.11. Let H be a separable Hilbert space. Let $Q\in L^1(H)$ with associated $q\in H\hat{\otimes}_\pi H$ and W be a Q-Brownian motion. Then for every continuous measurable process $z:\Omega\times[0,T]\longrightarrow(H\hat{\otimes}_\pi H)^*$ with associated operator (random) $Z=\{Z_t(\omega)\in L(H)\}$, for every $t\in[0,T]$, it holds

$$\int_0^t \langle z_s, d\widetilde{[W]}_s\rangle = \int_0^t Tr(Z_s\cdot d[W]^{dz}_s) = \int_0^t Tr(Z_s\cdot Q)ds = \int_0^t \langle z_s,q\rangle ds\ . \tag{6.17}$$

Proof. The result follows by definition of Bochner Lebesgue-Stieltjes integral. The equality can be shown first taking z and Z as elementary processes and using Proposition 6.9. □

6.2.3 The case of a stochastic integral with respect to a Q Brownian motion

Let Φ be in $\mathcal{N}^2_W(0,T;L^2(H_0;F))$ and M be the Brownian martingale defined by (3.6), i.e. $M := \int_0^t \Phi_s \cdot dW_s^{dz}$. M is a continuous $\mathcal{M}^2_T(F)$-valued process and we recall from Section 3.3.1 that the quadratic variation in the sense of Da Prato-Zabczyk is the $L^1(F)$-valued process of the form

$$[M]^{dz} = \int_0^{\cdot} (\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^* ds \qquad t \in [0,T].$$

We note that $(\Phi_s Q^{1/2})$ and $(\Phi_s Q^{1/2})^*$, $s \in [0,T]$ are respectively $L^2(H;F)$ and $L^2(F;H)$-valued processes, so that the process $(\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^*$, $s \in [0,T]$ is an $L^1(F)$-valued process.

Propositions 6.9 and 6.11 admit an extension to the cas that the Q-Wiener process is replaced by a Brownian martingale. We omit its proof.

Proposition 6.12. According to the notations above we have the following.

1. M admits a global quadratic variation.
2. For every continuous process $z : \Omega \times [0,T] \longrightarrow (H\hat{\otimes}_\pi H)^*$ with associated (random) operator $Z = (Z_s)$ with values in $L(H)$ it holds
$$\int_0^t \langle z_s, d\widetilde{[M]}_s \rangle = \int_0^t Tr(Z_s \cdot d[M]^{dz}_s) = \int_0^t Tr(Z_s \cdot (\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^*) ds \; . \qquad (6.18)$$
for every $t \in [0,T]$.

In fact the proposition above and our Ito's formula in Theorem 8.1 will provide a new proof of Ito's formula stated at Theorem 4.17 in [13] when M is an Hilbert valued process.

6.3 The general Banach space case

Let B be a Banach space and X be a B-valued process. For this general case we refer to Section 4.1. The classical stochastic integration theory goes back to Pellaumail-Metivier [46], see also Dinculeanu [19]. They introduced a notion of real and tensor quadratic variation. Those notions are similar, even if with another language, to the real and tensor quadratic variations introduced in Definition 4.1. The significant link with our approach was given in Proposition 4.25 which states the following. If X has a tensor quadratic variation, then X has a global quadratic variation in the sense that X has a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$. In this case, the two concepts of quadratic variation can be easily associated. By definition, if X admits a global quadratic variation $\widetilde{[X]}$ belongs to $\chi^* = (B\hat{\otimes}_\pi B)^{**}$. If X has a tensor quadratic variation then $\widetilde{[X]}$ even belongs to $B\hat{\otimes}_\pi B$ since the approximating sequences converges "strongly". We recall that, in general, $B\hat{\otimes}_\pi B$ may be a strict subspace of the bidual $(B\hat{\otimes}_\pi B)^{**}$, see Proposition 2.21.

Chapter 7

Stability of χ-quadratic variation and of χ-covariation

Let X be a real finite quadratic variation process and $f \in C^1(\mathbb{R})$. We recall that $f(X)$ is again a finite quadratic variation process. Something similar will be illustrated in the infinite dimensional framework. In this section we will first introduce the definition of a so-called $\chi-$covariation between two Banach valued processes X and Y and later on we will discuss about stability of the χ-covariation through a real function C^1 in the Fréchet sense.

7.1 The notion of χ-covariation

Let B_1, B_2 be two Banach spaces.

Definition 7.1. A closed linear subspace χ of $(B_1\hat{\otimes}_\pi B_2)^*$, endowed with its own norm, such that

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B_1\hat{\otimes}_\pi B_2)^*} \ , \tag{7.1}$$

will be called a **Chi-subspace of** $(B_1\hat{\otimes}_\pi B_2)^*$.

Notation 7.2. Let X (resp. Y) be B_1 (resp. B_2) valued stochastic process. Let χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$ and $\epsilon > 0$. We denote by $[X,Y]^\epsilon$, the following application

$$[X,Y]^\epsilon : \chi \longrightarrow \mathcal{C}([0,T]) \qquad \text{defined by} \qquad \phi \mapsto \left(\int_0^t {}_{\chi}\langle \phi, \frac{J\left((X_{s+\epsilon}-X_s)\otimes(Y_{s+\epsilon}-Y_s)\right)}{\epsilon} \rangle_{\chi^*} \, ds \right)_{t\in[0,T]}$$

where $J : B_1\hat{\otimes}_\pi B_2 \longrightarrow (B_1\hat{\otimes}_\pi B_2)^{**}$ is the canonical injection between a space and its bidual as introduced in subsection 2.1.

With the application $[X,Y]^\epsilon$ it is possible to associate another one, denoted by $\widetilde{[X,Y]}^\epsilon$, defined by

$$\widetilde{[X,Y]}^\epsilon(\omega,\cdot):[0,T]\longrightarrow\chi^* \qquad \text{given by} \qquad t\mapsto\left(\phi\mapsto\int_0^t {}_\chi\langle\phi,\frac{J\left((X_{s+\epsilon}(\omega)-X_s(\omega))\otimes(Y_{s+\epsilon}(\omega)-Y_s(\omega))\right)}{\epsilon}\rangle_{\chi^*}\,ds\right).$$

Remark 7.3. Let X (resp. Y) be B_1 (resp. B_2) valued stochastic process. With a slight abuse of notation, the tensor product $(X_{s+\epsilon}-X_s)\otimes(Y_{s+\epsilon}-Y_s)$ will be confused with the element in χ^* defined by $J\left((X_{s+\epsilon}-X_s)\otimes(Y_{s+\epsilon}-Y_s)\right)$, the injection J from $B_1\hat{\otimes}_\pi B_2$ to its bidual will be omitted.

Definition 7.4. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let X (resp. Y) be B_1 (resp. B_2) valued stochastic process. We say that X **and** Y **admit a** χ**-covariation** if

H1 For all (ϵ_n) it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k\int_0^T\sup_{\|\phi\|_\chi\le1}\left|\langle\phi,\frac{(X_{s+\epsilon_{n_k}}-X_s)\otimes(Y_{s+\epsilon_{n_k}}-Y_s)}{\epsilon_{n_k}}\rangle\right|ds=\sup_k\int_0^T\frac{\left\|(X_{s+\epsilon_{n_k}}-X_s)\otimes(Y_{s+\epsilon_{n_k}}-Y_s)\right\|_{\chi^*}}{\epsilon_{n_k}}ds\;<\infty\;a.s. \tag{7.2}$$

H2 (i) It exists an application $\chi\longrightarrow\mathcal{C}([0,T])$, denoted by $[X,Y]$, such that

$$[X,Y]^\epsilon(\phi)\xrightarrow[\epsilon\longrightarrow0_+]{ucp}[X,Y](\phi) \tag{7.3}$$

for every $\phi\in\chi\subset(B_1\hat{\otimes}_\pi B_2)^*$.

(ii) There is a measurable process $\widetilde{[X,Y]}:\Omega\times[0,T]\longrightarrow\chi^*$, such that

- for almost all $\omega\in\Omega$, $\widetilde{[X,Y]}(\omega,\cdot)$ is a (cadlag) bounded variation process,
- $\widetilde{[X,Y]}(\cdot,t)(\phi)=[X,Y](\phi)(\cdot,t)$ a.s. for all $\phi\in\chi$.

If X and Y admit a χ-covariation we will call χ-**covariation** of X and Y the χ^*-valued process $(\widetilde{[X,Y]})_{0\le t\le T}$ defined for every $\omega\in\Omega$ and $t\in[0,T]$ by $\phi\mapsto\widetilde{[X,Y]}(\omega,t)(\phi)=[X,Y](\phi)(\omega,t)$. By abuse of notation, $[X,Y]$ will also be often called χ-covariation and it will be confused with $\widetilde{[X,Y]}$.

Remark 7.5.

1. For every fixed $\phi\in\chi$, the processes $\widetilde{[X,Y]}(\cdot,t)(\phi)$ and $[X,Y](\phi)(\cdot,t)$ are indistinguishable. In particular the χ^*-valued process $\widetilde{[X,Y]}$ is weakly star continuous, i.e. $\widetilde{[X,Y]}(\phi)$ is continuous for every fixed ϕ.
2. In fact the existence of $\widetilde{[X,Y]}$ guarantees that $[X,Y]$ admits a proper version which allows to consider it as pathwise integral.

Definition 7.6. If the χ-covariation exists for $\chi=(B_1\hat{\otimes}_\pi B_2)^*$, we say that X and Y admit a **global covariation**.

Proposition 7.7. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let X and Y be two stochastic processes with values in B_1 and B_2 admitting a χ-covariation and H a continuous measurable process $H : \Omega \times [0,T] \longrightarrow \mathcal{V}$ where $\mathcal{V}$ is a closed separable subspace of χ. Then for every $t \in [0,T]$

$$\int_0^t {}_\chi\langle H(\cdot,s), d\widetilde{[X,Y]}^\epsilon(\cdot,s)\rangle_{\chi^*} \longrightarrow \int_0^t \langle H(\cdot,s), d\widetilde{[X,Y]}(\cdot,s)\rangle_{\chi^*} \tag{7.4}$$

in probability, when $\varepsilon \to 0$.

Proof. The proof follows the same lines as the one of Corollary 4.33, the fundamental tools being the fact that the pairing duality between χ and χ^* is compatible with the one between $\mathcal{V}$ and $\mathcal{V}^*$ and Proposition 4.31. $\square$

Remark 7.8.

1. The statement of Propositions 4.26 and 4.27 related to the χ-quadratic variation of Banach valued process X can be immediately extended to the case of $\chi-$covariation of two Banach valued processes X and Y. If χ is a finite direct sum of Chi-subspaces, for instance the space $\chi^2([-\tau,0]^2)$, we obtain sufficient conditions for the the existence of the χ-covariation.

2. Analogously, the statements of Corollaries 4.38 and 4.39 related to the χ-quadratic variation of a Banach valued process X can be extended to the case of $\chi-$covariation of two Banach valued processes X and Y. Their proofs make use of Theorem 4.35. We remark that when χ is separable, Condition **H2**(i) reduces to the convergence in probability of (7.3); the existence of a (χ^*-valued) bounded variation version $\widetilde{[X,Y]}$ of $[X,Y]$ is automatically guaranteed.

Our $\chi-$covariation methodology provides a simple property related to the covariation of real processes, which was not formally stated in the literature.

Proposition 7.9. Let X and Y be two real continuous processes such that

i) $[X,Y]$ exists and

ii) for every sequence $(\epsilon_n) \downarrow 0$, it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \frac{1}{\epsilon_{n_k}} \int_0^T \left|X_{s+\epsilon_{n_k}} - X_s\right| \cdot \left|Y_{s+\epsilon_{n_k}} - Y_s\right| ds \quad < +\infty \, . \tag{7.5}$$

Then the real covariation process $[X,Y]$ has bounded variation.

Proof. The processes X and Y take values in $B = \mathbb{R}$ and the (separable) space $\chi = (B\hat{\otimes}_\pi B)^*$ coincides with $\mathbb{R}$. Processes X and Y admit therefore a global covariation which coincides with the classical covariation $[X,Y]$ defined in Definition 2.4. Taking into account point 2. of Remark 7.8, it follows that $[X,Y]$ has bounded variation. $\square$

Remark 7.10. 1. A sufficient condition to ensure that $[X, Y]$ has bounded variation is that X, Y and $X + Y$ are finite quadratic variation processes.
In this case, the bilinearity of the real covariation implies that $[X, Y]$ is difference of increasing processes and has therefore bounded variation. However the mentioned condition is too strong. Consider for instance the following example. Let X be any continuous process and V be a bounded variation process; then $[X, V] = 0$ by point 1. of Proposition 2.14. On the other hand, is easy to show that (7.5) is verified even if X is not a finite quadratic variation process, so that Proposition 7.9 provides a new argument for $[X, V] = 0$.

2. If X, Y are two continuous processes such that (X, Y) has all its mutual covariations then conditions i) and ii) of Proposition 7.9 are fulfilled. In fact by Cauchy-Schwarz inequality we have

$$\frac{1}{\epsilon}\int_0^T |X_{s+\epsilon} - X_s| \cdot |Y_{s+\epsilon} - Y_s|\, ds \leq \sqrt{\int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon} ds}\sqrt{\int_0^T \frac{(Y_{s+\epsilon} - Y_s)^2}{\epsilon} ds} =: A(\epsilon)$$

where the sequence $A(\epsilon)$ converges in probability to $\sqrt{[X]_T[Y]_T}$. This implies of course (7.5).

In view of the next section, we proceed to the evaluation of some χ-covariations for $C([-\tau, 0])$-valued window processes, i.e. when $B_1 = B_2 = B = C([-\tau, 0])$. Spaces χ will be Chi-subspaces of $(B\hat{\otimes}_\pi B)^*$. The proof of the propositions below can be provided taking into account point 2. of Remark 7.8.

Proposition 7.11. Let X and Y be two real continuous processes such that (X, Y) has all its mutual covariations. Then

1. $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation, where $\chi = L^2([-\tau, 0]^2)$.
2. $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation for every $i \in \{0, \ldots, N\}$, where $\chi = L^2([-\tau, 0])\hat{\otimes}_h\mathcal{D}_i([-\tau, 0])$.
3. Let $\chi = \mathcal{D}_{i,j}([-\tau, 0]^2)$ for a given $i, j \in \{0, \ldots, N\}$ and suppose morevoer that the covariation $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists. Then $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation which equals

$$[X(\cdot), Y(\cdot)]_t(\mu) = \mu(\{a_i, a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}]_t, \forall \mu \in \chi, t \in [0, T]. \tag{7.6}$$

Proof. The proof is practically the same as the one of Proposition 5.7. □

When $\chi = \mathcal{D}_{0,0}([-\tau, 0]^2)$ for the existence of a χ -covariation between X and Y we can even relax the hypotheses.

Proposition 7.12. Let X, Y be continuous processes fulfilling i) and ii) of Proposition 7.9. Then $X(\cdot)$ and $Y(\cdot)$ admit a $\mathcal{D}_{0,0}([-\tau, 0]^2)$-covariation and

$$[X(\cdot), Y(\cdot)]_t(\mu) = \mu(\{0, 0\})[X, Y]_t\ .$$

Proof. The proof is again very similar to the one of Proposition 5.7. The only relevant difference consists in checking the validity of condition **H1**. This will be verified identically until (5.4); the next step will follow by (7.5). □

The two results below follow by point 1. of Remark 7.8.

Theorem 7.13. Let X and Y be two real continuous processes such that $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for every $i,j = 0,\dots,N$. Then $X(\cdot)$ and $Y(\cdot)$ admit the following $\chi^2([-\tau,0]^2)$-covariation

$$[X(\cdot),Y(\cdot)] : \chi^2([-\tau,0]^2) \longrightarrow \mathcal{C}([0,T]) \qquad\qquad \mu \mapsto \sum_{i,j=0}^{N} \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}] \,.$$

Theorem 7.14. Let X and Y be two real continuous processes such (X,Y) has all its mutual covariations. Then $X(\cdot)$ and $Y(\cdot)$ admit the following $\chi^0([-\tau,0]^2)$-covariation

$$[X(\cdot),Y(\cdot)] : \chi^0([-\tau,0]^2) \longrightarrow \mathcal{C}([0,T]) \qquad\qquad \mu \mapsto \mu(\{0,0\})[X,Y] \,.$$

Remark 7.15.

1. The existence of $\chi^0([-\tau,0]^2)$-covariation only requires the existence of the mutual covariations of (X,Y) (or even less). We do not need the existence of $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$, for every $i,j = 0,\dots,N$.

2. Let D be a real $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation and decomposition $M + A$, M being its $(\mathcal{F}_t)$-local martingale component and let N be a real $(\mathcal{F}_t)$-martingale. Then $D(\cdot)$ and $N(\cdot)$ admit χ^0-covariation given by $[D(\cdot),N(\cdot)](\mu) = \mu(\{0,0\})[M,N]$ for every $\mu \in \chi^0$. This follows from Theorem 7.14, because D and N are with finite quadratic variation processes and $[D,N] = [M,N]$.

3. Let D be a real $(\mathcal{F}_t)$-Dirichlet process with decomposition $M + A$, M being the $(\mathcal{F}_t)$-local martingale part and let N be a real $(\mathcal{F}_t)$-local martingale. Then $D(\cdot)$ and $N(\cdot)$ admit a χ^2-covariation given by $[D(\cdot),N(\cdot)](\mu) = \sum_{i,j=0}^{N} \mu(\{a_i,a_j\})[D_{\cdot+a_i}, N_{\cdot+a_j}]. = \sum_{i,j=0}^{N} \mu(\{a_i,a_j\})[M_{\cdot+a_i}, N_{\cdot+a_j}] = \sum_{i=0}^{N} \mu(\{a_i,a_i\})[M_{\cdot+a_i}, N_{\cdot+a_i}]$. This follows again from Theorem 7.14 and Proposition 2.12

7.2 The stability of the χ-covariation in the Banach space framework

In this section, we analyze the stability of χ-covariation for Banach valued processes transformed through C^1 Fréchet differentiable functions.
We first recall what happens in the finite dimensional case as far as stability is concerned, see for instance [24], Proposition 2.7. which even states the result in the case of higher power variations.

Proposition 7.16. Let $X = (X^1, \ldots, X^n)$ be a $\mathbb{R}^n$-valued process having all its mutual covariations $([X^*, X]_t)_{1\le i,j\le n} = [X^i, X^j]_t$ and $F,\, G \in C^1(\mathbb{R}^n)$. Then the covariation $[F(X), G(X)]$ exists and is given by

$$[F(X), G(X)]_\cdot = \sum_{i,j=1}^{n} \int_0^\cdot \partial_i F(X)\partial_j G(X) d[X^i, X^j] \tag{7.7}$$

This includes the case of Propostion 2.1 in [59], setting $n = 2$, $F(x, y) = f(x)$, $G(x, y) = g(y)$, $f, g \in C^1(\mathbb{R})$.
When the value space is a general Banach space, we need to recall some other preliminary results.

Proposition 7.17. Let E be a Banach space, $S, T : E \longrightarrow \mathbb{R}$ be linear continuous forms. There is a unique linear continuous forms from $E\hat{\otimes}_\pi E$ to $\mathbb{R}\hat{\otimes}_\pi\mathbb{R} \cong \mathbb{R}$, denoted by $S\otimes T$, such that $S\otimes T(e_1\otimes e_2) = S(e_1)\cdot T(e_2)$ and $\|S \otimes T\| = \|S\|\,\|T\|$.

Proof. See Proposition 2.3 in [63]. $\square$

Remark 7.18.

1. If $T = S$, we will denote $S \otimes S = S\otimes^2$.
2. Let B be a Banach space and $F\,, G : B \longrightarrow \mathbb{R}$ of class $C^1(B)$ in the Fréchet sense. If x and y are fixed, $DF(x)$ and $DF(y)$ are linear continuous form from B to $\mathbb{R}$. According to Proposition 7.17 and the notation introduced there, the symbol $DF(x) \otimes DF(y)$ denotes the unique linear continuous form from $B\hat{\otimes}_\pi B$ to $\mathbb{R}$. We insist on the fact that "a priori" $DF(x) \otimes DF(y)$ does not denote an element of some tensor product $B^* \otimes B^*$.

When E is an Hilbert space the application $S \otimes T$ of Proposition 7.17 can be further specified.

Proposition 7.19. Let E be a Hilbert space, $S,\, T \in E^*$ and $\mathcal{S},\, \mathcal{T}$ the associated elements in E via Riesz identification. $S \otimes T$ can be characterized as the continuous bilinear form

$$S \otimes T(x \otimes y) = \langle \mathcal{S}, \mathcal{T}\rangle_E \cdot \langle x, y\rangle_E = \langle \mathcal{S} \otimes \mathcal{T}, x \otimes y\rangle_{E\hat{\otimes}_h E}, \quad \forall x, y \in E. \tag{7.8}$$

In particular the linear form $S \otimes T$ belongs to $(E\hat{\otimes}_h E)^*$ and via Riesz it is identified with the tensor product $\mathcal{S} \otimes \mathcal{T}$. That Riesz identification will be omitted in the sequel.

Proof. The application ϕ defined in the right-side of (7.8) belongs in $(E\hat{\otimes}_h E)^*$ by construction. Since $(E\hat{\otimes}_h E)^* \subset (E\hat{\otimes}_\pi E)^*$, it also belongs to $(E\hat{\otimes}_\pi E)^*$. Moreover we have

$$\|\phi\|_{\mathcal{B}} = \sup_{\|f\|_E\le 1, \|g\|_E\le 1} |\phi(f, g)| = \sup_{\|f\|_E\le 1} |\langle \mathcal{S}, f\rangle| \sup_{\|g\|_E\le 1} |\langle \mathcal{T}, g\rangle| = \|S\|_{E^*} \|T\|_{E^*}\ .$$

By uniqueness in Proposition 7.17, ϕ must coincides with $S \otimes T$. $\square$

As application of Proposition 7.19, setting the Hilbert space $E = \mathcal{D}_a \oplus L^2([-\tau, 0])$, we state the following useful result that will be often used in Section 7.3 where we consider $C([-\tau, 0])$-valued window processes.

Example 7.20. Let F^1 and F^2 be two functions from $C([-\tau,0])$ to $\mathcal{D}_a \oplus L^2([-\tau,0])$ such that $\eta \mapsto F^j(\eta) = \sum_{i=0,\ldots N} \lambda_i^j(\eta)\delta_{a_i} + g^j(\eta)$ with $\eta \in C([-T,0])$, $\lambda_i^j : C([-\tau,0]) \longrightarrow \mathbb{R}$ and $g^j : C([-\tau,0]) \longrightarrow L^2([-T,0])$ continuous for $j=1,2$. Then for any (η_1,η_2), $(F^1 \otimes F^2)(\eta_1,\eta_2)$ will be identified with the true tensor product $F^1(\eta_1) \otimes F^2(\eta_2)$ which belongs to $\chi^2([-\tau,0]^2)$. In fact we have

$$F^1(\eta_1) \otimes F^2(\eta_2) = \sum_{i,j=0,\ldots,N} \lambda_i^1(\eta_1)\lambda_j^2(\eta_2)\delta_{a_i} \otimes \delta_{a_j} + g^1(\eta_1) \otimes \sum_{i=0,\ldots,N} \lambda_i^2(\eta_2)\delta_{a_i} +$$

$$+ \sum_{i=0,\ldots,N} \lambda_i^1(\eta_1)\delta_{a_i} \otimes g^2(\eta_2) + g^1(\eta_1) \otimes g^2(\eta_2) \tag{7.9}$$

We now state the stability result related to χ-covariation.

Theorem 7.21. Let B be a separable Banach space, χ a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X^1, X^2 two B-valued continuous stochastic process admitting a χ-covariation. Let $F^1, F^2 : B \longrightarrow \mathbb{R}$ be two functions of class C^1 in the Fréchet sense. We suppose moreover that the following applications

$$\begin{aligned} DF^i(\cdot) \otimes DF^j(\cdot) : B \times B &\longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \\ (x,y) &\mapsto DF^i(x) \otimes DF^j(y) \end{aligned}$$

are continuous for $i,j = 1,2$.
Then, for every $i,j \in \{1,2\}$, the covariation between $F^i(X^i)$ and $F^j(X^j)$ exists and is given by

$$[F^i(X^i), F^j(X^j)] = \int_0^{\cdot} \langle DF^i(X_s^i) \otimes DF^j(X_s^j), d\widetilde{[X^i, X^j]}_s \rangle \tag{7.10}$$

Remark 7.22. In view of an application of Proposition 7.7 in the proof of Theorem 7.21, we observe the following. Since B is separable and $DF^i(\cdot) \otimes DF^j(\cdot) : B \times B \longrightarrow \chi$ is continuous, the process $H_t = DF^i(X_t^i) \otimes DF^j(X_t^j)$ takes values in a separable closed subspace $\mathcal{V}$ of χ.

Corollary 7.23. Let us formulate the same assumptions as in Theorem 7.21. If there is a χ^*-valued stochastic process H such that $\widetilde{[X^i, X^j]}_s = \int_0^s H_u \, du$ in the Bochner sense then

$$[F^i(X^i), F^j(X^j)]_\cdot = \int_0^{\cdot} \langle DF^i(X_s^i) \otimes DF^j(X_s^j), H_s \rangle \, ds$$

Proof of Theorem 7.21. We make use in an essential manner of Proposition 7.7. Without restriction of generality we only consider the case $F^1 = F^2 = F$ and $X^1 = X^2 = X$. In this case Proposition 7.7 reduces to Corollary 4.33.
By definition of the quadratic variation of a real process in Definition 2.4, we know that $[F(X)]_\cdot$ is the limit in the ucp sense of the quantity

$$\int_0^{\cdot} \frac{\left(F(X_{s+\epsilon}) - F(X_s)\right)^2}{\epsilon} ds \,.$$

According to Lemma 2.1, it will be enough to show the convergence in probability for a fixed $t \in [0,T]$. Using a Taylor's expansion we have

$$\begin{aligned}\frac{1}{\epsilon}\int_0^t \left(F(X_{s+\epsilon}) - F(X_s)\right)^2 ds = \frac{1}{\epsilon}\int_0^t \Big(\langle DF(X_s), X_{s+\epsilon} - X_s\rangle + \\ + \int_0^1 \langle DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s), X_{s+\epsilon} - X_s\rangle \, d\alpha\Big)^2 ds = \\ = A_1(\epsilon) + A_2(\epsilon) + A_3(\epsilon) \; ;\end{aligned}$$

where

$$\begin{aligned}A_1(\epsilon) &= \frac{1}{\epsilon}\int_0^t \langle DF(X_s), X_{s+\epsilon} - X_s\rangle^2 ds = \\ &= \int_0^t \langle DF(X_s) \otimes DF(X_s), \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon}\rangle ds \\ A_2(\epsilon) &= \frac{2}{\epsilon}\int_0^t \langle DF(X_s), X_{s+\epsilon} - X_s\rangle \cdot \\ &\qquad \cdot \int_0^1 \langle DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s), X_{s+\epsilon} - X_s\rangle d\alpha \, ds = \\ &= 2\int_0^t \int_0^1 \langle DF(X_s) \otimes \left(DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s)\right), \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon}\rangle d\alpha \, ds \\ A_3(\epsilon) &= \frac{1}{\epsilon}\int_0^t \Big(\int_0^1 \langle DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s), X_{s+\epsilon} - X_s\rangle d\alpha\Big)^2 ds \leq \\ &\leq \frac{1}{\epsilon}\int_0^t \int_0^1 \langle DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s), X_{s+\epsilon} - X_s\rangle^2 d\alpha \, ds = \\ &= \int_0^t \int_0^1 \langle \left(DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s)\right) \otimes^2, \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon}\rangle d\alpha \, ds \; .\end{aligned}$$

According to Remark 7.22 and Corollary 4.33, it follows

$$A_1(\epsilon) \xrightarrow{\mathbb{P}} \int_0^t \langle DF(X_s) \otimes DF(X_s), d\widetilde{[X]}_s\rangle \; .$$

It remains to show the convergence in probability of $A_2(\epsilon)$ and $A_3(\epsilon)$ to zero.
About $A_2(\epsilon)$ the following decomposition holds:

$$DF(X_s)\otimes\left(DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s)\right) = DF(X_s)\otimes DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) - DF(X_s)\otimes DF(X_s); \tag{7.11}$$

concerning $A_3(\epsilon)$ we get

$$
\begin{aligned}
\left(DF\left((1-\alpha)X_s+\alpha X_{s+\epsilon}\right)-DF(X_s)\right)\otimes^2 = {} & DF\left((1-\alpha)X_s+\alpha X_{s+\epsilon}\right)\otimes^2 + \\
& + DF(X_s)\otimes DF(X_s) + \\
& - DF\left((1-\alpha)X_s+\alpha X_{s+\epsilon}\right)\otimes DF(X_s) + \\
& - DF(X_s)\otimes DF\left((1-\alpha)X_s+\alpha X_{s+\epsilon}\right) \ .
\end{aligned}
\tag{7.12}
$$

Using (7.11), we obtain

$$
\begin{aligned}
|A_2(\epsilon)| \leq {} & 2\int_0^t\int_0^1 \left| \langle DF(X_s)\otimes\left(DF\left((1-\alpha)X_s+\alpha X_{s+\epsilon}\right)-DF(X_s)\right), \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle \right| d\alpha\, ds \leq \\
\leq {} & \int_0^t\int_0^1 \left\|DF(X_s)\otimes DF\left((1-\alpha)X_s+\alpha X_{s+\epsilon}\right)-DF(X_s)\otimes DF(X_s)\right\|_{\chi} \left\|\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\right\|_{\chi^*} d\alpha\, ds \ .
\end{aligned}
\tag{7.13}
$$

For fixed $\omega\in\Omega$ we denote by $\mathcal{V}(\omega) := \{X_t(\omega);\ t\in[0,T]\}$ and

$$
\mathcal{U} = \mathcal{U}(\omega) = \overline{conv(\mathcal{V}(\omega))},
\tag{7.14}
$$

i.e. the set $\mathcal{U}$ is the closed convex hull of the compact subset $\mathcal{V}(\omega)$ of B. From (7.13) we deduce

$$
|A_2(\epsilon)| \leq \varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}(\epsilon)\int_0^t \left\|\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\right\|_{\chi^*} ds,
$$

where $\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}$ is the continuity modulus of the application $DF(\cdot)\otimes DF(\cdot) : B\times B \longrightarrow \chi$ restricted to $\mathcal{U}\times\mathcal{U}$. We recall that

$$
\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}(\delta) = \sup_{\|(x_1,y_1)-(x_2,y_2)\|_{B\times B}\leq\delta} \|DF(x_1)\otimes DF(y_1)-DF(x_2)\otimes DF(y_2)\|_{\chi}
$$

where the space $B\times B$ is equipped with the norm obtained summing the norms of the two components. According to Theorem 5.35 in [2], $\mathcal{U}(\omega)$ is compact, so the function $DF(\cdot)\otimes DF(\cdot)$ on $\mathcal{U}(\omega)\times\mathcal{U}(\omega)$ is uniformly continuous and $\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero.
Let (ϵ_n) converging to zero; Condition **H1** in the definition of χ-quadratic variation, implies the existence of a subsequence (ϵ_{n_k}) such that $A_2(\epsilon_{n_k})$ converges to zero a.s. This implies that $A_2(\epsilon)\rightarrow 0$ in probability.

