ZONAL POLYNOMIALS VIA STANLEY-FÉRAY FORMULA AND FREE CUMULANTS

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ABSTRACT. We study zonal characters which are defined as suitably normalized coefficients in the expansion of zonal polynomials in terms of power-sum symmetric functions. We show that zonal characters are explicitly given by an analogue of Stanley-Féray formula for character values of symmetric groups. We also study an analogue of Kerov polynomials, namely we express zonal characters as polynomials in free cumulants and we give an explicit combinatorial interpretation of their coefficients. In this way, we prove two recent conjectures of Lassalle for Jack polynomials in the special case of zonal polynomials.

1. INTRODUCTION

1.1. Zonal polynomials.

1.1.1. *Background.* Zonal polynomials were introduced by James [Jam60, Jam61] (who credits also Hua [Hua63]) in order to solve some problems from statistics and multivariate analysis. They quickly became a fundamental tool in this theory as well as in the random matrix theory (an overview can be found in the book of Muirhead [Mui82] or also in the introduction to the monograph of Takemura [Tak84]). They also play an important role in the representation theory: they appear as zonal spherical functions of the pairs $(\mathfrak{S}_{2n}, H_n)$ (where H_n is the hyperoctahedral group) and $(\mathrm{GL}_d(\mathbb{R}), O_d)$, which means that they describe canonical basis of the algebra of left and right H_n-invariant (resp. O_d-invariant) functions on $\mathfrak{S}_{2n}$ (resp. $\mathrm{GL}_d(\mathbb{R})$).

This last property shows that zonal polynomials can be viewed as an analogue of Schur symmetric functions since the latter are zonal spherical functions for the Gelfand pairs $(\mathfrak{S}_n \times \mathfrak{S}_n, \mathfrak{S}_n)$ and $(\mathrm{GL}_d(\mathbb{C}), U_d)$. Therefore, many of the properties of Schur functions can be extended to zonal polynomials and this article goes in this direction.

In this article we use a James' characterization of zonal polynomials [Jam61] as their definition. The elements needed in our development (including the precise definition of zonal polynomials) are given in Section 2.1. For a more complete introduction to the topic we refer to the Chapter VII of Macdonald's book [Mac95].

The main results of this article are new combinatorial formulas for zonal polynomials. Note that, as they are a particular case of Jack symmetric functions, there exists already a combinatorial interpretation for them in terms of ribbon tableaux (due to Stanley [Sta89]). But our formula is of different type: it gives a combinatorial interpretation to the coefficients of the zonal polynomial Z_λ expanded in the power-sum basis as a function of λ. In more concrete words, the combinatorial objects describing the coefficient of p_μ in Z_λ depend on μ, whereas the statistics on them depend on λ (in Stanley's result it is roughly the opposite). This kind of *dual* approach makes appear shifted symmetric functions [OO97] and is an analogue of recent developments concerning characters of the symmetric group: more details will be given in Section 1.3.

1.1.2. *Jack polynomials.* Jack [Jac71] introduced a family of symmetric functions $J_\lambda^{(\alpha)}$ depending on an additional parameter α. These functions are now called *Jack polynomials*. For some special values of α they coincide with some established families of symmetric functions. Namely, up to multiplicative constants, for $\alpha = 1$ Jack polynomials coincide with Schur polynomials, for $\alpha = 2$ they coincide with zonal polynomials, for $\alpha = \frac{1}{2}$ they coincide with symplectic zonal polynomials, for $\alpha = 0$ we recover the elementary symmetric functions and finally their highest degree component in α are the monomial symmetric functions. Moreover, some other specializations appear in different contexts: the case $\alpha = 1/k$, where k is an integer, has been considered by Kadell in relation with generalizations of Selberg's integral [Kad97]. In addition, Jack polynomials for $\alpha = -(k+1)/(r+1)$ verify some interesting annihilation conditions [FJMM02].

Jack polynomials for a generic value of the parameter α might appear rather artificial and invented since, unlike Schur polynomials or zonal polynomials, they do not seem to have a direct interpretation, for example in the context of the representation theory or as zonal spherical functions for some Gelfand pairs. Nevertheless, over the time it has been shown that several results concerning Schur and zonal polynomials can be generalized in a rather natural way to Jack polynomials (see, for example, the work of Stanley [Sta89]), therefore Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions at the same time.

An extensive numerical exploration and conjectures done by Lassalle [Las08, Las09] suggest that the kind of combinatorial formulas we establish in this paper have generalizations for any value of the parameter α. Unfortunately, we are not able yet to achieve this goal yet.

1.2. **The main result 1: a new formula for zonal polynomials.**

1.2.1. *In terms of pair-partitions.* The combinatorial objects we need are pair-partitions:

Definition 1.1. A pair-partition P of $[2n] = \{1, \dots, 2n\}$ is a set of pairwise disjoint two-element sets, such that their (disjoint) union is equal to $[2n]$. A pair-partition can be seen as an involution of $[2n]$ without fixpoints, which associates to each element its partner from the pair.

The simplest example is the *first* pair-partition, which will play a particular role in our article:

$$S = \big\{\{1,2\},\{3,4\},\dots,\{2n-1,2n\}\big\}.$$

When one has two pair-partitions S_1, S_2 of the same set $[2n]$, it is interesting to represent them graphically as a collection of edges connecting the corresponding elements of the pairs. Then the union of the graphical representations of S_1 and S_2 is a collection of loops which will be denoted by $\mathcal{L}(S_1, S_2)$. Let $2\ell_1 \geq 2\ell_2 \geq \cdots$ be the ordered lengths of these loops (notice that the length of each loop must be an even number). Then, if we view the pair-partitions as involutions, the lengths of the cycles of the permutation $S_1 \circ S_2$ are given by $\ell_1, \ell_1, \ell_2, \ell_2, \dots$. The partition $(\ell_1, \ell_2, \dots)$ is called the type of $\mathcal{L}(S_1, S_2)$. We define the sign of a union of pair-partitions as follows:

$$(-1)^{\mathcal{L}(S_1,S_2)} = (-1)^{(\ell_1-1)+(\ell_2-1)+\cdots} = (-1)^{n-|\mathcal{L}(S_1,S_2)|}$$

and the power-sum symmetric function

$$p_{\mathcal{L}(S_1,S_2)}(z_1, z_2, \dots) = p_{\ell_1,\ell_2,\dots}(z_1, z_2, \dots) = \prod_i \sum_j z_j^{\ell_i}. \tag{1}$$

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n; we consider the Young tableau T of shape $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ in which boxes are numbered consecutively along the rows. Permutations of $[2n]$ can be viewed as permutations of the boxes of T. Then a pair (S_1, S_2) is called T*-admissible* if $S \circ S_1$ preserves each column of T and S_2 preserves each row.

Theorem 1.2. *With the definitions above, the zonal polynomial is given by*

$$Z_\lambda = \sum_{(S_1,S_2)\ T\text{-admissible}} (-1)^{\mathcal{L}(S,S_1)}\, p_{\mathcal{L}(S_1,S_2)}.$$

We postpone the proof until Section 2.6.

Remark 1.3. This theorem is an analogue of a known result on Schur symmetric functions:

$$\frac{n! \cdot s_\lambda}{\dim(\lambda)} = \sum (-1)^{|\sigma_1|}\, p_{\mathrm{type}(\sigma_1\circ\sigma_2)}, \tag{2}$$

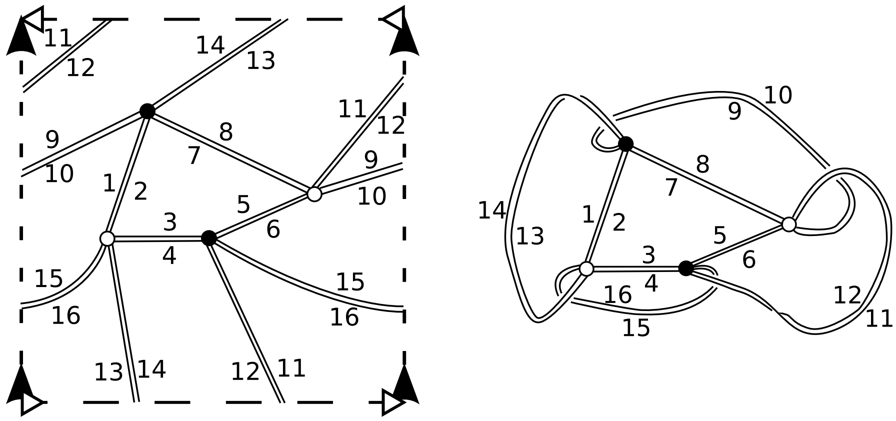

FIGURE 1. Example of a labeled map on Klein bottle.

where the sum runs over pairs of permutations (σ_1, σ_2) of the boxes of the diagram λ such that σ_1 (resp. σ_2) preserves the columns (resp. the rows) of λ and $\operatorname{type}(\sigma_1 \circ \sigma_2)$ denotes the partition describing the lengths of the cycles of $\sigma_1 \circ \sigma_2$. This formula can be found in [Han88, Equation (1.1)] (where the author tries unsuccessfully to generalize it to Jack symmetric functions). For a proof, see [FŚ07, Theorem 4].

1.2.2. *In terms of maps on surfaces.* A labeled bipartite graph drawn on a (not necessarily orientable and not necessarily connected) surface will be called a *map*. We will always assume that the surface is minimal in the sense that after removing the graph from the surface, the latter becomes a disjoint collection of open discs. If we draw an edge of such a graph with a fat pen and then take its boundary, this edge splits into two *edge-sides*. In the above definition of the map, by *labeled*, we mean that each edge-side is labeled with a number from the set $[2n]$ and each number from this set is used exactly once.

Example 1.4. Figure 1 shows two different representations of the same map. On the picture on the left-hand side, if one identifies the opposite sides of the square respecting the arrows, one obtains a Klein bottle and a graph drawn on it. On the right-hand side, we forget the surface and draw the graph on the plane respecting the order of the edge-sides around each vertex: one has to add crossings (information about which edge is above the other one is irrelevant) and (as the surface is non-orientable) twists.

