

# LIMIT AND EXTENDED LIMIT SETS OF MATRICES IN JORDAN NORMAL FORM

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ABSTRACT. In this note we describe the limit and the extended limit sets of every vector for a single matrix in Jordan normal form.

## 1. PRELIMINARIES AND BASIC NOTIONS

Limit and extended limit sets are in the center of interest in the study of dynamics of linear operators. To find them, even in relatively easy cases of operators, it is a difficult task. In this note we describe the limit and the extended limit sets for the simplest case which is the case of a single matrix in Jordan normal form. We use a method similar to the one used by N. H. Kuiper and J. W. Robbin in [6]. In this work Kuiper and Robbin dealt with the problem of the topological classification of linear endomorphisms and the main tool they used was the extended mixing limit sets of the exponential of the nilpotent part of a Jordan block. In the following we introduce the basic notions we use in the present work.

Let  $X$  be a complex Banach space and let  $T : X \rightarrow X$  be a bounded linear operator.

**Definition 1.1.** For every  $x \in X$  the sets

$$L(x) = \{y \in X : \text{there exists a strictly increasing sequence} \\ \text{of positive integers } \{k_n\} \text{ such that } T^{k_n}x \rightarrow y\}$$

$$J(x) = \{y \in X : \text{there exist a strictly increasing sequence of positive} \\ \text{integers } \{k_n\} \text{ and a sequence } \{x_n\} \subset X \text{ such that } x_n \rightarrow x \text{ and} \\ T^{k_n}x_n \rightarrow y\}$$

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and

$$J^{mix}(x) = \{y \in X : \text{there exists a sequence } \{x_n\} \subset X \text{ such that} \\ x_n \rightarrow x \text{ and } T^n x_n \rightarrow y\}$$

denote the limit set, the extended (prolongational) and the extended mixing limit set of  $x$  under  $T$  respectively.

The notions of the limit and extended limit sets are well known in the theory of topological dynamics, see [2]. Roughly speaking, we can say that the limit set of a vector  $x$  describes the limiting behavior of its orbit and the corresponding extended limit set  $J(x)$  describes the asymptotic behavior of all vectors nearby  $x$ . Let us see how these notions are connected, at a very first, naive level. To explain, we recall the following definition. A bounded linear operator  $T : X \rightarrow X$  acting on a complex separable Banach space  $X$  is called *hypercyclic* if there exists a vector  $x \in X$  so that the orbit of  $x$  under  $T$ , i.e. the set  $\{x, Tx, T^2x, \dots\}$ , is dense in  $X$ . A little thought shows that the orbit of  $x$  under  $T$  is dense if and only if  $L(x) = X$  and that  $T$  is hypercyclic if and only if  $J(x) = X$  for every  $x \in X$ , see [3]. For several examples of hypercyclic operators and an in depth study of several aspects of the notion of hypercyclicity we refer to the recent books [1], [5]. In [3], we localized the notion of hypercyclicity through the use of  $J$ -sets. To recall briefly, we say that a bounded linear operator  $T$  acting on a Banach space (not necessarily separable)  $X$  is locally hypercyclic or  $J$ -class if there exists a non-zero vector  $x \in X$  so that  $J(x) = X$ . Among other things we showed that there are locally hypercyclic, non-hypercyclic operators and that finite dimensional Banach spaces do not admit locally hypercyclic operators. The next proposition, which also appears in [4], gives a description of  $J$ -sets through the use of open sets. To keep the paper shelf contained we present the proof.

**Proposition 1.2.** *An equivalent definition for the sets  $J(x)$ ,  $J^{mix}(x)$  is the following.*

$$J(x) = \{y \in X : \text{for every pair of neighborhoods } U, V \text{ of } x, y \\ \text{respectively, there exists a positive integer } n, \\ \text{such that } T^n U \cap V \neq \emptyset\}.$$

and

$$J^{mix}(x) = \{y \in X : \text{for every pair of neighborhoods } U, V \text{ of } x, y \\ \text{respectively, there exists a positive integer } N, \\ \text{such that } T^n U \cap V \neq \emptyset \text{ for every } n \geq N\}.$$

*Proof.* We give the proof for the  $J^{mix}$ -sets, the proof for the  $J$ -sets is similar.