With similar arguments, using (7.12), we can show that $A_3(\epsilon) \to 0$ in probability. We observe in fact

$$\begin{aligned}
|A_3(\epsilon)| \leq & \int_0^t \int_0^1 \left\| DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) \otimes^2 - DF(X_s) \otimes DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) \right\|_\chi \cdot \\
& \cdot \left\| \frac{(X_{s+\epsilon} - X_s) \otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha \, ds + \\
& + \int_0^t \int_0^1 \left\| DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) \otimes DF(X_s) - DF(X_s) \otimes^2 \right\|_\chi \left\| \frac{(X_{s+\epsilon} - X_s) \otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha \, ds \leq \\
& \leq 2 \varpi^{\mathcal{U} \times \mathcal{U}}_{DF \otimes DF}(\epsilon) \int_0^t \left\| \frac{(X_{s+\epsilon} - X_s) \otimes^2}{\epsilon} \right\|_{\chi^*} ds \, .
\end{aligned}$$

The result is now established. □

Corollary 7.24. Let B be a separable Banach space and B_0 be a Banach space such that $B_0 \supset B$ continuously. Let $\chi = (B_0 \hat{\otimes}_\pi B_0)^*$ and X a B-valued stochastic process admitting a χ-quadratic variation. Let $F^1, F^2 : B \longrightarrow \mathbb{R}$ be functions of class C^1 Fréchet such that DF^i, $i = 1, 2$ are continuous as applications from B to B_0^*.
Then the covariation of $F^i(X)$ and $F^j(X)$ exists and it is given by

$$[F^i(X), F^j(X)]_\cdot = \int_0^\cdot \langle DF^i(X_s) \otimes DF^j(X_s), d\widetilde{[X]}_s \rangle \, . \tag{7.15}$$

Proof. It is clear that χ is a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$. For any given $x, y \in B$, $i, j = 1, 2$, by the characterization of $DF^i(x) \otimes DF^j(y)$ given in Proposition 7.17 and Remark 7.18, the following applications

$$DF^i(x) \otimes DF^j(y) : B_0 \hat{\otimes}_\pi B_0 \longrightarrow \mathbb{R}$$

are continuous for $i, j \in \{1, 2\}$. The result follows by Theorem 7.21. □

Remark 7.25. Under the same assumptions as Corollary 7.24 we suppose moreover that B_0 is a Hilbert space. For any $x, y \in B$, $DF(x) \otimes DG(y)$ belongs to $\left(B_0 \hat{\otimes}_h B_0\right)^*$ because of Proposition 7.19 and it will we associated to a true tensor product in the sense explained in the same proposition.

We discuss rapidly the finite dimensional framework.

Example 7.26. Let $X = (X^1, \cdots, X^n)$ be a $\mathbb{R}^n$-valued stochastic process admitting all its mutual covariations, and $F, G : \mathbb{R}^n \longrightarrow \mathbb{R} \in C^1(\mathbb{R}^n)$. We recall that, by Section 6.1, Proposition 6.2, X admits a global quadratic variation $\widetilde{[X]}$ which coincides with the tensor element associated to the matrix $([X^*, X])_{1 \leq i,j \leq n} = [X^i, X^j]$. We recall also that $\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n$ can be identified with the space of matrices $\mathbb{M}_{n \times n}(\mathbb{R})$.
The application of Theorem 7.21 to this context provides a new proof of Proposition 7.16.

If $D^{F,G}(X_s)$ denotes the matrix associated to the tensor product $DF(X_s) \otimes DG(X_s)$, the right-hand side of (7.10) equals

$$\int_0^{\cdot} Tr\left(D^{F,G}(X_s) \cdot d[X^*, X]_s\right)$$

which coincides with the right-hand side of (7.7).

7.3 Stability results for window Dirichlet processes with values in $C([-\tau,0])$

We formulate now some stability results involving $C([-\tau,0])$-valued window processes and some related Fukushima type decomposition. We first recall what happens in the finite dimensional case.

1. The class of real semimartingales with respect to a given filtration is known to be stable with respect to $C^2(\mathbb{R})$ transformations, as Theorem 2.13 implies. Proposition 7.16 says that finite quadratic variation processes are stable under $C^1(\mathbb{R})$ transformations. Also Dirichlet processes are stable with respect to $C^1(\mathbb{R})$ transformations and they admit a decomposition result. If $X = M + A$ is a real $(\mathcal{F}_t)$-Dirichlet process with M the $(\mathcal{F}_t)$-local martingale and A_t the zero quadratic variation process and F of class $C^1(\mathbb{R})$, then $F(X)$ is still a Dirichlet process with decomposition $F(X) = \tilde{M} + \tilde{A}$, where $\tilde{M}_t = F(X_0) + \int_0^t F'(X_s)dM_s$ and $\tilde{A}_t = F(X_t) - \tilde{M}_t$; see [4] and [62] for details.

2. In some applications, in particular to control theory (as illustrated in [31]), one often needs to know the nature of process $(F(t, D_t))$ where $F \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R})$ and D is a real continuous $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation. It was shown in [32], Proposition 3.10, that $\big(F(t, D_t)\big)$ is an $(\mathcal{F}_t)$-weak Dirichlet process.

Both results admit some generalizations in the infinite dimensional framework for the $C([-\tau,0])$-valued window processes. With the same notations on processes X and D we will show following statements.

1. Let $F : C([-\tau,0]) \longrightarrow \mathbb{R}$ be of class $C^1\big(C([-\tau,0])\big)$ in the Fréchet sense such that the first derivative DF at each point $\eta \in C([-\tau,0])$, belongs to $\mathcal{D}_0([-\tau,0]^2) \oplus L^2([-\tau,0])$. We suppose moreover that DF, with values in the mentioned space, is continuous. Then $F(X(\cdot))$ is a real Dirichlet process, as Theorem 7.33 says.

2. Let $F : [0,T] \times C([-\tau,0]) \longrightarrow \mathbb{R}$ be of class $C^{0,1}\big(\mathbb{R}^+ \times C([-\tau,0])\big)$ in the Fréchet sense such that the first derivative, at each point $(t,\eta) \in [0,T] \times C([-\tau,0])$, belongs to $\mathcal{D}_0([-\tau,0]^2) \oplus L^2([-\tau,0])$. We suppose again moreover that DF, with values in the mentioned space, is continuous. Similarly to [32] we cannot expect $\big(F(t, D_t(\cdot))\big)$ to be a Dirichlet process. In general it will not even be a finite quadratic variation process if the dependence on t is very irregular. However we will show in Theorem 7.36 that $\big(F(t, D_t(\cdot))\big)$ remains at least a weak Dirichlet process.

First we need a preliminary result on measure theory.

Lemma 7.27. Let E be a topological direct sum $E_1 \oplus E_2$ where E_1, E_2 are Banach spaces equipped with some norms $\|\cdot\|_{E_i}$. We denote by P_i the projectors $P_i : E \to E_i$, $i \in 1,2$.
Let $\tilde{g} : [0,T] \to E^*$ and we define $\tilde{g}_i : [0,T] \to E_i^*$ setting $\tilde{g}_i(t)(\eta) := \tilde{g}(t)(\eta)$ for all $\eta \in E_i$, i.e. the restriction of $\tilde{g}(t)$ to E_i^*. We suppose $\tilde{g}_i$ continuous with bounded variation, $i = 1,2$.
Let $f : [0,T] \to E$ measurable with projections $f_i := P_i(f)$ defined from $[0,T]$ to E_i.
Then the following statements hold:

1. f in $L^1_E(\tilde{g})$ if and only if f_i in $L^1_{E_i}(\tilde{g}_i), i = 1,2$ and yields

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t {}_{E_1}\langle f_1(s), d\tilde{g}_1(s)\rangle_{E_1^*} + \int_0^t {}_{E_2}\langle f_2(s), d\tilde{g}_2(s)\rangle_{E_2^*} \ . \tag{7.16}$$

2. If $\tilde{g}_2(t) \equiv 0$ and f_1 in $L^1_{E_i}(\tilde{g}_1)$ then

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t {}_{E_1}\langle f_1(s), d\tilde{g}_1(s)\rangle_{E_1^*} \ . \tag{7.17}$$

Proof.

1. By the hypothesis on $\tilde{g}_i$ we deduce that $\tilde{g} : [0,T] \to E^*$ has bounded variation. If $f : [0,T] \to E$ belongs to L^1_E, then $f_i = P_i(f) : [0,T] \to E_i$, $i = 1,2$ belong to $L^1_{E_i}$ by the property $\|P_i f\|_{E_i} \leq \|f\|_E$. We prove (7.16) for a step function $f : [0,T] \to E$ defined by $f(s) = \sum_{j=1}^N \phi_{A_j}(s) f_j$ with ϕ_{A_j} indicator functions of the subsets A_j of $[0,T]$ and $f_j \in E$. We have $f_j = f_{1j} + f_{2j}$ with $f_{ij} = P_i f_j$, $i = 1,2$, so

$$\begin{aligned}\int_0^T {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} &= \sum_{j=1}^N \int_{A_j} {}_E\langle f_j, d\tilde{g}(s)\rangle_{E^*} = \sum_{j=1}^N {}_E\langle f_j, \int_{A_j} d\tilde{g}(s)\rangle_{E^*} = \sum_{j=1}^N {}_E\langle f_j, d\tilde{g}(A_j)\rangle_{E^*} = \\ &= \sum_{j=1}^N {}_{E_1}\langle f_{1j}, d\tilde{g}_1(A_j)\rangle_{E_1^*} + \sum_{j=1}^N {}_{E_2}\langle f_{2j}, d\tilde{g}_2(A_j)\rangle_{E_2^*} = \\ &= \int_0^T {}_{E_1}\langle f_1(s), d\tilde{g}_1(s)\rangle_{E_1^*} + \int_0^T {}_{E_2}\langle f_2(s), d\tilde{g}_2(s)\rangle_{E_2^*} \ .\end{aligned}$$

   A general function f in $L^1_E(\tilde{g})$ is a sum of $f_1 + f_2$, $f_i \in L^1_{E_i}(\tilde{g}_i)$ for $i = 1,2$. Both f_1 and f_2 can be approximate by step functions. As we can see in Appendix B, vector integration $L^1_E(\tilde{g})$, as well as on $L^1_{E_i}(\tilde{g}_i)$, is defined by density on step functions. The result follows by an approximation argument.

2. It follows directly by 1.

□

A useful consequence of Lemma 7.27 is the following.

Proposition 7.28. Let $E_1 = \mathcal{D}_{i,j}([-\tau,0]^2)$ and E_2 be a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$ such that $E_1 \cap E_2 = \{0\}$.

- Let $\tilde{g} : [0,T] \to E^*$ such that $\tilde{g}(t)_{|E_2} \equiv 0$.
- We set $g_1 : [0,T] \to \mathbb{R}$ by $g_1(t) = {}_{E_1}\langle \delta_{(a_i,a_j)}, \tilde{g}_1(t)\rangle_{E_1^*}$, supposed continuous with bounded variation.
- Let $f : [0,T] \to E$ such that $t \to f(t)(\{(a_i,a_j\}) \in L^1(d|g_1|)$.

Then

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t f(s)(\{a_i,a_j\})dg_1(s)\ . \tag{7.18}$$

Remark 7.29. Let g_1 be the real function defined in the second item of the hypotheses.
Defining $\tilde{g}_1 : [0,T] \to E_1^*$ by $\tilde{g}_1(t) = g_1(t)\ \delta_{(a_i,a_j)}$, by construction it follows $\tilde{g}_1(t)(f) = \tilde{g}(t)(f)$ for every $f \in E_1, t \in [0,T]$. Since for $a, b \in [0,T]$, with $a < b$, we have

$$\|\tilde{g}(b) - \tilde{g}(a)\|_{E^*} = \|\tilde{g}_1(b) - \tilde{g}_1(a)\|_{E_1^*} = |g_1(b) - g_1(a)|$$

then the property g_1 continuous with bounded variation is equivalent to $\tilde{g}$ continuous with bounded variation.

Proof. We apply Lemma 7.27 2. Clearly we have $P_1(f) = f(\{a_i,a_j\})\delta_{(a_i,a_j)}$. It follows that

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t {}_{E_1}\langle f(s)(\{a_i,a_j\})\delta_{(a_i,a_j)}, d\tilde{g}_1(s)\rangle_{E_1^*}$$

Since $g_1(t) = {}_{E_1}\langle \delta_{(a_i,a_j)}, \tilde{g}_1(s)\rangle_{E_1^*}$ and because of Theorem 30 in Chapter 1, paragraph 2 of [19], previous expression equals the right-hand side of (7.18). □

Remark 7.30. Let E be a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ containing $\mathcal{D}_{i,j}([-\tau,0]^2)$. A typical example of application of Proposition 7.28 is given by $E_1 = \mathcal{D}_{i,j}([-\tau,0]^2)$ and $E_2 = \{\mu \in E \mid \mu(\{a_i,a_j\}) = 0\}$. Any $\mu \in E$ can be decomposed into $\mu_1 + \mu_2$, where $\mu_1 = \mu(\{a_i,a_j\})\delta_{(a_i,a_j)}$, which belongs to E_1, and $\mu_2 \in E_2$. This framework will be the one of proposition below where Proposition 7.27 will be applied considering $\tilde{g}$ as the χ-covariation of two processes $X(\cdot)$ and $Y(\cdot)$

Proposition 7.31. Let $i,j \in \{0,\ldots,N\}$ and let χ_2 be a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$ such that $\mu(\{a_i,a_j\}) = 0$ for every $\mu \in \chi_2$. We set $\chi = \mathcal{D}_{i,j}([-\tau,0]^2) \oplus \chi_2$.
Let X, Y be two real continuous processes such that $X(\cdot)$ and $Y(\cdot)$ admit a zero χ_2-covariation and a $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation. Then following properties hold.

1. $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists and the $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation is given by

$$[X(\cdot), Y(\cdot)] : \mathcal{D}_{i,j}([-\tau,0]^2) \longrightarrow \mathcal{C}([0,T]) \qquad\qquad \mu \mapsto \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}].$$

2. χ is a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$.

3. χ is a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$, with $B = C([-\tau,0])$.

4. $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation of the type

$$[X(\cdot),Y(\cdot)] : \chi \longrightarrow \mathcal{C}([0,T]) \qquad\qquad [X(\cdot),Y(\cdot)](\mu) = \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}]\ .$$

5. For every χ-valued process Z with locally bounded paths (for instance cadlag) we have

$$\int_0^\cdot \langle Z_s, d\widetilde{[X(\cdot),Y(\cdot)]}_s\rangle = \int_0^\cdot Z_s(\{a_i,a_j\})d[X_{\cdot+a_i}, Y_{\cdot+a_j}]_s\ . \tag{7.19}$$

Proof.

1. It is a consequence of the fact that $X(\cdot)$ and $Y(\cdot)$ admit a $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation, in particular of Condition **H2**.

2. It follows by Proposition 4.5.

3. It follows by previous point and Proposition 4.4.

4. We denote here $\chi_1 = \mathcal{D}_{i,j}([-\tau,0]^2)$; χ_1 and χ_2 are Chi-subspaces of $\mathcal{M}([-\tau,0]^2)$, $X(\cdot)$ and $Y(\cdot)$ admit χ_1-covariation and a χ_2-covariation. Remark 7.8 point 1. and Proposition 4.26 imply that $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation which can be determined from the χ_1-covariation and the χ_2-covariation. More precisely, for μ in χ with decomposition $\mu_1+\mu_2$, $\mu_1\in\chi_1$ and $\mu_2\in\chi_2$, with a slight abuse of notations, we have

$$\begin{aligned}
[X(\cdot),Y(\cdot)](\mu) &= [X(\cdot),Y(\cdot)](\mu_1) + [X(\cdot),Y(\cdot)](\mu_2) = \\
&= [X(\cdot),Y(\cdot)](\mu_1) = \\
&= \mu_1(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}] = \\
&= \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}]\ .
\end{aligned}$$

5. Since both sides of (7.19) are continuous processes, it is enough to show that they are equals a.s. for every fixed $t\in[0,T]$. This follows for almost all $\omega\in\Omega$ using Propositopn 7.28 where $f = Z(\omega)$ and $\tilde{g} = \widetilde{[X(\cdot),Y(\cdot)]}(\omega)$. We remark that here $\tilde{g}_1 = \widetilde{[X_{\cdot+a_i}(\cdot),Y_{\cdot+a_j}(\cdot)]}(\omega)$ and $g_1 = [X_{\cdot+a_i}, Y_{\cdot+a_j}](\omega)$.

□

Remark 7.32. Proposition 7.31 will be used in the sequel especially in the case $a_i = a_j = 0$.

Theorem 7.33. Let X be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $X = M+A$, where M is the $(\mathcal{F}_t)$-local martingale and A is a zero quadratic variation process with $A_0 = 0$. Let $F : C([-\tau,0]) \longrightarrow \mathbb{R}$ be a Fréchet differentiable function such that the range of DF is $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$. Moreover we suppose that $DF : C([-\tau,0]) \longrightarrow \mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$ is continuous.
Then $F(X(\cdot))$ is an $(\mathcal{F}_t)$-Dirichlet process with local martingale component equal to

$$\bar{M}_\cdot = F\big(X_0(\cdot)\big) + \int_0^\cdot D^{\delta_0}F\big(X_s(\cdot)\big)dM_s \ ,$$

where we recall Notation 2.28 that $D^{\delta_0}F(\eta) = DF(\eta)(\{0\})$.

Proof. We need to show that $[\bar{A}] = 0$ where $\bar{A} := F(X(\cdot)) - \bar{M}$. For simplicity of notations, in this proof we will denote $\alpha_0(\eta) = D^{\delta_0}F(\eta)$. By the linearity of the real covariation we have $[\bar{A}] = A_1 + A_2 - 2A_3$ where

$$\begin{aligned} A_1 &= [F(X_\cdot(\cdot))] \\ A_2 &= \left[\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right] \\ A_3 &= \left[F(X(\cdot)), \int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right] \ . \end{aligned}$$

Since X is a finite quadratic variation process, by Corollary 5.8, its window process $X(\cdot)$ admits $\chi^0([-\tau,0]^2)$-quadratic variation $[X(\cdot)]$. Moreover by Example 7.20 and Remark 7.25 the map $DF \otimes DF : C([-\tau,0]) \times C([-\tau,0]) \longrightarrow \chi^0([-\tau,0]^2)$ is a continuous application. Applying Theorem 7.21 and (7.19) of Proposition 7.31 we obtain

$$\begin{aligned} A_1 &= \int_0^\cdot \langle DF(X_s(\cdot)) \otimes DF(X_s(\cdot)), d\widetilde{[X_\cdot(\cdot)]}_s\rangle = \\ &= \int_0^\cdot \alpha_0^2(X_s(\cdot))d[X]_s = \int_0^\cdot \alpha_0^2(X_s(\cdot))d[M]_s \ . \end{aligned}$$

The term A_2 is the quadratic variation of an Itô's integral because the stochastic process $\alpha_0\big(X_s(\cdot)\big)$ is $(\mathcal{F}_s)$-adapted, so that

$$A_2 = \int_0^\cdot \alpha_0^2(X_s(\cdot))d[M]_s \ .$$

It remains to prove that $A_3 = \int_0^\cdot \alpha_0^2(X_s(\cdot))d[M]_s$. We define $G : C([-\tau,0]) \longrightarrow \mathbb{R}$ by $G(\eta) = \eta(0)$. We observe that $\bar{M} = G(\bar{M}(\cdot))$ where $\bar{M}(\cdot)$ denotes as usual the window process associated to $\bar{M}$. G is Fréchet differentiable and $DG(\eta) = \delta_0$, therefore DG is continuous from $C([-\tau,0])$ to $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$. Moreover by Example 7.20 we know that $DF \otimes DG : C([-\tau,0]) \times C([-\tau,0]) \longrightarrow \chi^0([-\tau,0]^2)$ is a continuous application. Remark 7.15 point 2. says that the $\chi^0([-\tau,0]^2)$-covariation between $X(\cdot)$ and $\bar{M}(\cdot)$ exists and it is given by

$$[X(\cdot), \bar{M}(\cdot)](\mu) = \mu(\{0,0\})[X, \bar{M}] \ . \tag{7.20}$$

By Remark 2.9 3) and the usual properties of stochastic calculus we have $[X,\bar{M}] = [M,\bar{M}] + [A,\bar{M}] = \big[M, \int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)dM_s\big] = \int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)d[M]_s$. Finally applying again Theorem 7.21, equation (7.19) in Proposition 7.31 and result (7.20) we obtain

$$
\begin{aligned}
A_3 &= [F(X(\cdot)), G(\tilde{M}(\cdot))] = \\
&= \int_0^{\cdot} \langle DF(X_s(\cdot)) \otimes DG(\bar{M}_s(\cdot)), d\widetilde{[X(\cdot), \bar{M}(\cdot)]}_s \rangle = \\
&= \int_0^{\cdot} \alpha_0(X_s(\cdot)) d[X,\bar{M}]_s = \int_0^{\cdot} \alpha_0^2(X_s(\cdot)) d[M]_s \,.
\end{aligned}
$$

The result is now established. □

Theorem 7.33 admits a small generalization.

Theorem 7.34. Let X be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $X = M + A$, M being the local martingale and A a zero quadratic variation process with $A_0 = 0$. Let $F : C([-\tau,0]) \longrightarrow \mathbb{R}$ be a Fréchet differentiable function such that $DF : C([-\tau,0]) \longrightarrow \mathcal{D}_a([-\tau,0]) \oplus L^2([-\tau,0])$ is continuous. We have the following.

1. $F(X(\cdot))$ is a finite quadratic variation process and

$$
[F(X(\cdot))] = \sum_{i=0,\dots,N} \int_0^t \left[D^{\delta_{a_i}} F(X_s(\cdot))\right]^2 d[M]_{s+a_i} \tag{7.21}
$$

2. $F(X(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $F(X(\cdot)) = \bar{M} + \bar{A}$, where $\bar{M}$ is the local martingale defined by

$$
\bar{M}_\cdot := F\left(X_0(\cdot)\right) + \int_0^{\cdot} D^{\delta_0} F(X_s(\cdot)) dM_s
$$

and $\bar{A}$ is the $(\mathcal{F}_t)$-martingale orthogonal process, see Definition 2.19.

3. Process $\bar{A}$ is a finite quadratic variation process and

$$
[\bar{A}]_t = \sum_{i=1,\dots,N} \int_0^t \left[D^{\delta_{a_i}} F(X_s(\cdot))\right]^2 d[M]_{s+a_i} \tag{7.22}
$$

4. In particular $\{F(X_t(\cdot)); t \in [0,-a_1]\}$ is a Dirichlet process with local martingale component $\bar{M}$.

Proof. In this proof $\alpha_i(\eta)$ will denote $D^{\delta_{a_i}} F(\eta) = DF(\eta)(\{a_i\})$.

1. By Example 7.20 we know that $DF \otimes DF : C([-\tau,0]) \times C([-\tau,0]) \longrightarrow \chi^2([-\tau,0]^2)$ and it is a continuous map. Applying Theorem 7.21, equation (7.19) in Proposition 7.31 and Example 5.15 point

4) we obtain

$$
\begin{aligned}
[F(X(\cdot))]_t &= \int_0^t \langle DF(X_s(\cdot)) \otimes DF(X_s(\cdot)), d\widetilde{[X_s(\cdot)]}\rangle = \\
&= \int_0^t \sum_{i,j=0\dots,N} \alpha_i(X_s(\cdot))\alpha_j(X_s(\cdot)) d[X_{\cdot+a_i}, X_{\cdot+a_j}]_s = \\
&= \sum_{i=0,\dots,N} \int_0^t \alpha_i^2(X_s(\cdot)) d[M_{\cdot+a_i}]_s
\end{aligned}
$$

and (7.21) is proved.

2. To show that $F(X(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process we need to show that $[F(X(\cdot))-\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s, N]$ is zero for every $(\mathcal{F}_t)$-continuous local martingale N. Again we set $G : C([-\tau,0]) \longrightarrow \mathbb{R}$ by $G(\eta) = \eta(0)$. It holds $N_t = G(N_t(\cdot))$. We remark that function G is Fréchet differentiable with $DG : C([-\tau,0]) \longrightarrow \mathcal{D}_0([-\tau,0])$ continuous and $DG(\eta) = \delta_0$. Example 7.20 says that $DF \otimes DG : C([-\tau,0]) \times C([-\tau,0]) \longrightarrow \chi^2([-\tau,0]^2)$ and it is a continuous map. Theorem 7.13 implies that $X(\cdot)$ and $N(\cdot)$ admit a $\chi^2([-\tau,0]^2)$-covariation which equals

$$
[X(\cdot), N(\cdot)](\mu) = \mu(\{0,0\})[M,N] \ . \tag{7.23}
$$

By Theorem 7.21 and (7.23) we have

$$
\begin{aligned}
[F(X(\cdot)), N]_t = [F(X(\cdot)), G(N(\cdot))]_t &= \int_0^t \langle DF(X_s(\cdot)) \otimes DG(N_s(\cdot)), d\widetilde{[X(\cdot), N(\cdot)]_s}\rangle = \\
&= \int_0^t \alpha_0(X_s(\cdot)) d[M,N]_s \ .
\end{aligned} \tag{7.24}
$$

On the other hand $\alpha_0\big(X_s(\cdot)\big)$ is $(\mathcal{F}_s)$-adapted and $\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s$ is an Itô's integral; so by Remark 2.9 3. and usual properties of stochastic calculus, it yields

$$
\left[\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s, N\right]_t = \int_0^t \alpha_0\big(X_s(\cdot)\big)d[M,N]_s
$$

and the result follows.

3. By bilinearity of the real covariation we have $[\bar{A}] = [F(X(\cdot))] + [\bar{M}] - 2[F(X(\cdot)), \bar{M}]$. The first bracket is equal to (7.21) and the second term gives

$$
\left[\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s \ .
$$

Setting $N_t = \int_0^t \alpha_0(X_s(\cdot))dM_s$, (7.24) gives

$$
\left[F(X(\cdot)), \int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s
$$

and (7.22) follows.

4. It is an easy consequence of (7.22) since $(\bar{A}_t)_{t\in[0,-a_1[}$ is a zero quadratic variation process.

□

Remark 7.35. 1. Theorem 7.34 gives a class of examples of $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation which are not necessarily $(\mathcal{F}_t)$-Dirichlet processes.

2. An example of $F : C([-\tau,0]) \longrightarrow \mathbb{R}$ Fréchet differentiable such that $DF : C([-\tau,0]) \longrightarrow \mathcal{D}_a([-\tau,0]) \oplus L^2([-\tau,0])$ continuously is, for instance, $F(\eta) = \sum_{i=0}^{N} f_i\big(\eta(a_i)\big)$, with $f_i \in C^1(\mathbb{R})$. We have $DF(\eta) = \sum_{i=0}^{N} f_i'\big(\eta(a_i)\big)\delta_{a_i}$.

3. Let $a \in [-\tau,0[$ and W be a classical $(\mathcal{F}_t)$-Brownian motion, process X defined as $X_t := W_{t+a}$ is an $(\mathcal{F}_t)$-weak Dirichlet process that is not $(\mathcal{F}_t)$-Dirichlet.
This follows from Theorem 7.34, point 2. and 3. taking $F(\eta) = \eta(a)$. In particular point 3. implies that the quadratic variation of the martingale orthogonal process is $[\bar{A}]_t = (t+a)^+$. This result was also proved directly in Proposition 4.11 in [11].

We now go on with a stability result concerning weak Dirichlet processes.

