Restricted mean value property for balayage spaces with jumps

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Abstract

It is shown that, for α-stable processes (Riesz potentials) or – more generally – for balayage spaces with jumps, "one-radius" results for harmonicity can be obtained under fairly weak assumptions.

1 Introduction

A continuous real function f on an open set D in $\mathbb{R}^d$, $d \geq 1$, is known to be harmonic if and only if, for every $x \in D$ and every $r > 0$ such that the closed ball $\overline{B}(x,r)$ of center x and radius r is contained in D, the volume mean $\lambda_{x,r}(f)$ of f on $B(x,r)$ (the surface mean $\sigma_{x,r}(f)$ of f on $\partial B(x,r)$, respectively) is equal to $f(x)$. It is also elementary to see that it is sufficient for the harmonicity of f on D to know that, for each $x \in D$, this mean value property holds for a sequence $(r_n(x))$ of radii converging to 0. The question, if it is sufficient to have the right averages, for each $x \in D$, for just *one* radius $r(x)$, has a history which goes back hundred years (see the survey articles [12, 4]).

For simplicity, let us assume that $0 \leq f \leq 1$. Then, for volume means $\lambda_{x,r(x)}$, the answer is YES for any dimension $d \geq 1$, any open set D in $\mathbb{R}^d$ (if $d = 2$, $D = \mathbb{R}^2$ or $\mathbb{R}^2 \setminus D$ non-polar) and any function $r\colon D \to (0,\infty)$ such that $r \leq \operatorname{dist}(\cdot, D^c)$, if $D \neq \mathbb{R}^d$, and $r \leq |\cdot| + M$, if $D = \mathbb{R}^d$ (see [7, 9, 10, 6]).

For surface means, there are some positive results under additional hypotheses on the function r. To get a picture of what can be said without any further assumption on r, let D be the unit ball $B(0,1)$ and $0 < r < 1 - |\cdot|$ or $D = \mathbb{R}^d$ and $0 < r \leq |\cdot| + M$. We suppose that $\sigma_{x,r(x)}(f) = f(x)$, for every $x \in D$. If $d = 1$, this does not imply that f is affinely linear, neither for $D = \mathbb{R}$ (consider $f\colon x \mapsto \sin x$) nor for $D = B(0,1)$ (see [3, p. 255]). For $D = \mathbb{R}^2$, we may conclude that f is constant (see [6] for an elementary proof). If, however, D is the unit disk in $\mathbb{R}^2$, the answer is NO (see [8, 5]). For $d \geq 3$, the problem is entirely open, both for $B(0,1)$ and $\mathbb{R}^d$.

In the following we shall see that for Riesz potentials (α-stable processes) or – more generally – for balayage spaces with jumps, it becomes, contrary perhaps to common belief, simpler to answer analogous questions. This is due to the fact that iterated sweeping generated by jumping from points in D on the complements of open neighborhoods may converge to a measure not charging the boundary ∂D.

The main results are the Theorems 2.4, 2.6, 3.1, Corollary 3.3, and Theorem 4.2.

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2 Balayage spaces with jumps

Let $(X, \mathcal{W})$ be a balayage space (see [1]) and let $\mathcal{C}(X)$ denote the set of all real functions on X which are continuous. Further, let $\mathcal{K}(X)$ be the set of all functions in $\mathcal{C}(X)$ having compact support and let $\mathcal{P}(X)$ be the set of all potentials in $\mathcal{C}(X)$. We recall that $\mathcal{W}$ is the set of all increasing limits of sequences in $\mathcal{P}(X)$. A real function f on X is called $\mathcal{P}(X)$-*bounded*, if $|f| \le p$ for some $p \in \mathcal{P}(X)$. It is $\mathcal{W} \cap \mathcal{C}(X)$-*bounded*, if $|f| \le w$ for some $w \in \mathcal{W} \cap \mathcal{C}(X)$.

For every open set U in X, let H_U denote the harmonic kernel for U, that is,

$$H_U(x, \cdot) = \begin{cases} \varepsilon_x^{U^c}, & \text{if } x \in U, \\ \varepsilon_x, & \text{if } x \in U^c, \end{cases}$$

or – equivalently – H_U is a kernel on X such that, for every $p \in \mathcal{P}(X)$,

$$H_U p = \inf\{w \in \mathcal{W} \colon w \ge p \text{ on } U^c\}. \tag{2.1}$$

If V is an open set in U, then $H_V H_U = H_U$. In particular, for every $\mathcal{W} \cap \mathcal{C}(X)$-bounded Borel measurable function f on X, the function $H_U f$ is harmonic on U.