Let us prove that

$$J^{mix}(x) \supset \{y \in X : \text{for every pair of neighborhoods } U, V \text{ of } x, y \\ \text{respectively, there exists a positive integer } N, \\ \text{such that } T^n U \cap V \neq \emptyset \text{ for every } n \geq N\}.$$

since the converse inclusion is obvious. Fix a vector

$$y \in \{y \in X : \text{for every pair of neighborhoods } U, V \text{ of } x, y \\ \text{respectively, there exists a positive integer } N, \\ \text{such that } T^n U \cap V \neq \emptyset \text{ for every } n \geq N\}.$$

Consider the open balls  $B(x, 1/n), B(y, 1/n)$  for  $n = 1, 2, \dots$ . Then there exists a strictly increasing sequence  $\{k_n\}$  of positive integers such that  $T^m B(x, 1/n) \cap B(y, 1/n) \neq \emptyset$  for every  $m \geq k_n, n = 1, 2, \dots$ . Therefore there exist  $x_{k_1}, x_{k_1+1}, \dots, x_{k_2-1} \in B(x, 1)$  such that  $\|T^m x_m - y\| < 1$  for every  $m = k_1, k_1 + 1, \dots, k_2 - 1$ . In a similar fashion we may find  $x_{k_2}, x_{k_2+1}, \dots, x_{k_3-1} \in B(x, 1/2)$  such that  $\|T^m x_m - y\| < 1/2$  for every  $m = k_2, k_2 + 1, \dots, k_3 - 1$ . Proceeding inductively we find a sequence  $\{x_n\}, n \geq k_1$  such that  $x_n \rightarrow x$  and  $T^n x_n \rightarrow y$ . This completes the proof.  $\square$

The above proposition will not be used in the sequel. The reason we included is that it gives a better understanding of how the  $J$ -sets behave. Limit sets, extended limit sets and extended mixing limit sets are closed and invariant [4]. Next proposition will be used later to simplify proofs.

**Proposition 1.3.** *Let  $T : X \rightarrow X$  be a bounded linear operator. Then  $J_{\lambda T}(0) = J_T(0)$  for every  $|\lambda| = 1$ .*

*Proof.* Let  $y \in J_{\lambda T}(0)$ . Then there exist a strictly increasing sequence of positive integers  $\{k_n\}$  and a sequence  $\{x_n\} \subset X$  such that  $x_n \rightarrow 0$  and  $\lambda^{k_n} T^{k_n} x_n \rightarrow y$ . Since  $|\lambda| = 1$  then  $\lambda^{k_n} x_n \rightarrow 0$  and since  $T^{k_n}(\lambda^{k_n} x_n) \rightarrow y$  it follows that  $y \in J_T(0)$ . Take now a vector  $y \in J_T(0)$ . Then there exist a strictly increasing sequence of positive integers  $\{k_n\}$  and a sequence  $\{x_n\} \subset X$  such that  $x_n \rightarrow 0$  and  $T^{k_n} x_n \rightarrow y$ . Since  $|\lambda| = 1$ , without loss of generality we may assume that  $\lambda^{k_n} \rightarrow \mu$  for some  $|\mu| = 1$ . Hence  $\lambda^{k_n} T^{k_n} \frac{x_n}{\mu} \rightarrow y$  and since  $\frac{x_n}{\mu} \rightarrow 0$  then  $y \in J_{\lambda T}(0)$ .  $\square$

## 2. LIMIT AND EXTENDED LIMIT SETS OF A MATRIX IN JORDAN NORMAL FORM

We mainly focus in the case  $A$  is a  $l \times l$  Jordan block over  $\mathbb{C}$ . This means that the main diagonal consists of  $\lambda$ 's, for some  $\lambda \in \mathbb{C}$ , the diagonal above the main diagonal consists of 1's and all the other entries of the matrix are filled with zeros. We shall then describe the limit and extended limit sets of every  $x \in \mathbb{C}^l$  under  $A$ . The general case of a matrix in a Jordan canonical form follows easily from the latter case, since we can "glue" the limit and extended limit sets of separate Jordan blocks. Finally, since every complex matrix  $B$  is similar to a matrix in Jordan canonical form, we are able to determine the limit and extended limit sets of every  $x \in \mathbb{C}^l$  under  $B$ . For the rest of this section  $A$  will be a  $l \times l$  Jordan block over  $\mathbb{C}$ .

### Proposition 2.1.

- (i) *If  $A$  is a Jordan block with an eigenvalue  $|\lambda| = 1$  then  $L(x) \neq \emptyset$  if and only if  $x_2, \dots, x_l = 0$ . In this case  $L(x) = \{Dx_1\} \times 0$ , where  $D$  is the closure of the set  $\{\lambda^n\}$ .*
- (ii) *If  $A$  is a Jordan block with an eigenvalue  $|\lambda| > 1$  then  $L(x) = \emptyset$  for every non-zero vector  $x \in \mathbb{C}^l$ .*
- (iii) *If  $A$  is a Jordan block with an eigenvalue  $|\lambda| < 1$  then  $L(x) = \{0\}$  for every  $x \in \mathbb{C}^l$ .*