Theorem 7.36. Let D be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation where M is the local martingale part. Let $F : [0,T] \times C([-\tau,0]) \longrightarrow \mathbb{R}$ continuous. We suppose moreover that $(t,\eta) \mapsto DF(t,\eta)$ exists with values in $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$ and $DF : [0,T] \times C([-\tau,0]) \longrightarrow \mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$ is continuous.
Then $F(\cdot, D_\cdot(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part

$$\bar{M}_t^F = F(0, D_0(\cdot)) + \int_0^t D^{\delta_0} F(s, D_s(\cdot)) dM_s \ . \tag{7.25}$$

Proof. In this proof we will denote real processes $\bar{M}^F$ simply by $\bar{M}$. We need to show that for any $(\mathcal{F}_t)$-continuous local martingale N

$$\big[F(\cdot, D(\cdot)) - \bar{M}, N_\cdot\big]_t = 0 \quad a.s. \tag{7.26}$$

Since the covariation of semimartingales coincides with the classical covariation, see Remark 2.9.3, it follows

$$\big[\bar{M}, N\big]_t = \int_0^t D^{\delta_0} F(s, D_s(\cdot)) d[M,N]_s \ . \tag{7.27}$$

It remains to check that, for every $t \in [0,T]$,

$$[F(\cdot, D(\cdot)), N]_t = \int_0^t D^{\delta_0} F(s, D_s(\cdot)) d[M,N]_s \ .$$

For this point we have to evaluate the ucp limit of

$$\int_0^t \Big(F(s+\epsilon, D_{s+\epsilon}(\cdot)) - F(s, D_s(\cdot))\Big)\frac{N_{s+\epsilon}-N_s}{\epsilon}ds \tag{7.28}$$

if it exists. (7.28) can be written as the sum of the two terms

$$\begin{aligned} I_1(t,\epsilon) &= \int_0^t \Big(F(s+\epsilon, D_{s+\epsilon}(\cdot)) - F(s+\epsilon, D_s(\cdot))\Big)\frac{N_{s+\epsilon}-N_s}{\epsilon}ds\,, \\ I_2(t,\epsilon) &= \int_0^t \Big(F(s+\epsilon, D_s(\cdot)) - F(s, D_s(\cdot))\Big)\frac{N_{s+\epsilon}-N_s}{\epsilon}ds\,. \end{aligned}$$

First we prove that $I_1(t,\epsilon)$ converges to $\int_0^t D^{\delta_0}F(s, D_s(\cdot))d[M,N]_s$.
If $G : C([-\tau,0]) \to \mathbb{R}$ is again the function $G(\eta) = \eta(0)$, then G is of class C^1 and $DG(\eta) = \delta_0$ for all $\eta \in C([-\tau,0])$ so that $DG : C([-\tau,0]) \longrightarrow \mathcal{D}_0([-\tau,0])$ is continuous. In particular it holds the equality $\eta(0) = G(\eta(\cdot)) = \langle \delta_0, \eta\rangle$. We express

$$\begin{aligned} I_1(t,\epsilon) &= \int_0^t \langle DF(s+\epsilon, D_s(\cdot)), (D_{s+\epsilon}(\cdot) - D_s(\cdot))\rangle \frac{N_{s+\epsilon}-N_s}{\epsilon}ds + R_1(t,\epsilon) \\ &= \int_0^t \langle DF(s+\epsilon, D_s(\cdot)), (D_{s+\epsilon}(\cdot) - D_s(\cdot))\rangle \frac{\langle \delta_0, N_{s+\epsilon}(\cdot)N_s(\cdot)\rangle}{\epsilon}ds + R_1(t,\epsilon), \end{aligned} \tag{7.29}$$

and

$$\begin{aligned} R_1(t,\epsilon) &= \int_0^t \Big[\int_0^1 \langle DF\big(s+\epsilon, (1-\alpha)D_s(\cdot) + \alpha D_s(\cdot)\big) - DF\big(s+\epsilon, D_s(\cdot)\big), \big(D_{s+\epsilon}(\cdot) - D_s(\cdot)\big)\rangle d\alpha\Big] \times \\ &\qquad\qquad \times \frac{\langle \delta_0, N_{s+\epsilon}(\cdot) - N_s(\cdot)\rangle}{\epsilon}ds = \\ &= \int_0^t \int_0^1 \langle DF\big(s+\epsilon, (1-\alpha)D_s(\cdot) + \alpha D_s(\cdot)\big)\otimes\delta_0 - DF\big(s+\epsilon, D_s(\cdot)\big)\otimes\delta_0, \\ &\qquad\qquad \frac{\big(D_{s+\epsilon}(\cdot) - D_s(\cdot)\big)\otimes\big(N_{s+\epsilon}(\cdot) - N_s(\cdot)\big)}{\epsilon}\rangle d\alpha\, ds\,. \end{aligned}$$

We will show that $R_1(\cdot,\varepsilon)$ converges ucp to zero. Since $L^2([-\tau,0])\otimes\mathcal{D}_0([-\tau,0])$ is a Hilbert space, making the proper Riesz identification for $t\in[0,T]$, $\eta_1,\eta_2 \in C([-\tau,0])$ the map $DF(t,\eta_1)\otimes DG(\eta_2)$ coincides with the tensor product $DF(t,\eta_1)\otimes\delta_0$, see Proposition 7.19. As in Example 7.20 map $DF\otimes\delta_0 : [0,T]\times C([-\tau,0])$ takes values in $\chi^0([-\tau,0]^2)$ and it is a continuous map.
We denote by $\mathcal{U} = \mathcal{U}(\omega)$ the closed convex hull of the compact subset $\mathcal{V}$ of $C([-\tau,0])$ defined, for every ω, by

$$\mathcal{V} = \mathcal{V}(\omega) := \{D_t(\omega);\, t\in[0,T]\}\,.$$

According to Theorem 5.35 from [2], $\mathcal{U}(\omega) = \overline{conv(\mathcal{V})(\omega)}$ is compact, so the function $DF(\cdot,\cdot)\otimes\delta_0$ on $[0,T]\times\mathcal{U}$ is uniformly continuous and we denote by $\varpi^{[0,T]\times\mathcal{U}}_{DF(\cdot,\cdot)\otimes\delta_0}$ the continuity modulus of the application

$DF(\cdot,\cdot)\otimes\delta_0$ restricted to $[0,T]\times\mathcal{U}$. $\varpi_{DF(\cdot,\cdot)\otimes\delta_0}^{[0,T]\times\mathcal{U}}$ is, as usual, a positive, increasing function on $\mathbb{R}^+$ converging to zero when the argument converges to zero. So we have

$$\sup_{t\in[0,T]}|R_1(t,\epsilon)| \leq \int_0^T \varpi_{DF(\cdot,\cdot)\otimes\delta_0}^{[0,T]\times\mathcal{U}}(\epsilon)\left\|\frac{\big(D_{s+\epsilon}(\cdot)-D_s(\cdot)\big)\otimes\big(N_{s+\epsilon}(\cdot)-N_s(\cdot)\big)}{\epsilon}\right\|_{\chi^0([-\tau,0]^2)} ds\ . \tag{7.30}$$

We recall by Remark 7.15, point 3. that $D(\cdot)$ and $N(\cdot)$ admit a $\chi^2([-\tau,0]^2)$-covariation. In particular using condition **H1** and (7.30) claim $R_1(t,\epsilon)\xrightarrow[\epsilon\longrightarrow 0]{ucp} 0$ follows.
On the other hand, the first addend in (7.29) can be rewritten as

$$\int_0^t \langle DF\big(s,D_s(\cdot)\big)\otimes\delta_0, \frac{\big(D_{s+\epsilon}(\cdot)-D_s(\cdot)\big)\otimes\big(N_{s+\epsilon}(\cdot)-N_s(\cdot)\big)}{\epsilon}\rangle ds + R_2(t,\epsilon) \tag{7.31}$$

where $R_2(t,\epsilon)\xrightarrow[\epsilon\longrightarrow 0]{ucp} 0$ arguing similarly as for $R_1(t,\epsilon)$.
In view of application of Proposition 7.7 we observe that, since $DF\otimes\delta_0 : [0,T]\times C([-\tau,0]) \longrightarrow \mathcal{D}_{0,0}([-\tau,0]^2)\oplus\mathcal{D}_0([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ is continuous, then the process $H_s = DF\big(s,D_s(\cdot)\big)\otimes\delta_0$ takes obviously values in the separable closed subspace $\mathcal{V}$ of $\chi^2([-\tau,0]^2)$ defined by $\mathcal{V} := \mathcal{D}_{0,0}([-\tau,0]^2)\oplus\mathcal{D}_0([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$. Using bilinearity and the Proposition mentioned above the integral in (7.31) converges then ucp to

$$\int_0^t \langle DF\big(s,D_s(\cdot)\big)\otimes\delta_0, d\widetilde{[D(\cdot),N(\cdot)]}_s\rangle\ . \tag{7.32}$$

By (7.19) in Proposition 7.31 in the case $a_i=a_j=0$, (7.32) equals

$$\int_0^t D^{\delta_0}F(s,D_s(\cdot))d[D,N]_s = \int_0^t D^{\delta_0}F(s,D_s(\cdot))d[M,N]_s\ . \tag{7.33}$$

It remains to show that $I_2(\cdot,\epsilon)\xrightarrow[\epsilon\longrightarrow 0]{ucp} 0$.
By stochastic Fubini's theorem we obtain

$$I_2(t,\epsilon) = \int_0^t \xi(\epsilon,r)dN_r$$

where

$$\xi(\epsilon,r) = \frac{1}{\epsilon}\int_{0\vee(r-\epsilon)}^r F(s+\epsilon,D_s(\cdot)) - F(s,D_s(\cdot))ds\ .$$

Proposition 2.26, chapter 3 of [41] says that $I_2(\cdot,\epsilon)\xrightarrow[\epsilon\longrightarrow 0]{ucp} 0$ if

$$\int_0^T \xi^2(\epsilon,r)d[N]_r \xrightarrow[\epsilon\longrightarrow 0]{} 0 \tag{7.34}$$

in probability. We fix $\omega\in\Omega$ and we even show that the convergence in (7.34) holds pointwise (so a.s.). We denote by $\varpi_F^{[0,T]\times\mathcal{U}}$ the continuity modulus of the application F restricted to the compact set $[0,T]\times\mathcal{U}$.

For every $r \in [0,T]$ we have

$$|\xi(\epsilon,r)| \leq \sup_{r\in[0,T]} |F(r+\epsilon, D_r(\cdot)) - F(r, D_r(\cdot))| \leq \varpi_F^{[0,T]\times\mathcal{U}}(\epsilon)$$

which converges to zero for ϵ going to zero since function F on $[0,T]\times\mathcal{U}$ is uniformly continuous on the compact set and $\varpi_F^{[0,T]\times\mathcal{U}}$ is, as usual, a positive, increasing function on $\mathbb{R}^+$ converging to zero when the argument converges to zero. By Lebesgue's dominated convergence theorem we finally obtain (7.34). $\square$

Chapter 8

Itô's formula

We are now able to state an Itô's formula for stochastic processes with values in a general Banach space.

Theorem 8.1. Let B be a separable Banach space, χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X a B-valued continuous process admitting a χ-quadratic variation. Let $F : [0,T] \times B \longrightarrow \mathbb{R}$ of class $C^{1,2}$ Fréchet. such that

$$D^2F : [0,T] \times B \longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \text{ continuously with respect to } \chi \tag{8.1}$$

Then for every $t \in [0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DF(s,X_s), d^- X_s\rangle_B$$

exists and following formula holds

$$F(t,X_t) = F(0,X_0) + \int_0^t \partial_t F(s,X_s)ds + \int_0^t {}_{B^*}\langle DF(s,X_s), d^- X_s\rangle_B + \frac{1}{2}\int_0^t {}_\chi\langle D^2F(s,X_s), d\widetilde{[X]}_s\rangle_{\chi^*} \ a.s. \tag{8.2}$$

Proof. We fix $t \in [0,T]$ and we observe that the quantity

$$I_0(\epsilon,t) = \int_0^t \frac{F(s+\epsilon, X_{s+\epsilon}) - F(s,X_s)}{\epsilon} ds \tag{8.3}$$

converges ucp for $\epsilon \to 0$ to $F(t,X_t) - F(0,X_0)$ since $\big(F(s,X_s)\big)_{s\geq 0}$ is continuous. At the same time, using Taylor's expansion, (8.3) can be written as the sum of the two terms:

$$I_1(\epsilon,t) = \int_0^t \frac{F(s+\epsilon, X_{s+\epsilon}) - F(s,X_{s+\epsilon})}{\epsilon} ds \tag{8.4}$$

and

$$I_2(\epsilon, t) = \int_0^t \frac{F(s, X_{s+\epsilon}) - F(s, X_s)}{\epsilon} ds \ . \tag{8.5}$$

First we prove that

$$I_1(\epsilon, t) \longrightarrow \int_0^t \partial_t F(s, X_s) ds \tag{8.6}$$

in probability. In fact

$$I_1(\epsilon, t) = \int_0^t \partial_t F(s, X_{s+\epsilon}) ds + R_1(\epsilon, t) \tag{8.7}$$

where

$$R_1(\epsilon, t) = \int_0^t \int_0^1 \partial_t F\big(s + \alpha\epsilon, X_{s+\epsilon}\big) - \partial_t F(s, X_{s+\epsilon}) d\alpha ds \ .$$

For $x \in \Omega$, we have

$$\sup_{t\in[0,T]} |R_1(\epsilon, t)| \leq T\, \varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$$

where $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus in ϵ of the application $\partial_t F : [0,T] \times B \longrightarrow \mathbb{R}$ restricted to $[0,T] \times \mathcal{U}$ and $\mathcal{U} = \mathcal{U}(\omega)$ is the (random) compact set defined in (7.14). From the continuity of the $\partial_t F$ as function from $[0,T] \times B$ to $\mathbb{R}$ follows that the restriction on $[0,T] \times \mathcal{U}$ is uniformly continuous and $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. Therefore we have proved that $R_1(\epsilon, \cdot) \to 0$ ucp as $\epsilon \to 0$.

On the other hand the first term in (8.7) can be rewritten as

$$\int_0^t \partial_t F(s, X_s) ds + R_2(\epsilon, t)$$

where $R_2(\epsilon, t) \to 0$ ucp arguing similarly as for $R_1(\epsilon, t)$ and so convergence (8.6) is established.

The second addend $I_2(\epsilon, t)$ in (8.5), can also be approximated by Taylor's expansion and it can be written as the sum of the following three terms:

$$I_{21}(\epsilon, t) = \int_0^t {}_{B^*}\langle DF(s, X_s), \frac{X_{s+\epsilon} - X_s}{\epsilon} \rangle_B ds \ ,$$

$$I_{22}(\epsilon, t) = \frac{1}{2} \int_0^t {}_{\chi}\langle D^2 F(s, X_s), \frac{(X_{s+\epsilon} - X_s)^{\otimes 2}}{\epsilon} \rangle_{\chi^*} ds \ ,$$

$$I_{23}(\epsilon, t) = \int_0^t \left[\int_0^1 \alpha\, {}_{\chi}\langle D^2 F\left(s, (1-\alpha) X_{s+\epsilon} + \alpha X_s\right) - D^2 F(s, X_s), \frac{(X_{s+\epsilon} - X_s)^{\otimes 2}}{\epsilon} \rangle_{\chi^*} \, d\alpha \right] ds \ .$$

Since $D^2 F : [0,T] \times B \longrightarrow \chi$ is continuous and B separable, we observe that the process H defined by $H_s = D^2 F(s, X_s)$ takes values in a separable closed subspace $\mathcal{V}$ of χ. Applying Corollary 4.33, it yields

$$I_{22}(\epsilon, t) \xrightarrow[\epsilon \to 0]{\mathbb{P}} \frac{1}{2} \int_0^t {}_{\chi}\langle D^2 F(s, X_s), d\widetilde{[X]}_s \rangle_{\chi^*}$$

for every $t \in [0,T]$.
We analyse now $I_{23}(\epsilon,t)$ and we show that $I_{23}(\epsilon,t) \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0$. In fact we have

$$\begin{aligned}|I_{23}(\epsilon,t)| &\leq \frac{1}{\epsilon}\int_0^t\int_0^1 \alpha \left| {}_{\chi}\langle D^2F\left(s,(1-\alpha)X_{s+\epsilon}+\alpha X_s\right) - D^2F(s,X_s), (X_{s+\epsilon}-X_s)\otimes^2\rangle_{\chi^*}\right| d\alpha\, ds \leq \\ &\leq \frac{1}{\epsilon}\int_0^t\int_0^1 \alpha \left\| D^2F\left(s,(1-\alpha)X_{s+\epsilon}+\alpha X_s\right) - D^2F(s,X_s)\right\|_{\chi} \left\|(X_{s+\epsilon}-X_s)\otimes^2\right\|_{\chi^*} d\alpha\, ds \leq \\ &\leq \varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)\int_0^t \sup_{\|\phi\|_{\chi}\leq 1}\left|\langle \phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle\right| ds\ ,\end{aligned}$$

where $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus of the application $D^2F : [0,T]\times B \longrightarrow \chi$ restricted to $[0,T]\times\mathcal{U}$ where $\mathcal{U}$ is the same random compact set introduced in (7.14). So again D^2F on $[0,T]\times\mathcal{U}$ is uniformly continuous and $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. Taking into account condition **H1** in the definition of χ-quadratic variation, $I_{23}(\epsilon,t) \to 0$ in probability when ϵ goes to zero.
Since $I_0(\epsilon,t)$, $I_1(\epsilon,t)$, $I_{22}(\epsilon,t)$ and $I_{23}(\epsilon,t)$ converge in probability for every fixed $t\in[0,T]$, it follows

$$I_{21}(\epsilon,t) \longrightarrow \int_0^t {}_{B^*}\langle DF(s,X_s), d^-X_s\rangle_B$$

in probability. This insures by definition that the forward integral exists.
This also in particular implies the so-called Itô's formula (8.2). □

As corollary of Theorem 8.1 we have the so-called time-homogeneous Itô's formula, i.e. without the dependence on the time variable t.

Corollary 8.2. Let B be a separable Banach space, χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X a B-valued continuous process admitting a χ-quadratic variation. Let $G : B \longrightarrow \mathbb{R}$ a function of class C^2 Fréchet such that

$$D^2G : B \longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \text{ continuously with respect to } \chi \tag{8.8}$$

Then for every $t\in[0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DG(X_s), d^-X_s\rangle_B$$

exists and following formula a.s. holds:

$$G(X_t) = G(X_0) + \int_0^t {}_{B^*}\langle DG(X_s), d^-X_s\rangle_B + \frac{1}{2}\int_0^t {}_{\chi}\langle D^2G(X_s), d\widetilde{[X]}_s\rangle_{\chi^*}. \tag{8.9}$$

We make now some operational remarks. The Chi-subspace χ of $(B\hat{\otimes}_\pi B)^*$ constitutes a degree of freedom in the statement of Itô's formula. In order to find the suitable expansion for $F(t,X_t)$ we may proceed as follows.

- Let $F : [0,T] \times B \longrightarrow \mathbb{R}$ of class $C^{1,1}([0,T] \times B)$ we compute the second order derivative D^2F if it exists.
- We look for the existence of a Chi-subspace χ of $(B\hat{\otimes}_\pi B)^*$ for which the range of $D^2F : [0,T] \times B \longrightarrow (B\hat{\otimes}_\pi B)^*$ is included in χ and it is continuous with respect to the topology of χ.
- We verify that X admits a χ-quadratic variation.

We observe that whenever X admits a global quadratic variation, i.e. $\chi = (B\hat{\otimes}_\pi B)^*$, previous points reduce to check that $F \in C^{1,2}([0,T] \times B)$. When X is a semimartingale we rediscover the classical Itô's formula for Banach valued processes, see [46].

We illustrate now an application of Corollary 8.2 for window processes $X(\cdot)$, where X is a real continuous finite quadratic variation process. $X(\cdot)$ can be reasonably observed under the two following perspectives:

a) $X(\cdot)$ is $C([-\tau,0])$-valued and χ has to be a Chi-subspace of $(C([-\tau,0])\hat{\otimes}_\pi C([-\tau,0]))^*$. Related examples of such χ are listed in Example 4.7.

b) $X(\cdot)$ is $L^2([-\tau,0])$-valued and χ has to be a Chi-subspace of $(L^2([-\tau,0])\hat{\otimes}_\pi L^2([-\tau,0]))^*$. Related examples of such χ are listed in Examples 4.12.

We illustrate this in a elementary situation.
Let $G : L^2([-\tau,0]) \longrightarrow \mathbb{R}$ be defined by

$$G(\eta) = \int_{-\tau}^{0} \eta^2(s)ds = \|\eta\|^2_{L^2([-\eta,0])} \; ; \tag{8.10}$$

G is a continuous function as well as its restriction F to $C([-\tau,0])$.
We have

$$D^2G : L^2([-\tau,0]) \longrightarrow Diag_{\mathcal{B}}([-\tau,0]^2) \; . \tag{8.11}$$

In fact it is constant and equal to twice the inner product in $L^2([-\tau,0])$, i.e. for every $\eta \in L^2([-\tau,0])$, $D^2G(\eta)$ is the bilinear map such that

$$(f,g) \mapsto 2\langle f,g\rangle_{L^2([-\tau,0])}.$$

Also the restriction F is C^2 Fréchet because

$$D^2F : C([-\tau,0]) \longrightarrow Diag([-\tau,0]^2) \tag{8.12}$$

is the constant Radon measure on $[-\tau,0]^2$, defined for every $\eta \in C([-\tau,0])$ by

$$D^2F(\eta) \mapsto \mu(dx,dy) = 2\mathbb{1}_{[-\tau,0]}(x)\delta_y(dx)dy \; . \tag{8.13}$$

Being constant, previous maps are both continuous with respect to the corresponding χ-topology. The proposition below gives in particular a representation of a forward type integral.

Proposition 8.3. Let $0 < \tau \le T$ and X be a continuous real process such that $[X]_t = t$. We set $B = C([-\tau, 0])$. Then for the B-valued window process $X(\cdot)$ it holds

$$2\int_0^t {}_{B^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_B = \|X_t(\cdot)\|^2_{L^2([-\tau,0])} - \int_0^{t\wedge\tau} (t-y)dy \; . \tag{8.14}$$

Proof. We apply Itô's formula stated in Corollary 8.2 to $F(X_t(\cdot))$. In this case, for η, h, h_1 and h_2 in $C([-\tau, 0])$, we have

$$DF(\eta)(h) = 2\int_{-\tau}^0 \eta(s)h(s)ds$$
$$D^2F(\eta)(h_1, h_2) = 2\int_{-\tau}^0 h_1(s)h_2(s)ds = 2\langle h_1, h_2\rangle_{L^2([-\eta,0])}$$

where D^2F was given in (8.13). In term of measures, it gives

$$\begin{aligned} D_{dx}F(\eta) &= 2\mathbb{1}_{[-\tau,0]}(x)\eta(x)dx \\ D^2_{dx\,dy}F(\eta) &= 2\mathbb{1}_{[-\tau,0]}(x)\delta_y(dx)dy \end{aligned} \tag{8.15}$$

We set $\chi = Diag([-\tau, 0]^2)$. Using Proposition 5.18 and the fact that $[X]_t = t$, the $X(\cdot)$ admits a χ-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} g(-x)(t-x)dx \; ,$$

where μ is a diagonal measure $\mu(dx, dy) = g(x)\delta_y(dx)dy$, $g \in L^\infty([-\tau, 0])$. In this case the second order derivative is given by the constant measure (8.13), then $g \equiv 2$. For every $t \in [0, T]$, Corollary 8.2 implies the existence of the forward integral

$$\int_0^t {}_{B^*}\langle DF(X_s(\cdot)), \frac{X_{s+\epsilon}(\cdot) - X_s(\cdot)}{\epsilon}\rangle_B ds \xrightarrow[\epsilon\to 0]{\mathbb{P}} 2\int_0^t {}_{B^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_B \; . \tag{8.16}$$

Moreover the second order derivative term in Itô's formula becomes a trivial case of Lebesgue-Stieltjes integral:

$$\frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(X_s(\cdot)), d\widetilde{[X(\cdot)]}_s\rangle_{\chi^*} = \frac{1}{2}\,{}_{\chi}\langle \mu, \widetilde{[X(\cdot)]}_t\rangle_{\chi^*} - \frac{1}{2}\,{}_{\chi}\langle \mu, \widetilde{[X(\cdot)]}_0\rangle_{\chi^*} = \frac{1}{2}[X(\cdot)]_t(\mu) = \int_0^{t\wedge\tau}(t-y)dy \; .$$

This concludes the proof. $\square$

Remark 8.4.

1. Let now $H = L^2([-\tau, 0])$. Expressing $G(X_t(\cdot))$ where $X(\cdot)$ is seen as a H-valued process and G is defined as in (8.10). By Corollary 8.2 we obtain

$$2\int_0^t {}_{H^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_H = \|X_t(\cdot)\|^2_{L^2([-\tau,0])} - \int_0^{t\wedge\tau}(t-x)dx \; . \tag{8.17}$$

In fact $X(\cdot)$ admits a $Diag_{\mathcal{B}}([-\tau, 0]^2)$-quadratic variation given by (5.32), see Proposition 5.30.

2. Remark 3.5 implies that

$$\int_0^t {}_{B^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_B = \int_0^t {}_{H^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_H$$

   so that point 1) provides another proof of Proposition 8.3.

Remark 8.5. In the case X is a classical Brownian motion W, formula (8.17) was established in Example 8.7 of [73]. Their techniques use Skorohod anticipating calculus and they only could be applied because X is Gaussian. In fact even when $X = W$, the forward integral $\int_0^t {}_{H^*}\langle W_s(\cdot), d^- W_s(\cdot)\rangle_H$ involves anticipating calculations. We observe that our considerations do not make any assumption on the law of X.

Chapter 9

A generalized Clark-Ocone formula

9.1 Preliminaries

We start with a technical definition.

Definition 9.1. Let $f : \mathbb{R} \to \mathbb{R}$. f will be called a **subexponential** if there exist $M > 0$ and $\gamma > 0$ such that $|f(y)| \le e^{\gamma |y|}$ for $|y| > M$.

Next proposition gives necessary and sufficient condition such that, given a Gaussian random variable ζ and a subexponential function f, $f(\zeta)$ belong to $L^p(\Omega)$, $p \ge 1$.

Proposition 9.2. Let ζ be a Gaussian non-degenerate random variable. Let $f : \mathbb{R} \to \mathbb{R}$ be a subexponential function.
Then $f(\zeta) \in L^p(\Omega)$ if and only if $f \in L^p_{loc}(\mathbb{R})$, $p \ge 1$.

Proof. This is a consequence of the fact that the Gaussian density is equivalent to Lebesgue measure on compact intervals. □

In this section we will consider $\tau = T$. Let $X = (X_t)_{t \in [0,T]}$ be a real continuous stochastic process such that $X_0 = 0$ which is, as usual, prolongated by continuity outside $[0, T]$ and such that $[X]_t = t$. Let $H : C([-T, 0]) \longrightarrow \mathbb{R}$ be a Borel functional; in this section we aim at representing the random variable

$$h = H(X_T(\cdot)) \ . \tag{9.1}$$

The main task will consist in looking for classes of functionals H for which there is $H_0 \in \mathbb{R}$ and a process ξ adapted with respect to the canonical filtration of X such that h admits the representation

$$h = H_0 + \int_0^T \xi_s d^- X_s \ . \tag{9.2}$$

Moreover we look for an explicit expression for H_0 and ξ.

Remark 9.3. If X is a classical Brownian motion W equipped with its canonical filtration $(\mathcal{F}_t)$, and $h \in L^2(\Omega)$, the martingale representation theorem states the existence of a predictable process $\xi \in L^2(\Omega \times [0,T])$ such that $h = \mathbb{E}[h] + \int_0^T \xi_s dW_s$.
If $h \in \mathbb{D}^{1,2}$ in the sense of Malliavin, the celebrated Clark-Ocone formula implies that $\xi_s = \mathbb{E}\left[D^m h|\mathcal{F}_s\right]$, so that

$$h = \mathbb{E}[h] + \int_0^T \mathbb{E}\left[D^m h|\mathcal{F}_s\right] dW_s \tag{9.3}$$

where D^m is the Malliavin gradient.
We remind that [68] obtains a generalization of (9.3) when $h \in L^2(\Omega)$ making use of predictable projections of a Wiener distributions in the sense of [72].

Example 9.4. We list some examples of processes X such that $[X]_t = c\,t$, c being a constant. As we will see there are several classes of such processes, Gaussian or non-Gaussian.

1. The most celebrated example is of course the classical Brownian motion $X = W$.
2. The first non-Brownian example can be obtained adding a zero quadratic variation process A, $X = W + A$. If A is $(\mathcal{F}_t)$-adapted where $(\mathcal{F}_t)$ is the canonical filtration of W then X is an $(\mathcal{F}_t)$-Dirichlet process.
3. We can consider a bifractional Brownian motion $X = B^{H,K}$ of parameters $H \in]0,1[$, $K \in]0,1[$ where $HK = 1/2$. In this case $c = 2^{1-K}$ and X is not a Dirichlet process with respect to its canonical filtration, see [56].
4. For fixed fixed $k \geq 1$, [28] construct a weak k-order Brownian motion X, which in general is not even Gaussian. We recall that X is a **weak k-order Brownian motion** if for every $0 \leq t_1 \leq \cdots \leq t_k < +\infty$, $(X_{t_1}, \cdots, X_{t_k})$ is distributed as $(W_{t_1}, \cdots, W_{t_k})$. If $k \geq 4$ then $[X]_t = t$.

In this paper we do not aim to achieve a full generality but to introduce a methodology which allows to represent a random variable h depending on the whole path of X. Following the same idea it would be possible to consider finite quadratic variation processes of the type $[X]_t = \int_0^t \sigma^2(s, X_s) ds$, where $\sigma : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is some suitable Borel function.

As we said, in this section we are interested in the case where X is a general process with $[X]_t = t$. For this we obtain representations for h when H smoothly depends on the path of X, see Theorem 9.41 and Corollary 9.45, or it is not smooth but it depends on some finite number of Wiener integrals of the type

$\int_0^T g(s)d^-X_s$ where g is of class $C^2([0,T];\mathbb{R})$, see Propositions 9.53 and 9.55.

We are also interested in new representation results even when X is a Brownian motion W if Clark-Ocone formula does not apply.

We start recalling a simple peculiar example, a sort of toy model, which, in spite of its simplicity, provides examples of representation of non-square integrable (and sometimes even not integrable) random variables as we will explain in Remark 9.7, point 1.

Proposition 9.5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with polynomial growth and $v \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T]\times\mathbb{R})$ which verifies

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\partial^2_{xx} v(t,x) = 0 \\ v(T,x) = f(x) \end{cases} \tag{9.4}$$

Then $h := f(X_T)$ can be represented according to (9.2), i.e.

$$h = H_0 + \int_0^T \xi_s d^-X_s \qquad \text{choosing} \qquad H_0 = v(0,X_0) \qquad \text{and} \qquad \xi_t = \partial_x v(t,X_t). \tag{9.5}$$

Proof. See [64, 3, 11]. □

Remark 9.6.
1. We observe that a solution of (9.4) always exists and it is given by

$$\begin{cases} v(t,x) = \int_{\mathbb{R}} q_{T-t}(x-y)f(y)dy \quad t \in [0,T[\ , \\ v(T,x) = f(x) \end{cases} \tag{9.6}$$

where q_t, $t \in]0,T[$, is the density of the Gaussian law $N(0,t)$.