EXAMPLE 2.1. *In this paper our standard example for a balayage space with jumps is given by α-stable processes or – equivalently – by Riesz potentials* (*see* [1]). *Here $X = \mathbb{R}^d$, $d \ge 1$, $\alpha \in (0, 2 \wedge d)$, and, defining $G\mu(x) := \int |x-y|^{\alpha-d}\, d\mu(y)$,*

$$\mathcal{P}(X) = \{p \in \mathcal{C}(X) \colon p = G\mu,\ \mu \text{ (positive) Radon measure on } \mathbb{R}^d\}.$$

For $x \in \mathbb{R}^d$ and $r > 0$, let $B(x, r)$ be the open ball with center x and radius $r > 0$, and let $\mu_{x,r}$ denote the α-harmonic measure for x and $B(x,r)$, *that is,*

$$H_{B(x,r)}(x, \cdot) := \mu_{x,r} := \varphi_{x,r} \lambda^d,$$

where λ^d denotes Lebesgue measure on $\mathbb{R}^d$, $\varphi_{x,r} = 0$ on $\overline{B}(x,r)$ and, for $|z-x| > r$,

$$\varphi_{x,r}(z) = a_\alpha \left(\frac{|z-x|^2}{r^2} - 1 \right)^{-\alpha/2} |z-x|^{-d}, \qquad a_\alpha := \pi^{-(\frac{d}{2}+1)} \Gamma(\tfrac{d}{2}) \sin \tfrac{\alpha\pi}{2}. \tag{2.2}$$

Functions which are harmonic with respect to α-harmonic measures may be called α-harmonic functions. The reader who is not familiar with general balayage spaces may assume that we are only dealing with Riesz potentials.

Let $\mathcal{M}(X)$ denote the convex cone of all (positive) Radon measures μ on X such that $\mu(p) < \infty$, for every $p \in \mathcal{P}(X)$. Let "$\prec$" denote the *specific order* on $\mathcal{M}(X)$, that is, $\mu \prec \nu$, if $\mu(p) \le \nu(p)$ for all $p \in \mathcal{P}(X)$ (or, equivalently, if $\mu(w) \le \nu(w)$ for all $w \in \mathcal{W}$). For example, for all open sets U in X and $x \in X$, $H_U(x, \cdot) \prec \varepsilon_x$.

GENERAL ASSUMPTION 2.2. *In the following let D be an open set in X and let f be a Borel measurable $\mathcal{W} \cap \mathcal{C}(X)$-bounded function on X. If D is not relatively compact, we assume, in addition, that f is even $\mathcal{P}(X)$-bounded or, more generally, that $f - H_D f$ is $\mathcal{P}(X)$-bounded. Moreover, let T be a kernel on X such that $T(x, \cdot) = \varepsilon_x$, if $x \in D^c$, and, for every $x \in D$,*

$$\varepsilon_x^{D^c} \prec T(x, \cdot) \prec \varepsilon_x, \qquad T(x, \cdot) \ne \varepsilon_x. \tag{2.3}$$

We immediately observe that

$$TH_D = H_D. \tag{2.4}$$

Indeed, given $p \in \mathcal{P}(X)$, there exists a sequence (p_n) in $\mathcal{P}(X)$ such that $p_n \downarrow H_D p$ on X and $p_n = p$ on D^c. Then, for every $n \in \mathbb{N}$, by (2.3),

$$H_D p = H_D p_n \le T p_n \le p_n$$

and therefore $H_D p \le T H_D p = \lim_{n\to\infty} T p_n \le \lim_{n\to\infty} p_n = H_D p$. This proves (2.4).

REMARK 2.3. Let us note that, by (2.3) and Bauer's minimum principle (see [1, I.2.2]), certainly $f = H_D f$ (and hence f is harmonic on D) provided $f - H_D f$ tends to 0 at ∂D. Clearly, the latter holds trivially if $D = X$ or if X is a discrete space (where $\mathcal{W}$ is given by a transient random walk).

THEOREM 2.4. *Suppose that the measures $\varepsilon_x^{D^c}$, $x \in D$, do not charge the boundary ∂D of D and that $\lim T^n p = H_D p$ for every $p \in \mathcal{P}(X)$.*

If $Tf = f$, then $f = H_D f$, and f is harmonic on D.

Proof. Let us consider $g := f - H_D f$. Clearly, $g = 0$ on D^c, $Tg = g$, and g is $\mathcal{W} \cap \mathcal{C}(X)$-bounded. If D is relatively compact, the latter implies that g is $\mathcal{P}(X)$-bounded. We intend to prove that $g = 0$, that is, $f = H_D f$, and hence f is harmonic on D.

To that end we fix $x \in D$ and $\eta > 0$. Since $g = 0$ on D^c and $\varepsilon_x^{D^c}(\overline{D}) = 0$, there exists a continuous $\mathcal{P}(X)$-bounded function $\varphi \ge 0$ that $|g| \le \varphi$ and $\varepsilon_x^{D^c}(\varphi) < \eta$. Of course, for every $n \in \mathbb{N}$,

$$|g| = |T^n g| \le T^n |g| \le T^n \varphi,$$

and therefore

$$|g(x)| \le \lim_{n\to\infty} T^n \varphi(x) = H_D \varphi(x) = \varepsilon_x^{D^c}(\varphi) < \eta$$

(where the first equality follows from our assumption on T). □

For all open sets U in X and all $w \in \mathcal{W}$, let

$$\hat{H}_U w(x) := \liminf_{y\to x} H_U w(y) \qquad (x \in X).$$

By [1, III.4.4], $\hat{H}_U$ maps $\mathcal{W}$ into $\mathcal{W}$. For applications of Theorem 2.4 it will be useful to note the following (which, unfortunately, has not been written down explicitly in [1]; cf. [1, III.6.1]).