There is a correspondence between maps and triplets (S, S_1, S_2) of pair-partitions, given as follows:

- the pair-partition S associates to the label of an edge-side the label of the other edge-side from the same edge;
- the pair-partition S_1 associates to the label of an edge-side the label of the other edge-side surrounding the same white corner;
- the pair-partition S_2 associates to the label of an edge-side the label of the other edge-side surrounding the same black corner.

Note that the combinatorics of the map can be easily recovered from the triplet of permutations, as follows. Each white (resp. black) vertex of degree d corresponds to two cycles of length d of $S \circ S_1$ (resp. $S \circ S_2$) or, equivalently, to a loop of length $2d$ of $\mathcal{L}(S, S_1)$ (resp. $\mathcal{L}(S, S_2)$). Each face of length $2d$ corresponds to two cycles of length d of $S_1 \circ S_2$ or, equivalently, to a loop of length $2d$ of $\mathcal{L}(S_1, S_2)$. Therefore, our formula from Theorem 1.2 can be restated as a summation over maps.

For instance, the map from Figure 1 is associated to the following triplet of pair-partitions:

$$\begin{aligned}
S &= \big\{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\},\{11,12\},\{13,14\},\{15,16\}\big\} \\
S_1 &= \big\{\{1,15\},\{2,3\},\{4,14\},\{13,16\},\{5,7\},\{6,10\},\{8,11\},\{9,12\}\big\} \\
S_2 &= \big\{\{1,10\},\{2,7\},\{8,13\},\{9,14\},\{3,5\},\{4,12\},\{6,15\},\{11,16\}\big\}
\end{aligned}$$

Note that the union $S \cup S_1$ has two loops: $(1, 2, 3, 4, 14, 13, 16, 15)$ and $(5, 6, 10, 9, 12, 11, 8, 7)$, which correspond to the two white vertices of the map.

The related combinatorics of maps which are not bipartite has been studied by Goulden and Jackson [GJ96].

Remark 1.5. It is well-known that bipartite graphs on oriented surfaces correspond to pairs of permutations. Therefore, Schur analogue of our formula (see Remark 1.3) can be restated as a summation over graphs on orientable surfaces.

1.3. **Zonal characters.** The above formula expresses zonal polynomials in terms of power-sum symmetric functions. In Section 3, we will extract the coefficient of a given power-sum. In this way we study an analogue of the coordinates of Schur polynomials in the power-sum basis of the symmetric function ring. These coordinates are known to be the irreducible characters of the symmetric group and have plenty of interesting properties. Some of them are (conjecturally) generalizable to the context where Schur functions are replaced by Jack polynomials and our results in the case of zonal polynomials go in this direction.

1.3.1. *Characters of symmetric groups.* For a Young diagram λ we denote by ρ^λ the corresponding irreducible representation of the symmetric group

$\mathfrak{S}_n$ with $n = |\lambda|$. Any partition μ such that $|\mu| = n$ can be viewed as a conjugacy class in $\mathfrak{S}_n$. Let $\pi_\mu \in \mathfrak{S}_n$ be any permutation from this conjugacy class; we will denote by $\operatorname{Tr} \rho^\lambda(\mu) := \operatorname{Tr} \rho^\lambda(\pi_\mu)$ the corresponding irreducible character value. If $m \leq n$, any permutation $\pi \in \mathfrak{S}_m$ can be also viewed as an element of $\mathfrak{S}_n$, we just have to add $n - m$ additional fixpoints to π; for this reason

$$\operatorname{Tr} \rho^\lambda(\mu) = \operatorname{Tr} \rho^\lambda \left(\mu\, 1^{|\lambda|-|\mu|}\right)$$

makes sense also when $|\mu| \leq |\lambda|$.

Normalized characters of the symmetric group were defined by Ivanov and Kerov [IK99] as follows:

$$\Sigma^{(1)}_\mu(\lambda) = \underbrace{n(n-1)\cdots(n-|\mu|+1)}_{|\mu| \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(\mu)}{\text{dimension of } \rho^\lambda} \tag{3}$$

(the meaning of the superscript in the notation $\Sigma^{(1)}_\mu(\lambda)$ will become clear later on). The novelty of the idea was to view the character as a function $\lambda \mapsto \Sigma^{(1)}_\mu(\lambda)$ on the set of Young diagrams (of any size) and to keep the conjugacy class fixed. The normalization constants in (3) were chosen in such a way that the normalized characters $\lambda \mapsto \Sigma^{(1)}_\mu(\lambda)$ form a linear basis (when μ runs over the set of all partitions) of the algebra $\Lambda^\star$ of shifted symmetric functions introduced by Okounkov and Olshanski [OO97], which is very rich in structure (this property is, for example, the key point in a recent approach to study asymptotics of random Young diagrams under Plancherel measure [IO02]). In addition, recently a combinatorial description of the quantity (3) has been given [Sta06, Fér10], which is suitable for study of asymptotics of character values [FŚ07].

Thanks to Frobenius' formula for characters of the symmetric groups [Fro00], definition (3) can be rephrased using Schur functions. We expand the Schur polynomial s_λ in the base of the power-sum symmetric functions (p_ρ) as follows:

$$\frac{n!\, s_\lambda}{\dim(\lambda)} = \sum_{\substack{\rho: \\ |\rho|=|\lambda|}} \theta^{(1)}_\rho(\lambda)\, p_\rho \tag{4}$$

for some numbers $\theta^{(1)}_\rho(\lambda)$. Then

$$\Sigma^{(1)}_\mu(\lambda) = \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} z_\mu\, \theta^{(1)}_{\mu, 1^{|\lambda|-|\mu|}}(\lambda), \tag{5}$$

where

$$z_\mu = \mu_1 \mu_2 \cdots\, m_1(\mu)!\, m_2(\mu)! \cdots$$

and $m_i(\mu)$ denotes the multiplicity of i in the partition μ.

1.3.2. *Zonal and Jack characters.* In this paragraph, we will define analogues of the quantity $\Sigma^{(1)}_{\mu}(\lambda)$ via Jack polynomials. First of all, as there are several of them, we have to fix a normalization for Jack polynomials. In our context, the best is to use the functions denoted by J in the book of Macdonald [Mac95, VI, (10.22)]. With this normalization, one has

$$J^{(1)}_{\lambda} = \frac{n!\, s_\lambda}{\dim(\lambda)},$$
$$J^{(2)}_{\lambda} = Z_\lambda.$$

If in (4), we replace the left-hand side by Jack polynomials:

$$J^{(\alpha)}_{\lambda} = \sum_{\substack{\rho:\\ |\rho|=|\lambda|}} \theta^{(\alpha)}_{\rho}(\lambda)\, p_\rho \tag{6}$$

then in analogy to (5), we define

$$\Sigma^{(\alpha)}_{\mu}(\lambda) = \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)}\, z_\mu\, \theta^{(\alpha)}_{\mu,1^{|\lambda|-|\mu|}}(\lambda). \tag{7}$$

This quantities are called *Jack characters*. Notice that for $\alpha = 1$ we recover the usual normalized character values of the symmetric groups. The case $\alpha = 2$ is of central interest in this article, since then the left-hand side of (6) is equal to the zonal polynomial; for this reason $\Sigma^{(2)}_{\mu}(\lambda)$ will be called *zonal character*.

Study of Jack characters has been initiated by Lassalle [Las08, Las09]. Just like the usual normalized characters $\Sigma^{(1)}_{\mu}$, they are (α-)shifted symmetric functions [Las08, Proposition 2] as well, which is a good hint that they might be an interesting generalization of the characters. The names *zonal characters* and *Jack characters* are new; we decided to introduce them because quantities $\Sigma^{(\alpha)}_{\mu}(\lambda)$ are so interesting that they deserve a separate name. One could argue that this name is not perfect since zonal and Jack characters are not *sensu stricto* characters in the sense of the representation theory. On the other hand, as we shall see, zonal and Jack characters share many interesting properties with the usual characters of symmetric groups, therefore the former can be viewed as generalizations of the latter which justifies to some extent their new name.

1.4. **The main result 2: combinatorial formulas for zonal characters.**

1.4.1. *Zonal characters in terms of numbers of colorings functions.* Let S_0, S_1, S_2 be three pair-partitions of the set $[2k]$. We consider the following function on the set of Young diagrams:

Definition 1.6. $N^{(1)}_{S_0,S_1,S_2}(\lambda)$ is the number of functions f from $[2k]$ to the boxes of the Young diagram λ such that for every $l \in [2k]$:

(Q0) $f(l) = f(S_0(l))$, in other words f can be viewed as a function on the set of pairs constituting S_0;
(Q1) $f(l)$ and $f(S_1(l))$ are in the same column;
(Q2) $f(l)$ and $f(S_2(l))$ are in the same row.

Note that $\lambda \mapsto N^{(1)}_{S_0,S_1,S_2}(\lambda)$ is, in general, not a shifted symmetric function, so it cannot be expressed in terms of zonal characters. On the other hand, the zonal characters have a very nice expression in terms of the N functions:

Theorem 1.7. *Let μ be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of the set $[2k]$ whose union has type μ. Then one has the following equality between functions on the set of Young diagrams:*

$$\Sigma^{(2)}_\mu = \frac{1}{2^{\ell(\mu)}} \sum_{S_0} (-1)^{\mathcal{L}(S_0,S_1)}\ 2^{|\mathcal{L}(S_0,S_1)|}\ N^{(1)}_{S_0,S_1,S_2}, \tag{8}$$

where the sum runs over pair-partitions of $[2k]$ and $\ell(\mu)$ denotes the number of parts of partition μ.

We postpone the proof to Sections 3.1–3.4. This formula is an intermediate step towards Theorem 1.8, but we wanted to state it as an independent result because its analogue for the usual characters [FŚ07, Theorem 2] has been quite useful in some contexts (see [FŚ07, Fér09]).

1.4.2. *Zonal characters in terms of Stanley's coordinates.* The notion of Stanley's coordinates was introduced by Stanley [Sta04] who found a nice formula for normalized irreducible character values of the symmetric group corresponding to rectangular Young diagrams. In order to generalize this result, he defined, given two sequences $\mathbf{p}$ and $\mathbf{q}$ of positive integers of same size ($\mathbf{q}$ being non-increasing), the partition:

$$\mathbf{p} \times \mathbf{q} = (\underbrace{q_1, \dots, q_1}_{p_1 \text{ times}}, \dots, \underbrace{q_l, \dots, q_l}_{p_l \text{ times}}).$$

Then he suggested to consider the quantity $\Sigma^{(1)}_\mu(\mathbf{p} \times \mathbf{q})$ as a polynomial in $\mathbf{p}$ and $\mathbf{q}$. An explicit combinatorial interpretation of the coefficients was conjectured in [Sta06] and proved in [Fér10].