2. We recall that in that case

$$\int_0^T \xi_s d^-X_s$$

denotes the improper forward integral, $\lim_{t\to T} \int_0^t \xi_s d^-X_s$, i.e. is the limit in probability for $t \to T$ of the forward integral whenever it exists, see Definition 2.3.

This toy model will be rewritten in an infinite dimensional framework in Section 9.3 setting $H : C([-T,0]) \longrightarrow \mathbb{R}$, by $H(\eta) = f(\eta(0))$.

Remark 9.7.

1. Representation (9.5) holds even if h does not belong to $L^1(\Omega)$. For instance it is enough to consider in fact $f(x) = x$ and $X_t = W_t + t\,G$, where G is a non-negative r.v. such that $\mathbb{E}[G] = +\infty$.

2. We observe however that if X is the Brownian motion W, then $h = f(W_T)$, with f continuous with polynomial growth always belongs to $L^p(\Omega)$, with $p \geq 1$, see Proposition 9.2. In Proposition 9.10, we show that the methodology developed to obtain (9.5) can be adapted to represent $h = f(W_T)$ with $h \in L^1(\Omega)$ but f not necessarily continuous. In this case $v \notin C^0([0,T] \times \mathbb{R})$.

3. A similar phenomenon appears when $h = f(G)$ where G is a random variable which smoothly depends on the Brownian path as for instance $G = \int_0^T W_s ds$, $h \in L^1(\Omega)$. h admits in such a case a representation even if f is not continuous, this case being treated in details in Theorem 9.20.

After those preliminaries we emphasize that our idea is to obtain the representation formula by expressing h as

$$\lim_{t \uparrow T} u(t, X_t(\cdot)) \tag{9.7}$$

where $u \in C^{1,2}\left([0,T[\times C([0,T])\right)$ solves an infinite dimensional partial differential equation.

1) When X is not a Brownian motion and H is continuous then u also belongs to $C^0\left([0,T] \times C([0,T])\right)$ and we will be able to show that (9.7) exists by continuity. In this case, only pathwise considerations intervene and there is no need to suppose that the law of X is Wiener measure.
In particular in several different situations, we show the existence of $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ of class $C^{1,2}\left([0,T[\times C([-T,0]);\mathbb{R}\right) \cap C^0\left([0,T] \times C([-T,0]);\mathbb{R}\right)$ such that (9.2) holds with $H_0 = u(0, X_0(\cdot))$ and $\xi_t = D^{\delta_0}u(t, X_t(\cdot))$, where $D^{\delta_0}u(t,\cdot) = Du(t,\cdot)(\{0\})$. For the whole chapter we will explain the validity of a metatheorem which postulates that u can be chosen as a solution of an infinite dimensional PDE problem of the type

$$\begin{cases} \partial_t u(t,\eta) + \text{“}\displaystyle\int_{-t}^0 D^{ac}u\,(t,\eta)\, d\eta\text{''} + \dfrac{1}{2}\langle D^2 u\,(t,\eta)\ ,\ \mathbb{1}_D\rangle = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{9.8}$$

where $\mathbb{1}_D(x,y) := \begin{cases} 1 & \text{if } x = y,\ x,y \in [-T,0] \\ 0 & \text{otherwise} \end{cases}$ and $D^{ac}u\,(t,\eta)$ is the absolute continuous part of the measure $Du\,(t,\eta)$. The integral “ $\int_{-t}^0 D^{ac}u\,(t,\eta)\, d\eta$” has to be suitably defined and term $\langle D^2 u\,(t,\eta)\ ,\ \mathbb{1}_D\rangle$ indicates the evaluation of the second order derivative on the diagonal of the square $[-T,0]^2$.

The program of the chapter consists in illustrating first four particular cases in Sections 9.3, 9.4, 9.5, 9.6 where we will develop explicitly some calculus with Itô formula (8.2) for path dependent functionals of the process. Then we will observe that, in those cases, it is possible to find a function u which solves an infinite dimensional PDE and which gives at the same time the representation result. At that point in Section 9.7 we state a central result. Corollary 9.28 says essentially that if we have a function u solving an infinite dimensional PDE of type (9.8) then $h = u(0, X_0) + \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t$. Sections 9.8 and 9.9 are devoted to give sufficient condition on H to solve the PDE in more general situations. The path dependent functionals of the process is actually motivated by hedging of path dependent options. Whenever it is possible, we will retrieve the terms appearing in the Clark-Ocone's formula.

2) If X is a Brownian motion W, the limit (9.7) may exist in some cases even if H is not continuous by making use of probabilistic technology, as for instance Lemma 9.8. In that case we remark that naturally appear improper forward integrals. This technicality will appear in Proposition 9.10 and in the case treated in Section 9.6. In that context we need a preliminary result.

Lemma 9.8. Let $(\mathcal{F}_t)$ be a Brownian filtration. For any real cadlag $(\mathcal{F}_t)$-martingale $(M_t)_{t\in[0,T]}$ it holds $M_{T^-} := \lim_{t\uparrow T} M_t = M_T$ a.s.

Remark 9.9.
1. We recall that any martingale admits a a cadlag version, even if $(\mathcal{F}_t)$ is any filtration fulfilling the usual conditions.
2. If M is square integrable, it has a continuous version because of martingale representation theorem.

Proof of Lemma 9.8. We set $h = M_T$, so it holds $M_t = \mathbb{E}\left[h|\mathcal{F}_t\right]$. We can easily reduce the problem to the case $h \geq 0$, decomposing $h = h^+ - h^-$ and operating by linearity. Let $N > 0$ be a fixed number. We set $h^N = h \wedge N$.
Since the martingale $\mathbb{E}[h|\mathcal{F}_t]$ admits a cadlag version, it exists a random variable denoted by M_{T^-} such that

$$\mathbb{E}\left[h|\mathcal{F}_t\right] \longrightarrow M_{T^-} \quad \text{a.s.} \tag{9.9}$$

We want to compare $h = M_T$ and M_{T^-} and to show that they are equal a.s. We can rewrite that difference as the sum

$$h - M_{T^-} = I_1 + I_2(t) + I_3(t) + I_4(t)$$

where

$$\begin{aligned}
I_1 &= h - h^N \\
I_2(t) &= h^N - \mathbb{E}\left[h^N|\mathcal{F}_t\right] \\
I_3(t) &= \mathbb{E}\left[h^N|\mathcal{F}_t\right] - \mathbb{E}\left[h|\mathcal{F}_t\right] \\
I_4(t) &= \mathbb{E}\left[h|\mathcal{F}_t\right] - M_{T^-}
\end{aligned}$$

We observe that $\lim_{t\to T}|I_4(t)| = 0$ a.s. because of (9.9). Since h^N is bounded, $h^N \in L^2(\Omega)$ so $\lim_{t\to T} I_2(t) = 0$, because $\mathbb{E}\left[h^N|\mathcal{F}_t\right]$ admits a continuous version by martingale representation theorem. Therefore

$$\begin{aligned} |h - M_{T^-}| = &\leq |I_1| + \lim_{t\to T}|I_2(t)| + \lim\inf_{t\to T}|I_3(t)| + \lim_{t\to T}|I_4(t)| \leq \\ &\leq \left|h - h^N\right| + \lim\inf_{t\to T}\mathbb{E}\left[h^N - h|\mathcal{F}_t\right] \end{aligned} \tag{9.10}$$

Taking the expectation of the left and right-hand side of (9.10) and using Fatou's lemma we obtain

$$\mathbb{E}\left[|h - M_{T^-}|\right] \leq \mathbb{E}\left[\left|h^N - h\right|\right]$$

By Lebesgue dominated convergence theorem, letting $N \to +\infty$ we obtain that previous expectation vanishes. $\square$

9.2 A first Brownian example

Proposition 9.10. Let W be a Brownian motion equipped with its natural filtration $(\mathcal{F}_t)$ and $f : \mathbb{R} \to \mathbb{R}$ be a subexponential function such that $f(W_T) \in L^1(\Omega)$ (or equivalently $f \in L^1_{loc}(\mathbb{R})$). Let $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{cases} v(t,x) = \int_{\mathbb{R}} q_{T-t}(x-y)f(y)dy & t \in [0,T[\ , \\ v(T,x) = f(x) \end{cases}$$

where q_t, $t \in [0,T[$, is the density of Gaussian law $N(0,t)$. Then

$$h := f(W_T) = v(0,W_0) + \int_0^T \partial_x v(s,W_s)d^- W_s, \tag{9.11}$$

where the last integral is an improper forward integral.

Proof. We consider the $(\mathcal{F}_t)$-martingale $M_t = \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right]$ and we apply Lemma 9.8. Using properly Lebesgue dominated convergence theorem and the assumptions on f it is possible to show that $v \in C^{1,2}([0,T[\times\mathbb{R})$. We apply Itô's formula to $v(t,W_t)$ for $t < T$ and we have

$$v(t,W_t) = u(0,W_0) + \int_0^t \partial_x v(s,W_s)dW_s\ . \tag{9.12}$$

If f is not continuous then $v \notin C^0([0,T] \times \mathbb{R})$. On the left-hand side of (9.12), taking the limit a.s. (and so in probability) and recalling that by construction $v(t,W_t) = \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right]$, we obtain

$$\lim_{t\to T} v(t,W_t) = \lim_{t\to T}\mathbb{E}\left[f(W_T)|\mathcal{F}_t\right] = M_{T^-} = f(W_T) \quad \text{a.s.}$$

by Lemma 9.8. This forces the convergence for $t \to T$ of the right-hand side of (9.12) obtaining $v(0,W_0) + \int_0^T \partial_x v(s,W_s)d^- W_s$ which is the right-hand side of (9.11). The result is now established. $\square$

9.3 The toy model revisited

The toy model seen in Proposition 9.5 is reinterpreted in an infinite dimensional framework. Using the same notation we set $H(\eta) = f(\eta(0))$, so that $h = H(X_T(\cdot)) = f(X_T)$. We give a first result about the solution of an infinite dimensional PDE, which constitutes a first adaptation of (9.8).

Proposition 9.11. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with polynomial growth and $v \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ which verifies (9.4). We define $H : C([-T,0]) \longrightarrow \mathbb{R}$, by $H(\eta) = f(\eta(0))$.
Then function $u : [0,T]\times C([-T,0]) \longrightarrow \mathbb{R}$ defined by $u(t,\eta) := v(t,\eta(0))$ belongs to $C^{1,2}\left([0,T[\times C([-T,0]);\mathbb{R}\right)\cap C^0\left([0,T] \times C([-T,0]);\mathbb{R}\right)$ and solves

$$\begin{cases} \partial_t u(t,\eta) + \dfrac{1}{2}\langle D^2 u\,(t,\eta)\ ,\ \mathbb{1}_D\rangle = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{9.13}$$

where $\langle D^2 u\,(t,\eta)\ ,\ \mathbb{1}_D\rangle$ is the measure $D^2 u(t,\eta)$ evaluated on the diagonal D of of the square $[-T,0]^2$.

Remark 9.12. The system (9.13) is a “particular case” of (9.8); in this case, as we will show in the proof, $D^{ac}\, u(t,\eta) \equiv 0$; so $\int_{-t}^{0} D^{ac} u\,(t,\eta)\; d\eta$ has an obvious interpretation.

Proof. It holds $u(T,\eta) = v(T,\eta(0)) = f(\eta(0)) = H(\eta)$ by (9.4). Moreover we have

$$\begin{aligned} \partial_t u\,(t,\eta) &= \partial_t v\,(t,\eta(0)) \\ D_{dx} u\,(t,\eta) &= \partial_x v\,(t,\eta(0))\, \delta_0(dx) \\ D^2_{dx\,dy} u\,(t,\eta) &= \partial^2_{x\,x} v\,(t,\eta(0))\delta_{0,\,0}(dx,dy). \end{aligned}$$

In particular we observe that $D^{ac}_{dx} u\,(t,\eta) \equiv 0$. Using again (9.4), we obtain $\partial_t u\,(t,\eta) + \frac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = \partial_t v\,(t,\eta(0)) + \frac{1}{2}\partial^2_{x\,x} v\,(t,\eta(0)) = 0$. Consequently u solves the infinite dimensional PDE

$$\begin{cases} \partial_t u\,(t,\eta) + \dfrac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{9.14}$$

and (9.13) is fulfilled. □

As a corollary we rediscover the representation result (9.5) already stated in Proposition 9.5.

Corollary 9.13. Let X be a real continuous stochastic process such that $X_0 = 0$ and $[X]_t = t$. There exists a continuous function $u : [0,T[\times C([-T,0] \to \mathbb{R}$ which belongs to class $C^{1,2}([0,T[\times C([-T,0])$ such that

- u solves (9.13) and

- $h := H(X_T(\cdot))$ admits a (9.2) representation, i.e. $h = H_0 + \int_0^T \xi_s d^- X_s$, with $H_0 = u(0, X_0(\cdot))$ and $\xi_t = D^{\delta_0} u(t, X_t(\cdot))$.

Proof. The result follows Propositions 9.5 and 9.11. In fact, using the notations of those propositions, we have $u(0, X_0(\cdot)) = v(0, X_0)$ and $D^{\delta_0} u(t, X_t(\cdot)) = \partial_x v(t, X_t)$. □

Remark 9.14. We could have shown the same result by applying our Banach valued Itô formula (8.2) to function $u(t, X_t(\cdot))$. In fact $D^2 u(t, \eta) \in \mathcal{D}_{0,0}$ and the window process $X(\cdot)$ associated to a finite quadratic variation process X admits a $\mathcal{D}_{0,0}-$quadratic variation given by (5.12).

9.4 A motivating path dependent example

We consider now the functional $H : C([-T, 0]) \longrightarrow \mathbb{R}$ defined by

$$H(\eta) = \left(\int_{-T}^{0} \eta(s) ds \right)^2 . \tag{9.15}$$

The random variable $h = H(W_T(\cdot))$ is $\mathcal{F}_T$-measurable and it belongs to $\mathbb{D}^{1,2}$. We compute first the Malliavin's derivative of h denoted by $D^m h$; it gives

$$D_t^m h = D_t^m \left(\int_0^T W_s ds \right)^2 = 2(T - t) \int_0^T W_s ds \ .$$

Consequently, using usual properties of conditional expectation, we have

$$\mathbb{E}\left[D_t^m h | \mathcal{F}_t \right] = \mathbb{E}\left[2(T - t) \int_0^T W_s ds | \mathcal{F}_t \right] = 2(T - t) \int_0^t W_s ds + 2(T - t)^2 W_t \ .$$

Computing the expectation of h we obtain

$$\mathbb{E}[h] = \mathbb{E}\left[\left(\int_0^T W_s ds \right)^2 \right] = \frac{T^3}{3} \ .$$

Finally Clark-Ocone formula stated in Proposition 2.33 gives

$$h = H(W_T(\cdot)) = \mathbb{E}[h] + 2 \int_0^T (T - t) \left(\int_0^t W_s \, ds \right) dW_t + 2 \int_0^T (T - t)^2 W_t dW_t \ . \tag{9.16}$$

We look now a function $u : [0, T] \times C([-T, 0]) \to \mathbb{R}$ for which we can express $V_t = \mathbb{E}[H(W_T(\cdot)) | \mathcal{F}_t]$ as $u(t, W_t(\cdot))$. Again by usual properties of conditional expectation, we obtain

$$V_t = \mathbb{E}[h | \mathcal{F}_t] = \left(\int_{-t}^{0} W_t(s) ds + W_t(0)(T - t) \right)^2 + \frac{(T - t)^3}{3} \ .$$

Setting u as

$$u(t,\eta)=\left(\int_{-T}^{0}\eta(s)ds+\eta(0)(T-t)\right)^{2}+\frac{(T-t)^{3}}{3},t\in[0,T],\eta\in C([0,T]),\tag{9.17}$$

we have the required property $V_t=u(t,W_t(\cdot))$ for any $t\in[0,T]$. In particular $h=H(W_T(\cdot))=V_T=u(T,W_T(\cdot))$ and trivially $u(0,W_0(\cdot))=T^3/3=\mathbb{E}\left[h\right]$. We stress that we could have chosen other u with the same property, for example setting the inferior extreme of the integral in (9.17) equal to t; that choice will be treated in Example 9.31.
Let now X be a real continuous stochastic process such that $X_0=0$ and $[X]_t=t$. In order to to apply Itô formula (8.2) for the function $u(t,X_t(\cdot))$ we observe that $u\in C^{1,2}([0,T]\times C([-T,0]))$ and we evaluate the corresponding derivatives obtaining

$$\begin{aligned}
\partial_t u(t,\eta)&=-2\eta(0)\left(\int_{-T}^{0}\eta(s)ds+\eta(0)(T-t)\right)-(T-t)^2\\
D_{dx}u(t,\eta)&=D^{ac}_{x}u(t,\eta)dx+D^{\delta_0}u(t,\eta)\delta_0(dx)\\
&\quad\text{where}\\
D^{ac}_{x}u(t,\eta)&=2\left(\int_{-T}^{0}\eta(s)ds+\eta(0)(T-t)\right)\mathbb{1}_{[-T,0]}(x)\\
D^{\delta_0}u(t,\eta)&=2\left(\int_{-T}^{0}\eta(s)ds+\eta(0)(T-t)\right)(T-t)\\
&\quad\text{and}\\
D^2_{dx\,dy}u(t,\eta)&=2\mathbb{1}_{[-T,0]^2}(x,y)dx\,dy+\\
&\quad+2(T-t)\mathbb{1}_{[-T,0]}(x)dx\,\delta_0(dy)+\\
&\quad+2(T-t)\delta_0(dx)\,\mathbb{1}_{[-T,0]}(y)dy+\\
&\quad+2(T-t)^2\delta_0(dx)\,\delta_0(dy)\ .
\end{aligned}\tag{9.18}$$

We observe that for any (t,η) in $[0,T]\times C([-T,0])$ the first Fréchet derivative $Du(t,\eta)$ is the sum of a measure absolute continuous with respect to Lebegue, denoted by $D^{ac}u(t,\eta)$, and a multiple of a Dirac measure at 0, denoted by $D^{\delta_0}u(t,\eta)$, see Notation 2.28. In particular $Du(t,\eta)$ belongs to $\mathcal{D}_0([-T,0])\oplus L^2([-T,0])$. Moreover for any (t,η), $D^2u(t,\eta)$ belongs to $\chi^0([-T,0]^2)$ and $D^2u:[0,T]\times C([-T,0])\to\chi^0([-T,0]^2)$ is continuous. Corollary 5.8 point 7) says that any finite quadratic variation process admits a $\chi^0([-T,0]^2)$-quadratic variation. Therefore Itô formula (8.2) for $u(T,X_T(\cdot))$ gives

$$u(T,X_T(\cdot))=I_0+I_1+I_2+I_3\tag{9.19}$$

where

$$I_0 = u(0, X_0(\cdot)) = \frac{T^3}{3}$$
$$I_1 = \int_0^T \partial_t u(t, X_t(\cdot))dt$$
$$I_2 = \int_0^T \langle Du(t, X_t(\cdot)), d^- X_t(\cdot)\rangle$$
$$I_3 = \frac{1}{2}\int_0^T \langle D^2 u(t, X_t(\cdot)), d\widetilde{[X(\cdot)]}_t\rangle \ .$$

We get

$$\begin{aligned} I_1 &= -2\int_0^T X_t \int_{-T}^0 X_t(s)ds\, dt - 2\int_0^T X_t^2(T-t)dt - \int_0^T (T-t)^2 dt \\ &= -2\int_0^T X_t \left(\int_0^t X_u du\right) dt - 2\int_0^T X_t^2(T-t)dt - \int_0^T (T-t)^2 dt \ . \end{aligned}$$

Concerning I_2 it holds $I_2 = I_{21} + I_{22}$ with

$$I_{21} = \int_0^T \langle D^{ac} u(t, X_t(\cdot)), d^- X_t(\cdot)\rangle = \lim_{\epsilon\longrightarrow 0}\int_0^T \langle D^{ac} u(t, X_t(\cdot)), \frac{X_{t+\epsilon}(\cdot) - X_t(\cdot)}{\epsilon}\rangle dt = \lim_{\epsilon\longrightarrow 0} I_{21}(\epsilon)$$
$$I_{21}(\epsilon) = 2\int_0^T \left(\int_0^t X_s\, ds\right)\left(\int_0^t \frac{X_{s+\epsilon} - X_s}{\epsilon} ds\right) dt + 2\int_0^T (T-t)X_t \left(\int_0^t \frac{X_{s+\epsilon} - X_s}{\epsilon} ds\right) dt \ ;$$
$$I_{22} = \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t \tag{9.20}$$

provided that I_{21} and I_{22} exist.
Since

$$\int_0^t \frac{X_{s+\epsilon} - X_s}{\epsilon} ds \xrightarrow[\epsilon\longrightarrow 0]{a.s.} X_t - X_0 = X_t \ ,$$

by Lebesgue dominated convergence theorem we get the convergence of $I_{21}(\epsilon)$ to I_{21}

$$I_{21}(\epsilon) \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} I_{21} := 2\int_0^T X_t \left(\int_0^t X_u du\right) dt + 2\int_0^T X_t^2 (T-t)dt \ .$$

Since I_2 and I_{21} exist, so does I_{22}. We recall the $\chi^0([-T,0]^2)$-quadratic variation of $X(\cdot)$ was given by (5.12). Proposition 7.31 implies that

$$I_3 = \frac{1}{2}\int_0^T 2(T-t)^2 dt = \int_0^T (T-t)^2 dt \ .$$

We observe that $I_1 = -I_2 - I_3$ so that (9.19) gives a representation for $h = H(X_T(\cdot)) = u(T, X_T(\cdot))$ in the form (9.2)

$$h = H_0 + \int_0^T \xi_t\, d^- X_t \tag{9.21}$$

with $H_0 = u(0, X_0(\cdot)) = T^3/3$ and $\xi_t = D^{\delta_0} u\,(t, X_t(\cdot)) = 2(T-t)\int_0^t X_s\,ds + 2(T-t)^2 X_t$.

Remark 9.15. Let us suppose $X = W$.

1. By Remark 2.9 2. the forward integral $\int_0^T \xi_t\, d^- W_t$ coincides with the Itô integral $\int_0^T \xi_t\, dW_t$.
2. As expected, the representation of the random variable $h = H(W_T(\cdot))$ given in (9.21) coincides with the Clark-Ocone representation (9.16), because of point 1. and the fact that ξ coincides with the expression provided by Clark-Ocone formula, i.e. $\xi_t = \mathbb{E}\left[D_t^m h|\mathcal{F}_t\right]$ and $H_0 = \mathbb{E}[h]$.
3. If X is a finite quadratic variation process in general $H_0 \neq \mathbb{E}[h]$ since $\mathbb{E}\left[\int_0^T \xi_t d^- X_t\right]$ does not generally vanish. In fact $\mathbb{E}[h]$ will specifically depend on the unknown law of X.

9.5 A more singular path-dependent example

This example is relatively simple and explicit, but it is not located in the application framework of Corollary 9.28, which configures a fairly general situation. In that case in fact the representing process V_t such that $V_T = h$, is of the form $u(t, X_t(\cdot))$ where $D^2 u(t, \eta)$ takes values in $\chi^0([-T, 0]^2)$; it will not be the case here.

Let X be, as usual, a process such that $X_0 = 0, [X]_t = t$ and h be a random variable of the type $h = H(X_T(\cdot))$ where $H : C([-T, 0]) \longrightarrow \mathbb{R}$ is the functional defined by $H(\eta) = \|\eta\|^2_{L^2}$, i.e.

$$h = H(X_T(\cdot)) = \int_{-T}^0 X_T(s)^2 ds = \int_0^T X_s^2 ds.$$

Suppose for a moment that $X = W$ is a classical Wiener process equipped with its canonical filtration $(\mathcal{F}_t)$. The random variable $h = H(W_T(\cdot))$ is $\mathcal{F}_T-$measurable and belongs to $\mathbb{D}^{1,2}$, so, by Clark-Ocone formula (2.42), we have

$$h = \mathbb{E}\left[h\right] + \int_0^T \mathbb{E}\left[D_t^m h|\mathcal{F}_t\right] dW_t \tag{9.22}$$

where the Malliavin's derivative $D_t^m h$ can be easily calculated as follows

$$D_t^m h = D_t^m \left(\int_0^T W_s^2 ds\right) = \int_t^T D_t^m (W_s^2) ds = \int_t^T 2W_s D_t^m (W_s) ds = \int_t^T 2W_s ds\ .$$

Consequently, by usual properties of the conditional expectation,

$$\mathbb{E}\left[D_t^m h|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T 2W_s ds|\mathcal{F}_t\right] = 2\int_t^T \mathbb{E}\left[W_s|\mathcal{F}_t\right] ds = 2W_t(T-t)\ .$$

Then (9.22) gives

$$h = \frac{T^2}{2} + 2\int_0^T W_t(T-t)dW_t \tag{9.23}$$

since $\mathbb{E}[h] = \frac{T^2}{2}$.

As in Section 9.4, we look for $u : [0,T] \times C([-T,0]) \to \mathbb{R}$ so that

$$V_t = \mathbb{E}\left[h|\mathcal{F}_t\right] = \mathbb{E}\left[H(W_T(\cdot))|\mathcal{F}_t\right] = u(t, W_t(\cdot)). \tag{9.24}$$

The evaluation of the conditional expectation in (9.24) yields

$$V_t = \int_0^t W_s^2 ds + W_t^2(T-t) + \frac{(T-t)^2}{2} = \int_{-t}^0 W_t^2(u)du + W_t^2(0)(T-t) + \frac{(T-t)^2}{2}\ .$$

Setting

$$u(t,\eta) = \int_{-T}^0 \eta^2(s)ds + \eta(0)^2(T-t) + \frac{(T-t)^2}{2} \tag{9.25}$$

it holds effectively $V_t = u(t, W_t(\cdot))$. We observe again that we could have chosen other functionals u which verifies $V_t = u(t, W_t(\cdot))$. For instance, as we will see in Remark 9.18, we can choose another $\tilde{u}$ with the same property which provides a solution for an infinite dimensional PDE of the type (9.8). We have in particular $H(W_T(\cdot)) = V_T = u(T, W_T(\cdot))$ and $\mathbb{E}\left[h\right] = \mathbb{E}\left[\int_0^T W_s^2 ds\right] = T^2/2$.

Formula (9.23) extends to the case where W is no longer a Brownian motion but a general finite quadratic variation process X, in fact previous considerations suggest the following statement.

Proposition 9.16. Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ defined by $H(\eta) = \|\eta\|^2_{L^2([-T,0])}$. Let X be a process such that $[X]_t = t$ and $X_0 = 0$ and $h = \|X_T(\cdot)\|^2_{L^2([-T,0])}$. Then

$$h = H_0 + \int_0^T \xi_t d^- X_t \tag{9.26}$$

with $H_0 = \frac{T^2}{2}$ and $\xi_t = 2X_t(T-t)$.
Moreover let $u : [0,T] \times C([-T,0]) \to \mathbb{R}$ defined by (9.25) it holds $H_0 = u(0, X_0(\cdot))$ and $\xi_t = D^{\delta_0}u(t, X_t(\cdot))$.