LEMMA 2.5. *For every $m \in \mathbb{N}$, let V_m be an open set in D such that $\overline{V}_m \subset D$ or V_m is regular, that is, $\hat{H}_{V_m} p = H_{V_m} p$, for all $p \in \mathcal{P}(X)$. Further, let $\bigcup_{m\in\mathbb{N}} V_m = D$ and let (U_n) be a sequence in $\{V_m : m \in \mathbb{N}\}$ containing every V_m infinitely many times. Then, for every $p \in \mathcal{P}(X)$,*

$$\lim_{n\to\infty} H_{U_1} H_{U_2} \dots H_{U_n} p = \lim_{n\to\infty} \hat{H}_{U_1} \hat{H}_{U_2} \dots \hat{H}_{U_n} p = H_D p.$$

Proof. Let $p \in \mathcal{P}(X)$, $f_n := H_{U_1} H_{U_2} \dots H_{U_n} p$, and $w_n := \hat{H}_{U_1} \hat{H}_{U_2} \dots \hat{H}_{U_n} p$, $n \in \mathbb{N}$. Since $p \geq H_{U_n} p \geq \hat{H}_{U_n} p$, the sequences (f_n) and (w_n) are decreasing and

$$f := \inf f_n \geq \inf w_n =: g.$$

Since $w_n \in \mathcal{W}$ and $g \geq p$ on D^c, $n \in \mathbb{N}$, we see, by (2.1), that $g \geq H_D p$. Moreover, for every $m \in \mathbb{N}$, $H_{V_m} H_{V_m} = H_{V_m}$, and therefore $H_{V_m} f = f$. So, by [1, III.4.4], f is α-harmonic on D. Since $f = p$ on U^c, (2.1) and the minimum principle (see [1, III.6.6]) imply that $H_D p \geq f$. Since $f \geq g$, the proof is finished. □

Our next aim is the following (which once more covers the case of a discrete space X).

THEOREM 2.6. *Suppose that the measures $\varepsilon_x^{D^c}$, $x \in D$, do not charge ∂D.*
If $Tf = f$ and f is continuous in D, then $f = H_D f$, and f is harmonic on D.

To prove Theorem 2.6 we need some preparations. If $\mu \in \mathcal{M}$, then the measure μT, defined by $(\mu T)(\varphi) := \mu(T\varphi)$, $\varphi \in \mathcal{K}(X)$, is contained in $\mathcal{M}$ (since, for every $p \in \mathcal{P}(X)$, $\mu T(p) \leq \mu(p) < \infty$). Given a sequence (μ_n) in $\mathcal{M}$ which is decreasing with respect to the specific order, there exists a unique measure $\mu \in \mathcal{M}$ such that, for every $p \in \mathcal{P}(X)$, $\lim_{n\to\infty} \mu_n(p) = \mu(p)$. Of course, $\mu \prec \mu_n$, $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} \mu_n(\varphi) = \mu(\varphi)$, for every continuous $\mathcal{P}(X)$-bounded function on X. In particular, $\lim_{n\to\infty} \mu_n(\varphi) = \mu(\varphi)$, for every $\varphi \in \mathcal{K}(X)$, that is, (μ_n) converges weakly to μ as $n \to \infty$.

We are interested in subsets $\mathcal{N}$ of $\mathcal{M}$ having the following stability properties:

(i) If $\nu \in \mathcal{N}$, then $\nu T \in \mathcal{N}$.

(ii) If (ν_n) is a specifically decreasing sequence in $\mathcal{N}$, then $\lim_{n\to\infty} \nu_n \in \mathcal{N}$.

As we noted above, $\mathcal{M}$ itself satisfies (i) and (ii). Given $\mu \in \mathcal{M}$, let $\mathcal{M}_\mu(T)$ denote the *smallest* subset of $\mathcal{M}$ which contains μ and satisfies (i) and (ii). Since $\mu \prec \mu$ and the set of all $\nu \in \mathcal{M}$ such that $\nu \prec \mu$ satisfies (i) and (ii), we know that $\nu \prec \mu$ for every $\nu \in \mathcal{M}_\mu(T)$.

LEMMA 2.7. *For every $\mu \in \mathcal{M}$, $\mathcal{M}_\mu(T)$ is totally ordered* (*with respect to "$\prec$"*).

Proof. Simple modification of the (lengthy) proof of [7, Proposition 3.2]. □

For every $x \in X$, let $\mathcal{M}_x(T) := \mathcal{M}_{\varepsilon_x}(T)$.