*Proof.* (i) We give the proof for the case  $l = 3$ . The general case follows easily. Let  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3) \in L(x)$ . Then there exists a strictly increasing sequence of positive integers  $\{k_n\}$  such that  $A^{k_n}x \rightarrow y$ . Setting  $y_n = (y_{n1}, y_{n2}, y_{n3}) = A^{k_n}x$  we get

$$\begin{aligned} y_{n1} &= \lambda^{k_n}x_1 + k_n\lambda^{k_n-1}x_2 + \frac{k_n(k_n-1)}{2}\lambda^{k_n-2}x_3 \\ y_{n2} &= \lambda^{k_n}x_2 + k_n\lambda^{k_n-1}x_3 \\ y_{n3} &= \lambda^{k_n}x_3. \end{aligned}$$

Using the second equation we conclude that  $x_3 = y_3 = 0$  since the sequences  $\{y_{n2}\}$  and  $\{\lambda^{k_n}x_2\}$  are bounded. So we have the following system of linear equations.

$$\begin{aligned} y_{n1} &= \lambda^{k_n}x_1 + k_n\lambda^{k_n-1}x_2 \\ y_{n2} &= \lambda^{k_n}x_2. \end{aligned}$$

Using the same argument for  $x_2$  we have  $x_2 = y_2 = 0$ . Hence, the only remaining equation is  $y_{n1} = \lambda^{k_n}x_1$  and the proof of the proposition is completed.

The proof of items (i) and (ii) follows easily and is omitted.  $\square$

In the following we describe the extended limit sets of the zero-vector.

**Theorem 2.2.** *Let  $A$  be a Jordan block with an eigenvalue  $|\lambda| = 1$ .*

- (i) *If  $l = 1$  then  $J(0) = \{0\}$ . If  $l > 1$  and  $l$  is of the form  $l = 2r$  or  $l = 2r - 1$  then  $J(0) = \mathbb{C}^r \times 0$ .*
- (ii) *A point  $y \in J(0)$  if and only if there exists a sequence  $\{x_n\} \subset \mathbb{C}^l$  such that  $A^n x_n \rightarrow y$ , hence  $J(0) = J^{\text{mix}}(0)$ .*
- (iii) *For every linear map  $B : \mathbb{C}^l \rightarrow \mathbb{C}^l$  with eigenvalues of modulus 1 the set  $J(0)$  is a proper linear subspace of  $\mathbb{C}^l$ .*

*Proof.* We give the proof for the cases  $l = 3$  and  $l = 4$ . For the general case we may use the same technics as in [6]. Since, by Proposition 1.3,  $J_{\lambda A}(0) = J_A(0)$  for every  $\lambda$  of modulus 1 we may assume that  $\lambda = 1$ .

*Case  $l = 3$ :* Let  $x_n = (x_{n1}, x_{n2}, x_{n3}) \rightarrow (0, 0, 0)$  and  $A^{k_n} x_n \rightarrow y = (y_1, y_2, y_3)$ . If  $y_n = (y_{n1}, y_{n2}, y_{n3}) = A^{k_n} x_n$ , then

$$\begin{aligned} y_{n1} &= x_{n1} + k_n x_{n2} + \frac{k_n(k_n-1)}{2} x_{n3} \\ y_{n2} &= x_{n2} + k_n x_{n3} \\ y_{n3} &= x_{n3}. \end{aligned}$$

Since  $x_{n3} \rightarrow 0$  it follows that  $y_{n3} \rightarrow 0$ , hence  $y_3 = 0$ . Using the second equation we have  $x_{n3} = \frac{y_{n2} - x_{n2}}{k_n}$ . Therefore dividing the first equation by  $k_n$  and substitute  $x_{n3}$  we have

$$\frac{y_{n1}}{k_n} = \frac{x_{n1}}{k_n} + x_{n2} + \frac{k_n - 1}{2k_n} (y_{n2} - x_{n2})$$

Since  $\frac{y_{n1}}{k_n}$ ,  $\frac{x_{n1}}{k_n}$  and  $x_{n2}$  have limit 0 and  $\frac{k_n-1}{2k_n} \rightarrow \frac{1}{2}$  then  $y_{n2} - x_{n2} \rightarrow 0$ . Since  $x_{n2} \rightarrow 0$  it follows that  $y_{n2} \rightarrow 0$ , therefore  $y_2 = 0$ . Till now we have proved that  $J(0) \subset \mathbb{C} \times 0$ . Next we show the inverse inclusion. Let  $y = (y_1, 0, 0)$ . We put  $x_{n1} = 0$ ,  $x_{n2} = 0$ ,  $y_{n1} = y_1$  and then we solve the system of the linear equations. So, we have

$$\begin{aligned} x_{n1} &= 0, & x_{n2} &= 0, & x_{n3} &= \frac{2y_1}{n(n-1)} \\ y_{n1} &= y_1, & y_{n2} &= \frac{2y_1}{n-1}, & y_{n3} &= \frac{2y_1}{n(n-1)}. \end{aligned}$$

Now it is easy to check that  $x_n \rightarrow 0$  and  $y_n \rightarrow y$ . Note that we have also proved (ii).