Proof. The idea consists again in applying Itô's formula (8.2) to $u(T, X_T(\cdot))$.
We remark that $u \in C^{1,2}([0,T] \times C([-T,0]))$ and we evaluate the corresponding derivatives obtaining

$$\begin{aligned}
\partial_t u(t,\eta) &= -\eta^2(0) - (T-t)\ ; \\
D_{dx}u(t,\eta) &= D_x^{ac}u(t,\eta)dx + D^{\delta_0}u(t,\eta)\,\delta_0(dx) \qquad\qquad \text{where} \\
D_x^{ac}u(t,\eta) &= 2\eta(x)\ , \\
D^{\delta_0}u(t,\eta) &= 2\eta(0)(T-t)\ ; \\
D^2_{dx\,dy}u(t,\eta) &= 2\delta_y(dx)\,dy + 2(T-t)\delta_0(dx)\delta_0(dy) = 2\delta_x(dy)\,dx + 2(T-t)\delta_0(dx)\delta_0(dy)\ .
\end{aligned}$$

We observe that $D^2u(t,\eta)$ belongs to the Chi-subspace $Diag \oplus \mathcal{D}_{0,0}$ of $\mathcal{M}([-T,0]^2)$ and $(t,\eta) \to D^2u(t,\eta)$ is continuous from $[0,T] \times C([-T,0])$ into $Diag \oplus \mathcal{D}_{0,0}$. Corollary 5.21 says that a window process $X(\cdot)$ associated to a finite quadratic variation process X admits a $Diag \oplus \mathcal{D}_{0,0}$-quadratic variation given by

$$\begin{aligned} &[X.(\cdot)]_t : Diag \oplus \mathcal{D}_{0,0} \longrightarrow \mathcal{C}([0,T]) \\ &\qquad\qquad \mu_1 + \mu_2 \longrightarrow \int_{-t}^{0} g(y)(t+y)dy + \alpha[X]_t = \int_{-t}^{0} g(y)(t+y)dy + \alpha t \end{aligned} \tag{9.27}$$

where $\mu_1(dx,dy) = g(y)\delta_y(dx)\,dy$, with $g \in L^\infty([-T,0])$ is a general diagonal measure and $\mu_2(dx,dy) = \alpha\delta_0(dx)\delta_0(dy)$, $\alpha \in \mathbb{R}$, is a general Dirac's measure on $\{0,0\}$. Therefore

$$\langle \mu_1 + \mu_2 \, , \, d\widetilde{[X(\cdot)]_t} \rangle = d_t \left(\int_{-t}^{0} g(y)(t+y)dy \right) + \alpha dt = \int_{-t}^{0} g(y)dydt + \alpha dt.$$

Applying Itô's formula (8.2) to $u(T, X_T(\cdot))$ we obtain

$$u(T, X_T(\cdot)) = I_0 + I_1 + I_2 + I_3 \tag{9.28}$$

where

$$\begin{aligned} I_0 &= u(0, X_0(\cdot)) = \frac{T^2}{2} \\ I_1 &= \int_0^T \partial_t u(t, X_t(\cdot))dt = \int_0^T (t - T - X_t^2)dt = \int_0^T (t - X_t^2)\,dt - T^2 \\ I_2 &= \int_0^T \langle Du(t, X_t(\cdot)), d^- X_t(\cdot) \rangle \\ I_3 &= \frac{1}{2} \int_0^T \langle D^2 u(t, X_t(\cdot)), d\widetilde{[X(\cdot)]_t} \rangle. \end{aligned}$$

Concerning the second term we have that $I_2 = I_{21} + I_{22}$ where

$$\begin{aligned} I_{21} &= \int_0^T \langle D^{ac} u(t, X_t(\cdot)), d^- X_t(\cdot) \rangle = \lim_{\epsilon \to 0} \int_0^T \langle D^{ac} u(t, X_t(\cdot)), \frac{X_{t+\epsilon}(\cdot) - X_t(\cdot)}{\epsilon} \rangle dt = \lim_{\epsilon \to 0} I_{21}(\epsilon) \, , \\ I_{21}(\epsilon) &= 2 \int_0^T \int_{-t}^0 X_t(r) \frac{X_{t+\epsilon}(r) - X_t(r)}{\epsilon} dr\,dt = 2 \int_0^T \int_0^t X_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds\,dt \, , \\ I_{22} &= \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t \end{aligned}$$

provided that I_{21} and I_{22} exist. We observe that

$$\int_0^t X_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds \xrightarrow{ucp} \int_0^t X_s d^- X_s$$

in the ucp sense, because of Theorem 2.13. Consequently $I_{21}(\epsilon)$ converges to

$$I_{21} = 2 \int_0^T \int_0^t X_s d^- X_s dt = \int_0^T (X_t^2 - t)\,dt.$$

Since I_2 and I_{21} exist, so does I_{22}.
Coming back to (9.27) and replacing function $g \in L^\infty([-T,0])$ by the constant function $g = 2$ and $\alpha(t,\eta) = 2(T-t)$ we obtain

$$I_3 = \int_0^T d_t \left(\int_{-t}^0 (t+y)dy \right) + \int_0^T (T-t)dt = \int_0^T tdt + \int_0^T (T-t)dt = T^2$$

because $d_t \left(\int_{-t}^0 (t+y)dy \right) = tdt$. Finally (9.28) gives

$$h = H_0 + \int_0^T \xi_t d^- X_t \tag{9.29}$$

where $H_0 = u(0, X_0(\cdot))$ and $\xi_t = D^{\delta_0} u(t, X_t(\cdot)) = 2X_t(T-t)$. □

Remark 9.17. If $X = W$ we observe that the forward integral $\int_0^T \xi_t d^- W_t$ coincides with Itô integral $\int_0^T \xi_t dW_t$, see Remark 2.9 2.; process ξ coincides with the process given by the classical Clark-Ocone formula and $H_0 = \mathbb{E}[h]$. Again, as expected, representation (9.29) is the same as in Clark-Ocone (9.23).

In the following remark we exhibit another function u, denoted by $\tilde{u}$, which fits the statement of Proposition 9.16. However $\tilde{u}$ will solve again, in some suitable sense the infinite dimensional PDE problem (9.8).

Remark 9.18. Let H defined by $H(\eta) = \|\eta\|^2_{L^2([-T,0])}$, X be a process such that $[X]_t = t$ and $X_0 = 0$, $h := H(X_T(\cdot))$ and u be the function defined by (9.25). We define in a slight different way another function $\tilde{u} : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ by

$$\tilde{u}(t,\eta) = \int_{-t}^0 \eta^2(s)ds + \eta(0)^2(T-t) + \frac{(T-t)^2}{2} \tag{9.30}$$

We observe that for a process X such that $X_0 = 0$ following result hold.

1. $$u(t, X_t(\cdot)) = \tilde{u}(t, X_t(\cdot)) \qquad \forall\, t \in [0,T] \qquad \text{a.s.}$$

   in particular

   $$\tilde{u}(0, X_0(\cdot)) = u(0, X_0(\cdot)) \qquad \text{and} \qquad \tilde{u}(T, X_T(\cdot)) = u(T, X_T(\cdot))$$

2. The function $\tilde{u}$ belongs to $C^{1,2}([0,T] \times C([-T,0]))$ and

   $$\begin{aligned}
   \partial_t \tilde{u}(t,\eta) &= \eta^2(-t) - \eta^2(0) + (t-T) \\
   D_{dx}\tilde{u}(t,\eta) &= D^{ac}_x \tilde{u}(t,\eta)dx + D^{\delta_0}\tilde{u}(t,\eta)\,\delta_0(dx) \\
   D^{ac}_x \tilde{u}(t,\eta) &= 2\,\mathbb{1}_{]-t,0]}(x)\eta(x) \\
   D^{\delta_0}\tilde{u}(t,\eta) &= 2\,\eta(0)(T-t) \\
   D^2_{dx\,dy}\tilde{u}(t,\eta) &= 2\,\mathbb{1}_{]-t,0]}(x)\delta_x(dy)\,dx + 2(T-t)\delta_0(dx)\delta_0(dy)
   \end{aligned}$$

3. Moreover

$$D^{\delta_0}\tilde{u}(t, X_t(\cdot)) = D^{\delta_0}u(t, X_t(\cdot))$$

4. Interpreting "$\int_{-t}^{0} D_x^{ac}\tilde{u}(t,\eta)\, d\eta(x)$" in the spirit of an inverse Itô formula where η mimics a Brownian motion, gives

$$\text{“}\int_{-t}^{0} D_x^{ac}\tilde{u}(t,\eta)\, d\eta(x)'' = \text{“}2\int_{-t}^{0} \eta(x)\, d\eta(x)'' = \eta^2(0) - \eta^2(-t) - t. \tag{9.31}$$

With this convention we can say that $\tilde{u}$ is again a solution of the infinite dimensional PDE (9.8), which confirms again the validity of the meta-thereom stated in (9.8). We observe that $D_x^{ac}\tilde{u}(t,\eta) = 2\,\mathbb{1}_{]-t,0]}(x)\eta(x)$ is not of bounded variation. In the sequel a "strict" solution of the infinite dimensional PDE (9.8) will be given when $D_x^{ac}\tilde{u}(t,\eta)$ has bounded variation, so that the left-hand side of (9.31) can be defined via an integration by parts, see Notation 9.26. In the present case $\tilde{u}$ cannot be considered as a solution to (9.8) in that sense. It is legitimate to consider it as a solution only admitting identity (9.31).

5. Previous points 1. and 3. confirm that the representation of Proposition 9.16 through H_0 and ξ holds with u replaced by $\tilde{u}$.

9.6 A more general path dependent Brownian random variable

9.6.1 Some notations

Notation 9.19. We denote by $\sigma : [0,T] \longrightarrow \mathbb{R}_+,\ t \mapsto \sigma_t = \sqrt{\frac{(T-t)^3}{3}}$. σ is a differentiable function such that $\sigma_T = 0$ and $\sigma_t > 0$ for all $t \in [0,T[$. Its derivative in t will be denoted by σ_t'. We define $p_\sigma : [0,T[\times\mathbb{R} \to \mathbb{R}$ as

$$p_\sigma(t,x) = \frac{1}{\sigma_t\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_t^2}}$$

For the real function p_σ we clearly have

$$\partial_t p_\sigma(t,x) = \sigma_t \sigma_t' \partial_{xx}^2 p_\sigma(t,x). \tag{9.32}$$

We also define the measure valued function $p : [0,T] \longrightarrow \mathcal{M}(\mathbb{R})$, as

$$p(t,dx) = \begin{cases} p_\sigma(t,x)dx & \text{if } t \in [0,T[\\ \delta_0(dx) & \text{if } t = T; \end{cases} \tag{9.33}$$

for $t \in [0,T]$, $p(t,dx)$ is the law of a Gaussian random variable with expected value 0 and variance given by σ_t^2. In the case $t = T$, $p(T,dx) = \delta_0(dx)$ is the law of the degenerated Gaussian law $N(0,0)$. We remark that σ_t is the standard deviation of the random variable $\int_t^T (W_r - W_t)dr$. It holds in particular

$$\partial_t p_\sigma(t,x) = \left[-\frac{(T-t)^2}{2}\right] \partial_{xx}^2 p_\sigma(t,x) \tag{9.34}$$

9.6.2 The example

Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ defined by

$$H(\eta) = f\left(\int_{-T}^0 \eta(s)ds\right) \tag{9.35}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a Borel, subexponential function such that

$$h = f\left(\int_0^T W_s ds\right) \in L^1(\Omega), \tag{9.36}$$

where (W_t) is a classical Wiener process. By Proposition 9.2, condition (9.36) could be replaced by $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R})$. Since h does not belong to $L^2(\Omega)$, a priori, neither Clark-Ocone formula nor its extensions to Wiener distribution apply.

The representation result obtained here is in principle new even if the underlying process is a Brownian motion.

We remark that

$$\mathbb{E}\left[f\left(\int_0^T W_s ds\right)\right] = \int_{\mathbb{R}} f(y)\, p_\sigma(0,y)\, dy\ .$$

Theorem 9.20. Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ such that (9.35) holds and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel subexponential function with $f \in L^1_{loc}(\mathbb{R})$.
Let $u : [0,T[\times C([-T,0]) \longrightarrow \mathbb{R}$ defined by

$$u(t,\eta) = \int_{\mathbb{R}} f\left(\int_{-T}^0 \eta(r)dr + \eta(0)(T-t) + x\right) p_\sigma(t,x)dx\ , \tag{9.37}$$

where we recall that $\sigma_t = \sqrt{\frac{(T-t)^3}{3}}$.
Then the random variable $h := H(W_T(\cdot))$ admits the following representation

$$h = H_0 + \int_0^T \xi_t d^- W_t \tag{9.38}$$

with the following properties.

1. $u(t, W_t(\cdot)) = \mathbb{E}[h|\mathcal{F}_t]$ for $t \in [0, T[$. In particular $H_0 = u(0, W_0(\cdot)) = \mathbb{E}[h]$;

2. $$D^{\delta_0} u(t, \eta) = (T - t) \int_{\mathbb{R}} f\left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T - t) + x\right) \partial_x p_\sigma(t, x)dx \; ; \tag{9.39}$$

3. $(\xi_t)_{t\in[0,u]}$ is the process defined by
$$\xi_t = D^{\delta_0} u(t, W_t(\cdot)) \qquad t \in [0, T[\; . \tag{9.40}$$

Remark 9.21. 1. Operating an affine change of variable $z = \left(\int_{-T}^{0} \eta(r)dr + \eta(0)(T - t) + x\right)$ we obtain a slight different expression of u, which gives
$$u(t, \eta) = \int_{\mathbb{R}} f(z) p_\sigma\left(t, z - \int_{-T}^{0} \eta(r)dr - \eta(0)(T - t)\right) dz; \tag{9.41}$$
This shows that u does not depend on the (Lebesgue) representative of the class to which belongs f. This also allows to show that u is in $C^{1,2}([0, T[\times C([-T, 0]))$.

2. Since f is not continuous we cannot expect that $u \in C^0([0, T] \times C([-T, 0]))$.

3. The stochastic integral in (9.38) is an improper stochastic integral, see Definition 2.3. In fact $D^{\delta_0}u$, and even u itself, may have a very singular behaviour when $t \to T$ so that we may not have $\int_0^T \xi_s^2 ds < \infty$ a.s.

4. When X is a Brownian motion, the random variable considered in (9.15) belongs trivially to class of random variables considered in this example considering $f(x) = x^2$.

Proof. We have
$$h = H(W_T(\cdot)) = f\left(\int_{-T}^{0} W_T(s)ds\right)$$
Let $(\mathcal{F}_t)$ be the associated Brownian filtration. We consider the real martingale
$$V_t = \mathbb{E}[h|\mathcal{F}_t] \qquad\qquad t \in [0, T]$$
It gives indeed, for $t \in [0, T[$,
$$V_t = u(t, W_t(\cdot)) \qquad\qquad t \in [0, T[\; .$$
In fact we have
$$\mathbb{E}[h|\mathcal{F}_t] = \mathbb{E}\left[f\left(\int_{-T}^{0} \eta(r)dr + \eta(0)(T - t) + \int_t^T (W_r - W_t)dr\right)\right]_{|\eta = W_t(\cdot)} \; . \tag{9.42}$$

In particular, it holds

$$u(0,W_0(\cdot)) = \mathbb{E}\left[f\left(\int_{-T}^0 W_0(s)ds + W_0T + \int_0^T (W_s - W_0)ds\right)\right] = \mathbb{E}\left[f\left(\int_0^T W_s ds\right)\right] = \mathbb{E}\left[h\right] \ . \quad (9.43)$$

The main idea of the proof consists in applying the Banach valued Itô's formula (8.2) from 0 to $s<T$. By Remark 9.21, $u \in C^{1,2}\left([0,T[\times C([-T,0])\right)$ and we have an exploitable expression of it. Evaluating the different derivatives in (9.41), for $t\in[0,T[, \eta \in C([-T,0])$, we obtain

$$\begin{aligned}
\partial_t u(t,\eta) &= \int_{\mathbb{R}} f(z)\eta(0)\partial_x p_\sigma\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz + \\
&\quad + \int_{\mathbb{R}} f(y)\partial_t p_\sigma\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \ ; \\
D_{dx}u(t,\eta) &= D^{ac}_x u(t,\eta)dx + D^{\delta_0}u(t,\eta)\,\delta_0(dx) \qquad\qquad \text{where} \\
D^{ac}_x u(t,\eta) &= -\int_{\mathbb{R}} f(z)\partial_x p_\sigma\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \cdot \left(1\!\!1_{[-T,0]}(x)\right) \ , \\
D^{\delta_0}u(t,\eta) &= -\int_{\mathbb{R}} f(z)\partial_x p_\sigma\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \cdot (T-t) \ ; \\
D^2_{dx\,dy}u(t,\eta) &= \int_{\mathbb{R}} f(z)\partial^2_{xx} p_\sigma\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \cdot \left(A_1 + A_2 + A_3 + A_4\right)
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= 1\!\!1_{[-T,0]^2}(x,y)dx\,dy \ , \\
A_2 &= (T-t)1\!\!1_{[-T,0]}(x)dx\,\delta_0(dy) \ , \\
A_3 &= (T-t)\delta_0(dx)\,1\!\!1_{[-T,0]}(y)dy \ , \\
A_4 &= (T-t)^2\delta_0(dx)\,\delta_0(dy) \ .
\end{aligned}$$

By (9.34) we recall that $\partial_x p_\sigma(t,x) = \frac{1}{\sigma_t\sqrt{2\pi}}\left(-\frac{x^2}{\sigma_t^2}\right)e^{-\frac{x^2}{2\sigma_t^2}}$ and $\partial_t p_\sigma(t,x) = \left[-\frac{(T-t)^2}{2}\right]\partial^2_{xx}p_\sigma(t,x)$.
In fact $D^2u : [0,T[\times C([-T,0]) \longrightarrow \chi^0([-T,0]^2)$ and it is continuous. Corollary 5.8 point 7) says that $W(\cdot)$ admits a $\chi^0([-T,0]^2)$-quadratic variation given by (5.12). In particular the $\chi^0([-T,0]^2)$-quadratic variation is determined only by the $\mathcal{D}_{0,0}$ component of the measure, which equals

$$A_4 \cdot \int_{\mathbb{R}} f(z)\partial^2_{xx} p\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \ .$$

Applying Itô's formula (8.2) for u from 0 to $s<T$ we obtain

$$u(s,W_s(\cdot)) = I_0 + I_1(s) + I_2(s) + I_3(s) \quad (9.44)$$

where

$$
\begin{aligned}
I_0 &= u(0, W_0(\cdot)) = \mathbb{E}\,[h] \\
I_1(s) &= \int_0^s \partial_t u(t, W_t(\cdot)) dt \\
I_2(s) &= \int_0^s \langle Du(t, W_t(\cdot)), d^- W_t(\cdot) \rangle \\
I_3(s) &= \frac{1}{2} \int_0^s \langle D^2 u(t, W_t(\cdot)), d\widetilde{[W(\cdot)]}_t \rangle .
\end{aligned}
$$

For convenience we make another change of variable $x = \left(z - \int_{-T}^0 \eta(r) dr - \eta(0)(T-t) \right)$. Concerning the first term $I_1(s)$, (9.34) allows to obtain

$$
\begin{aligned}
I_1(s) = & \int_0^s \int_{\mathbb{R}} f \left(\int_{-T}^0 W_t(r) dr + W_t(T-t) + x \right) W_t \, \partial_x p_\sigma(t,x) dx \, dt + \\
& + \int_0^s \int_{\mathbb{R}} f \left(\int_{-T}^0 W_t(r) dr + W_t(T-t) + x \right) \partial_t p_\sigma(t,x) dx \, dt = \\
& = \int_0^s \int_{\mathbb{R}} f \left(\int_{-T}^0 W_t(r) dr + W_t(T-t) + x \right) W_t \, \partial_x p_\sigma(t,x) dx \, dt + \\
& - \frac{1}{2} \int_0^s \int_{\mathbb{R}} f \left(\int_{-T}^0 W_t(r) dr + W_t(T-t) + x \right) (T-t)^2 \partial^2_{xx} p_\sigma(t,x) dx \, dt.
\end{aligned}
$$

We go on with the second term $I_2(s)$ obtaining

$$
\begin{aligned}
I_2(s) &= I_{21}(s) + I_{22}(s) \\
I_{22}(s) &= \int_0^s D^{\delta_0} u(t, W_t(\cdot)) d^- W_t = \\
&= - \int_0^s \left[\int_{\mathbb{R}} f \left(\int_{-T}^0 W_t(r) dr + W_t(T-t) + x \right) \partial_x p_\sigma(t,x) dx \right] (T-t) dW_t \; ;
\end{aligned}
$$

$I_{22}(s)$ is an Itô integral because of Remark 2.9.2, in fact the integrand process $D^{\delta_0} u(t, W_t(\cdot))$ is $(\mathcal{F}_t)$-adapted. On the other hand,

$$
\begin{aligned}
I_{21}(s) &= \lim_{\epsilon \to 0} I_{21}(s, \epsilon) \qquad \text{where} \\
I_{21}(s, \epsilon) &= \int_0^s \langle D^{ac} u(t, W_t(\cdot)), \frac{W_{t+\epsilon}(\cdot) - W_t(\cdot)}{\epsilon} \rangle dt = \\
&= - \int_0^s \left[\int_{\mathbb{R}} f \left(\int_{-T}^0 W_t(r) dr + W_t(T-t) + x \right) \partial_x p_\sigma(t,x) dx \right] \left[\int_0^t \frac{W_{u+\epsilon} - W_u}{\epsilon} du \right] dt \; .
\end{aligned}
$$

By Lebesgue dominated convergence theorem $I_{21}(s, \epsilon)$ converges to

$$
I_{21}(s) := - \int_0^s \left[\int_{\mathbb{R}} f \left(\int_{-T}^0 W_t(r) dr + W_t(T-t) + x \right) \partial_x p_\sigma(t,x) dx \right] W_t \, dt.
$$

Finally, concerning the term $I_3(s)$, we obtain

$$I_3(s) = \frac{1}{2}\int_0^s (T-t)^2 \left[\int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial^2_{xx} p_\sigma(t,x)dx\right] dt$$

So (9.44) gives explicitly

$$V_s = u(s, W_s(\cdot)) = \mathbb{E}[h] + \int_0^s \xi_t \, dW_t \tag{9.45}$$

where $(\xi_t)_{t\in[0,s]}$ is the process defined by

$$\xi_t = D^{\delta_0} u\left(t, W_t(\cdot)\right) ,$$

and (9.40) is verified. Using Lemma 9.8, with the Brownian martingale $M_t := V_t = \mathbb{E}[h|\mathcal{F}_t]$ we can pass (9.45) to the limit a.s. for $s \to T$ and we finally obtain the result and in particular (9.38). $\square$

Remark 9.22. If f were continuous, then u would also be continuous, so in (9.45) we could have passed to the limit when $s \to T$ a.s. for $u(s, W_s(\cdot))$ obtaining h without explicitly making use of the fact that W is a Brownian motion. Since u is not continuous, we can go to the limit making use of the Lemma 9.8 because (V_t) is a Brownian martingale.

Remark 9.23.

1. Proposition 9.2 gives sufficient conditions on function f, for instance, such that $h = f\left(\int_0^T W_s ds\right) \in L^1(\Omega)$. In fact $\int_0^T W_s ds$ is a mean zero Gaussian r.v. with variance $T^3/3$.
2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely continuous function such that f' is in $L^2_{loc}(\mathbb{R})$ and subexponential. In this case $h = f\left(\int_0^T W_s ds\right) \in \mathbb{D}^{1,2}$. Uniqueness of the representation of h implies that

$$\xi_t = \mathbb{E}\left[D^m_t h|\mathcal{F}_t\right].$$

Clearly the expression can be also obtained via the usual rules of Malliavin calculus.
As a particular case, if $f \in C^1_{pol}(\mathbb{R})$ then $h = f\left(\int_0^T W_s ds\right) \in L^2(\Omega)$, since f is subexponential and by Proposition 9.2 it follows that $f \in L^2_{loc}(\mathbb{R})$.

Remark 9.24.

1. Choosing

$$\tilde{u}(t,\eta) = \int_{\mathbb{R}} f\left(\int_{-t}^0 \eta(r)dr + \eta(0)(T-t) + x\right) p_\sigma(t,x)dx,$$

it is possible to show that

$$\partial_t \tilde{u}(t,\eta) + \int_{-t}^0 D^{ac}\tilde{u}\left(t,\eta\right) d\eta + \frac{1}{2}\langle D^2\tilde{u}\left(t,\eta\right) , \ \mathbb{1}_D\rangle = 0 ,$$

so that the metatheorem stated in (9.8) is again partially confirmed. The only problem is related to the final condition $\tilde{u}(t,\eta) = f\left(\int_{-T}^0 \eta(r)dr\right)$, which is only verified if f is continuous.

2. Indeed, if f is continuous, we can the limit when $t \to T$ in expression (9.37) and obtain

$$\lim_{t\to T} u(t,\eta) = \int_{\mathbb{R}} f\left(\int_{-T}^{0} \eta(r)dr + x\right) p_\sigma(T,dx) = \int_{\mathbb{R}} f\left(\int_{-T}^{0} \eta(r)dr + x\right) \delta(dx) = f\left(\int_{-T}^{0} \eta(r)dr\right).$$

9.7 A general representation result

9.7.1 An infinite dimensional partial differential equation

This subsection is devoted to a relatively general representation theorem for a path dependent random variable when the underlying is a general process $(X_t)_{t\geq 0}$ with $X_0 = 0$ a.s. and $[X]_t = t$. We make the usual convention of prolongation by continuity for $t \leq 0$. As in previous subsections, but with a more general formalism here, we aim at representing

$$h = H(X_T(\cdot)) \qquad \text{where} \qquad H : C([-\tau, 0]) \longrightarrow \mathbb{R}$$

in the form

$$h = H_0 + \int_0^T \xi_s d^- X_s \tag{9.46}$$

under reasonable sufficient conditions on function H.

The first step will be Corollary 9.28 which provides a precise link between a solution of an infinite dimensional partial differential equation and that representation.

It is comfortable to introduce the following notation.

Notation 9.25. If $\eta \in C([-T, 0])$ and $g : [-T, 0] \longrightarrow \mathbb{R}$ has bounded variation we denote

$$\int_{]-t,0]} g\, d\eta = g(0)\eta(0) - g(-t)\eta(-t) - \int_{]-t,0]} \eta\, dg\ .$$

Notation 9.26. If $g \in C^{1,2}([0,T[\times C([-T,0]))$ such that $x \mapsto D_x^{ac} g\,(t,\eta)$ has bounded variation, with the help of Notation 9.25, we define

$$\mathcal{L}g\,(t,\eta) = \partial_t g(t,\eta) + \int_{]-t,0]} D^{ac} g(t,\eta)\, d\eta + \frac{1}{2} D^2 g\,(t,\eta)(\{0,0\}) \qquad t \in [0,T],\ \eta \in C([-T,0])\ . \tag{9.47}$$

A consequence of the infinite dimensional Banach space valued Itô formula (8.2) is the following.

Proposition 9.27. Let $a \in]0,T[$ and $u \in C^{1,2}([0,a] \times C([-T,0]))$ such that $x \mapsto D_x^{ac} u\,(t,\eta)$ has bounded variation, for any $t \in [0,a]$, $\eta \in C([-T,0])$. We suppose moreover that $D^2 u\,(t,\eta) \in \chi^0([-T,0]^2)$ for every $t \in [0,T]$, $\eta \in C([-T,0])$.
Let X be a real continuous finite quadratic variation process with $[X]_t = t$ and $X_0 = 0$.
Then for every $t \in [0,a]$ it holds

$$u(t, X_t(\cdot)) = u(0, X_0(\cdot)) + \int_0^t D^{\delta_0} u\,(s, X_s(\cdot)) d^- X_s + \int_0^t \mathcal{L}u\,(s, X_s(\cdot)) ds \tag{9.48}$$

Proof. The proof follows applying our Banach valued Itô formula (8.2) to $u(s, X_s(\cdot))$ from 0 to $t < T$. $\square$

Corollary 9.28. Let $H : C([-T, 0]) \longrightarrow \mathbb{R}$ be a Borel functional. Let $u \in C^{1,2}([0, T[\times C([-T, 0])) \cap C^0([0, T] \times C([-T, 0]))$ such that $x \mapsto D_x^{ac} u(t, \eta)$ has bounded variation, for any $t \in [0, T]$, $\eta \in C([-T, 0])$. We suppose moreover that $D^2 u(t, \eta) \in \chi^0([-T, 0]^2)$ for every $t \in [0, T]$, $\eta \in C([-T, 0])$.
Suppose that u is a solution of

$$\begin{cases} \mathcal{L}u(t, \eta) = 0 \\ u(T, \eta) = H(\eta). \end{cases} \tag{9.49}$$

Let X be a real continuous finite quadratic variation process with $[X]_t = t$ and $X_0 = 0$.
Then the random variable $h := H(X_T(\cdot))$ admits the following representation

$$h = u(T, X_T(\cdot)) = H_0 + \int_0^T \xi_t d^- X_t \tag{9.50}$$

with $H_0 = u(0, X_0(\cdot))$, $\xi_t = D^{\delta_0} u(s, X_s(\cdot))$ and $\int_0^T \xi_t d^- X_t$ is an improper forward integral.

Remark 9.29. In particular u will be shown also to be a solution of (9.49) since $\langle D^2 u(t, \eta), \mathbb{1}_D \rangle = D^2 u(t, \eta)(\{0, 0\})$.

Remark 9.30. Since $H(\eta) = u(T, \eta)$, we observe that H is automatically continuous by hypothesis $u \in C^0([0, T] \times C([-T, 0]))$.

Proof of Corollary 9.28. Let $t < T$, applying Proposition 9.27 we obtain (9.48). By $\mathcal{L}(u)(t, \eta) = 0$ in (9.49) we have

$$u(t, X_t(\cdot)) = u(0, X_0(\cdot)) + \int_0^t D^{\delta_0} u(s, X_s(\cdot)) d^- X_s \tag{9.51}$$

Now for every fixed ω, since $u \in C^0([0, T] \times C([-T, 0]))$ and X is continuous the left-hand side converges, i.e.

$$\lim_{t \to T} u(t, X_t(\cdot)) = u(T, X_T(\cdot)),$$

which equals $H(X_T(\cdot))$ by (9.49). This forces the right-hand side of (9.51) to converge, so that the result follows. $\square$

Example 9.31. We come back to the example given in Section 9.4, where $H(\eta) = \left(\int_{-T}^0 \eta(s) ds \right)^2$. We exhibited in (9.17) a function $u : [0, T] \times C([-T, 0]) \longrightarrow \mathbb{R}$ for which $h = H(X_T(\cdot)) = H_0 + \int_0^T \xi_s d^- X_s$ where $H_0 = u(0, X_0(\cdot))$, $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$. In principle u does not verify the partial differential equation (9.8) of the metatheorem. However, similarly as in example treated in Section 9.5, we can define a function

$\tilde{u} : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ having the same representation property (9.21) and solving (9.8). We define it by

$$\tilde{u}\,(t,\eta) = \left(\int_{-t}^{0} \eta(s)ds + \eta(0)(T-t)\right)^2 + \frac{(T-t)^3}{3}\,. \tag{9.52}$$

We have indeed

$$\tilde{u}(t, X_t(\cdot)) = u(t, X_t(\cdot)). \qquad \text{a.s.} \qquad t \in [0,T]$$

In particular $\tilde{u}(0, X_0(\cdot)) = u(0, X_0(\cdot))$, which provides H_0, and $\tilde{u}(T, X_T(\cdot)) = u(T, X_T(\cdot))$, which equals $H(X_T(\cdot))$.
We can show that $\tilde{u}$ fulfills the hypotheses of Corollary 9.28. In fact $\tilde{u} \in C^{1,2}\left([0,T[\times C([-T,0])\right) \cap C^0\left([0,T] \times C([-T,0])\right)$. The computation of the different derivatives gives

$$\begin{aligned}
\partial_t \tilde{u}(t,\eta) &= 2\left(\eta(-t) - \eta(0)\right)\left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t)\right) - (T-t)^2 \\
D_{dx}\tilde{u}(t,\eta) &= D_x^{ac}\tilde{u}\,(t,\eta)dx + D^{\delta_0}\tilde{u}\,(t,\eta)\delta_0(dx) \\
D_x^{ac}\tilde{u}\,(t,\eta) &= 2\left(\int_{-t}^{0} \eta(s)ds + \eta(0)(T-t)\right)\mathbb{1}_{[-t,0]}(x) \\
D^{\delta_0}\tilde{u}\,(t,\eta) &= 2\left(\int_{-t}^{0} \eta(s)ds + \eta(0)(T-t)\right)(T-t) \\
D^2_{dx\,dy}\tilde{u}\,(t,\eta) &= 2\mathbb{1}_{[-t,0]^2}(x,y)dx\,dy + \\
&\quad + 2(T-t)\mathbb{1}_{[-t,0]}(x)dx\,\delta_0(dy) + \\
&\quad + 2(T-t)\delta_0(dx)\,\mathbb{1}_{[-t,0]}(y)dy + \\
&\quad + 2(T-t)^2\delta_0(dx)\,\delta_0(dy)\,.
\end{aligned}$$

In particular $x \mapsto D_x^{ac}\tilde{u}\,(t,\eta)$ has bounded variation and $\tilde{u}$ solves the infinite dimensional PDE (9.49). We observe that function $\tilde{u}$ solves even the infinite dimensional PDE (9.8) stated in the metatheorem since $D^2\tilde{u}\,(t,\eta) \in \chi^0([-T,0]^2)$.