LEMMA 2.8. *Let $x \in X$. Then, for every $\nu \in \mathcal{M}_x(T)$, $\varepsilon_x^{D^c} \prec \nu$.*

Proof. Let $\mathcal{N} := \{\nu \in \mathcal{M}_x(T) \colon \varepsilon_x^{D^c} \prec \nu\}$. Obviously, $\varepsilon_x \in \mathcal{N}$ and $\mathcal{N}$ satisfies (ii). To prove that $\mathcal{N}$ satisfies (i) as well, we fix $\nu \in \mathcal{N}$ and $p \in \mathcal{P}(X)$. There exist $p_n \in \mathcal{P}(X)$ such that $p_n \downarrow H_D p$ and $p_n = p$ on D^c, $n \in \mathbb{N}$. Then

$$\nu(H_D p) = \lim_{n\to\infty} \nu(p_n) \geq \lim_{n\to\infty} \varepsilon_x^{D^c}(p_n) = \lim_{n\to\infty} H_D p_n(x) = H_D p(x) = \varepsilon_x^{D^c}(p).$$

Moreover, by (2.4),

$$(\nu T)(p) = \nu(Tp) \geq \nu(T H_D p) = \nu(H_D p).$$

Therefore $(\nu T)(p) \geq \varepsilon_x^{D^c}(p)$, that is, $\nu T \in \mathcal{N}$, $\mathcal{N}$ satisfies (i). Thus $\mathcal{N} = \mathcal{M}_x(T)$. □

PROPOSITION 2.9. *For every $x \in X$, $\varepsilon_x^{D^c} \in \mathcal{M}_x(T)$.*

Proof. Let p_0 be a strict potential in $\mathcal{P}(X)$, $x \in X$, and

$$\gamma := \inf\{\nu(p_0) \colon \nu \in \mathcal{M}_x(T)\}.$$

We choose $\nu_n \in \mathcal{M}_x(T)$, $n \in \mathbb{N}$, such that $\nu_n(p_0) \downarrow \gamma$ as $n \to \infty$. By Lemma 2.7, (ν_n) is specifically decreasing. Let $\nu := \lim_{n\to\infty} \nu_n$. Since $\nu T \in \mathcal{M}_x(T)$, we see that

$$\gamma \le (\nu T)(p_0) = \nu(Tp_0) \le \nu(p_0) = \lim_{n\to\infty} \nu_n(p_0) = \gamma.$$

Hence $\nu(Tp_0) = \nu(p_0)$. This implies that ν is supported by D^c, since $Tp_0 < p_0$ on D, by (2.3). If $p \in \mathcal{P}(X)$ and $p_n \in \mathcal{P}(X)$ such that $p_n = p$ on D^c and $p_n \downarrow H_D p$, then

$$\nu(p) = \lim_{n\to\infty} \nu(p_n) \le \lim_{n\to\infty} p_n(x) = H_D p(x) = \varepsilon_x^{D^c}(p).$$

Therefore $\nu \prec \varepsilon_x^{D^c}$. Thus, by Lemma 2.8, $\varepsilon_x^{D^c} = \nu \in \mathcal{M}_x(T)$. □

Proof of Theorem 2.6. Let $x \in D$ and g, η, φ as in the proof of Theorem 2.4. We define

$$\psi := \begin{cases} g, & \text{on } D, \\ -\varphi, & \text{on } D^c, \end{cases} \qquad \text{and} \qquad \mathcal{N} := \{\nu \in \mathcal{M}_x(T) \colon \nu(\psi) \le \psi(x)\}.$$

Then ψ is lower semicontinuous and $\mathcal{P}(X)$-bounded, and therefore $\mathcal{N}$ satisfies (ii). Moreover, $T\psi \le \psi$, and hence $\mathcal{N}$ satisfies (i). Of course, $\varepsilon_x \in \mathcal{N}$. So $\mathcal{N} = \mathcal{M}_x(T)$. In particular, $\varepsilon_x^{D^c} \in \mathcal{N}$. Thus

$$g(x) = \psi(x) \ge \varepsilon_x^{D^c}(\psi) = \varepsilon_x^{D^c}(-\varphi) > -\eta.$$

This shows that $g \ge 0$. Similarly, $-g \ge 0$. Hence $g = 0$, $f = H_D f$, f is harmonic. □

3 Application to Riesz potentials

Let us now apply the results of Section 2 to Riesz potentials, where $X := \mathbb{R}^d$, $d \ge 1$, $\alpha \in (0, 2 \wedge d)$, and $\mathcal{P}(X)$ is the set of all continuous real α-potentials $G\mu$ (see Example 2.1).

Again let D be an open set in X and let f be a Borel measurable $\mathcal{W} \cap \mathcal{C}(X)$-bounded function on X, $\mathcal{P}(X)$-bounded, if D is not relatively compact. By Theorems 2.4 and 2.6, we have the following general result.

THEOREM 3.1. *Suppose that the measures $\varepsilon_x^{D^c}$, $x \in D$, do not charge ∂D and that T is a kernel on X such that $T(x, \cdot) = \varepsilon_x$, if $x \in D^c$, and, for every $x \in D$, $\varepsilon_x^{D^c} \prec T(x, \cdot) \prec \varepsilon_x$, $T(x, \cdot) \ne \varepsilon_x$. Suppose that $Tf = f$ and that f is continuous in D or $\lim_{n\to\infty} T^n p = H_D p$, for every $p \in \mathcal{P}(X)$.*

Then $f = H_D f$ and f is α-harmonic on D.[1]

[1]If $D = \mathbb{R}^d$, then the $\mathcal{P}(X)$-boundedness of f implies that $f = 0$.