It is easy to deduce from the expansion of $\Sigma^{(2)}_\mu$ in terms of the N functions a combinatorial description of the polynomial $\Sigma^{(2)}_\mu(\mathbf{p} \times \mathbf{q})$.

Theorem 1.8. *Let μ be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of $[2k]$ whose union has type μ. Then, one has:*

$$\Sigma_\mu^{(2)}(\mathbf{p}\times\mathbf{q}) = \frac{(-1)^k}{2^{\ell(\mu)}}\sum_{S_0}\left[\sum_{\phi:\mathcal{L}(S_1,S_0)\to\mathbb{N}^\star}\prod_{l\in\mathcal{L}(S_1,S_0)}(p_{\varphi(l)})\cdot\prod_{l'\in\mathcal{L}(S_2,S_0)}(-2q_{\psi(l')})\right] \tag{9}$$

where $\psi(l') := \max_l \varphi(w)$ with l running over the loops of $\mathcal{L}(S_0,S_1)$ having at least one element in common with l'.

Proof. It is a direct application of Lemma 3.6 to Theorem 1.7. □

1.4.3. *Lassalle's conjecture.* The following result shows that a conjecture of Lassalle [Las08] who investigated the expansion of Jack characters in terms of Stanley's coordinates holds true in the case $\alpha = 2$.

Corollary 1.9. *The quantity $(-1)^{|\mu|}\Sigma_\mu^{(2)}(\mathbf{p}\times\mathbf{q})$ is a polynomial in $\mathbf{p}$ and $-\mathbf{q}$ with nonnegative integer coefficients.*

Moreover, the coefficient of $p_i^{|\mu|}(-q_i)^{\ell(\mu)}$ is equal to 1 for any i (Lassalle conjectured that at least one of the coefficients is equal to 1).

Proof. Except for integrality of the coefficients this is an immediate consequence of Theorem 1.8. In order to prove integrality of the coefficients, we need another, but closely related, formula, namely Theorem 1.10. Then, we just apply Lemma 3.6 and obtain an analogue of Theorem 1.8 with a formula which does not have the division by $2^{\ell(\mu)}$. □

1.5. **Combinatorial interpretation of main result 2.**

1.5.1. *Gluings of polygons.* Similarly as Theorem 1.2, also Theorem 1.7 and hence Theorem 1.8 have a nice combinatorial interpretation.

Let μ be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of the set $[2k]$ whose union has type μ. We consider a collection of bipartite polygons: each polygon corresponds to a loop of $\mathcal{L}(S_1,S_2)$ and the number of sides of the polygon is equal to the size of the loop. The edges are numbered with the integers $1,\dots,2k$ such that S_1 (resp. S_2) associates to the label of an edge the label of the other edge with the same white (resp. black) extremity. As pair-partitions S_1 and S_2 are fixed, this is equivalent to the choice of one marked edge per polygon.

Then the third pair-partition S_0 will be seen as a way to glue in pairs the external sides of the edges of our polygons (we always glue edges such that their white, resp. black, extremities are glued together). After performing these gluings we obtain a map M. This map is exactly the one associated to the triplet (S_0, S_1, S_2) in Section 1.2.2. Note that the polygons become the faces of M.

It is easy to see that the function $N^{(1)}_{S_0,S_1,S_2}$ depends only on the bipartite graph underlying the map M associated to (S_0, S_1, S_2). Therefore, Eq. (8) can be read as a sum over all ways to glue a collection of labeled polygons of the values of some function associated to the corresponding graphs.

Furthermore, since $|\mathcal{L}(S_0, S_1)|$ is equal to the number of white vertices in M, the factor $2^{|\mathcal{L}(S_0,S_1)|}$ can be interpreted as the number of choices of a local orientation at each of the white vertices (we recall that orientation at a point of a surface is by definition the same as choosing which one of the two possible directions of rotation will be called clockwise). Therefore (8) can be equivalently written as

$$\Sigma^{(2)}_{\mu} = \frac{1}{2^{\ell(\mu)}} \sum_{S_0} \sum_{\substack{\text{choice of orientation} \\ \text{at every white vertex of map}}} (-1)^{\mathcal{L}(S_0,S_1)}\ N^{(1)}_{S_0,S_1,S_2}. \tag{10}$$

Notice that after ungluing the polygons which formed the map M with a fixed orientations at white vertices we obtain the original family of polygons with additional information about the orientations around the white vertices. Therefore (10) can be equivalently written as

$$\Sigma^{(2)}_{\mu} = \frac{1}{2^{\ell(\mu)}} \sum_{\substack{\text{choice of orientation} \\ \text{at every white vertex} \\ \text{of the polygons}}} \sum_{S_0} (-1)^{\mathcal{L}(S_0,S_1)}\ N^{(1)}_{S_0,S_1,S_2}, \tag{11}$$

where the sum over S_0 is restricted to pair-partitions with a property that the orientations of all white vertices which are glued together by S_0 must match.

1.5.2. *Gluings of unlabeled polygons and Lassalle's conjecture.* Let us change now our combinatorial setup: we remove all the labels from the edges of the considered family of polygons and, instead, we mark one white vertex per polygon. We shall consider the ways to glue the edges of such polygons and to choose orientations at the white vertices. The best way to do this will be to consider the corresponding problem for the polygons with labeled edges and then identify the combinatorial objects which differ just by the choice of labeling.

For a given polygon we consider its axial reflection around the axis passing through the marked white vertex and its antipodal vertex. These reflections commute and hence they generate the group isomorphic to $\mathbb{Z}_2^{\ell(\mu)}$, which describes the symmetry of our setup. This group $\mathbb{Z}_2^{\ell(\mu)}$ acts on the set of pairs

$$(\text{choice of orientations}, S_0) \tag{12}$$

which contribute to (11); each orbit of this action contains combinatorial objects which differ only by a change of the labeling of the edges. In other words, the orbits are the wanted ways of gluing edges and choosing orientations for unlabeled polygons.

The axes of reflection pass through white vertices and each such a reflection flips the orientation of the marked white vertex; it follows immediately that the stabilizer of any non-identity element with respect to this action is trivial hence every orbit has $2^{\ell(\mu)}$ elements. In this way we proved the following result.

Theorem 1.10. *Let μ be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of the set $\{1,\dots,2k\}$ whose union has type μ. Then*

(13) $$\Sigma^{(2)}_\mu = \sum_\Omega (-1)^{\mathcal{L}(S_0(\Omega),S_1)}\ N^{(1)}_{S_0(\Omega),S_1,S_2},$$

where the sum runs over the orbits of $\mathbb{Z}_2^{\ell(\mu)}$ acting on (12) *and where $S_0(\Omega)$ denotes any representative of orbit Ω. Alternatively, the summation can be taken over all ways of gluing edges of unlabeled polygons (with one marked white vertex per polygon) and choosing the orientations at the white vertices.*

Remark 1.11. The analogous result in the case $\alpha = 1$ [FŚ07, Theorem 2] can be reformulated in the following way which is very similar to Theorem 1.10:

$$\Sigma^{(1)}_\mu = \sum_\Omega (-1)^{\mathcal{L}(S_0(\Omega),S_1)}\ N^{(1)}_{S_0(\Omega),S_1,S_2},$$

where the summation is over all ways of gluing edges of unlabeled polygons (with one marked white vertex per polygon) and over choices of orientations at white vertices in such a way that any two vertices in a connected component have the same orientation. Alternatively, this can be reformulated in a simpler way which, unfortunately, makes the analogy less apparent: the summation is over all ways of gluing edges of unlabeled polygons in such a way that the resulting map is orientable and over the choices of the orientation in each connected component.

1.6. **Kerov polynomials.**

1.6.1. *Free cumulants.* For a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots)$ and an integer $s \geq 1$ we consider the dilated Young diagram

$$D_s\lambda = (\underbrace{s\lambda_1,\dots,s\lambda_1}_{s \text{ times}}, \underbrace{s\lambda_2,\dots,s\lambda_2}_{s \text{ times}}, \dots).$$

If we interpret the Young diagrams geometrically as collections of boxes then the dilated diagram $D_s\lambda$ is just the image of λ under scaling by factor s.

Following Biane [Bia98] (who used a different, but equivalent definition), for a Young diagram λ we define its *free cumulants* $R_2(\lambda), R_3(\lambda), \dots$ by the formula

$$R_k(\lambda) = \lim_{s\to\infty} \frac{1}{s^k} \Sigma^{(1)}_{k-1}(D_s\lambda).$$

In other words, each free cumulant $R_k(\lambda)$ is asymptotically the dominant term of the character on a cycle of length $k-1$ in the limit when the Young diagram tends to infinity. It is natural to generalize this definition using Jack characters:

$$R^{(\alpha)}_k(\lambda) = \lim_{s\to\infty} \frac{\alpha}{s^k} \Sigma^{(\alpha)}_{k-1}(D_s\lambda).$$

In fact, the general α case can be expressed simply in terms of the usual free cumulant thanks to [Las09, Theorem 7] (notice that our normalization of free cumulants differs from the one in Lassalle's papers by some normalization factors):

$$R^{(\alpha)}_k(\lambda) = R_k(\alpha\lambda),$$

where

$$\alpha\lambda = (\alpha\lambda_1, \alpha\lambda_2, \dots)$$

is the Young diagram stretched anisotropically only along the OX axis. The quantities $R^{(\alpha)}_k(\lambda)$ are called α-anisotropic free cumulants of the Young diagram λ.

With this definition free cumulants might seem to be rather abstract quantities, but in fact they could be equivalently defined in a very explicit way using the shape of the diagram and linked to free probability, whence their name, see [Bia98]. The equivalence of these two descriptions makes them very useful parameters for describing Young diagrams. Moreover, they form a homogeneous algebraic basis of the ring of shifted symmetric functions, so many interesting functions can be written in terms of free cumulants. These features make free cumulants a perfect tool in the study of asymptotic problems in representation theory, see for example [Bia98, Śni06].