We come back to the general process $(X_t)_{t\geq 0}$ such that $[X]_t = t$ with the usual convention of prolongation by continuity for $t \leq 0$.

In Section 9.8 and 9.9 we will provide different reasonable sufficient conditions on $H : C([-T,0]) \longrightarrow \mathbb{R}$ such that there is a solution u of (9.49). Therefore, applying Corollary 9.28, we have a representation result for $h := H(X_T(\cdot))$ in term of function u. In Section 9.8 we will require an $L^2([-T,0])$-regularity on $H : C([-T,0]) \subset L^2([-T,0]) \longrightarrow \mathbb{R}$ and in Section 9.9 we will consider a non smooth but $L^2([-T,0])$-finitely based functional H.

9.8 The infinite dimensional PDE with smooth Fréchet terminal condition

9.8.1 About a Brownian stochastic flow

Firstly we need now to develop some technical preliminaries. In this section $\omega \in \Omega$ will be fixed. Let consider a standard Brownian motion W and its canonical filtration $(\mathcal{F}_t)$.

Definition 9.32. For $0 < s < t < T$, $\eta \in C([-T,0])$ we define the **stochastic flow**

$$Y_t^{s,\eta}(x) = \begin{cases} \eta(x+t-s) & x \in [-T, s-t] \\ \eta(0) + W_t(x) - W_s & x \in [s-t, 0]. \end{cases} \tag{9.53}$$

$(Y_t^{s,\eta})_{0\le s\le t\le T,\, \eta\in C([-T,0])}$ is a $C([-T,0])$-valued random field.

Remark 9.33. We have

$$Y_T^{t,\eta}(x) = \begin{cases} \eta(x+T-t) & x \in [-T, t-T] \\ \eta(0) + \overline{W}_{T-t}(x) & x \in [t-T, 0] \end{cases}$$

where $\overline{W}$ is a standard Brownian motion.

The following lemma gives a “flow property”.

Lemma 9.34. For $0 < s < t < r < T$, the following flow property holds

$$Y_r^{s,\eta} = Y_r^{t,Y_t^{s,\eta}} \tag{9.54}$$

Proof. For fixed $\omega \in \Omega$, we inject $\tilde{\eta} = Y_s^{t,\eta}$ into $Y_r^{t,\tilde{\eta}}$ obtaining

$$Y_r^{t,Y_s^{t,\eta}}(x) = \left\{ \begin{array}{ll} \eta(x+r-s) & x \in [-T, s-r] \\ \eta(0) + W_t(x+r-t) - W_s & x \in [s-r, t-r] \\ \eta(0) + (W_t - W_s) + W_r(x) - W_t & x \in [t-r, 0] \end{array} \right\} = Y_r^{s,\eta}(x)$$

which concludes the proof of the Lemma. $\square$

Next proposition concerns the continuity of the stochastic flow with respect to its three variables.

Proposition 9.35. $(Y_t^{s,\eta})_{0\le s\le t\le T,\, \eta\in C([-T,0])}$ is is a continuous random field.

Proof. As usual in this section $\omega \in \Omega$ is fixed and ϖ_η (resp. $\varpi_{W(\omega)}$) is respectively the modulus of continuity of η (resp. the Brownian path $W(\omega)$).
Let (s,t,η) such that $0 \le s \le t \le T, \eta \in C([-T,0])$ and a sequence (s_n, t_n, η_n) also such that $0 \le s_n \le t_n \le T, \eta_n \in C([-T,0])$ with

$$\lim_{n\to\infty} (|s-s_n| + |t-t_n| + \|\eta-\eta_n\|_\infty) = 0.$$

We have to show that $Y^{s_n,\eta_n}_{t_n} \longrightarrow Y^{s,\eta}_t$ in $C([0,T]$, when $n \to \infty$ i.e. uniformly. For $x \in [0,T]$, we evaluate

$$|Y^{s_n,\eta_n}_{t_n} - Y^{s,\eta}_t|(x) \leq (I_1(n) + I_2(n) + I_3(n))(x),$$

where

$$\begin{aligned} I_1(n)(x) &= |Y^{s_n,\eta_n}_{t_n} - Y^{s_n,\eta}_{t_n}|(x) \\ I_2(n)(x) &= |Y^{s,\eta}_{t_n} - Y^{s,\eta}_t|(x) \\ I_3(n)(x) &= |Y^{s_n,\eta}_{t_n} - Y^{s,\eta}_{t_n}|(x). \end{aligned}$$

By Definition 9.32, it is easy to see that

$$\begin{aligned} \|I_1(n)\|_\infty &\leq \|\eta - \eta_n\|_\infty + |\eta_n(0) - \eta(0)| \\ &\leq 2\|\eta - \eta_n\|_\infty. \end{aligned}$$

Since $I_3(n)$ behaves similarly to $I_2(n)$, we only show that

$$\lim_{n\to\infty} I_2(n) = 0. \tag{9.55}$$

Without restriction to generality, we will suppose that $t_n \leq t$ for any n, since the case when the sequence (t_n) is greater or equal than t, could be treated analogously. We observe that following equality holds:

$$\begin{aligned} (Y^{s,\eta}_{t_n} - Y^{s,\eta}_t)(x) &= \eta(x + t_n - s)\mathbb{1}_{[-T,s-t_n]}(x) - \eta(x+t-s)\mathbb{1}_{[-T,s-t]}(x) + \\ &\quad + (\eta(0) + W_{t_n}(x) - W_s)\,\mathbb{1}_{[s-t_n,0]}(x) - (\eta(0) + W_t(x) - W_s)\,\mathbb{1}_{[s-t,0]}(x) = \\ &= (\eta(x+t_n-s) - \eta(x+t-s))\,\mathbb{1}_{[-T,s-t]}(x) + \\ &\quad + (\eta(x+t_n-s) - \eta(0) - W_t(x) + W_s)\,\mathbb{1}_{[s-t,s-t_n]}(x) \\ &\quad + (W_{t_n}(x) - W_t(x))\,\mathbb{1}_{[s-t_n,0]}(x)\ . \end{aligned} \tag{9.56}$$

Using (9.56) to evaluate $\|I_2(n)\|_\infty$ we obtain

$$\begin{aligned} \sup_{x\in[-T,0]} |Y^{s,\eta}_{t_n}(x) - Y^{s,\eta}_t(x)| &\leq \sup_{x\in[-T,0]} |\eta(x+t_n-s) - \eta(x+t-s)| + \\ &\quad + \sup_{x\in[s-t,s-t_n]} |\eta(x+t_n-s) - \eta(0)| + \sup_{x\in[s-t,s-t_n]} |W_t(x) - W_s| + \\ &\quad + \sup_{x\in[-T,0]} |W_{t_n}(x) - W_t(x)| \leq \\ &\leq 2\,\varpi_\eta(|t_n - t|) + 2\,\varpi_{W(\omega)}(|t_n - t|) \xrightarrow[n\longrightarrow+\infty]{} 0. \end{aligned}$$

Since η and $W(\omega)$ are uniformly continuous on the compact set $[0,T]$ both modulus of continuity converge to zero when $t_n \to t_0$. $\square$

At this point we make some simple observations that will be often used in the sequel.

Remark 9.36.

1. The stochastic flow is obviously $L^2([-T,0])$-continuous, being continuous with respect to the stronger topology $C([-T,0])$.

2. There are universal constants C_1, C_2, C_3 and C_4 such that for every $t \in [0,T], \epsilon$ with $t+\epsilon \in [0,T]$ such that

$$\left\|Y_T^{t,\eta}\right\|_\infty \le C_1 \left(1+\|\eta\|_\infty + \sup_{t\in[0,T]}|W_t|\right) \;;\quad \left\|Y_T^{t+\epsilon,\eta}\right\|_\infty \le C_2 \left(1+\|\eta\|_\infty + \sup_{t\in[0,T]}|W_t|\right) \tag{9.57}$$

and

$$\left\|Y_0^{T-t,\eta}\right\|_\infty \le C_3 \left(1+\|\eta\|_\infty + \sup_{t\in[0,T]}|W_t|\right) . \tag{9.58}$$

(9.57) implies that, for any $\alpha \in [0,1], t\in[0,T], \epsilon$ with $t+\epsilon\in[0,T]$, it holds

$$\left\|\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\right\|_\infty \le C_4 \left(1+\|\eta\|_\infty + \sup_{t\in[0,T]}|W_t|\right). \tag{9.59}$$

3. For any $\alpha\in[0,1], t\in[0,T]$ it holds

$$\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}} \xrightarrow[\epsilon\longrightarrow 0]{C([-T,0])} Y_T^{t,\eta} . \tag{9.60}$$

In fact developing term $Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$, which equals $Y_T^{t,\eta}$, we obtain

$$\left\|\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}} - Y_T^{t,\eta}\right\|_\infty = \alpha \left\|Y_T^{t+\epsilon,\eta} - Y_T^{t,\eta}\right\|_\infty .$$

The right-hand side converges to zero because of Proposition 9.35.

4. In the sequel we will make an explicit use of the expression below:

$$\left(Y_T^{t+\epsilon,\eta} - Y_T^{t,\eta}\right)(x) = \begin{cases} \eta(x+T-t+\epsilon) - \eta(x+T-t) & x\in[-T,t-T] \\ \eta(x+T-t+\epsilon) - \eta(0) - W_T(x) + W_t & x\in[t-T,t-T+\epsilon] \\ W_t - W_{t+\epsilon} & x\in[t-T+\epsilon,0] \end{cases} \tag{9.61}$$

We continue applying the properties of previous stochastic flow to the evaluation of conditional expectations.

Given $H : L^2([-T,0]) \longrightarrow \mathbb{R}$, we express

$$\mathbb{E}\left[H(W_T(\cdot))|\mathcal{F}_t\right] = u(t, W_t(\cdot)) \tag{9.62}$$

where $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$. Clearly Lemma 9.34 implies $W_T(\cdot) = Y_T^{t,W_t(\cdot)}$, so

$$V_t = \mathbb{E}\left[H\big(Y_T^{t,Y_t^{0,0}}\big)|\mathcal{F}_t\right] = \mathbb{E}\left[H\big(Y_T^{t,W_t(\cdot)}\big)|\mathcal{F}_t\right] = u(t, W_t(\cdot))$$

with

$$u(t,\eta) = \mathbb{E}\left[H\big(Y_T^{t,\eta}\big)\right]. \tag{9.63}$$

In the sequel η will always be a generic function in $C([-T,0])$.
That function u will play a crucial role in this section. In particular, given a real continuous process X, we will evaluate again an Itô's type expansion of $u(t, X_t(\cdot))$.

Remark 9.37. By Definition 9.32 it follows the following homogeneity property.

$$u(t,\eta) = \mathbb{E}\left[H\big(Y_{T-t}^{0,\eta}\big)\right] \tag{9.64}$$

Next results links Fréchet and Malliavin derivatives. Those tools will be used in the proof of the Theorem 9.41.

Lemma 9.38. Let $s > 0$. Let $G : C([-s,0]) \longrightarrow \mathbb{R}$ of class C^1 such the Fréchet derivative DG has polynomial growth.
Then $G(W_s(\cdot))$ belongs to $\mathbb{D}^{1,2}$ and

$$D_x^m G(W_s(\cdot)) = \int_{]x-T,0]} D_{dy} G(W_s(\cdot)). \tag{9.65}$$

Proof. The proof of this result needs some boring technicalities involving the approximation of a continuous function by its polynomial approximation. Formula (9.65) is stated in a particular case for instance in [48], Example 1.2.1. □

A careful investigation allows to show the following.

Lemma 9.39. Let $H : L^2([-T,0]) \longrightarrow \mathbb{R}$ of class C^2 Fréchet, $\zeta \in L^2([-T,0])$. Let $\eta \in C([-T,0])$ be fixed and $G_\eta : C([-T,0]) \longrightarrow C([-T,0])$ defined by

$$G_\eta(\gamma)(x) = \begin{cases} \eta(x+T-t) & x \in [-T, t-T] \\ \eta(0) + \gamma(T-t) & x \in]t-T, 0] \end{cases}. \tag{9.66}$$

We denote $\mathcal{G} : C([-T,0]) \to \mathbb{R}$ by $\mathcal{G}(\gamma) = \langle DH(G_\eta(\gamma)), \zeta\rangle$.
Then $\mathcal{G}$ is C^1 Fréchet and

$$\langle D\mathcal{G}(\gamma), \zeta_1\rangle = \langle D^2 H, \mathbb{1}_{]t-T,0]}\zeta_1 \otimes \zeta\rangle, \qquad \zeta_1 \in L^2([-T,0]).$$

Remark 9.40. If $D^2H \in L^2([-T,0]^2)$ then

$$\langle D\mathcal{G}(\gamma), \zeta_1 \rangle = \int_{]t-T,0]\times]-T,0]} D_x D_y H(G_\eta(\gamma))\zeta_1(x)\zeta(y)\, dx\, dy\ . \tag{9.67}$$

In other words

$$(D_x G)(\gamma) = \int_{]-T,0]} \zeta(y) D_x D_y H(G_\eta(\gamma))\, dy \quad \text{a.e.} \tag{9.68}$$

9.8.2 An infinite dimensional partial differential equation

In the theorem below we will give conditions on the function H such that u solves the PDE stated on (9.49). We are aware that for the moment the assumptions are not optimal, but we decided however to formulate a reasonable framework, not too heavy, in which a Clark-Ocone type formula is valid.

Theorem 9.41. Let $H \in C^3(L^2([-T,0]))$ such that the second order Fréchet derivative D^2H belongs to $L^2([-T,0]^2)$ and D^3H has polynomial growth (for instance bounded). Let u be defined by $u(t,\eta) = \mathbb{E}\left[H\big(Y_T^{t,\eta}\big)\right] = \mathbb{E}\left[H\big(Y_{T-t}^{0,\eta}\big)\right]$.

1) Then $u \in C^{0,2}([0,T] \times C([-T,0]))$.

2) Suppose moreover

   i) $DH(\eta) \in H^1([-T,0])$, i.e. function $x \mapsto D_xH(\eta)$ is in $H^1([-T,0])$, every fixed η;

   ii) DH has polynomial growth in $H^1([-T,0])$, i.e. there is $p \geq 1$ such that

$$\eta \mapsto \|DH(\eta)\|_{H^1} \leq const\,(\|\eta\|_\infty^p + 1)\ . \tag{9.69}$$

   iii) The map

$$\eta \mapsto DH(\eta) \qquad \text{considered} \qquad C([-T,0]) \to H^1([-T,0]) \qquad \text{is continuous.} \tag{9.70}$$

Then $u \in C^{1,2}([0,T] \times C([-T,0]))$. is given by (9.98) and u is a solution of (9.49), i.e. it holds

$$\begin{cases} \partial_t u(t,\eta) + \displaystyle\int_{]-t,0]} D_x^{ac} u(t,\eta)\, d\eta(x) + \frac{1}{2} D_{0,0}^2 u(t,\eta) = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{9.71}$$

where $D^{ac}u$ is the absolutely continuous term of measure $Du\,(t,\eta)$ and $D_{0,0}^2 u(t,\eta) = D^2u(t,\eta)\,(\{0,0\})$.

Remark 9.42.

1. Assumption (9.69) implies in particular that DH has polynomial growth in $C([-T,0])$, i.e. there is $p \geq 1$ such that

$$\eta \mapsto \sup_{x\in[-T,0]} |D_x H(\eta)| = \|DH(\eta)\|_\infty \leq const\,(\|\eta\|_\infty^p + 1) \tag{9.72}$$

It is well known in fact that $H^1([-T,0]) \hookrightarrow C([-T,0])$ and for a function $f \in H^1$ it holds $\|f\|_\infty \leq \|f\|_{H^1}$.

2. By a Taylor's expansion, given for instance by Theorem 5.6.1 in [8], the fact that D^3H has polynomial growth implies that H, DH and D^2H have also polynomial growth.

3. $Du(t,\eta)$, $D^2u(t,\eta)$ and $\partial_t u(t,\eta)$ will be explicitly expressed in term of H at (9.75), (9.78) and (9.98).

Proof. By definition (9.63) it is obvious that $u(T,\eta) = H(\eta)$.
Proof of 1)

- **Continuity of function u with respect to time t.**

We consider a sequence (t_n) in $[0,T]$ such that $t_n \xrightarrow[n\to\infty]{} t_0$. By Assumption, $H \in C^0(L^2([-T,0]))$ and so also $H \in C^0(C([-T,0]))$. Consequently, by Proposition 9.35

$$H\big(Y^{0,\eta}_{T-t_n}\big) \xrightarrow[n\to\infty]{a.s.} H\big(Y^{0,\eta}_{T-t_0}\big)\ . \tag{9.73}$$

By Remark 9.42.2. H has also polynomial growth, therefore there is $p \geq 1$ such that

$$|H(\zeta)| \leq const\left(1 + \sup_{x\in[-T,0]} |\zeta(x)|^p\right) \qquad\qquad \forall\, \zeta \in C([-T,0])\ .$$

By (9.58), we observe that

$$\begin{aligned}|H(Y^{0,\eta}_{T-t})| &\leq const\left(1 + \left\|Y^{0,\eta}_{T-t}\right\|_\infty^p\right) \leq \\ &\leq const\left(1 + \sup_{x\in[-T,0]} |\eta(x)|^p + \sup_{t\leq T} |W_t|^p\right)\ .\end{aligned}$$

By Lebesgue dominated convergence theorem, the fact that $\sup_{t\leq T}|W_t|^p$ is integrable and (9.73), it follows that

$$u(t_n,\eta) = \mathbb{E}\left[H\big(Y^{0,\eta}_{T-t_n}\big)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[H\big(Y^{0,\eta}_{T-t_0}\big)\right] = u(t_0,\eta)\ . \tag{9.74}$$

The continuity is now established by Remark 9.37.

- **First Fréchet derivative.**

We express now the derivatives of u with respect to derivatives of H. We start with $Du : [0,T] \times C([-T,0]) \longrightarrow \mathcal{M}([-T,0])$. We have

$$D_{dx}u(t,\eta) = D^{\delta_0}u\,(t,\eta)\delta_0(dx) + D^{ac}_x u\,(t,\eta)dx \tag{9.75}$$

where

$$D^{\delta_0}u\,(t,\eta) = \mathbb{E}\left[\int_{t-T}^{0} D_s H\big(Y_T^{t,\eta}\big)ds\right] \tag{9.76}$$

and

$$D^{ac}_x u(t,\eta) = \mathbb{E}\left[D_{x-T+t}H\big(Y_T^{t,\eta}\big)\right]\mathbb{1}_{[-t,0]}(x) = \begin{cases} 0 & x\in[-T,-t] \\ \mathbb{E}\left[D_{x-T+t}H\big(Y_T^{t,\eta}\big)\right] & x\in\,]-t,0] \end{cases}. \tag{9.77}$$

Remark 9.43. We observe that $x \mapsto D^{ac}_x u(t,\eta)$ has bounded variation on $[-T,0]$, in particular (9.71) has to be understood in the sense introduced in Notation 9.25.

- **Second Fréchet derivative.**

We discuss the second derivative

$$D^2u\,: [0,T]\times C([-T,0]) \longrightarrow (C([-T,0])\hat{\otimes}_\pi C([-T,0]))^* \cong \mathcal{B}(C([-T,0]),C([-T,0])).$$

For every fixed (t,η), in fact $D^2u\,(t,\eta)$ belongs to $(\mathcal{D}_0\oplus L^2([-T,0]))\hat{\otimes}_h^2 = \chi^0([-T,0]^2)$:

$$\begin{aligned} D^2_{dx,dy}u(t,\eta) = {}& \mathbb{E}\Big[D_{y-T+t}D_{x-T+t}H\big(Y_T^{t,\eta}\big)\Big]\,\mathbb{1}_{[-t,0]}(x)\,\mathbb{1}_{[-t,0]}(y)dx\,dy+ \\ &+\mathbb{E}\left[\int_{t-T}^{0} D_s D_{x-T+t}H\big(Y_T^{t,\eta}\big)ds\right]\mathbb{1}_{[-t,0]}(x)dx\,\delta_0(dy)+ \\ &+\mathbb{E}\left[\int_{t-T}^{0} D_{y-T+t}D_s H\big(Y_T^{t,\eta}\big)ds\right]\mathbb{1}_{[-t,0]}(y)dy\,\delta_0(dx)+ \\ &+\mathbb{E}\left[\int_{[t-T,0]^2} D_{s_1}D_{s_2}H\big(Y_T^{t,\eta}\big)ds_1\,ds_2\right]\delta_0(dx)\,\delta_0(dy)\,. \end{aligned} \tag{9.78}$$

It is possible to show that all the terms in the first and the second derivative are well defined and continuous using similar techniques used in the first part of the proof. We omit these technicalities for simplicity.

Proof of 2)

We will denote by $D'H(\eta)$ the derivative in x of $x\mapsto D_xH(\eta)$, where $DH(\eta)$ is the first Fréchet derivative in $L^2([-T,0])$ of H, for every fixed η.

- **Derivability with respect to time t.**

Let $t \in [0,T], \eta \in C([-T,0])$. We need to consider ϵ such that $t+\epsilon \in [0,T]$ and evaluate the limit (if it exists) of

$$\frac{u(t+\epsilon,\eta)-u(t,\eta)}{\epsilon} \tag{9.79}$$

when $\epsilon \to 0$. Without restriction of generality we will suppose here $\epsilon > 0$; considerning the case $\epsilon < 0$ would bring similar calculations.
The flow property (9.54) gives $Y_T^{t,\eta} = Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$, which allows to write

$$u(t,\eta) = \mathbb{E}\left[H\big(Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\big)\right] . \tag{9.80}$$

We go on with the evaluation of the limit of (9.79). By (9.80) and by differentiability of H in $L^2([-T,0])$ we have

$$\begin{aligned} H\big(Y_T^{t+\epsilon,\eta}\big) - H\big(Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\big) &= \langle DH\left(Y_T^{t,\eta}\right), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\rangle + \\ &+ \int_0^1 \langle DH\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right) - DH\left(Y_T^{t,\eta}\right), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\rangle d\alpha = \\ &= \int_{-T}^0 D_x H\left(Y_T^{t,\eta}\right)\left(Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}(x)\right) dx + S(\epsilon,t,\eta) \end{aligned} \tag{9.81}$$

where

$$S(\epsilon,t,\eta) = \int_0^1 \langle DH\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right) - DH\left(Y_T^{t,\eta}\right), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\rangle d\alpha .$$

We need to evaluate

$$Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon,\gamma}(x) \qquad x \in [-T,0] \qquad\qquad \text{setting} \qquad \gamma = Y_{t+\epsilon}^{t,\eta} . \tag{9.82}$$

(9.82) gives

$$Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon,\gamma}(x) = \begin{cases} \eta(x+T-t-\epsilon) - \gamma(x+T-t-\epsilon) & x \in [-T, t-T+\epsilon] \\ \eta(0) - \gamma(0) = -W_{t+\epsilon}(0) + W_t & x \in [t-T+\epsilon, 0] \end{cases} , \tag{9.83}$$

where $\gamma(0) = Y_{t+\epsilon}^{t,\eta}(0) = \eta(0) + W_{t+\epsilon}(0) - W_t$. Moreover, by (9.53), we have

$$\gamma(x+T-t-\epsilon) = Y_{t+\epsilon}^{t,\eta}(x+T-t-\epsilon) = \begin{cases} \eta(x+T-t) & x \in [-T, t-T] \\ \eta(0) + W_T(x) - W_t & x \in [t-T, t-T+\epsilon] \end{cases} .$$

Finally we obtain an explicit expression for (9.82); indeed (9.83) gives

$$Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon,\gamma}(x) = \begin{cases} \eta(x+T-t-\epsilon) - \eta(x+T-t) & x \in [-T, t-T] \\ \eta(x+T-t-\epsilon) - \eta(0) - W_T(x) + W_t & x \in [t-T, t-T+\epsilon] \\ W_t - W_{t+\epsilon} & x \in [t-T+\epsilon, 0] \end{cases} . \tag{9.84}$$

Consequently, using (9.80), (9.81) and (9.84), the quotient (9.79) appear as sum of four terms:

$$\frac{u(t+\epsilon,\eta)-u(t,\eta)}{\epsilon} = \frac{1}{\epsilon}\mathbb{E}\left[H\big(Y_T^{t+\epsilon,\eta}\big) - H\big(Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\big)\right] = I_1(\epsilon,t,\eta)+I_2(\epsilon,t,\eta)+I_3(\epsilon,t,\eta)+\frac{1}{\epsilon}\mathbb{E}\left[S(\epsilon,t,\eta)\right] \tag{9.85}$$

where

$$\begin{aligned}
I_1(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-T}^{t-T} D_xH\left(Y_T^{t,\eta}\right)\frac{\eta(x+T-t-\epsilon)-\eta(x+T-t)}{\epsilon}dx\right] = \\
&= -\mathbb{E}\left[\int_{-t}^{0} D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x)-\eta(x-\epsilon)}{\epsilon}dx\right]\\
I_2(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_xH\left(Y_T^{t,\eta}\right)\frac{\eta(x+T-t-\epsilon)-\eta(0)-W_T(x)+W_t}{\epsilon}dx\right] + \\
&- \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_xH\left(Y_T^{t,\eta}\right)\frac{W_t-W_{t+\epsilon}}{\epsilon}dx\right]\\
&= \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_xH\left(Y_T^{t,\eta}\right)\frac{\eta(x+T-t-\epsilon)-\eta(0)-W_T(x)+W_{t+\epsilon}}{\epsilon}dx\right]\\
I_3(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{t-T}^{0} D_xH\left(Y_T^{t,\eta}\right)\frac{W_t-W_{t+\epsilon}}{\epsilon}dx\right]
\end{aligned}$$

and $\frac{1}{\epsilon}\mathbb{E}\left[S(\epsilon,t,\eta)\right]$ is equal to

$$\frac{1}{\epsilon}\int_0^1 \mathbb{E}\left[\int_{-T}^{0}\left(D_xH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right)-D_xH\left(Y_T^{t,\eta}\right)\right)\left(Y_T^{t+\epsilon,\eta}(x)-Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}(x)\right)dx\right]d\alpha\,. \tag{9.86}$$

• First we prove that $I_1(\epsilon,t,\eta)\xrightarrow[\epsilon\to 0]{} I_1(t,\eta):=I_{11}(t,\eta)+I_{12}(t,\eta)+I_{13}(t,\eta)$ where

$$\begin{aligned}
I_{11}(t,\eta) &= \mathbb{E}\left[D_{-T}H\left(Y_T^{t,\eta}\right)\eta(-t)\right]\\
I_{12}(t,\eta) &= \mathbb{E}\left[\int_{-t}^{0} D'_{x-T+t}H\left(Y_T^{t,\eta}\right)\eta(x)dx\right]\\
I_{13}(t,\eta) &= -\mathbb{E}\left[D_{t-T}H\left(Y_T^{t,\eta}\right)\eta(0)\right]\ .
\end{aligned}$$

In fact $I_1(\epsilon,t,\eta)$ can be rewritten as sum of three terms

$$\begin{aligned}
I_{11}(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-t}^{-t+\epsilon} D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\right]\\
I_{12}(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-t}^{0}\frac{D_{x+\epsilon-T+t}H\left(Y_T^{t,\eta}\right)-D_{x-T+t}H\left(Y_T^{t,\eta}\right)}{\epsilon}\eta(x)dx\right]\\
I_{13}(\epsilon,t,\eta) &= -\mathbb{E}\left[\int_{0}^{\epsilon} D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\right]\ .
\end{aligned}$$

By hypothesis, function $x \mapsto D_x H(\eta)$ belongs to H^1, for every fixed η. We recall that its derivative in the sense of distribution is denoted by $D'H(\eta)$; in particular $x \mapsto D_x H(\eta)$ is a continuous function. By application of finite increments theorem and dominated convergence theorem the following limit $I_{1i}(\epsilon, t, \eta) \xrightarrow[\epsilon \to 0]{} I_{1i}(t, \eta)$ for $i = 1, 2, 3$ holds.
In particular we observe that $I_1(t, \eta)$ equals $-\int_{]-t,0]} D^{ac} u\,(t, \eta)\, d\eta$ in the sense given by Notation 9.25.

• We can prove that $I_2(\epsilon, t, \eta)$ converges to zero when $\epsilon \to 0$. In fact, using Cauchy-Schwarz inequality we obtain

$$|I_2(\epsilon, t, \eta)|^2 \leq \frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_x H\left(Y_T^{t,\eta}\right)^2 dx\right] \frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} (\eta(x+T-t-\epsilon) - \eta(0) - W_T(x) + W_{t+\epsilon})^2\, dx\right] . \tag{9.87}$$

We recall that given any Brownian motion $\bar{W}$, $\sup_{x \leq T} |\bar{W}_x|$ has all moments; using (9.72), Lebesgue dominated convergence theorem and finite increments theorem, it follows that the first integral converges to $\mathbb{E}\left[D_{t-T} H\left(Y_T^{t,\eta}\right)^2\right]$ and the second integral to zero.