For an application of Theorem 3.1 we shall use the following consequence of Lemma 2.5.

PROPOSITION 3.2. *Let $r\colon D \to \mathbb{R}$ be Borel measurable such that $r \le \operatorname{dist}(\cdot, D^c)$ and, for every compact set K in D, $\inf r(K) > 0$. Moreover, let $T(x,\cdot) = \mu_{x,r(x)}$, if $x \in D$, and $T(x,\cdot) := \varepsilon_x$, if $x \in D^c$. Then, for every $p \in \mathcal{P}(X)$, $\lim_{n\to\infty} T^n p = H_D p$.*

Proof. There is a covering of D by open balls B_m, $m \in \mathbb{N}$, such that $\overline{B}_m \subset B(x, r(x))$, whenever $x \in B_m$. Let (U_n) be a sequence in $\{B_m : m \in \mathbb{N}\}$ containing every B_m infinitely many times. For all $p \in \mathcal{P}(X)$ and $m \in \mathbb{N}$, $Tp \le H_{B_m} p$, and hence, for every $n \in \mathbb{N}$,

$$H_D p \le T^n p \le H_{U_1} H_{U_2} \dots H_{U_n} p.$$

By Lemma 2.5, the right side converges to $H_D p$ as $n \to \infty$. □

COROLLARY 3.3. *Suppose that the measures $\varepsilon_x^{D^c}$, $x \in D$, do not charge ∂D and that r is a real function on D such that $0 < r \le \operatorname{dist}(\cdot, D^c)$ and $\mu_{x,r(x)}(f) = f$, for every $x \in D$. Moreover, let f be continuous in D or $\inf r(K) > 0$, for every compact set K in D. Then $f = H_D f$ and f is α-harmonic on D.*

Proof. By the arguments used for [13, Proposition 2.1 and Remark 2.2] (where we have to replace the application of the monotone convergence theorem by $\mathcal{L}^1$-convergence of a uniformly integrable sequence), we may assume that r is Borel measurable. Then the result follows immediately, by Theorem 3.1 and Proposition 3.2. □

Obviously, the measures $\varepsilon_x^{D^c}$, $x \in D$, charge ∂D, if the boundary ∂D has strictly positive Lebesgue measure (since $\varepsilon_x^{D^c}(\partial D) \ge \mu_{x,r}(\partial D)$ provided $B(x,r) \subset D$). This may happen, even if ∂D has Lebesgue measure zero. Indeed, if, for example, $d \ge 3$, $\alpha \in (1,2)$, and $D := \{x \in \mathbb{R}^d : x_1 \neq 0\}$, then ∂D is the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$ which is not α-polar (see [1, VI.5.4.4]). And, for every $x \in D$, the support of $\varepsilon_x^{D^c}$ is the Lebesgue null set ∂D.

We shall see, however, that the harmonic measures $\varepsilon_x^{D^c}$ do not charge ∂D provided $\mathbb{R}^d \setminus \overline{D}$ is not too small near ∂D (Proposition 3.4). If D is bounded and satisfies an outer cone condition, the following result is part of [2, Lemma 6] (obtained by a probabilistic proof which would also be valid in our more general situation).

PROPOSITION 3.4. *Let us suppose that, for every $R > 0$, there exists $r_0 > 0$ and $\beta \in (0,1)$, such that, for all $z \in B_R \cap \partial D$ and $r \in (0, r_0)$,*

$$\lambda^d(B(z,r) \setminus \overline{D}) \ge \beta r^d. \tag{3.1}$$

Then $\lambda^d(\partial D) = 0$ and the measures $\varepsilon_x^{D^c}$, $x \in D$, are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d \setminus \overline{D}$. In particular, they do not charge ∂D.

Proof. By Lebesgue's density theorem, $\lambda^d(\partial D) = 0$. Let $x \in D$, $p_0 \in \mathcal{P}(X)$, $p_0 > 0$, and let A be a Borel set in D^c such that $\lambda^d(A) = 0$.

1. Let us first assume that D is bounded. We define $r(y) := \mathrm{dist}(y, D^c)$ and $T(y,\cdot) := \mu_{y,r(y)}$, if $y \in D$, $T(y,\cdot) := \varepsilon_y$, if $y \in D^c$. By (3.1) and (2.2), a straightforward estimate yields that there exists $\gamma \in (0,1)$ such that $T(1_{\mathbb{R}^d\setminus D}\, p_0) > \gamma p_0$ on D, and hence

$$T(1_D p_0) < (1-\gamma)p_0 \quad \text{on } D. \tag{3.2}$$

We define probability measures ν_n on $\mathbb{R}^d$ by $\nu_n := T^n(x,\cdot)$, $n \in \mathbb{N}$, that is

$$\nu_0 := \varepsilon_x \quad \text{and} \quad \nu_{n+1} := \nu_n T = \int T(y,\cdot)\, d\nu_n(y).$$