1.6.2. *Jack Kerov polynomials.* The following observation is due to Lassalle [Las09]. Let $k \geq 1$ be a fixed integer and let α be fixed. Since $\Sigma^{(\alpha)}_k$ is an α-shifted symmetric function and the anisotropic free cumulants $(R^{(\alpha)}_l)_{l\geq 2}$ form an algebraic basis of the ring of α-shifted symmetric functions, there exists a polynomial $K^{(\alpha)}_k$ such that, for any Young diagram λ,

$$\Sigma^{(\alpha)}_k(\lambda) = K^{(\alpha)}_k\big(R^{(\alpha)}_2(\lambda), R^{(\alpha)}_3(\lambda), \dots\big).$$

This polynomial is called *Jack Kerov polynomial*.

Thus Jack Kerov polynomials express Jack characters on cycles in terms of free cumulants. For more complicated conjugacy classes it turns out to be more convenient to express not directly the characters $\Sigma^{(\alpha)}_{(k_1,\dots,k_\ell)}$ but rather *cumulant*

$$(-1)^{\ell-1}\kappa^{\text{id}}(\Sigma^{(\alpha)}_{k_1},\dots,\Sigma^{(\alpha)}_{k_\ell}). \tag{14}$$

This gives rise to *generalized Jack Kerov polynomials* $K^{(\alpha)}_{(k_1,\dots,k_\ell)}$. In the classical context $\alpha = 1$ these quantities have been introduced by one of us and Rattan [RŚ08]; in the Jack case they have been studied by Lassalle [Las09]. We skip the definitions and refer to the above papers for details since generalized Kerov polynomials are not of central interest for this paper.

1.6.3. *Classical Kerov polynomials.* For $\alpha = 1$ these polynomials are called simply *Kerov polynomials*. This case has a much longer history and it was initiated by Kerov [Ker00] and Biane [Bia03] who proved that in this case the coefficients are in fact integers and conjectured their non-negativity. This conjecture has been proved by the first-named author in [Fér09], also for generalized Kerov polynomials. Then, an explicit combinatorial interpretation has been given by the authors, together with Dołęga, in [DFŚ10], using a new method.

These polynomials have a deep structure, from a combinatorial and analytic point of view, and there are still open problems concerning them. For a quite comprehensive bibliography on this subject we refer to [DFŚ10].

Most of properties of Kerov polynomials seem to be generalizable in the case of a general value of the parameter α, although not much has been proved for the moment (see [Las09]).

1.7. **The main result 3: Kerov's polynomials for zonal characters.** We will show the following combinatorial interpretation of the coefficients of zonal Kerov polynomials, analogous to the one from [DFŚ10].

Theorem 1.12. *Let μ be a partition and let $s_2, s_3, \dots$ be a sequence of non-negative integers with only finitely many non-zero elements.*

The rescaled coefficient

$$(-2)^{\ell(\mu)}(-1)^{|\mu|+2s_2+3s_3+\cdots}\left[\left(R_2^{(\alpha)}\right)^{s_2}\left(R_3^{(\alpha)}\right)^{s_3}\cdots\right]K_\mu^{(2)}$$

of the (generalized) zonal Kerov polynomial is equal to the number of pairs (M,q) with the following properties:

- *(a) M is a map created by gluing pairs of edges of a collection of bipartite polygons with the numbers of edges $2\mu_1, 2\mu_2, \dots$ with one marked edge per polygon;*
- *(b) the number of the black vertices of M is equal to $s_2 + s_3 + \cdots$;*
- *(c) the number of all vertices of M is equal to $2s_2 + 3s_3 + 4s_4 + \cdots$;*

(d) *q is a function from the set of the black vertices of M to the set $\{2,3,\dots\}$; we require that each number $i \in \{2,3,\dots\}$ is used exactly s_i times;*
(e) *for every subset $A \subset V_\bullet(M)$ of black vertices of M which is nontrivial (i.e., $A \neq \emptyset$ and $A \neq V_\bullet(M)$) there are more than $\sum_{v\in A}\big(q(v)-1\big)$ white vertices of M which are connected to at least one vertex from A.*

Condition (e) can be reformulated in a number of equivalent ways [DFŚ10]. This result will be proved in Section 4.

Remark 1.13. From the viewpoint of the above result it would be more aesthetically appealing to change the definition of Jack characters and rather consider quantities $\alpha^{\ell(\mu)}\Sigma_\mu^{(\alpha)}$ instead since then the coefficients of zonal Kerov polynomials would be integers.

1.8. **Symplectic zonal polynomials.** As mentioned above, the case $\alpha=\frac{1}{2}$ is also special for Jack polynomials, as we recover the so-called symplectic zonal polynomials. These polynomials appear in a quaternionic analogue of James' theory, see [Mac95, VII.6].

Our formulas have immediate analogues in this case thanks to the duality formula for Jack characters (see [Mac95, Chapter VI, equation (10.30)]):

$$\theta_\rho^{(\alpha)}(\lambda) = (-\alpha)^{|\rho|-\#\text{parts of }\rho}\, \theta_\rho^{(\alpha^{-1})}(\lambda'), \tag{15}$$

where λ' is the partition conjugate to λ. For instance, the analogue of Theorem 1.2 is:

$$J_\lambda^{(1/2)} = \sum_{(S_1,S_2)\ T'\text{-admissible}} (-1)^{\mathcal{L}(S,S_2)} \left(-\frac{1}{2}\right)^{|\mathcal{L}(S_1,S_2)|} p_{\mathcal{L}(S_1,S_2)}, \tag{16}$$

where the sum runs over pairs (S_1,S_2) of pair-partitions which are T'-admissible with respect to the Young tableau T' of shape $(2\lambda)'$ in which cases are numbered consecutively along the rows.

The combinatorial interpretation of Stanley's and Kerov's polynomials for zonal characters have of course also analogue in the symplectic zonal case. In particular, for the symplectic zonal Kerov polynomials, one has the following proposition which is a special case of [Las09, Conjecture 14.1]:

Proposition 1.14. *Recall that the generalized symplectic zonal Kerov polynomials are defined by:*

$$(-1)^{\ell-1}\kappa^{id}(\Sigma_{k_1}^{(1/2)},\dots,\Sigma_{k_\ell}^{(1/2)}) = K_{k_1,\cdots,k_\ell}^{(1/2)}(R_2,R_3,\dots). \tag{17}$$

Then $K_{k_1,\cdots,k_\ell}^{(1/2)}$ is a non-homogeneous polynomial with non-negative coefficients of degree $k_1+\cdots+k_\ell-\ell+2$.

Proof. This is a consequence of Theorem 1.12 and Eq. (15). □

1.9. **Link with Weingarten function.** Zonal characters are linked with Weingarten functions (see [Mat10, Section 8]). These functions appear in the computation of the integral of a monomial in the coordinates over the orthogonal or unitary group. Our combinatorial description of zonal characters is a good tool in this context: we will use it in a forthcoming article to show in a natural way that the asymptotics of Weingarten functions for both unitary and orthogonal groups are the same.

1.10. **Maps and zonal characters: the dual picture.** It should be stressed that a previous result linking maps on not necessarily orientable surfaces and zonal characters can be found in the work of Goulden and Jackson [GJ96]. But their result goes in the reverse direction than ours: they count maps using zonal characters, while we express zonal characters using maps. The same picture exists for maps on orientable surface and the usual characters. It would be nice to understand the link between these two dual approaches.

1.11. **Overview of the paper.** The paper is organized according to our three main results. Sections 2, 3 and 4 are respectively devoted to the proofs of main results 1, 2 and 3.

2. Formulas for zonal polynomials

The main result of this section is Theorem 1.2, which gives a combinatorial formula for zonal polynomials.

2.1. **Preliminaries.** In this paragraph we give the characterization of zonal polynomials, which is the starting point of our proof of Theorem 1.2. This characterization is due to James and the content of this paragraph can be found with more details and proofs in his paper [Jam61].

For $Z = v_1 \otimes \cdots \otimes v_{2n} \in (\mathbb{R}^d)^{\otimes 2n}$ we define a homogeneous polynomial function of degree $2n$ on the set $\mathcal{M}_d(\mathbb{R})$ of $d \times d$ matrices by

$$\phi_Z(X) = \langle X^T v_1, X^T v_2 \rangle \cdots \langle X^T v_{2n-1}, X^T v_{2n} \rangle \quad \text{for } X \in \mathcal{M}_d(\mathbb{R})$$

and for general tensors $Z \in (\mathbb{R}^d)^{\otimes 2n}$ by linearity. Clearly,

$$\phi_Z(XO) = \phi_Z(X) \qquad \text{for any } O \in O_d(\mathbb{R});$$

in other words ϕ_Z is invariant under the right action of the orthogonal group $O_d(\mathbb{R})$.

The action of $\mathrm{GL}_d(\mathbb{R})$ on $(\mathbb{R}^d)^{\otimes 2n}$ is defined on elementary tensors by

$$L(v_1 \otimes \cdots \otimes v_{2n}) = Lv_1 \otimes \cdots \otimes Lv_{2n}. \tag{18}$$

The group $\mathrm{GL}_d(\mathbb{R})$ acts also canonically on the polynomial functions on $\mathcal{M}_d(\mathbb{R})$ as follows:

$$(Lf)(X) = f(L^T X).$$

In this way both $(\mathbb{R}^d)^{\otimes 2n}$ and the space of polynomial functions on $\mathcal{M}_d(\mathbb{R})$ are representations of $\mathrm{GL}_d(\mathbb{R})$ and it is easy to check that ϕ is an intertwiner, i.e.

$$L\phi_Z = \phi_{LZ}.$$

James proved that, for a given Young diagram λ, there is a unique (up to a multiplicative constant) element $z_\lambda \in (\mathbb{R}^d)^{\otimes 2n}$ such that:

(a) ϕ_{z_λ} is non-zero,
(b) z_λ (and hence ϕ_{z_λ}) is invariant under the left action of the orthogonal group $O_d(\mathbb{R}) \subset \mathrm{GL}_d(\mathbb{R})$,
(c) z_λ (and hence ϕ_{z_λ}) belongs to the subrepresentation of $\mathrm{GL}_d(\mathbb{R})$ corresponding to the highest weight 2λ.