• As third step we prove that

$$I_3(\epsilon, t, \eta) \xrightarrow[\epsilon \to 0]{} -\mathbb{E}\left[\int_{[t-T,0]^2} D_y D_x H\big(Y_T^{t,\eta}\big) dx\, dy\right] =: I_3(t, \eta) .$$

By Lemma 9.38 and Lemma 9.39, it follows that $\mathcal{Z} := \langle DH(Y_{T-t}^{0,\eta}), \mathbb{1}_{[t-T,0]}\rangle$ belongs to $\mathbb{D}^{1,2}$ and

$$D_r^m \mathcal{Z} = \int_{r-T}^0 \int_{t-T}^0 D_y D_x \left(Y_{T-t}^{0,\eta}\right) dx\, dy = \int_{]r-T,0] \times]t-T,0]} D_y D_x \left(Y_{T-t}^{0,\eta}\right) dx\, dy .$$

Using Skorohod integral formulation we obtain

$$I_3(\epsilon, t, \eta) = -\frac{1}{\epsilon}\mathbb{E}\left[\langle DH(Y_{T-t}^{0,\eta}), \mathbb{1}_{[t-T,0]}\rangle \cdot \int_t^{t+\epsilon} \delta W_s\right] = -\frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z} \cdot \int_t^{t+\epsilon} \delta W_s\right] . \tag{9.88}$$

By integration by parts on Wiener space (2.40), Fubini's theorem between r and y and then integrating with respect to r, (9.88) becomes

$$\begin{aligned}
-\frac{1}{\epsilon}\mathbb{E}\left[\int_t^{t+\epsilon} D_r^m \mathcal{Z}\, dr\right] &= -\frac{1}{\epsilon}\mathbb{E}\left[\int_t^{t+\epsilon} \int_{r-T}^0 \int_{t-T}^0 D_y D_x H\left(Y_T^{t,\eta}\right) dx\, dy\, dr\right] = \\
&= -\frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^0 \int_t^{t+\epsilon} \int_{t-T}^0 D_y D_x H\left(Y_T^{t,\eta}\right) dx\, dr\, dy\right] = \\
&= -\mathbb{E}\left[\int_{t-T}^0 \int_{t-T}^0 D_y D_x H\left(Y_T^{t,\eta}\right) dx dy\right] .
\end{aligned}$$

• We study now the term

$$\frac{1}{\epsilon}\mathbb{E}\left[S(\epsilon, t, \eta)\right].$$

By (9.84), the a.s. equality $Y_T^{t,\eta} = Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$ and the fact that $H \in C^2(L^2([-T,0]))$, (9.86) can be rewritten as the sum of the following terms

$$
\begin{aligned}
A_1(\epsilon,t,\eta) &= \int_0^1 \mathbb{E}\left[\int_{-T}^{t-T} \left(D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_x H\left(Y_T^{t,\eta}\right)\right) \cdot \right. \\
&\qquad\qquad \left. \cdot \frac{\eta(x+T-t-\epsilon) - \eta(x+T-t)}{\epsilon} dx\right] d\alpha \\
A_2(\epsilon,t,\eta) &= \int_0^1 \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} \left(D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_x H\left(Y_T^{t,\eta}\right)\right) \cdot \right. \\
&\qquad\qquad \left. \cdot \frac{\eta(x+T-t-\epsilon) - \eta(0) - W_T(x) + W_{t+\epsilon}}{\epsilon} dx\right] d\alpha \\
A_3(\epsilon,t,\eta) &= A_{31}(\epsilon,t,\eta) + A_{32}(\epsilon,t,\eta) + A_{33}(\epsilon,t,\eta) + A_{34}(\epsilon,t,\eta)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{31}(\epsilon,t,\eta) &= \frac{1}{2}\mathbb{E}\left[\langle D^2 H\left(Y_{T-t}^{0,\eta}\right), \mathbb{1}_{[t-T,0]} \otimes \mathbb{1}_{[t-T,0]}\rangle \cdot \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\right] = \\
&= \frac{1}{2}\mathbb{E}\left[\int_{[t-T,0]^2} D_y D_x H\left(Y_{T-t}^{0,\eta}\right) dy\, dx \cdot \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\right] \\
A_{32}(\epsilon,t,\eta) &= \int_0^1 \mathbb{E}\left[\langle \left(D^2 H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D^2 H\left(Y_T^{t,\eta}\right)\right), \mathbb{1}_{[t-T,0]^2}\rangle \cdot \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\right] = \\
&= \int_0^1 \mathbb{E}\left[\int_{[t-T,0]^2} \left(D_y D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_x D_y H\left(Y_T^{t,\eta}\right)\right) dy\, dx \cdot \right. \\
&\qquad\qquad \left. \cdot \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\right] d\alpha \\
A_{33}(\epsilon,t,\eta) &= \int_0^1 \mathbb{E}\left[\int_{t-T}^0 \int_{-T}^{t-T} \left(D_y D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_y D_x H\left(Y_T^{t,\eta}\right)\right) \cdot \right. \\
&\qquad\qquad \left. \cdot \frac{\eta(y+T-t+\epsilon) - \eta(y+T-t)}{\epsilon}(W_t - W_{t+\epsilon}) dy\, dx\right] d\alpha \\
A_{34}(\epsilon,t\eta) &= \int_0^1 \mathbb{E}\left[\int_{t-T}^0 \int_{t-T}^{t-T+\epsilon} \left(D_y D_x H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_y D_x H\left(Y_T^{t,\eta}\right)\right) \cdot \right. \\
&\qquad\qquad \left. \cdot \frac{\eta(y+T-t-\epsilon) - \eta(0) - W_T(y) + W_{t+\epsilon}}{\epsilon}(W_t - W_{t+\epsilon}) dy\, dx\right] d\alpha
\end{aligned}
$$

• Similarly to $I_1(\epsilon,t,\eta)$, term $A_1(\epsilon,t,\eta)$ can be developed in the sum of terms given below.

$$A_{11}(\epsilon,t,\eta) = \mathbb{E}\left[\int_0^1\int_{-t}^{-t+\epsilon} D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\,d\alpha\right]$$

$$A_{12}(\epsilon,t,\eta) = \mathbb{E}\left[\int_0^1\int_{-t}^{0}\frac{D_{x+\epsilon-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right)}{\epsilon}\eta(x)dx\,d\alpha\right] +$$
$$- \mathbb{E}\left[\int_0^1\int_{-t}^{0}\frac{D_{x+\epsilon-T+t}H\left(Y_T^{t,\eta}\right) - D_{x-T+t}H\left(Y_T^{t,\eta}\right)}{\epsilon}\eta(x)dx\,d\alpha\right]$$

$$A_{13}(\epsilon,t,\eta) = -\mathbb{E}\left[\int_0^1\int_{-\epsilon}^{0} D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_{x-T+t}H\left(Y_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\,d\alpha\right] .$$

• We show now that $A_{11}(\epsilon,t,\eta)$ converges to zero.
By Cauchy-Schwarz inequality we have

$$[A_{11}(\epsilon,t,\eta)]^2 \le \int_{-t}^{-t+\epsilon}\frac{\eta^2(x-\epsilon)}{\epsilon}dx \,\times$$
$$\times\,\mathbb{E}\left[\int_0^1\int_{-t}^{-t+\epsilon}\frac{1}{\epsilon}\left[D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_{x-T+t}H\left(Y_T^{t,\eta}\right)\right]^2 dx\,d\alpha\right]$$

The integral $1/\epsilon\int_{-t}^{-t+\epsilon}\eta^2(x-\epsilon)dx$ converges to $\eta^2(-t)$ by the finite increments theorem.
By hypothesis (9.70) and (9.60) we have

$$\left\|DH\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - DH\left(Y_T^{t,\eta}\right)\right\|_{H^1([-T,0])} \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0 . \tag{9.89}$$

Because of (9.89), it follows that

$$D_yH\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_xH\left(Y_T^{t,\eta}\right) \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0 \qquad \forall\, y\in[-T,0] . \tag{9.90}$$

Since $x\mapsto D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_{x-T+t}H\left(Y_T^{t,\eta}\right)$ is a continuous function for $x\in[-t,-t+\epsilon]$, the finite increments theorem and (9.90) imply that

$$\int_0^1\int_{-t}^{-t+\epsilon}\frac{1}{\epsilon}\left[D_{x-T+t}H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}\right) - D_{x-T+t}H\left(Y_T^{t,\eta}\right)\right]^2 dx\,d\alpha \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0 .$$

Using (9.72), (9.59), (9.57) and the fact that given any Brownian motion $\bar W$, $\sup_{x\le T}|\bar W_x|$ has all moments and Lebesgue dominated convergence theorem it follows that $A_{11}(\epsilon,t,\eta)$ converges to zero.
• Using the same technique we also obtain that $A_{13}(\epsilon,t,\eta)$ converges to zero.
• We show that $A_{12}(\epsilon,t,\eta)$ converges to zero.
For every fixed continuous function ζ we can develop

$$D_{x-T+t+\epsilon}H(\zeta) - D_{x-T+t}H(\zeta) = \int_{x-T+t}^{x+\epsilon-T+t} D'_uH(\zeta)\,du .$$

It follows that $A_{12}(\epsilon,t,\eta)$ can be rewritten as

$$\mathbb{E}\left[\int_0^1\int_{-t}^0 \frac{1}{\epsilon}\int_{x-T+t}^{x-T+t+\epsilon} \left[D'_u H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_u H\left(Y_T^{t,\eta}\right)\right]\eta(x)\,du\,dx\,d\alpha\right]. \tag{9.91}$$

Taking the absolute value and considering the fact that $|\eta(x)|\le\|\eta\|_\infty$ we obtain

$$|A_{12}(\epsilon,t,\eta)|\le\mathbb{E}\left[\int_0^1\int_{-t}^0 \frac{1}{\epsilon}\int_{x-T+t}^{x-T+t+\epsilon} \left|D'_u H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_u H\left(Y_T^{t,\eta}\right)\right|\,du\,dx\,d\alpha\right]\|\eta\|_\infty .$$

By Fubini's theorem it follows

$$|A_{12}(\epsilon,t,\eta)|\le\mathbb{E}\left[\int_0^1\int_{-T}^{-T+t} \left|D'_u H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_u H\left(Y_T^{t,\eta}\right)\right|\,du\,d\alpha\right]\|\eta\|_\infty .$$

Now using Cauchy-Schwarz inequality we have

$$\begin{aligned}|A_{12}(\epsilon,t,\eta)|^2&\le T\,\mathbb{E}\left[\int_0^1\int_{-T}^{-T+t} \left(D'_u H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D'_u H\left(Y_T^{t,\eta}\right)\right)^2\,du\,d\alpha\right]\ \|\eta\|_\infty^2\le\\ &\le T\,\mathbb{E}\left[\int_0^1 \left\|D' H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D' H\left(Y_T^{t,\eta}\right)\right\|^2_{L^2([-T,0])}d\alpha\right]\ \|\eta\|_\infty^2 .\end{aligned}$$

Convergence (9.89) implies in particular

$$\left\|D' H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D' H\left(Y_T^{t,\eta}\right)\right\|^2_{L^2([-T,0])}\xrightarrow[\epsilon\longrightarrow 0]{a.s.}0 .$$

Again using (9.72), (9.59), (9.57) the fact that given any Brownian motion $\bar W$, $\sup_{x\le T}|\bar W_x|$ has all moments and Lebesgue dominated convergence theorem we have that $A_{12}(\epsilon,t,\eta)$ converges to zero.

- This concludes the proof of $A_1(\epsilon,t,\eta)$ convergence.
- Concerning $A_2(\epsilon,t,\eta)$, Cauchy-Schwarz implies that

$$\begin{aligned}|A_2(\epsilon,t,\eta)|^2\le\int_0^1\frac{1}{\epsilon}\mathbb{E}&\left[\int_{t-T}^{t-T+\epsilon}\left(D_x H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_x H\left(Y_T^{t,\eta}\right)\right)^2dx\right]\cdot\\ &\cdot\frac{1}{\epsilon}\mathbb{E}\left[\int_{t-T}^{t-T+\epsilon}\left(\eta(x+T-t-\epsilon)-\eta(0)-W_T(x)+W_{t+\epsilon}\right)^2dx\right]d\alpha .\end{aligned}$$

The continuity of DH (see (9.70)), the fact that it has polynomial growth by Remark 9.42.2, (9.59) and Lebesgue dominated convergence theorem imply that the first expectation converges to zero. The second expectation converges to zero by the same arguments together with the fact that $\sup_{x\le T}|\bar W_x|$ has all moments.

- We show now that $A_{31}(\epsilon,t,\eta)$ converges to

$$\frac{1}{2}\mathbb{E}\left[\int_{[t-T,0]^2}D_yD_xH\left(Y_T^{t,\eta}\right)dy\,dx\right]=:A_{31}(t,\eta) .$$

In fact the term $A_{31}(\epsilon, t, \eta)$ can be written as follows

$$\frac{1}{2}\mathbb{E}\left[\mathcal{Z} \cdot \frac{(W_{t+\epsilon} - W_t)^2}{\epsilon}\right] \tag{9.92}$$

where

$$\mathcal{Z} := \langle D^2 H\left(Y^{0,\eta}_{T-t}\right), \mathbb{1}_{[t-T,0]} \otimes \mathbb{1}_{[t-T,0]}\rangle = \langle D^2 H\left(Y^{0,\eta}_{T-t}\right), \mathbb{1}_{[t-T,0]^2}\rangle \, .$$

At this level we need a technical result.

Lemma 9.44. The random variable $B(\epsilon) := \frac{(W_{t+\epsilon} - W_t)^2}{\epsilon}$ weakly converges in $L^2(\Omega)$ to 1 when $\epsilon \to 0$.

Proof. In fact, $\mathbb{E}\left[B(\epsilon)^2\right] = 3$, so that $(B(\epsilon))$ is bounded in $L^2(\Omega)$. Therefore it exists a subsequence (ϵ_n) such that $(B(\epsilon_n))$ converges weakly to some square integrable variable B_0. In order to show that $B_0 = 1$ and to conclude the proof of the lemma it is enough to prove that

$$\mathbb{E}\left[B(\epsilon) \cdot Z\right] \longrightarrow \mathbb{E}[Z] \tag{9.93}$$

for any r.v. Z of a dense subset $\mathcal{D}$ of $L^2(\Omega)$. We choose $\mathcal{D}$ ad the r.v. Z such that $Z = \mathbb{E}[Z] + \int_0^T \xi_s dW_s$ where $(\xi_s)_{s\in[0,T]}$ is a bounded previsible process. We have

$$\mathbb{E}\left[B(\epsilon) \cdot Z\right] = \mathbb{E}\left[B(\epsilon)\right] \mathbb{E}\left[Z\right] + \mathbb{E}\left[\frac{(W_{t+\epsilon} - W_t)^2}{\epsilon} \int_0^T \xi_s dW_s\right]$$

Since $\mathbb{E}\left[B(\epsilon)\right] \mathbb{E}\left[Z\right] = \mathbb{E}\left[Z\right]$, we only need to show that

$$\mathbb{E}\left[\frac{(W_{t+\epsilon} - W_t)^2}{\epsilon} \int_0^T \xi_s dW_s\right] \xrightarrow[\epsilon \longrightarrow 0]{} 0 \, . \tag{9.94}$$

Since $\int_0^T \xi_s dW_s$ is a Skorohod integral, integration by parts on Wiener space (2.40) implies that the left-hand side of (9.94) equals

$$\mathbb{E}\left[\frac{2}{\epsilon}\int_0^T \xi_s (W_{t+\epsilon} - W_t)\mathbb{1}_{[t,t+\epsilon]}(s) ds\right] = \mathbb{E}\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} \xi_s ds \, (W_{t+\epsilon} - W_t)\right] \; ;$$

this converges to zero since ξ is bounded. □

By an immediate application of Lemma 9.44, term $A_{31}(\epsilon, t, \eta)$ expressed in (9.92) converges to $\frac{1}{2}\mathbb{E}[\mathcal{Z}]$ which equals $A_{31}(t, \eta)$.

• Concerning term $A_{32}(\epsilon, t, \eta)$, using Cauchy-Schwarz we obtain

$$\mathbb{E}\left[\langle D^2 H\left(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha) Y_T^{t+\epsilon, Y^{t,\eta}_{t+\epsilon}}\right) - D^2 H\left(Y_T^{t,\eta}\right), \mathbb{1}_{[t-T,0]^2}\rangle \cdot \frac{(W_{t+\epsilon} - W_t)^2}{\epsilon}\right] \leq$$

$$\leq \sqrt{\mathbb{E}\left[\left\|\langle D^2 H\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right)-D^2H\left(Y_T^{t,\eta}\right), \mathbb{1}_{[t-T,0]^2}\rangle\right\|^2_{L^2([-T,0]^2)}\right]} \cdot \sqrt{3}$$

The last term converges to zero because $D^2H \in C^0(L^2([-T,0]))$ and D^2H has polynomial growth as we have seen in Remark 9.42.2.

• We show that $A_{33}(\epsilon,t,\eta)$ converges to zero. We rewrite $A_{33}(\epsilon,t,\eta)$ as $A_{331}(\epsilon,t,\eta)-A_{332}(\epsilon,t,\eta)$, where

$$A_{331}(\epsilon,t,\eta)=\mathbb{E}\left[\int_{t-T}^0\int_{-T}^{t-T} D_yD_xH\left(Y_T^{t,\eta}\right)\frac{\eta(y+T-t+\epsilon)-\eta(y+T-t)}{\epsilon}(W_{t+\epsilon}-W_t)dy\,dx\right]$$

$$A_{332}(\epsilon,t,\eta)=\int_0^1\mathbb{E}\left[\int_{t-T}^0\int_{-T}^{t-T} D_yD_xH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\right)\cdot\right.$$
$$\left.\cdot\frac{\eta(y+T-t+\epsilon)-\eta(y+T-t)}{\epsilon}(W_{t+\epsilon}-W_t)dy\,dx\right]d\alpha\,.$$

We will show that both $A_{331}(\epsilon,t,\eta)$ and $A_{332}(\epsilon,t,\eta)$ converge to zero.
Let us consider

$$\mathcal{Z}:=\langle D^2H\left(Y_T^{t,\eta}\right)\ ,\ \mathbb{1}_{[t-T,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\left[\eta(y+T-t+\epsilon)-\eta(y+T-t)\right]\rangle\,. \tag{9.95}$$

Using Lemma 9.38 and Lemma 9.39 and the fact that $\mathcal{Z}$ is Fréchet differentiable, since $H\in C^3(L^2([-T,0]))$, it follows that $\mathcal{Z}$ belongs to $\mathbb{D}^{1,2}$ and

$$\begin{aligned}D_r^m\mathcal{Z}&=\int_{r-T}^0 D_z\mathcal{Z}\,dz=\langle D\mathcal{Z}\ ,\ \mathbb{1}_{[r-T,0]}(z)\rangle=\\&=\langle D^3H\left(Y_T^{t,\eta}\right)\ ,\ \mathbb{1}_{[t-T,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\left[\eta(y+T-t+\epsilon)-\eta(y+T-t)\right]\otimes\mathbb{1}_{[r-T,0]}(z)\rangle\,.\end{aligned} \tag{9.96}$$

Using (9.95), Skorohod integral, integration by parts on Wiener space (2.40), (9.96) and successively Fubini's theorem between r and z and then integrating with respect to r, we obtain

$$\begin{aligned}A_{331}(\epsilon,t,\eta)&=\frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot(W_{t+\epsilon}-W_t)\right]=\frac{1}{\epsilon}\mathbb{E}\left[\mathcal{Z}\cdot\int_t^{t+\epsilon}\delta W_u\right]=\\&=\frac{1}{\epsilon}\mathbb{E}\left[\int_t^{t+\epsilon}D_r^m\mathcal{Z}\,dr\right]=\frac{1}{\epsilon}\mathbb{E}\left[\int_t^{t+\epsilon}\int_{r-T}^0 D_z\mathcal{Z}\,dz\,dr\right]\\&=\mathbb{E}\left[\int_{t-T}^0 D_z\mathcal{Z}\,dz\right]=\\&=\mathbb{E}\left[\langle D^3H\left(Y_T^{t,\eta}\right)\ ,\ \mathbb{1}_{[t-T,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\left[\eta(y+T-t+\epsilon)-\eta(y+T-t)\right]\otimes\mathbb{1}_{[t-T,0]}(z)\rangle\right]\,.\end{aligned} \tag{9.97}$$

The third order Fréchet derivative of H, denoted by D^3H, is an operator from $L^2([-T,0])$ to the dual of the triple projective tensor product of $L^2([-T,0])$, i.e. $\left(L^2([-T,0])\hat{\otimes}_\pi^3\right)^*$. We recall that, given a general Banach space E equipped with its norm $\|\cdot\|_E$ and x,y,z three elements of E, then the norm of an elementary element of the tensor product $x \otimes y \otimes z$ which belongs to $E \otimes^3$ is $\|x\|_E \cdot \|y\|_E \cdot \|z\|_E$. Since $\left\|\mathbb{1}_{[-T,t-T]}(\cdot)\left[\eta(\cdot+T-t+\epsilon)-\eta(\cdot+T-t)\right]\right\|_{L^2([-T,0])} = \left\|\mathbb{1}_{[-t,0]}(\cdot)\left[\eta(\cdot+\epsilon)-\eta(\cdot)\right]\right\|_{L^2([-T,0])}$ obviously converges to zero, we obtain the following.

$$|\langle D^3 H\left(Y_T^{t,\eta}\right)\ ,\ \mathbb{1}_{[t-T,0]}(x) \otimes \mathbb{1}_{[-t,0]}(y)\left[\eta(y+\epsilon)-\eta(y)\right] \otimes \mathbb{1}_{[t-T,0]}(z)\rangle| \leq$$

$$\leq \left\|D^3 H\left(Y_T^{t,\eta}\right)\right\|_{\left(L^2([-T,0])\hat{\otimes}_\pi^3\right)^*}\left\|\mathbb{1}_{[t-T,0]}(\cdot)\right\|^2_{L^2([-T,0])}\left\|\mathbb{1}_{[-t,0]}(\cdot)\left[\eta(\cdot+\epsilon)-\eta(\cdot)\right]\right\|_{L^2([-T,0])} \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0\ .$$

By the polynomial growth of D^3H, (9.57), the fact that for any given Brownian motion $\bar{W}$, $\sup_{x\leq T}|\bar{W}_x|$ has all moments and finally the Lebesgue dominated convergence theorem we conclude that (9.97) converges to zero, therefore $A_{331}(\epsilon,t,\eta)$ converges to zero.
At this point we should establish the convergence to zero of $A_{332}(\epsilon,t,\eta)$. This can be done using, again as above, integration by parts on Wiener space (2.40). However there are several technicalities that we omit.
• We show finally that $A_{34}(\epsilon,t,\eta)$ converges to zero.
Using the finite increments theorem, for every $\alpha\in[0,1]$, $\omega\in\Omega$ a.s., it follows that

$$\frac{1}{\epsilon}\int_{t-T}^{0}\int_{t-T}^{t-T+\epsilon}\left(D_yD_xH\left(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}\right)-D_yD_xH\left(Y_T^{t,\eta}\right)\right)$$
$$\left[\eta(y+T-t-\epsilon)-\eta(0)-W_T(y)+W_{t+\epsilon}\right](W_t-W_{t+\epsilon})dy\,dx$$

converges to zero. By polynomial growth of D^2H, (9.59), the usual property that given any Brownian motion $\bar{W}$, $\sup_{x\leq T}|\bar{W}_x|$ has all moments and applying Lebesgue dominated convergence theorem we conclude that $A_{34}(\epsilon,t,\eta)$ converges to zero.
• We are now able to express $\partial_t u : [0,T]\times C([-T,0]) \longrightarrow \mathbb{R}$. For $t\in[0,T]$, it gives $\partial_t u(t,\eta) = I_1(t,\eta)+I_3(t,\eta)+A_{31}(t,\eta)$, i.e.

$$\begin{aligned}\partial_t u(t,\eta) = {} & \mathbb{E}\left[D_{-T}H\left(Y_T^{t,\eta}\right)\eta(-t)\right]+\mathbb{E}\left[\int_{-t}^{0}D'_{x-T+t}H\left(Y_T^{t,\eta}\right)\eta(x)dx\right]-\mathbb{E}\left[D_{t-T}H\left(Y_T^{t,\eta}\right)\eta(0)\right]+\\ & -\frac{1}{2}\mathbb{E}\left[\langle D^2H\left(Y_T^{t,\eta}\right),\mathbb{1}_{[t-T,0]^2}\rangle\right].\end{aligned} \tag{9.98}$$

Taking into account (9.77) and Notation 9.25, it finally follows that u solves (9.71). □

As consequence of previous theorem we obtain the following.

Corollary 9.45. Let H which satisfies the assumptions of Theorem 9.41 and $u(t,\eta)=\mathbb{E}\left[H\left(Y_T^{t,\eta}\right)\right], t\in[0,T], \eta\in C([-T,0])$ defined as in (9.63). Let X a real continuous process with $[X]_t=t$ and $X_0=0$. Then the random variable h defined by $h := H(X_T(\cdot))$ admits the representation

$$h = H_0+\int_0^T \xi_t d^- X_t \tag{9.99}$$

where $H_0=u(0,X_0(\cdot))$, $\xi_t = D^{\delta_0}u(t,X_t(\cdot))$ and $\int_0^T \xi_t d^-X_t$ is a proper forward integral.

Proof. The proof is a consequence of Theorem 9.41 and Corollary 9.28. The forward is proper because because $u \in C^{1,2}([0,T] \times C([-T,0]))$. □

Remark 9.46. If X is a continuous semimartingale such that $[X]_t = t$ previous Corollary 9.45 applies and the forward integral in (9.99) is in fact an Itô integral, see Remark 2.9.2. We recall to this purpose that the process ξ is continuous, since $D^{\delta_0} u\,(t,\eta)$ is continuous.

We repeat that Corollary 9.45 constitutes a generalization of Clark-Ocone formula. Suppose that $X = W$ is the classical Brownian motion. If $h \in \mathbb{D}^{1,2}$, the classical Clark-Ocone formula recalled in (9.3) holds. Next proposition shows that (9.3) has a robust form which does not depend on the law of $W(\cdot)$, i.e. Wiener measure, at least if has a smooth Fréchet dependence on the underlying process.

Proposition 9.47. Let $u(t,\eta) = \mathbb{E}\left[H\big(Y_T^{t,\eta}\big)\right], t \in [0,T], \eta \in C([-T,0])$, defined as in (9.63), fulfilling assumption of Theorem 9.41 and $X = W$ the Brownian motion equipped with its canonical filtration $(\mathcal{F}_t)$, $h = H(W_T(\cdot))$. Then

$$D^{\delta_0} u(t, W_t(\cdot)) = \mathbb{E}\left[D_t^m H(W_T(\cdot)) | \mathcal{F}_t\right]. \tag{9.100}$$

In particular

$$\int_0^T D^{\delta_0} u(t, W_t(\cdot)) d^- W_t = \int_0^T \mathbb{E}\left[D_t^m h | \mathcal{F}_t\right] dW_t \tag{9.101}$$

Proof.

1. Remark 2.9.2. says that the forward integral from 0 to $t \in [0,T[$ coincides with Itô integral; the result follows by uniqueness of the representation of $h \in L^2(\Omega, \mathcal{F}_T)$ in the Brownian case.

2. On the other hand it is possible to show (9.100) directly. In fact using Lemma 9.38 and the fact that $h = H(W_T(\cdot))$ we have

$$D_t^m h = \int_{t-T}^0 D_s H(W_T(\cdot)) ds.$$

Taking the expectation with respect to $(\mathcal{F}_t)$ we obtain

$$\mathbb{E}\left[D_t^m h | \mathcal{F}_t\right] = \mathbb{E}\left[\int_{t-T}^0 D_s H\big(Y_T^{t,W_t(\cdot)}\big) ds | \mathcal{F}_t\right] = \Gamma(W_t(\cdot))$$

where

$$\Gamma(\eta) = \mathbb{E}\left[\int_{t-T}^0 D_s H\big(Y_T^{t,\eta}\big) ds\right].$$

We observe that $\Gamma(\eta) = D^{\delta_0} u\,(t,\eta)$ by (9.76).

□

9.8.3 Some considerations about a martingale representation theorem

Suppose that $X = M$ is a square integrable martingale equipped with its canonical filtration $(\mathcal{G}_t)$ and $h = H(M_T(\cdot))$ with $H : C([-T, 0]) \longrightarrow \mathbb{R}$ having linear growth. We are interested in sufficient conditions so that

$$h = \mathbb{E}[h] + \int_0^T \xi_s dM_s \tag{9.102}$$

where (ξ_s) is an explicit previsible process.

We state a result which belongs to the same family as Corollary 9.45. In fact in that corollary if process X is a continuous semimartingale, the Clark-Ocone type formula stated at (9.99) holds and of course the forward integral is a Itô integral.
The proposition below is a consequence of Theorem 7.36. We recall that $\mathcal{D}_0 \oplus L^2$ denotes $\mathcal{D}_0([-T, 0]) \oplus L^2([-\tau, 0])$.

Proposition 9.48. Let $u : [0, T] \times C([-T, 0]) \longrightarrow \mathbb{R}$ continuous such that $(t, \eta) \mapsto Du(t, \eta)$ exists with values in $\mathcal{D}_0 \oplus L^2$ and $Du : [0, T] \times C([-T, 0]) \longrightarrow \mathcal{D}_0 \oplus L^2$ is continuous. If moreover

$$\mathbb{E}\left[h|\mathcal{G}_t\right] = u(t, M_t(\cdot)) \qquad\qquad \forall\, t \in [0, T[\text{ a.s.} \tag{9.103}$$

then

$$h = \mathbb{E}[h|\mathcal{G}_0] + \int_0^T D^{\delta_0} u(s, M_s(\cdot)) dM_s\ . \tag{9.104}$$

Proof. We observe that u verifies the assumptions of Theorem 7.36, then $u(\cdot, M_\cdot(\cdot))$ is a $(\mathcal{G}_t)$-weak Dirichlet process with martingale part, according to (7.25), given by

$$M^u_t = u(0, M_0(\cdot)) + \int_0^t D^{\delta_0} u(s, M_s(\cdot)) dM_s\ . \tag{9.105}$$

By (9.103), $u(\cdot, M_\cdot(\cdot))$ is obviously a $(\mathcal{G}_t)$-martingale being a conditional expectation with respect to filtration $(\mathcal{G}_t)$. By uniqueness of the decomposition of $(\mathcal{G}_t)$-weak Dirichlet processes, see Remark 3.5 in [32], it follows

$$u(t, M_t(\cdot)) = u(0, M_0(\cdot)) + \int_0^T D^{\delta_0} u(s, M_s(\cdot)) dM_s\ .$$

In particular the $(\mathcal{G}_t)$-martingale orthogonal process is zero. Since $h = u(T, M_T(\cdot))$ and $u(0, M_0(\cdot)) = \mathbb{E}[h|\mathcal{G}_0]$ the result follows. ☐

9.9 The infinite dimensional PDE with an $L^2([-T,0])$-finitely based terminal condition

As mentioned earlier, this subsection gives sufficient conditions on H to solves the infinite dimensional PDE in Corollary 9.28 involving much less regularity on H with respect to Section 9.8.