Since $\mu_{y,r(y)}(A) = 0$, if $y \in D$, and $T(y,\cdot) = \varepsilon_y$, if $y \in D^c$, we see, by induction, that

$$\nu_n(A) = 0 \quad \text{and} \quad 1_{D^c}\nu_n \le 1_{D^c}\nu_{n+1} \qquad (n \in \mathbb{N}). \tag{3.3}$$

Hence, for every $n \in \mathbb{N}$, by (3.2),

$$\nu_{n+1}(1_D p_0)(x) = \int_D T(1_D p_0)(y)\, d\nu_n(y) < (1-\gamma)\nu_n(1_D p_0).$$

So $\lim_{n\to\infty} \nu_n(D) = 0$. By Proposition 3.2, $\lim_{n\to\infty} \nu_n(p) = H_U p(x)$, for every $p \in \mathcal{P}(X)$, and hence the sequence (ν_n) converges weakly to $\varepsilon_x^{D^c}$, by standard approximation (see [1, I.1.2]). Hence $\varepsilon_x^{D^c}$ is the limit of the increasing sequence $(1_{D^c}\nu_n)$, and therefore $\varepsilon_x^{D^c}(A) = 0$, by (3.3).

2. Let us now consider the case, where D is unbounded, $D \ne \mathbb{R}^d$, and let $z \in D^c$, $E := A \cap \overline{B(z,1)}$, $\delta > 0$. We claim that $\varepsilon_x^{D^c}(E) \le \delta$ (which implies that $\varepsilon_x^{D^c}(A) = 0$). Indeed, let $R := \max\{|x-z|+1, \delta^{1/(\alpha-d)}\}$ and $B := B(z,R)$. Assuming without loss of generality that $\beta \le 1/2$, the bounded set $\tilde{D} := D \cap B$ obviously satisfies (3.1). So, by the first part of the proof, $\varepsilon_x^{\tilde{D}^c}(A) = 0$. Moreover, the measure $\nu := 1_D \varepsilon_x^{\tilde{D}^c}$ is supported by B^c and, for every $y \in B^c$, $\varepsilon_y^E(E) \le R^{\alpha-d} \le \delta$. Therefore $\nu^E(E) \le \delta$. Now, by [1, VI.9.4],

$$\varepsilon_x^{D^c} = \varepsilon_x^{\tilde{D}^c}|_{D^c} + \nu^{D^c} \quad \text{and} \quad \nu^{D^c}(E) \le \nu^E(E) \le \delta.$$

Hence $\varepsilon_x^{D^c}(E) \le \delta$ finishing the proof. □

4 A result based on martingale convergence

In this section, we shall apply the method used in [11] to Riesz potentials. We assume that $D \ne \emptyset$ is an open set in $\mathbb{R}^d$, $d \ge 1$, that $\alpha \in (0, 2\wedge d)$, $c \ge 1$, and that $r\colon D \to (0,\infty)$ is such that, for every $x \in D$, $\overline{B(x,r(x))} \subset D$ and

$$c^{-1}r(x) \le r \le c\,r(x) \quad \text{on } B(x,r(x)). \tag{4.1}$$

REMARK 4.1. We note that (4.1) is satisfied, if there exist $\varepsilon_0 > 0$ and a function $\rho\colon D \to (0,\infty)$ (for example, $\rho = \mathrm{dist}(\cdot, D^c)$, if $D \ne \mathbb{R}^d$, or $\rho = M + |\cdot|$, if $D = \mathbb{R}^d$) such that $|\rho(x) - \rho(y)| \le |x-y|$, $x, y \in D$, and

$$\varepsilon_0 \rho \le r \le (1-\varepsilon_0)\rho. \tag{4.2}$$

Indeed, (4.2) implies that, for all $x \in D$ and $y \in B(x, r(x))$,

$$r(y) \le \rho(y) \le |y - x| + \rho(x) \le (1 + (1/\varepsilon_0))r(x),$$

$$(1/\varepsilon_0)r(y) \ge \rho(y) \ge \rho(x) - |y - x| \ge \rho(x) - r(x) \ge \varepsilon_0\rho(x) \ge \varepsilon_0 r(x).$$

Our aim in this section is the following.

THEOREM 4.2. *Let f be a bounded Borel measurable function on $\mathbb{R}^d$ such that $\mu_{x,r(x)}(f) = f(x)$, $x \in D$. Then f is α-harmonic on D.*

In the following let

$$A := \{z \in \mathbb{R}^d \colon c + 1 < |z| < c + 2\} \quad \text{and} \quad \delta := \frac{a_\alpha \lambda^d(A)}{2(c+3)^{d+2\alpha}}. \tag{4.3}$$

LEMMA 4.3. *Let $x \in B(0, 1)$, $c^{-1} \le r \le c$, and let B be a Borel set in A such that $\mu_{x,r}(B) < \delta$. Then $\lambda^d(B) < (1/2)\lambda^d(A)$.*