By definition, its image ϕ_{z_λ} is a polynomial function on the set $\mathcal{M}_d(\mathbb{R})$ which is invariant under the left and right action of the orthogonal group. Therefore, the image $\phi_{z_\lambda}(X)$ only depends on the multiset $\mathrm{Sp}(XX^T) = \{z_1, \dots, z_d\}$ of eigenvalues of XX^T. As ϕ_{z_λ} is a polynomial function of degree $2n$, there exists a symmetric polynomial $Z_\lambda^{(d)}$ of degree n such that:

$$\phi_{z_\lambda}(X) = Z_\lambda(\mathrm{Sp}(XX^T)).$$

If $d \geq 2n$ (inequality that we assume from now on), the polynomial $Z_\lambda^{(d)}$ is unique up to a multiplicative constant, which we fix by asking that the coefficient in the expansion in the power-sum symmetric functions is equal to 1:

$$[p_1^{|\lambda|}] Z_\lambda^{(d)} = 1.$$

The collection of symmetric polynomials $Z_\lambda^{(d)}$ defines a symmetric function Z_λ, which is, by definition, the zonal symmetric function.

In the following paragraphs we will use this definition of the zonal symmetric function to prove Theorem 1.2. More precisely, we will exhibit an element $z_\lambda \in (\mathbb{R}^d)^{\otimes 2n}$ having properties (a), (b) and (c) and compute the corresponding symmetric function Z_λ.

2.2. **Pair-partitions and tensors.** If P is a pair-partition of the ground set $[2n]$, we will associate to it the tensor

$$\Psi_P = \sum_{1 \leq i_1, \dots, i_{2n} \leq d} \delta_P(i_1, \dots, i_{2n})\, e_{i_1} \otimes \cdots \otimes e_{i_{2n}} \in (\mathbb{R}^d)^{\otimes 2n},$$

where $\delta_P(i_1, \dots, i_{2n})$ is equal to 1 if $i_k = i_l$ for all $\{k, l\} \in P$ and is equal to zero otherwise. The symmetric group $\mathfrak{S}_{2n}$ acts on the set of pair-partitions (when we interpret the latter as permutations in $\mathfrak{S}_{2n}$, the action is given by conjugation) as well as on the set of tensors $(\mathbb{R}^d)^{\otimes 2n}$ and it is easy to check that $P \mapsto \Psi_P$ is an intertwiner with respect to these two actions.

Recall that we have defined

$$S = \big\{\{1,2\},\{3,4\},\dots,\{2n-1,2n\}\big\}. \tag{19}$$

Notice that with this choice of S, one has the following lemma

Lemma 2.1. *Let $Z \in (\mathbb{R}^d)^{\otimes 2n}$. Then*

$$\phi_Z(X) = \langle Z, X^{\otimes 2n}\Psi_S\rangle$$

with respect to the standard scalar product in $(\mathbb{R}^d)^{\otimes 2n}$.

Proof. We can assume by linearity that $Z = v_1 \otimes \ldots \otimes v_{2n}$. The right-hand side becomes:

$$\langle Z, X^{\otimes 2n}\Psi_S\rangle = \sum_{i_1,\dots,i_n} \Big\langle v_1 \otimes \cdots \otimes v_{2n}, Xe_{i_1} \otimes Xe_{i_1} \otimes \cdots \otimes Xe_{i_n} \otimes Xe_{i_n}\Big\rangle$$

$$= \sum_{1\le i_1,\dots,i_n\le d} \prod_{j=1}^{n} \langle v_{2j-1}, Xe_{i_j}\rangle \cdot \langle v_{2j}, Xe_{i_j}\rangle$$

$$= \prod_{j=1}^{n}\left[\sum_{1\le i\le d} \langle X^T v_{2j-1}, e_i\rangle \cdot \langle X^T v_{2j}, e_i\rangle\right] = \prod_{j=1}^{n} \langle X^T v_{2j-1}, X^T v_{2j}\rangle. \quad \square$$

Lemma 2.2. *Let P be a pair-partition of $[2n]$ and S, as before, the pair-partition of the same set given by* (19)*. Then*

$$\phi_{\Psi_P}(X) = \langle \Psi_P, X^{\otimes 2n}\Psi_S\rangle = \operatorname{Tr}\left[(XX^T)^{\ell_1}\right] \operatorname{Tr}\left[(XX^T)^{\ell_2}\right]\cdots = p_{\mathcal{L}(P,S)}(\operatorname{Sp}(XX^T)),$$

where $2\ell_1, 2\ell_2, \dots$ are the lengths of the loops of $\mathcal{L}(P,S)$.

Proof. Let us consider the case where $\mathcal{L}(P,S)$ has only one loop of length 2ℓ. Up to a renumbering of P which leaves S invariant one can assume that $P = \big\{\{2,3\},\{4,5\},\dots,\{2\ell-2,2\ell-1\},\{2\ell,1\}\big\}$. Such a renumbering corresponds to the action on Ψ_P of an orthogonal operator which commutes with $X^{\otimes 2\ell}$ and leaves Ψ_S invariant, and thus does not change the value of $\phi_{\Psi_P}(X)$. In this case

$$\Psi_P = \sum_{1\le j_1,\dots,j_\ell\le d} e_{j_\ell} \otimes e_{j_1} \otimes e_{j_1} \otimes \cdots \otimes e_{j_{\ell-1}} \otimes e_{j_{\ell-1}} \otimes e_{j_\ell}.$$

Therefore one has:

$$\begin{aligned}\phi_{\Psi_P}(X) &= \sum_{1\le j_1,\dots,j_\ell\le d} \langle X^T e_{j_\ell}, X^T e_{j_1}\rangle \cdot \langle X^T e_{j_1}, X^T e_{j_2}\rangle \cdots \langle X^T e_{j_{\ell-1}}, X^T e_{j_\ell}\rangle \\ &= \sum_{1\le j_1,\dots,j_\ell\le d} \langle XX^T e_{j_\ell}, e_{j_1}\rangle \cdot \langle XX^T e_{j_1}, e_{j_2}\rangle \cdots \langle XX^T e_{j_{\ell-1}}, e_{j_\ell}\rangle \\ &= \sum_{1\le j_1,\dots,j_\ell\le d} (XX^T)_{j_1,j_\ell} \cdot (XX^T)_{j_2,j_1} \cdots (XX^T)_{j_\ell,j_{\ell-1}} \\ &= \operatorname{Tr}(XX^T)^\ell.\end{aligned}$$

The general case is simply obtained by multiplication of the one-loop case. □

It follows that $X \mapsto \phi_{\Psi_P}(X)$ is invariant under the left action of the orthogonal group $O_d(\mathbb{R})$. The above discussion shows that if P is a pair-partition (or, more generally, a formal linear combination of pair-partitions) then condition (b) is fulfilled for $z_\lambda = \Psi_P$. For this reason we will look for candidates for z_λ corresponding to zonal polynomials in this particular form.

2.3. **Young symmetriser.** Let a partition λ be fixed; we denote $n = |\lambda|$. We consider the Young tableau T of shape 2λ in which boxes are numbered consecutively along the rows. This tableau was chosen in such a way that if we interpret the pair-partition S as a pairing of the appropriate boxes of T then a box in the column $2i-1$ is paired with the box in the column $2i$ in the same row, where i is a positive integer (these two boxes will be called neighbors in the Young diagram 2λ).

Tableau T allows us to identify boxes of the Young diagram 2λ with the elements of the set $[2n]$. In particular, permutations from $\mathfrak{S}_{2n}$ can be interpreted as permutations of the boxes of 2λ. We denote

$$\begin{aligned}P_{2\lambda} =& \{\sigma \in \mathfrak{S}_{2n} \colon \sigma \text{ preserves each row of } 2\lambda\}, \\ Q_{2\lambda} =& \{\sigma \in \mathfrak{S}_{2n} \colon \sigma \text{ preserves each column of } 2\lambda\}\end{aligned}$$

and define

$$\begin{aligned}a_{2\lambda} =& \sum_{\sigma\in P_{2\lambda}} \sigma \in \mathbb{C}[\mathfrak{S}_{2n}], \\ b_{2\lambda} =& \sum_{\sigma\in Q_{2\lambda}} (-1)^{|\sigma|}\sigma \in \mathbb{C}[\mathfrak{S}_{2n}], \\ c_{2\lambda} =& b_{2\lambda} a_{2\lambda}.\end{aligned}$$

The element $c_{2\lambda}$ is called *Young symmetriser* and it is well-known that there exists some non-zero scalar $\alpha_{2\lambda}$ such that $\alpha_{2\lambda} c_{2\lambda}$ is a projection. Its image $\mathbb{C}[\mathfrak{S}_{2n}]\alpha_{2\lambda} c_{2\lambda}$ under multiplication from the right on the left-regular representation gives an irreducible representation $\rho^{2\lambda}$ of the symmetric group (where the symmetric group acts by left multiplication) associated to the Young diagram 2λ.