Notation 9.49. In this section, if $g, \ell : [a,b] \to \mathbb{R}$ are càdlàg and g has bounded variation we will use the following notation

$$\int_{[a,b]} g\, d\ell = g(b)\, \ell(b) - g(a^-)\, \ell(a^-) - \int_{[a,b]} \ell\, dg\ . \tag{9.106}$$

We introduce the functional H. For all $i = 1, \ldots, n$, let $\varphi_i : [0,T] \longrightarrow \mathbb{R}$ be $C^2([0,T];\mathbb{R})$. it exists $\dot{\varphi}_i \in L^2([0,T])$ For technical reasons we extend for every i, $\varphi_i(t) = 0$ for $t \notin [0,T]$. Obviously we have $\varphi_i(0^-) = 0$ and $\varphi_i(T^+) = 0$.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable and with linear growth. We consider the functional

$$H : C([-T,0]) \to \mathbb{R}$$

defined by

$$H(\eta) = f\left(\int_{[-T,0]} \varphi_1(u+T) d\eta(u), \ldots, \int_{[-T,0]} \varphi_n(u+T) d\eta(u)\right)\ . \tag{9.107}$$

Let X be again a real continuous process such that $X_0 = 0$ and $[X]_t = t$. According to previous Notation 9.49, the random variable $h := H(X_T(\cdot))$ can be expressed as follows.

$$\begin{aligned} h = H(X_T(\cdot)) &= f\left(\int_{[-T,0]} \varphi_1(u+T) d^- X_T(u), \ldots, \int_{[-T,0]} \varphi_n(u+T) d^- X_T(u)\right) = \\ &= f\left(\int_0^T \varphi_1(s) d^- X_s, \ldots, \int_0^T \varphi_n(s) d^- X_s\right)\ . \end{aligned} \tag{9.108}$$

For every $i \in \{1, \ldots, n\}$, integration by parts (2.9) for stochastic processes implies

$$\begin{aligned} \int_0^t \varphi_i(s) d^- X_s &= \int_{-t}^0 \varphi_i(u+t) d^- X_t(u) = \varphi_i(t) X_t(0) - \varphi_i(0) X_t(-t) - \int_{[-t,0]} X_t(u) d\varphi_i(u+t) = \\ &= \varphi_i(t) X_t - \int_{[-t,0]} X_s d\varphi_i(s), \end{aligned} \tag{9.109}$$

so that previous integrals can be characterized pathwise.
We formulate the following assumption.

Assumption 1. For $t \in [0,T]$, we denote Σ_t the matrix in $\mathbb{M}_{n\times n}(\mathbb{R})$ defined by

$$(\Sigma_t)_{1\le i,j\le n} = \left(\int_t^T \varphi_i(s)\varphi_j(s)ds\right)_{1\le i,j\le n} .$$

We suppose

$$\det(\Sigma_t) > 0 \qquad \forall\, t \in [0,T[\ . \tag{9.110}$$

Remark 9.50.

1. We observe that, by continuity of function $t \mapsto \det(\Sigma_t)$, there is always $\tau > 0$ such that $\det(\Sigma_t) \neq 0$ for all $t \in [0,\tau[$.
2. It is not restrictive to consider $\det(\Sigma_0) \neq 0$ since it is always possible to orthogonalise $(\varphi_i)_{i=1,\dots,n}$ in $L^2([0,T])$ via a Gram-Schmidt procedure.
3. When the family is orthogonal, Σ_0 is a diagonal invertible matrix in $M_{n\times n}(\mathbb{R})$.

In view of defining a functional $u : [0,T]\times C([-T,0]) \longrightarrow \mathbb{R}$, we suppose for a while that X is a classical Wiener W process equipped with its canonical filtration $(\mathcal{F}_t)$. We set $h = H(W_T(\cdot))$ and we evaluate the conditional expectation $\mathbb{E}[h|\mathcal{F}_t]$. It gives

$$\begin{aligned}
\mathbb{E}[h|\mathcal{F}_t] &= \mathbb{E}\left[f\left(\int_0^T \varphi_i(s)dW_s,\dots,\int_0^T \varphi_n(s)dW_s\right)|\mathcal{F}_t\right] = \\
&= \Psi\left(t, \int_0^t \varphi_1(s)dW_s,\dots,\int_0^t \varphi_n(s)dW_s\right) = \\
&= \Psi\left(t, \int_{-t}^0 \varphi_1(u+t)dW_t(u),\dots,\int_{-t}^0 \varphi_n(u+t)dW_t(u)\right) = \\
&= \Psi\left(t, \int_{-T}^0 \varphi_1(u+t)dW_t(u),\dots,\int_{-T}^0 \varphi_n(u+t)dW_t(u)\right) ,
\end{aligned} \tag{9.111}$$

where the function $\Psi : [0,T]\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by

$$\Psi(t,y_1,\dots,y_n) = \mathbb{E}\left[f\left(y_1 + \int_t^T \varphi_1(s)dW_s,\dots\dots,y_n + \int_t^T \varphi_n(s)dW_s\right)\right], \tag{9.112}$$

for any $t\in[0,T], y_1,\dots,y_n \in \mathbb{R}$. In particular

$$\Psi(T,y_1,\dots,y_n) = f(y_1,\dots\dots,y_n) \ . \tag{9.113}$$

We simplify expression (9.112) introducing the density function p of the Gaussian vector

$$\left(\int_t^T \varphi_1(s)dW_s,\dots,\int_t^T \varphi_n(s)dW_s\right)$$

whose variance-covariance matrix equals to Σ_t. Function $p : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is characterized by

$$p(t, z_1, \ldots, z_n) = \sqrt{\frac{1}{(2\pi)^n \det(\Sigma_t)}} \exp\left\{ -\frac{(z_1, \ldots, z_n)\Sigma_t^{-1}(z_1, \ldots, z_n)^*}{2} \right\} ,$$

and function Ψ becomes

$$\Psi(t, y_1, \ldots, y_n) = \begin{cases} \int_{\mathbb{R}^n} f(\tilde{z}_1, \ldots, \tilde{z}_n) \, p(t, \tilde{z}_1 - y_1, \ldots, \tilde{z}_n - y_n) d\tilde{z}_1 \cdots d\tilde{z}_n & \text{if } t \in [0, T[\\ f(y_1, \ldots \ldots, y_n) & \text{if } t = T\ . \end{cases} \tag{9.114}$$

Remark 9.51.

1. We remark that, at time $t = T$, $\Psi(T, \cdot)$ is a function which strictly depends on the representative of f and not only on its Lebesgue a.e. representative. So Ψ, as a class does not admit a restriction to $t = T$.

2. Function p is a solution $C^{3,\infty}([0,T[\times\mathbb{R}^n)$ of

$$\partial_t p(t, z_1, \ldots, z_n) = -\frac{1}{2} \sum_{i,j=1}^{n} \varphi_i(t)\varphi_j(t)\partial^2_{ij} p(t, z_1, \ldots, z_n)\ . \tag{9.115}$$

Therefore function Ψ is $C^{1,2}([0,T[\times\mathbb{R}^n)$ and solves

$$\partial_t \Psi(t, z_1, \ldots, z_n) = -\frac{1}{2} \sum_{i,j=1}^{n} \varphi_i(t)\varphi_j(t)\partial^2_{ij} \Psi(t, z_1, \ldots, z_n). \tag{9.116}$$

We define now function $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ by

$$u(t, \eta) = \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t) d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t) d\eta(s)\right) , \tag{9.117}$$

where $\Psi(t, y_1, \ldots, y_n)$ is defined by (9.114).
By the fact that, for every i, function φ_i are C^2 bounded variation functions and $\varphi_i(0^-) = 0$ we can write, according to Notation 9.49,

$$\int_{[-t,0]} \varphi_i(s+t) d\eta(s) = \eta(0)\varphi_i(t) - \int_{[0,t]} \eta(s-t)\dot{\varphi}_i(s) ds\ .$$

Obviously $\int_{[0,t]} \eta(s-t)\dot{\varphi}_i(s)ds = \int_{]0,t]} \eta(s-t)\dot{\varphi}_i(s)ds$ being a Lebesgue integral.

Remark 9.52. By construction we have

$$u(t, W_t(\cdot)) = \mathbb{E}[h|\mathcal{F}_t]$$

and in particular $u(0, W_0(\cdot)) = \mathbb{E}[h]$.

We state now the first proposition related to the section.

Proposition 9.53. Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ be defined by (9.107) and $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ be defined by (9.117).

1. Function u belongs to $C^{1,2}\left([0,T[\times C([-T,0])\right)$ and it solves (9.49), i.e.

$$\begin{cases} \mathcal{L}u\,(t,\eta) = \partial_t u(t,\eta) + \displaystyle\int_{]-t,0]} D^{ac}_x u(t,\eta)\, d\eta(x) + \frac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = 0 \\ u(T,\eta) = H(\eta) \end{cases}$$

2. If f is continuous then we have moreover $u \in C^0\left([0,T] \times C([-T,0])\right)$.

Proof. We first evaluate the derivative $\partial_t u\,(t,\eta)$, for a given $(t,\eta) \in [0,T] \times C([-T,0)$:

$$\begin{aligned} \partial_t u(t,\eta) =& \partial_t \Psi\left(t, \int_{[-t,0]} \varphi(s+t) d\eta(s)\right) + \\ & + \sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s)\right)\right) \cdot \left(\partial_t \int_{[-t,0]} \varphi_i(s+t)d\eta(s)\right) = \\ & = \partial_t \Psi\left(t, \int_{[-t,0]} \varphi(s+t) d\eta(s)\right) + \\ & + \sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s)\right)\right) \cdot \left(\int_{]-t,0]} \dot{\varphi}_i(s+t)d\eta(s)\right) . \end{aligned} \tag{9.118}$$

The last equality holds through integration by parts, Notation 9.49 we obtain

$$\begin{aligned} \partial_t \left(\int_{[-t,0]} \varphi_i(s+t)d\eta(s)\right) &= \partial_t \left(\eta(0)\varphi_i(t) - \int_{[-t,0]} \eta(s)\dot{\varphi}_i(s+t)ds\right) = \\ &= \eta(0)\dot{\varphi}_i(t) - \eta(-t)\dot{\varphi}_i(0^+) - \int_{[-t,0]} \eta(s)\ddot{\varphi}_i(s+t)ds = \\ &= \eta(0)\dot{\varphi}_i(t) - \eta(-t)\dot{\varphi}_i(0^+) - \int_{]-t,0]} \eta(s)\ddot{\varphi}_i(s+t)ds = \\ &= \int_{]-t,0]} \dot{\varphi}_i(s+t)d\eta(s) . \end{aligned} \tag{9.119}$$

We go on with the evaluation of the derivatives with respect to η. For every $t \in [0,T]$, $\eta \in C([-T,0])$, the first derivative Du evaluated at (t,η) is the measure on $[-T,0]$ defined by

$$\begin{aligned} D_{dx} u(t,\eta) &= D^{ac}_x u(t,\eta)\, dx + D^{\delta_0} u(t,\eta)\delta_0(dx) \\ D^{ac}_x u(t,\eta) &= -\sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s)\right)\right) \cdot \left(\mathbb{1}_{]-t,0]}(x)\dot{\varphi}_i(x+t)\right) \\ D^{\delta_0} u(t,\eta) &= \sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s)\right)\right) \cdot \varphi_i(t) . \end{aligned}$$

(9.120)

We observe that $x \mapsto D_x^{ac} u\,(t,\eta)$ has bounded variation.
For every $t \in [0,T]$, $\eta \in C([-T,0])$, the second order derivative $D^2 u$ evaluated at (t,η) gives

$$D^2_{dx,dy}u(t,\eta) = \sum_{i,j=1}^{n} \left(\partial^2_{i,j}\Psi \left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s) \right) \right) \cdot$$
$$\cdot \Bigg(\varphi_i(t)\varphi_j(t)\delta_0(dx)\delta_0(dy) - \varphi_i(t)\mathbb{1}_{[-t,0]}(x)\dot{\varphi}_j(dx+t)\delta_0(dy) +$$
$$- \varphi_j(t)\mathbb{1}_{[-t,0]}(y)\dot{\varphi}_i(dy+t)\delta_0(dx) + \mathbb{1}_{[-t,0]}(x)\mathbb{1}_{[-t,0]}(y)\dot{\varphi}_i(dx+t)\dot{\varphi}_j(dy+t) \Bigg) . \tag{9.121}$$

We also observe that $D^2 u : [0,T] \times C([-T,0]) \to \chi^0([-T,0]^2)$ continuously.
Using (9.116) we obtain that

$$\mathcal{L}u\,(t,\eta) = \sum_{i=1}^{n} \left(\partial_i \Psi \left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s) \right) \right) \cdot I_i$$

where

$$I_i = \left(\int_{]-t,0]} \dot{\varphi}_i(x+t)d\eta(x) - \int_{]-t,0]} \mathbb{1}_{]-t,0]}(x)\dot{\varphi}_i(x+t)d\eta(x) \right) = 0 \,. \tag{9.122}$$

We conclude that $\mathcal{L}u\,(t,\eta) = 0$.
Condition $u(T,\eta) = H(\eta)$ is trivially verified by definition. This concludes the proof of point 1.
Point 2. is immediate. □

Remark 9.54. In this example we have introduced the concept of integral on a closed interval

$$\int_{[-t,0]} \varphi_i(s+t)d\eta(s) \,. \tag{9.123}$$

It is applied to $\eta = X_t(\cdot)$. Since $X_0 = 0$ we have

$$\int_{[-t,0]} \varphi_i(s+t)d\eta(s)|_{\eta = X_t(\cdot)} = \int_{]-t,0]} \varphi_i(s+t)d\eta(s)|_{\eta = X_t(\cdot)}$$

The choice of (9.123) is justified since

$$t \mapsto \int_{]-t,0]} \varphi_i(s+t)d\eta(s)$$

is not differentiable.

We can now state the main theorem of the section.

Proposition 9.55. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Borel function with linear growth. Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ be defined by (9.107) and $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ be defined by (9.117). Let X be a continuous finite quadratic variation process such that $X_0 = 0$ and $[X]_t = t$. Let h be the random variable $H(X_T(\cdot))$. Suppose that one of the following assumptions holds:

1. f is continuous with linear growth.

2. X is a classical Brownian motion W and f is Borel subexponential.

Then

$$h = H_0 + \int_0^T \xi_t d^- X_t \tag{9.124}$$

with $H_0 = u(0, X_0(\cdot))$ and $\xi_t = D^{\delta_0} u(t, X_t(\cdot))$

Proof.

1. follows by Proposition 9.53 and Corollary 9.28.

2. We apply Proposition 9.27 from 0 to $s < T$ and Remark 2.9.2 which gives

$$u(s, W_s(\cdot)) = H_0 + \int_0^s \xi_t dW_t,$$

where $\xi_t = D^{\delta_0} u(t, X_t(\cdot))$ and $D^{\delta_0} u(t, \eta)$ is given by (9.120). Clearly the process

$$\partial_i \Psi \left(t, \int_{-t}^0 \varphi_1(s+t) dW_t(s), \ldots, \int_{-t}^0 \varphi_n(s+t) dW_t(s) \right) \varphi_i(t)$$

is $(\mathcal{F}_t)$-adapted, so the forward integral coincides with the classical Itô integral since $s < T$. To conclude we need to take the limit when $s \longrightarrow T$. Since $u(s, W_s(\cdot))$ is the Brownian martingale $\mathbb{E}[h|\mathcal{F}_s]$, the result follows by Lemma 9.8. We have therefore

$$h = u(T, W_T(\cdot)) = H_0 + \int_0^T \xi_t d^- W_t$$

where

$$\xi_t = \sum_{i=1}^n \partial_i \Psi \left(t, \int_0^t \varphi_1(s) dW_s, \ldots, \int_0^t \varphi_n(s) dW_s \right) \varphi_i(t) = D^{\delta_0} u(t, W_t(\cdot))$$

and $H_0 = u(0, W_0(\cdot)) = \mathbb{E}[h]$ since Remark 9.52.

□

Remark 9.56. 1. If f is Lipschitz then $\xi_t = \mathbb{E}\left[D_t^m h|\mathcal{F}_t\right]$ since $h \in \mathbb{D}^{1,2}$. This follows either by uniqueness of the representation of square integrable random variable or by a direct computation of Malliavin derivatives and conditional expectation. In fact

$$\begin{aligned}D_t^m h = D_t^m &\left[f\left(\int_0^T \varphi_1(s)dW_s, \ldots, \int_0^T \varphi_n(s)dW_s\right)\right] = \\ &= \sum_{i=1}^n \partial_i f\left(\int_0^T \varphi_1(s)dW_s, \ldots, \int_0^T \varphi_n(s)dW_s\right)\varphi_i(t)\end{aligned}$$

and

$$\mathbb{E}\left[D_t^m h|\mathcal{F}_t\right] = \sum_{i=1}^n \mathbb{E}\left[\partial_i f\left(\int_0^T \varphi_1(s)dW_s, \ldots, \int_0^T \varphi_n(s)dW_s\right)|\mathcal{F}_t\right]\varphi_i(t) \, .$$

By the definition of Ψ in (9.114), for every $i = 1, \ldots, n$, we can show that

$$\mathbb{E}\left[\partial_i f\left(\int_0^T \varphi_i(s)dW_s, \ldots, \int_0^T \varphi_n(s)dW_s\right)|\mathcal{F}_t\right] = \partial_i \Psi\left(t, \int_0^t \varphi_1(s)dW_s, \ldots, \int_0^t \varphi_n(s)dW_s\right) \tag{9.125}$$

2. In particular we observe that $\mathbb{E}\left[D_t^m h|\mathcal{F}_t\right]$ only depends on the derivatives of Ψ which may exists also when f is not differentiable.

3. We emphasize again that when $X = W$ the improper forward integral in the representation (9.124) is not a classical Itô integral. In fact Ψ not differentiable in T.

Appendix A

Bochner and Pettis Integral

As a main reference we mention [19] and [18].
Those integrals are generalization of Lebesgue integral to Banach-valued functions, i.e. are used for integrations of functions f from some finite measure space $(\Omega, \mathcal{F}, \mu)$ to a Banach space F equipped with a norm $\|\cdot\|_F$. Both the integrals are F-valued. We recall some definitions and properties.

A function $f : \Omega \longrightarrow F$ is **weakly measurable** if the scalar function $g \circ f : \Omega \longrightarrow \mathbb{R}$, also denoted by ${}_{F^*}\langle g\,,\, f\rangle_F$, is measurable for every $g \in F^*$.

A weakly measurable function f is **Pettis integrable** if for every $A \in \mathcal{F}$ there is an element of F, denoted $\int_A f d\mu$, such that for every $g \in F^*$. ${}_{F^*}\langle g\,,\, f\rangle_F$ belongs to $L^1(d\mu)$

$$
{}_{F^*}\langle g\,,\, \int_A f d\mu\rangle_F = \int_A {}_{F^*}\langle g\,,\, f\rangle_F d\mu.
$$

A function $f : \Omega \longrightarrow F$ is **strongly measurable** with respect to μ if is μ-a.e. the limit in the norm topology of F of a sequence of simple functions (f_n), i.e. if $\|f_n - f\|_F \xrightarrow[n\longrightarrow+\infty]{} 0$.

A strongly measurable function $f : \Omega \longrightarrow F$ is **Bochner integrable** with respect to μ, or μ-Bochner integrable, if $\int_\Omega \|f\|_F d\mu < +\infty$. This is equivalent to have $\int_\Omega \|f_n - f\|_F d\mu \xrightarrow[n\longrightarrow+\infty]{} 0$ where the integral on the left-hand side is an ordinary Lebesgue integral. In this case the Bochner integral of f exists and is an element of F defined by $\lim_{n\longrightarrow 0} \int_\Omega f_n\, d\mu$.

The space of μ-Bochner integrable functions $f : \Omega \longrightarrow F$ defined μ-a.e. on Ω is denoted by $L^1(\Omega, \mathcal{F}, \mu;\ F)$ or even by $L^1_F(\mu)$.

For $f \in L^1_F(\mu)$ we define a seminorm, called **Bochner norm**, defined by $\|f\|_1 = \|f\|_{L^1_F(\mu)} := \int_\Omega \|f\|_F d\mu$. Then $L^1_F(\mu)$ is complete for this seminorm. The set of equivalence classes of μ-measurable functions still denoted by $L^1_F(\mu)$ is a Banach space.

An important theorem discussing the relation between the strongly and weakly measurability is the Pettis Measurability Theorem which says that a function is strongly measurable if and only if it is weakly measurable and there is a μ-null set which has separable range, i.e. there exists a set $N \in \mathcal{F}$ with $\mu(N) = 0$ such that the range set $\{f(x);\ x \in \Omega \setminus N\} \subset F$ is separable. For more details about those arguments we refer to [74].

In this paper generally we consider $(\Omega, \mathcal{F}, \mu) = ([0,T], \mathcal{B}([0,T]), \mu)$ where $\mathcal{B}([0,T])$ denotes the Borel algebra on $[0,T]$ and μ denotes the Lebesgue measure on $[0,T]$.

We recall the construction of the Bochner integral.

We recall that let a set Ω, a **ring** is a a set of subsets of Ω closed under union $A \cup B$ and difference $A \setminus B$, for all possible $A, B \subseteq \Omega$. A **δ-ring** is ring closed under countable intersections. We denote by $\mathcal{D}$ the the δ-ring of the sets $A \in \mathcal{F}$ with $\mu(A) < \infty$. In the definition of the Bochner integral only the restriction of the measure μ to the δ-ring $\mathcal{D}$ is influent. In fact if $\mathcal{R} \subset \mathcal{D}$ is a ring generating the δ-ring $\mathcal{D}$, then the set of $\mathcal{R}$-step function is dense in $L^1_F(\mu)$. For a $\mathcal{D}$-step function $f = \sum_{i \in I} \phi_{A_i} f_i$ with $A_i \in \mathcal{D}$ mutually disjoint such that $\bigcup_{i \in I} A_i = \Omega$ and $(f_i)_{i \in I} \in F$, we have

$$\int f d\mu = \sum_{i \in I} \mu(A_i) f_i \in F$$

and

$$\left\| \int f d\mu \right\|_F = \left\| \sum_{i \in I} \mu(A_i) f_i \right\|_F \leq \sum_{i \in I} \mu(A_i) \|f_i\|_F = \int_\Omega \|f\|_F \, d\mu = \|f\|_1$$

So the mapping $f \longrightarrow \int f d\mu$ from the subspace of the F-valued $\mathcal{D}$-step functions, into the space F, is continuous for the seminorm $\|f\|_1$, therefore it can be extended uniquely to a linear, continuous mapping from the whole space $L^1_F(\mu)$ into F. The value of the extension for a function $f \in L^1_F(\mu)$ is denoted by $\int f d\mu$ and is called the **Bochner integral** of f with respect to μ.

We still have $\left\| \int f d\mu \right\|_F \leq \int \|f\|_F \, d\mu \leq \|f\|_1$ for $f \in L^1_F(\mu)$.

If $f \in L^1_F(\mu)$ and $A \in \mathcal{F}$, then $\phi_A f \in L^1_F(\mu)$ and we denote $\int_A f d\mu := \int \phi_A f d\mu$. The mapping $A \longrightarrow \int_A f d\mu$ from $\mathcal{F}$ into F is a σ-additive measure.

Let $L^p_F(\mu)$ the set of μ-measurable function $f : \Omega \longrightarrow F$ with $\|f\| \in L^p(\mu)$ (in the classical sense). We define on $L^p_F(\mu)$ the seminorm $\|f\|_p = \left(\int \|f\|^p\right)^{1/p} = \|\,|f|\,\|_p$ if $1 \leq p < \infty$. Then $L^p_F(\mu)$ is complete for the seminorm $\|f\|_p$. For $1 \leq p < \infty$, the set of equivalence class of μ-measurable functions still denoted by $L^p_F(\mu)$ is a Banach space.

We state a useful result about Bochner integral. Let E, F and G be Banach spaces.

Proposition A.1. Assume $E \subseteq L(F, G)$. If $f \in L^1_F(\mu)$ and $g \in E$, then $f \circ g$, denoted also by ${}_E\langle g\,,\, f\rangle_F$, belongs to $L^1_G(\mu)$ and we have

$$ {}_E\langle g\,,\, \int f d\mu\rangle_F = \int {}_E\langle g\,,\, f\rangle_F d\mu \in G\ . $$

In particular, if $f \in L^1_F(\mu)$ and $g \in F^*$, then $\langle f\,,\, g\rangle \in L^1_{\mathbb{R}}(\mu)$ and we have

$$ {}_{F^*}\langle g\,,\, \int f d\mu\rangle_F = \int {}_{F^*}\langle g\,,\, f\rangle_F d\mu\ . $$

In particular if f is a Bochner integrable function then it is naturally Pettis integrable and the Pettis integral exists and equals the Bochner integral.

Appendix B

Integration with respect to vector measure with finite variation

We are interested in the integral $\int f dm$, where m is a vector measure with finite variation and f is vector-valued.
The framework consists of δ-ring $\mathcal{D}$ of subsets of Ω, three Banach space E, F, G with $E \subset L(F, G)$ and a σ-additive **vector valued measure** $m : \mathcal{D} \longrightarrow E$ with finite variation $|m|$. We shall reduce integrability of vector-valued functions $f : \Omega \longrightarrow E$ with respect to m, to the Bochner integrability of f with respect to the variation $|m|$. We suppose $(\Omega, \mathcal{F}, |m|)$ the measure space with variation of the vector measure $|m|$.

Definition B.1. We say that a set $A \subset \Omega$ is m**-negligible** (resp. m-measurable) if it is $|m|$-negligible (resp. $|m|$-measurable). We say a function $f : \Omega \longrightarrow F$ is m-negligible, m-measurable, m**-integrable** if it has the same property with respect to the variation $|m|$ in the case of the classical Bochner integral.
For $1 \leq p < \infty$ we denote $L^p_F(m) := L^p_F(|m|)$ (in the Bochner sense) and endow $L^p_F(m)$ with the seminorm of $L^p_F(|m|)$, i.e.

$$\|f\|_p = \left(\int \|f\|^p_F d|m| \right)^{1/p}$$

if $1 \leq p < \infty$.

If $1 \leq p < \infty$, then $L^p_F(m)$ contains all the characteristic functions of the sets $A \in \mathcal{F}$ with $|m|(A) < +\infty$. We have the following properties: $L^p_F(m)$ is complete; if $1 \leq p < \infty$ and if $\mathcal{R}$ is a ring generating the δ-ring $\mathcal{D}$, then the $\mathcal{R}$-step functions $f : \Omega \longrightarrow F$ are dense in $L^p_F(m)$. In particular the $\mathcal{D}$-step functions are dense in $L^p_F(m)$. If $1 \leq p < \infty$ the Vitali and the Lebesgue convergence theorems are valid in $L^p_F(m)$.
For the m-measurable functions $f : \Omega \longrightarrow F$ the following assertions are equivalent: f is m-integrable; f is $|m|$-integrable; $|f|$ is $|m|$-integrable.

We define the integral for a $\mathcal{D}$-step function $f = \sum_{i\in I} \phi_{A_i} f_i$ with $A_i \in \mathcal{D}$ mutually disjoints and $f_i \in F$ by

$$\int f dm = \sum_{i\in I} {}_E\langle m(A_i)\, ,\, f_i\rangle_F \ \in G$$

and it holds

$$\left\|\int f dm\right\|_G = \left\|\sum_{i\in I} {}_E\langle m(A_i)\, ,\, f_i\rangle_F\right\|_G \le \sum_{i\in I} \|m(A_i)\|_E \, \|f_i\|_F \le \sum_{i\in I} |m|(A_i)\, \|f_i\|_F = \int_\Omega \|f\|_F d|m| = \|f\|_1 \ .$$

Therefore, the mapping $f \longrightarrow \int f dm$ is linear and continuous from the set of F-valued $\mathcal{D}$-step functions into G with respect to the norm $\|f\|_1$. Since the set of F-valued $\mathcal{D}$-step functions is dense in $L^1_F(m)$, we can extend uniquely the map $f \longrightarrow \int f dm$ to a linear continuous mapping on the whole space $L^1_F(m)$ with values in G. The value of this extension for a function $f \in L^1_F(m)$ is denoted by $\int f dm$ and is called the **integral of f with respect to m.**

We still have $\left\|\int f dm\right\|_G \le \int \|f\|_F d|m| = \|f\|_1$ for $f \in L^1_F(m)$. If $f \in L^1_F(m)$ and $A \in \mathcal{F}$, then $\phi_A f \in L^1_F(m)$ and we denote $\int_A f dm := \int \phi_A f dm$.

The following properties hold by construction:

If $f_n \xrightarrow[n\longrightarrow+\infty]{L^1_F(m)} f$ then $\int f_n dm \xrightarrow[n\longrightarrow+\infty]{G} \int f dm$.

If $f \in L^1_F(m)$, then the mapping $A \longrightarrow \int_A f dm$ from Σ into G is σ-additive and $\lim_{|m|(A)\to 0} \int_A |f| d|m| = 0$.

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