Proof. If $z \in A$, then $c < |z - x| < c + 3$ and hence

$$\varphi_{x,r}(z) > \alpha_\alpha r^{-\alpha}|z - x|^{-(d+\alpha)} \ge a_\alpha(c+3)^{-(d+2\alpha)} = 2\delta/\lambda^d(A).$$

Thus

$$\frac{2\delta}{\lambda^d(A)}\,\lambda^d(B) \le \int_B \varphi_{x,r}(z)\,d\lambda^d(z) = \mu_{x,r}(B) < \delta$$

proving that $\lambda^d(B) < (1/2)\lambda^d(A)$. □

LEMMA 4.4. *Let $x_1, x_2 \in D$ such that $B(x_1, r(x_1)) \cap B(x_2, r(x_2)) \ne \emptyset$, and let C_1, C_2 be Borel sets in $\mathbb{R}^d$ such that $\mu_{x_j,r(x_j)}(C_j) > 1 - \delta$, $j = 1, 2$. Then $C_1 \cap C_2 \ne \emptyset$.*

Proof. Let $x \in B(x_1, r(x_1)) \cap B(x_2, r(x_2))$. By translation and scaling invariance, we may assume that $x = 0$ and $r(x) = 1$. Let $j \in \{1, 2\}$. By (4.1), $c^{-1} \le r(x_j) \le c$. Since $\mu_{x_j,r(x_j)}(\mathbb{R}^d) = 1$, we see that $\mu_{x_j,r(x_j)}(A \setminus C_j) < \delta$. So $\lambda^d(A \setminus C_j) < (1/2)\lambda^d(A)$, by Lemma 4.3. Thus $\lambda^d(A \setminus (C_1 \cap C_2)) \le \lambda^d(A \setminus C_1) + \lambda^d(A \setminus C_2) < \lambda^d(A)$ showing that $C_1 \cap C_2 \ne \emptyset$. □

Now Theorem 4.2 can be obtained using essentially the arguments in [11] (leading to restricted mean value results in classical potential theory) almost word by word replacing Brownian motion by the symmetric α-stable process $(\Omega, X_t, \theta_t, \mathbb{P}^x)$ on $\mathbb{R}^d$. Additional care, however, is needed to deal with the jumps of (X_t) from D to D^c. Again, we may assume that r is Borel measurable.

We recursively define stopping times τ_n, $n \ge 0$, by

$$\tau_0 := 0, \qquad \tau_{n+1} := \inf\{t \ge \tau_n \colon |X_t - X_{\tau_n}| \ge r(X_{\tau_n})\},$$

where we take $r(x) := 0$, if $x \in D^c$ (so that $\{\tau_{n+1} > \tau_n\} = \{X_{\tau_n} \in D\}$). Then, for every $x \in \mathbb{R}^d$, the sequence $(f(X_{\tau_n}))_{n \ge 0}$ is a $\mathbb{P}^x$-martingale.

For $\omega \in \Omega$, let $F(\omega) := \lim_{n\to\infty} f(X_{\tau_n}(\omega))$, if the limit exists, and $F(\omega) := 0$ otherwise. We shall say that a statement involving $\omega \in \Omega$ holds almost surely (a.s.

for short) provided that, for each $x \in \mathbb{R}^d$, it holds for $\mathbb{P}^x$-almost every $\omega \in \Omega$. By the martingale convergence theorem,

$$\lim_{n\to\infty} f(X_{\tau_n}) = F \quad \text{a.s..} \tag{4.4}$$

We fix an open ball U with $\overline{U} \subset D$ and define

$$\eta := \inf\{t \geq 0\colon X_t \in U^c\}.$$

By (4.4) and the strong Markov property,

$$\lim_{n\to\infty} f(X_{\tau_n} \circ \theta_\eta) = F \circ \theta_\eta \quad \text{a.s.} \tag{4.5}$$

Moreover, for every $x \in \mathbb{R}^d$, $\mathbb{E}^x(F) = \mathbb{E}^x(f(X_{\tau_0})) = f(x)$ and $\mathbb{E}^x(F \circ \theta_\eta) = \mathbb{E}^x(E^{X_\eta}(F)) = \mathbb{E}^x(f(X_\eta)) = H_U f(x)$. To see that f is harmonic on D it hence suffices to show that

$$F = F \circ \theta_\eta \quad \mathbb{P}^x\text{-a.s.,} \tag{4.6}$$

for every $x \in U$ (in fact, it would be sufficient to prove (4.6), if x is the center of U).