Recall that there is also a central projection in $\mathbb{C}[\mathfrak{S}_{2n}]$, denoted $\mathfrak{p}_{2\lambda}$, whose image $\mathbb{C}[\mathfrak{S}_{2n}]\mathfrak{p}_{2\lambda}$ under multiplication from the right (or, equivalently, from the left) on the left-regular representation is the sum of all irreducible representations of type $\rho^{2\lambda}$ contributing to $\mathbb{C}[\mathfrak{S}_{2n}]$. It follows that $\mathbb{C}[\mathfrak{S}_{2n}]c_{2\lambda}$ is a subspace of $\mathbb{C}[\mathfrak{S}_{2n}]\mathfrak{p}_{2\lambda}$. It follows that there is an inequality

$$\alpha_{2\lambda} c_{2\lambda} \leq \mathfrak{p}_{2\lambda} \tag{20}$$

between projections in $\mathbb{C}[\mathfrak{S}_{2n}]$, i.e.

$$\alpha_{2\lambda} c_{2\lambda} \mathfrak{p}_{2\lambda} = \mathfrak{p}_{2\lambda} \alpha_{2\lambda} c_{2\lambda} = \alpha_{2\lambda} c_{2\lambda}.$$

2.4. **Schur-Weyl duality.** The symmetric group $\mathfrak{S}_{2n}$ acts on the vector space $(\mathbb{R}^d)^{\otimes 2n}$ by permuting the factors and the linear group $\mathrm{GL}_d(\mathbb{R})$ acts on the same space by the diagonal action (18). The two action commute and Schur-Weyl duality asserts that, as a representation of $\mathfrak{S}_{2n} \times \mathrm{GL}_d(\mathbb{R})$, one has:

$$(\mathbb{R}^d)^{\otimes 2n} \simeq \bigoplus_{\mu \vdash 2n} V_\mu \times U_\mu,$$

where V_μ (resp. U_μ) is the irreducible representation of $\mathfrak{S}_{2n}$ (resp. $\mathrm{GL}_d(\mathbb{R})$) indexed by μ (as we assumed in Section 2.1 that $d \geq 2n$, the representation U_μ does always exist). But $\mathfrak{p}_{2\lambda}(V_\mu) = \delta_{\mu,2\lambda} V_\mu$, therefore the image $\mathfrak{p}_{2\lambda}\left((\mathbb{R}^d)^{\otimes 2n}\right)$ of the projection $\mathfrak{p}_{2\lambda}$ is, as representation of $\mathrm{GL}_d(\mathbb{R})$, a sum of some number of copies of the irreducible representation of $\mathrm{GL}_d(\mathbb{R})$ associated with the highest weight 2λ. Using inequality (20), we know that $\alpha_{2\lambda} c_{2\lambda}\left((\mathbb{R}^d)^{\otimes 2n}\right)$ is a subspace of $\mathfrak{p}_{2\lambda}\left((\mathbb{R}^d)^{\otimes 2n}\right)$. In this way, we proved that $\alpha_{2\lambda} c_{2\lambda}\left((\mathbb{R}^d)^{\otimes 2n}\right)$ is a representation of $\mathrm{GL}_d(\mathbb{R})$ which is a sum of some number of copies of the irreducible representation of $\mathrm{GL}_d(\mathbb{R})$ associated with the highest weight 2λ.

2.5. **A tensor satisfying James' conditions.** The formal linear combination of pair-partitions

$$c_{2\lambda} \cdot S$$

can be identified with the tensor

$$z_\lambda := \Psi_{c_{2\lambda} \cdot S} = c_{2\lambda} \Psi_S \in (\mathbb{R}^d)^{\otimes 2n}$$

which obviously fulfills conditions (b) and (c); the condition (a) will be verified later on, after our calculation of zonal polynomials is completed and it will become obvious that they are non-zero polynomials.

Therefore there exists a constant C_λ such that

$$\phi_{c_{2\lambda}\Psi_S}(X) = \frac{1}{C_\lambda} Z_\lambda(\mathrm{Sp}(X)).$$

The left-hand side can be transformed as follows:

$$\begin{aligned} \phi_{c_{2\lambda}\Psi_S}(X) &= \sum_{\sigma_1 \in Q_{2\lambda}} \sum_{\sigma_2 \in P_{2\lambda}} (-1)^{\sigma_1} \langle \Psi_{\sigma_1\sigma_2\cdot S}, X^{\otimes 2n}\Psi_S \rangle \\ &= \sum_{\sigma_1 \in Q_{2\lambda}} \sum_{\sigma_2 \in P_{2\lambda}} (-1)^{\sigma_1}\, p_{\mathcal{L}(\sigma_1\sigma_2\cdot S,S)}(\mathrm{Sp}(X)), \end{aligned} \tag{21}$$

where the power-sum symmetric functions p should be understood as in (1). Finally, one has the following formula for zonal polynomials:

$$Z_\lambda = C_\lambda \sum_{\sigma_1 \in Q_{2\lambda}} \sum_{\sigma_2 \in P_{2\lambda}} (-1)^{\sigma_1}\, p_{\mathcal{L}(\sigma_1\sigma_2\cdot S,S)}. \tag{22}$$

Lemma 2.3. *Let σ be a permutation of the boxes of 2λ which preserves each column. Then*

$$(-1)^\sigma = (-1)^{\mathcal{L}(\sigma\cdot S,S)}.$$

Proof. Young diagram 2λ can be viewed as a concatenation of rectangular Young diagrams of size $i \times 2$ (i parts, all of them equal to 2); for this reason it is enough to proof the lemma for the case when $2\lambda = i \times 2$. Permutation σ can be viewed as a pair $(\sigma^{(1)}, \sigma^{(2)})$ where $\sigma^{(j)} \in \mathfrak{S}_i$ is the permutation of j-th column. Then

$$(-1)^\sigma = (-1)^{\sigma^{(1)}}(-1)^{\sigma^{(2)}} = (-1)^{\sigma^{(1)}\left(\sigma^{(2)}\right)^{-1}} = (-1)^{(\ell_1-1)+(\ell_2-1)+\cdots},$$

where $\ell_1, \ell_2, \dots$ are the lengths of the cycles of the permutation $\sigma^{(1)}\left(\sigma^{(2)}\right)^{-1}$.

Let $(\square[c,r])$ denote the box of the Young diagram in the column c and the row r. Then

$$\begin{aligned} \sigma S \sigma^{-1} S(\square[1,i]) &= \sigma S \sigma^{-1}(\square[2,i]) = \sigma S\big(\square[2,(\sigma^{(2)})^{-1}(i)]\big) \\ &= \sigma\big(\square[1,(\sigma^{(2)})^{-1}(i)]\big) = \square[1, \sigma^{(1)}\left(\sigma^{(2)}\right)^{-1}(i)]. \end{aligned}$$

So $\sigma S \sigma^{-1} S = (\sigma \cdot S)S$ permutes the first column and its restriction to the first column has cycles of length $\ell_1, \ell_2, \dots$. The same is true for the second column. It follows that $(\sigma\cdot S)S$ has cycles of length $\ell_1, \ell_1, \ell_2, \ell_2, \dots$ or, equivalently, the lengths of the loops of $\mathcal{L}(\sigma\cdot S, S)$ are equal to $2\ell_1, 2\ell_2, \dots$ which finishes the proof. ∎

2.6. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. The right-hand side of (22) shows that we need to study the (signed) collection of conjugacy classes of the permutations

$$(\sigma_1\sigma_2 \cdot S) \circ S = \sigma_1\sigma_2 S\sigma_2^{-1}\sigma_1^{-1}S$$

which is conjugate to

$$(\sigma_1^{-1}S\sigma_1)(\sigma_2 S\sigma_2^{-1}) = S_1 S_2,$$

where $S_1 = \sigma_1 S\sigma_1^{-1} = \sigma_1 \cdot S$ and $S_2 = \sigma_2^{-1}S\sigma_2 = \sigma_2^{-1} \cdot S$ are pair-partitions. In the above calculation and whenever it does not lead to confusion we denote the composition of permutations simply by $\pi\sigma := \pi \circ \sigma$.

When σ_2 varies over $P_{2\lambda}$, pair-partition S_2 varies over all pair-partitions of the boxes of the Young diagram such that each pair of connected boxes lies in the same row of the Young diagram (we fixed the Young tableau T, so pair-partitions of the set $[2n]$ can be viewed as pair-partitions of the boxes of the Young diagram). Furthermore, each such a pair-partition is obtained for the same number of permutations σ_2 (that is for $2^n n!$ permutations). Thus replacing the summation over σ_2 by summation over pair-partitions S_2 for which each pair of connected boxes lies in the same row only changes the numerical constant.

Analogously, we can replace summation over σ_1 in $Q_{2\lambda}$ by summation over all pair-partitions S_1 with a property that the boxes belonging to each cycle of $S_1 \circ S$ are in one column. Indeed, if we fix a pair-partition S_1 verifying this condition, it can be written as $\sigma_1 \cdot S$ in $2^n n!$ different ways. Lemma 2.3 shows that the sign $(-1)^{\sigma_1}$ in each of these different writings is the same, equal to $(-1)^{\mathcal{L}(S,S_1)}$.

Finally, our zonal polynomial is equal to

$$Z_\lambda = C'_\lambda \sum_{S_1} \sum_{S_2} (-1)^{\mathcal{L}(S,S_1)} p_{\mathcal{L}(S_1,S_2)}, \tag{23}$$

where the sum runs over T-admissible (S_1, S_2). Recall that T-admissible means that S_2 preserves each row of T and $S \circ S_1$ preserves each columns.

The numerical value of $C'_\lambda = (2^n n!)^2 C_\lambda$ is easy to determine as, by definition,

$$[p_1^n] Z_\lambda = 1.$$

But the only pairs of T-admissible pair-partitions (S_1, S_2) such that $\mathcal{L}(S_1, S_2)$ is a union of n loops (the latter implies automatically that $S_1 = S_2$) is (S, S). Therefore the coefficient of p_1^n on the right-hand side of (23) is C'_λ, which must be equal to 1. □

3. FORMULAS FOR ZONAL CHARACTERS

This section is devoted to formulas for zonal characters; in particular we will prove Theorem 1.7.

3.1. Reformulation of Theorem 1.7.

Let S_0, S_1, S_2 be three pair-partitions of the set $[2k]$. We consider the following function on the set of Young diagrams:

Definition 3.1. $N^{(2)}_{S_0,S_1,S_2}(\lambda)$ is the number of functions f from $[2k]$ to the boxes of the Young diagram 2λ such that:

- (P0) $f(l)$ and $f(S_0(l))$ are neighbors in the Young diagram 2λ *i.e.*, if $f(l)$ is in the $2i+1$-th column (resp. $2i+2$-th column), $f(S_0(l))$ is the box in the same row but in the $2i+2$-th column (resp. $2i+1$-th column);
- (P1) $f(l)$ and $f(S_0 \circ S_1(l))$ are in the same column;
- (P2) $f(l)$ and $f(S_2(l))$ are in the same row.

Lemma 3.2. *Let S_0, S_1, S_2 be pair-partitions. Then*

$$N^{(2)}_{S_0,S_1,S_2} = 2^{|\mathcal{L}(S_0,S_1)|} N^{(1)}_{S_0,S_1,S_2}.$$

Proof. Let λ be a Young diagram and let f be a function $f : [2k] \to 2\lambda$ verifying properties (P0), (P1) and (P2). We consider the projection $p : 2\lambda \to \lambda$, which consists of forgetting the separations between the neighbors in 2λ. More precisely, the boxes $(2i-1,j)$ and $(2i,j)$ of 2λ are both sent to the box (i,j) of λ. It is easy to check that the composition $\overline{f} = p \circ f$ fulfills (Q0), (Q1), (Q2).