For $n \geq 0$, let $\sigma_n := \eta + \tau_n \circ \theta_\eta$ and

$$Y_n := X_{\tau_n}, \qquad Z_n := Y_n \circ \theta_\eta = X_{\sigma_n}.$$

Defining $\tau_D := \inf\{t \geq 0\colon X_t \in D^c\}$ we claim that

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \sigma_n = \tau_D \ \text{ a.s.,} \qquad \lim_{n\to\infty} Y_n = \lim_{n\to\infty} Z_n = X_{\tau_D} \ \text{ a.s. on } \{\tau_D < \infty\} \tag{4.7}$$

(cf. the proof of Lemma 1 in [11]). Indeed, it is easily seen that $\tau := \lim_{n\to\infty} \tau_n \leq \tau_D$. By the quasi-left continuity, $\lim_{n\to\infty} Y_n = X_\tau$ almost surely on $\{\tau < \infty\}$. Suppose that $\omega \in \Omega$ and $\tau(\omega) < \tau_D(\omega)$. Then $y := X_\tau(\omega) \in D$. If $n \geq 0$ such that $Y_n(\omega) \in B(y, r(y))$, then, by (4.1), $|Y_{n+1}(\omega) - Y_n(\omega)| \geq r(Y_n(\omega)) \geq c^{-1} r(y)$. Therefore the sequence $(Y_n(\omega))_{n\geq 0}$ does not converge to $X_\tau(\omega)$. Thus $\tau = \tau_D$ almost surely and $\lim_{n\to\infty} Y_n = X_{\tau_D}$ almost surely on $\{\tau_D < \infty\}$. Since $\eta + \tau_D \circ \theta_\eta = \tau_D$ and hence $X_{\tau_D} \circ \theta_\eta = X_{\tau_D}$, the remaining part of (4.7) follows by the strong Markov property.

Let Ω_0 denote the set of all $\omega \in \Omega$ satisfying the equalities in (4.4), (4.5), (4.7) (the last equalities only if $\tau_D(\omega) < \infty$) and such that the trajectory $t \mapsto X_t(\omega)$ is unbounded (which also holds almost surely).

Let Ω_1 denote the set of all $\omega \in \Omega_0$ such that the set

$$\Gamma(\omega) := \{X_t(\omega) : 0 \leq t < \tau_D(\omega)\}$$

is relatively compact in D. We note that, for all $\omega \in \Omega$, $\Gamma(\omega) \setminus U \subset \Gamma(\theta_\eta(\omega)) \subset \Gamma(\omega)$.

Let us first consider $\omega \in \Omega_1$. Then $\Gamma(\omega)$ is relatively compact in D. Therefore $\tau_D(\omega) < \infty$, and (4.7) implies that there exist $k, m \geq 0$ such that $Y_k(\omega), Z_m(\omega) \in D^c$. Then $Y_n(\omega) = Y_k(\omega)$, $n \geq k$, and $Z_n(\omega) = Z_m(\omega)$, $n \geq m$. Thus $F(\omega) = X_{\tau_D}(\omega) = F(\theta_\eta(\omega))$, by (4.4), (4.5), and (4.7).

We now fix $x \in U$. It remains to show that (4.6) holds $\mathbb{P}^x$-almost surely on $\Omega_0 \setminus \Omega_1$. We take $\varepsilon > 0$ and define, for every $n \geq 0$,

$$A_n := \{|f(Y_{n+1}) - f(Y_n)| < \varepsilon\} \quad \text{and} \quad B_n := \{|f(Z_{n+1}) - f(Z_n)| < \varepsilon\}.$$

For every $y \in \mathbb{R}^d$, let $C(y) := \{y' \in \mathbb{R}^d : |f(y') - f(y)| < \varepsilon\}$, $P(y, \cdot) := \mu_{y,r(y)}$ if $y \in D$, and $P(y, \cdot) := \varepsilon_y$ if $y \in D^c$. Then, $\mathbb{P}^x$-almost surely,

$$\lim_{n\to\infty} P(Y_n, C(Y_n)) = \lim_{n\to\infty} \mathbb{P}^x(A_n|Y_n) = 1,$$
$$\lim_{n\to\infty} P(Z_n, C(Z_n)) = \lim_{n\to\infty} \mathbb{P}^x(B_n|Z_n) = 1.$$

Let us consider $\omega \in \Omega_0 \setminus \Omega_1$ such that $y_n := Y_n(\omega)$, $z_n := Z_n(\omega)$ satisfy

$$\lim_{n\to\infty} P(y_n, C(y_n)) = \lim_{n\to\infty} P(z_n, C(z_n)) = 1.$$

There exists $k \in \mathbb{N}$ such that $|f(y_m) - F(\omega)| < \varepsilon$, $|f(z_m) - F \circ \theta_\eta(\omega)| < \varepsilon$, $P(y_m, C(y_m)) > 1 - \delta$, and $P(z_m, C(z_m)) > 1 - \delta$, for every $m \geq k$. By (4.7), the set $\Gamma(\omega)$ (which is not relatively compact in D) is covered by the balls $B(y_n, r(y_n)$, $n \geq 0$, and $\Gamma(\omega) \setminus U$ is covered by the balls $B(z_n, r(z_n))$, $n \geq 0$. Hence there are $m, n \geq k$ such that $B(y_m, r(y_m)) \cap B(z_n, r(z_n)) \neq \emptyset$. Then, by Lemma 4.4, $C(y_m) \cap C(z_n) \neq \emptyset$. Using a point $x' \in C(y_m) \cap C(z_n)$, we may finally conclude, by the triangle inequality, that $|F(\omega) - F \circ \theta_\eta(\omega)| < 4\varepsilon$.

This shows that (4.6) holds $\mathbb{P}^x$-almost surely on $\Omega_0 \setminus \Omega_1$, and the proof is finished.

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