But a function $g : [2k] \to \lambda$ verifying (Q0), (Q1), (Q2) can be written as $\overline{f}$ for exactly $2^{|\mathcal{L}(S_0,S_1)|}$ different functions f. Indeed, we can choose independently, for one number i in each loop of $\mathcal{L}(S_0,S_1)$, which of the two possible values should be assigned to $f(i)$. The function f is then entirely determined by these choices because of properties (P0) and (P1). □

The above lemma shows that in order to show Theorem 1.7 it is enough to prove the following, equivalent statement:

Theorem 3.3. *Let μ be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of the set $[2k]$ whose union has type μ. Then one has the following equality between functions on the set of Young diagrams:*

$$\Sigma^{(2)}_{\mu} = \frac{1}{2^{\ell(\mu)}} \sum_{S_0} (-1)^{\mathcal{L}(S_0,S_1)} N^{(2)}_{S_0,S_1,S_2}, \tag{24}$$

where the sum runs over pair-partitions of $[2k]$.

We will prove it in Sections 3.2–3.4.

3.2. **Extraction of the coefficients.** Firstly, let us consider the case where $|\mu| = |\lambda|$. If we look at the coefficients of a given power-sum function p_μ in J_λ, using Theorem 1.2, one has:

$$[p_\mu]Z_\lambda = \sum_{\substack{(S_1,S_2)\ T\text{-admissible} \\ \text{type}\,\mathcal{L}(S_1,S_2)=\mu}} (-1)^{\mathcal{L}(S,S_1)}. \tag{25}$$

This equation has been proved in the case where T and S are, respectively, the canonical Young tableaux and the first pair-partition, but the same proof works for any filling T of 2λ by the elements of $[2|\lambda|]$ and pair-partition S as long as S matches the labels of the pairs of neighbors of 2λ. As they are $(2|\lambda|)!$ fillings T and one corresponding pair-partition $S = S(T)$ per filling, one has:

$$[p_\mu]Z_\lambda = \frac{1}{(2|\lambda|)!} \sum_T \sum_{\substack{(S_1,S_2)\ T\text{-admissible} \\ \text{type}\,\mathcal{L}(S_1,S_2)=\mu}} (-1)^{\mathcal{L}(S(T),S_1)}, \tag{26}$$

where the first sum runs over all bijective fillings of the diagram 2λ. We can change the order of summation and obtain:

$$[p_\mu]Z_\lambda = \frac{1}{(2|\lambda|)!} \sum_{\substack{S_1,S_2 \\ \text{type}(S_1,S_2)=\mu}} \left(\sum_T (-1)^{\mathcal{L}(S(T),S_1)} \ [(S_1,S_2) \text{ is } T\text{-admissible}] \right),$$

where $[\text{condition}]$ is equal to 1 if the condition is true and is equal to zero otherwise. The next step is to show that the expression in the parenthesis does not depend on (S_1, S_2). This is a consequence of the following lemma:

Lemma 3.4. *If we consider the diagonal action of $\mathfrak{S}_{2n}$ on couples of pair-partitions given by*

$$\sigma \cdot (S_1, S_2) = (\sigma \cdot S_1, \sigma \cdot S_2) = (\sigma S_1 \sigma^{-1}, \sigma S_2 \sigma^{-1}),$$

then the set of pairs of pair-partitions of a given type is exactly an orbit of this action.

Proof. Let us represent the pair (S_1, S_2) as a graph as explained in Section 1.2.1, except that edges corresponding to pairs in S_1 (resp. S_2) are colored in blue (resp. red). Then two pairs have the same type if and only if their graphs are isomorphic, which is equivalent to the existence of a permutation σ sending one couple to the other. □

The lemma also helps to count the number of couples of a given type μ. Indeed, the stabilizer of a couple (S_1, S_2) is easy to describe: σ can permute the loops of (S_1, S_2) which gives rise to the factor $m_1(\mu)! m_2(\mu)! \cdots$, where $m_i(\mu)$ is the multiplicity of the part i in μ. After we have fixed which loop is sent to which loop, we can choose the image of one element per loop (the images of the other elements of the loop are then uniquely determined). Finally, the cardinality of the stabilizer is

$$\prod_i m_i(\mu)! \cdot \prod_j (2\mu_j) =: 2^{\ell(\mu)} z_\mu.$$

So the number of couples (S_1, S_2) of type μ is equal to $(2|\lambda|)!/(2^{\ell(\mu)} z_\mu)$ and, if $|\mu| = |\lambda|$, one has:

$$\Sigma^{(2)}_\mu(\lambda) = z_\mu\,[p_\mu] Z_\lambda = \frac{1}{2^{\ell(\mu)}} \sum_T (-1)^{\mathcal{L}(S(T),S_1)}\,[(S_1,S_2) \text{ is } T\text{-admissible}],$$

where (S_1, S_2) is any fixed couple of pair-partitions of type μ.

This last equality can be rewritten in the following form. If $|\mu| = |\lambda|$, one has:

$$\Sigma^{(2)}_\mu(\lambda) = \frac{1}{2^{\ell(\mu)}} \sum_{S_0} (-1)^{\mathcal{L}(S_0,S_1)} \widehat{N}^{(2)}_{S_0,S_1,S_2}(\lambda), \tag{27}$$

where $\widehat{N}^{(2)}_{S_0,S_1,S_2}(\lambda)$ is the number of bijective fillings T of the Young diagram 2λ such that (S_1, S_2) is T-admissible and $S_0 = S(T)$. Note that this corresponds exactly to the conditions (P0), (P1) and (P2) in the case $k = n$.

3.3. **Extending the formula to any size.** Let us now look at the case where $|\mu| = k \le n = |\lambda|$. We denote $\widetilde{\mu} = \mu 1^{n-k}$. Then, using the formula above for $[p_{\widetilde{\mu}}] Z_\lambda$, one has:

$$\begin{aligned}\Sigma^{(2)}_\mu(\lambda) &= z_\mu \binom{n-k+m_1(\mu)}{m_1(\mu)} [p_{\widetilde{\mu}}] Z_\lambda \\ &= \frac{1}{2^{\ell(\mu)+n-k}\,(n-k)!} \sum_{\widetilde{S_0}} (-1)^{\mathcal{L}(\widetilde{S_0},\widetilde{S_1})} \widehat{N}^{(2)}_{\widetilde{S_0},\widetilde{S_1},\widetilde{S_2}}(\lambda),\end{aligned}$$

where $(\widetilde{S}_1, \widetilde{S}_2)$ is any fixed couple of pair-partitions of type $\widetilde{\mu}$. We can choose it in the following way. Let (S_1, S_2) be a couple of pair-partitions of the set $\{1, \dots, 2k\}$ of type μ and define $\widetilde{S}_1$ and $\widetilde{S}_2$ by, for $i = 1, 2$:

$$\widetilde{S}_i = S_i \cup \big\{\{2k+1, 2k+2\}, \dots, \{2n-1, 2n\}\big\}.$$

With this choice of $(\widetilde{S}_1, \widetilde{S}_2)$, it is quite obvious that $\widehat{N}^{(2)}_{\widetilde{S_0},\widetilde{S_1},\widetilde{S_2}}(\lambda) = 0$ unless

$$\widetilde{S}_0\big|_{\{2k+1,\dots,2n\}} = \big\{\{2k+1, 2k+2\}, \dots, \{2n-1, 2n\}\big\}. \tag{28}$$

Indeed, for any $l \ge k$ condition (P1) shows that $f(2l+1)$ and $f(S_0(2l+2))$ are in the same column; furthermore condition (P0) shows that $f(2l+2)$ and $f(S_0(2l+2))$ are neighbors and hence they are in the same row and condition (P2) shows that $f(2l+1)$ and $f(2l+2)$ are in the same row. In this way we proved that $f(2l+1)$ and $f(S_0(2l+2))$ are in the same row and column hence $f(2l+1) = f(S_0(2l+2))$ therefore $2l+1 = S_0(2l+2)$ which proves (28).

In the case where (28) is fulfilled, we denote $S_0 = \widetilde{S_0}\big|_{\{1,\dots,2k\}}$. The bijective fillings T counted in $\widehat{N}^{(2)}_{\widetilde{S_0},\widetilde{S_1},\widetilde{S_2}}(\lambda)$ are obtained as follows:

- the boxes corresponding to the numbers $1, 2, \dots, 2k$ are given by an injective function f verifying conditions (P0), (P1) and (P2) (with respect to (S_0, S_1, S_2));
- the only condition on the places of the numbers $2k+1, \dots, 2n$ is that numbers $2i-1$ and $2i$ (for $k < i \leq n$) must be in neighboring boxes of the diagram 2λ. Therefore there are $2^{n-k}(n-k)!$ ways to place these numbers in the remaining boxes.

Therefore

$$\widehat{N}^{(2)}_{\widetilde{S_0},\widetilde{S_1},\widetilde{S_2}}(\lambda) = 2^{n-k}(n-k)!\, \widehat{N}^{(2)}_{S_0,S_1,S_2}(\lambda),$$

where $\widehat{N}^{(2)}_{S_0,S_1,S_2}(\lambda)$ is the number of injective functions $f : [2k] \hookrightarrow 2\lambda$ verifying conditions (P0), (P1) and (P2) (in this definition, the ground set of S_0, S_1 and S_2 is $[2k]$). Notice that this definition of $\widehat{N}^{(2)}$ is an extension of the one given at the end of Section 3.2 which corresponds to the case where k is the size of λ (and thus the function f must be a bijective filling T).

The above discussion shows that

$$\Sigma^{(2)}_{\mu} = \frac{1}{2^{\ell(\mu)}} \sum_{\substack{S_0 \text{ pair-partition} \\ \text{of } \{1,\dots,2|\mu|\}}} (-1)^{\mathcal{L}(S_0,S_1)} \widehat{N}^{(2)}_{S_0,S_1,S_2}, \tag{29}$$

where (S_1, S_2) is any couple of pair-partitions of type μ.

3.4. **Forgetting injectivity.**

In this section we will prove Theorem 3.3 (and thus finish the proof of Theorem 1.7). In other terms, we prove that Eq. (29) is still true if we replace in each term of the sum $\widehat{N}^{(2)}_{S_0,S_1,S_2}$ by $N^{(2)}_{S_0,S_1,S_2}$. To do this, we have to check that, for any *non-injective* function $f : [2|\mu|] \to 2\lambda$, the total contribution

$$\sum_{\substack{S_0 \text{ pair-partition} \\ \text{of } [2|\mu|]}} (-1)^{\mathcal{L}(S_0,S_1)} \left[f \text{ fulfills (P0), (P1) and (P2)}\right] \tag{30}$$

of f to the right-hand side of Eq. (29) is equal to zero.

Let us fix a couple (S_1, S_2) of pair-partitions of type μ.

Lemma 3.5. *Let $f : [2|\mu|] \to 2\lambda$ with $f(i) = f(j)$. Then*

a) conditions (P0), (P1) and (P2) are fulfilled for S_0 if and only if they are fulfilled for $S_0' = (i\ j) \cdot S_0 = (i\ j)S_0(i\ j)$;

b) *if these conditions are fulfilled, then*

$$(-1)^{\mathcal{L}(S_0,S_1)} + (-1)^{\mathcal{L}(S_0',S_1)} = 0.$$

From the discussion above, it is clear that the lemma allows us to group the terms in (30) into canceling pairs and finishes the proof of Theorem 3.3.

Proof. Recall that S_0' is exactly the same pairing as S_0 except that i and j have been exchanged. Thus the part a) is obvious from the definitions.

For the part b) let us consider two different cases.

- If i and j are in different loops of the union $\mathcal{L}(S_0, S_1)$, then $\mathcal{L}(S_0', S_1)$ is obtained from $\mathcal{L}(S_0, S_1)$ by joining the loops containing i and j. Thus $\mathcal{L}(S_0', S_1)$ has one loop less than $\mathcal{L}(S_0, S_1)$ and the result follows.
- If i and j are in the same loop of the union $\mathcal{L}(S_0, S_1)$, note that they must also be in the same cycle of $S_0 \circ S_1$ (recall that each loop of $\mathcal{L}(S_0, S_1)$ is composed of two cycles of $S_0 \circ S_1$; conditions (P0), (P1) imply that one of these cycles is mapped by f to boxes in the even columns and the other is mapped to boxes in the odd columns of 2λ). In this case, $\mathcal{L}(S_0', S_1)$ is obtained from $\mathcal{L}(S_0, S_1)$ by cutting the loop containing i and j into two parts. Thus $\mathcal{L}(S_0', S_1)$ has one loop more than $\mathcal{L}(S_0, S_1)$ and the result follows. □

3.5. **Proof of Theorem 1.8.** In this paragraph we express the N functions in terms of Stanley's coordinates $\mathbf{p}$ and $\mathbf{q}$. This is quite easy and shows the equivalence between Theorems 1.7 and 1.8.

Function $f : [2k] \to \lambda$ can be alternatively viewed as a function on the set of edge-sides of the map M associated to pair-partitions S_0, S_1, S_2. Condition (Q0) implies that this function f is, in fact, well-defined as a function on the edges of the underlying graph G. It is easy to check that in this setup conditions (Q1) and (Q2) take the following equivalent form:

(Q1') If two edges have the same white extremity in G, their images by f are in the same column of λ.

(Q2') If two edges have the same black extremity in G, their images by f are in the same row of λ.

For a given bipartite graph G will denote by $N_G^{(1)}(\lambda)$ the number of the functions $f : E_G \to \lambda$ which fulfill conditions (Q1') and (Q2'). This definition was chosen in such a way that $N_G^{(1)} = N_{S_1,S_2,S_3}^{(1)}$ when G is the bipartite graph underlying the maps associated to pair-partitions S_1, S_2, S_3.

Lemma 3.6. *Let G be a bipartite graph, $V_\circ(G)$ (resp. $V_\bullet(G)$) its set of white (resp. black) vertices. One has:*

$$N_G^{(1)}(\mathbf{p} \times \mathbf{q}) = \sum_{\varphi: V_\bullet(G) \to \mathbb{N}^\star} \prod_{b \in V_\bullet(G)} p_{\varphi(b)} \prod_{w \in V_\circ(G)} q_{\psi(w)},$$

where $\psi(w) = \max_b \varphi(b)$, w running over b which are neighbors of w in G.

Proof. Let $g : E_G \to \lambda$ be a function verifying conditions (Q1') and (Q2'). As g fulfills (Q1'), all the edges leaving a vertex $b \in V_\bullet(G)$ have their image by g in the same row r_b. We define $\varphi(b)$ as the integer i such that

$$p_1 + \cdots + p_{i-1} < r_b \leq p_1 + \cdots + p_i.$$

This associates to g a function $\varphi : V_\bullet(G) \to \mathbb{N}^\star$. The number of pre-images of a given function φ can be computed as follows:

- we have to choose, for each black vertex b, the value of r_b. Due to the equation above, one has $p_{\varphi(b)}$ choices for each black vertex b;
- then we have to choose, for each white vertex w, the value of c_w, the common column of the images by g of the edges leaving w. This value can not be greater than $q_{\psi(w)}$, otherwise the image of the edge linking w with its black neighbor which maximizes φ would be outside the Young diagram λ. Finally, one has $q_{\psi(w)}$ choices for each white vertex w.
- we have no more choices as a function $g : E_G \to \lambda$ verifying (Q1') and (Q2') is uniquely determined by the two collection of numbers $(c_w)_{w \in V_\circ(G)}$ and $(r_b)_{c \in V_\bullet(G)}$. □

The above lemma shows that Theorem 1.7 implies Theorem 1.8.

4. Kerov polynomials

4.1. General formula for Kerov polynomials.

Our analysis of zonal Kerov polynomials will be based on the following general result.

Lemma 4.1. *Let $\mathcal{G}$ be a finite collection of connected bipartite graphs and let $\mathcal{G} \ni G \mapsto m_G$ be a scalar-valued function on it. We assume that*

$$F(\lambda) = \sum_{G \in \mathcal{G}} m_G N_G^{(1)}(\lambda)$$

is a polynomial function on the set of Young diagrams; in other words F can be expressed as a polynomial in free cumulants.

Let $s_2, s_3, \dots$ be a sequence of non-negative integers with only finitely many non-zero elements; then

$$[R_2^{s_2} R_3^{s_3} \cdots]\, F = (-1)^{s_2+s_3+\cdots+1} \sum_{G \in \mathcal{G}} \sum_q m_G,$$

where the sums runs over G and q such that:

- (b) *the number of the black vertices of G is equal to $s_2 + s_3 + \cdots$;*
- (c) *the number of all vertices of G is equal to $2s_2 + 3s_3 + 4s_4 + \cdots$;*
- (d) *q is a function from the set of the black vertices to the set $\{2, 3, \dots\}$; we require that each number $i \in \{2, 3, \dots\}$ is used exactly s_i times;*
- (e) *for every subset $A \subset V_\bullet(G)$ of black vertices of G which is nontrivial (i.e., $A \neq \emptyset$ and $A \neq V_\bullet(G)$) there are more than $\sum_{v\in A} \big(q(v) - 1\big)$ white vertices which are connected to at least one vertex from A.*

This result was proved in our previous paper with Dołęga [DFŚ10] in the special case when $F = \Sigma_n^{(1)}$ and $\mathcal{G}$ is the collection of bipartite maps corresponding to all factorizations of a cycle, however it is not difficult to verify that the proof presented there works without any modifications also in this more general setup.

4.2. **Proof of Theorem 1.12.**

Proof of Theorem 1.12. We consider for simplicity the case when $\mu = (k)$ has only one part. Theorem 1.7 can be rewritten in the form

$$F(\lambda) := \Sigma_k^{(2)} \left(\frac{1}{2}\lambda\right) = \frac{1}{2} \sum_{S_0} (-1)^{|\mu| + |\mathcal{L}(S_0,S_1)|} \; N^{(1)}_{S_0,S_1,S_2} \left(\frac{1}{2}\lambda\right).$$

Function F is a polynomial function on the set of Young diagrams [Las08]. Then the map corresponding to S_0, S_1, S_2 is connected and Lemma 4.1 can be applied. Since $s_1 + s_2 + \cdots$ is equal to the number of black vertices and $|\mathcal{L}(S_0, S_1)|$ is equal to the number of white vertices,

$$[R_2^{s_2} R_3^{s_3} \cdots]\, F = \frac{1}{2} (-1)^{1 + |\mu| + 2s_2 + 3s_3 + 4s_4 + \cdots} \sum_{S_0} \sum_{q} 1,$$

where the sum runs over S_0 and q such that the corresponding map M_{S_0,S_1,S_2} and q fulfill the assumptions of Lemma 4.1. Under a change of variables $\widetilde{\lambda} = \frac{1}{2}\lambda$ we have $\Sigma_\mu^{(2)}(\widetilde{\lambda}) = F(\lambda)$ and $R_i = R_i(\lambda) = R_i^{(2)}(\widetilde{\lambda})$ which finishes the proof.

Consider now the general case $\mu = (k_1, \dots, k_\ell)$. In an analogous way as in [DFŚ10] one can show that $\kappa^{\text{id}}(\Sigma_{k_1}^{(\alpha)}, \dots, \Sigma_{k_\ell}^{(\alpha)})$ is equal to the right-hand side of (8), where S_1, S_2 are chosen so that $\operatorname{type}(S_1, S_2) = \mu$ and the summation runs over S_0 with a property that the corresponding map

M_{S_0,S_1,S_2} is connected. Therefore

$$F(\lambda) := (-1)^{\ell-1}\kappa^{\text{id}}(\Sigma^{(\alpha)}_{k_1},\dots,\Sigma^{(\alpha)}_{k_\ell})\left(\frac{1}{2}\lambda\right) = \frac{1}{2^{\ell(\mu)}}(-1)^{\ell-1}\sum_{S_0}(-1)^{|\mu|+|\mathcal{L}(S_0,S_1)|}\ N^{(1)}_{S_0,S_1,S_2}\left(\frac{1}{2}\lambda\right).$$

The remaining part of the proof follows in an analogous way. □

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