*Case  $l = 4$ :* Let  $x_n = (x_{n1}, x_{n2}, x_{n3}, x_{n4}) \rightarrow (0, 0, 0, 0)$  and  $A^{k_n} x_n \rightarrow y$ . Setting  $y = (y_1, y_2, y_3, y_4)$  and  $y_n = (y_{n1}, y_{n2}, y_{n3}, y_{n4}) = A^{k_n} x_n$  we get

$$\begin{aligned} y_{n1} &= x_{n1} + k_n x_{n2} + \frac{k_n(k_n-1)}{2} x_{n3} + \frac{k_n(k_n-1)(k_n-2)}{6} x_{n4} \\ y_{n2} &= x_{n2} + k_n x_{n3} + \frac{k_n(k_n-1)}{2} x_{n4} \\ y_{n3} &= x_{n3} + k_n x_{n4} \\ y_{n4} &= x_{n4}. \end{aligned}$$

Observe that the last three equations are exactly the same as in the case where  $l = 3$  hence  $y_3 = y_4 = 0$ . Therefore  $J(0) \subset \mathbb{C}^2 \times 0$ . Next

we show the inverse inclusion. Let  $y = (y_1, y_2, 0, 0)$ . We put  $x_{n1} = 0$ ,  $x_{n2} = 0$ ,  $y_{n1} = y_1$ ,  $y_{n2} = y_2$  and then we solve the system of the linear equations. So, we have

$$\begin{aligned} x_{n1} &= 0, & x_{n2} &= 0 \\ x_{n3} &= \frac{2(3y_1 - (n-2)y_2)}{n(n+1)}, & x_{n4} &= \frac{2(3(n-1)y_2 - 6y_1)}{n(n-1)(n+1)} \\ y_{n1} &= y_1, & y_{n2} &= y_2 \\ y_{n3} &= \frac{2(3y_1 - (n-2)y_2)}{n(n+1)} + k_n \frac{2(3(n-1)y_2 - 6y_1)}{n(n-1)(n+1)}, & y_{n4} &= \frac{2(3(n-1)y_2 - 6y_1)}{n(n-1)(n+1)}. \end{aligned}$$

Now it is easy to check that  $x_n \rightarrow 0$  and  $y_n \rightarrow y$ . Again we have proved simultaneously (ii). Item (iii) is obtained by (i) and (ii), since we can glue the  $J(0)$ -sets of the Jordan blocks of  $B$ .  $\square$

**Theorem 2.3.** *Let  $A$  be a Jordan block with an eigenvalue  $|\lambda| > 1$ . Then the following hold.*

- (i)  $J(0) = \mathbb{C}^l$ .
- (ii) For every point  $y \in \mathbb{C}^l$  there exists a sequence  $\{x_n\} \subset \mathbb{C}^l$  such that  $A^n x_n \rightarrow y$ , hence  $J(0) = J^{mix}(0)$ .
- (iii) For every linear map  $B : \mathbb{C}^l \rightarrow \mathbb{C}^l$  with eigenvalues of modulus greater than 1 it holds that  $J(0) = \mathbb{C}^l$ .

*Proof.* We prove the theorem for the case  $l = 3$ . Let  $y = (y_1, y_2, y_3)$  and  $y_n = (y_{n1}, y_{n2}, y_{n3}) = A^n x_n$ . We put  $y_{n1} = y_1$ ,  $y_{n2} = y_2$ ,  $y_{n3} = y_3$  and we solve again the corresponding system of the linear equations:

$$\begin{aligned} y_{n1} &= \lambda^n x_{n1} + n\lambda^{n-1}x_{n2} + \frac{n(n-1)}{2}\lambda^{n-2}x_{n3} \\ y_{n2} &= \lambda^n x_{n2} + n\lambda^{n-1}x_{n3} \\ y_{n3} &= \lambda^n x_{n3}. \end{aligned}$$

Hence, we have

$$\begin{aligned} x_{n1} &= \frac{2\lambda^2 y_1 - n2\lambda y_2 + n(n+1)y_3}{2\lambda^{n+2}} \\ x_{n2} &= \frac{\lambda y_2 - n y_3}{\lambda^{n+1}} \\ x_{n3} &= \frac{y_{n3}}{\lambda^n}. \end{aligned}$$

Now it is trivial to check that  $x_n \rightarrow 0$  and  $y_n \rightarrow y$ . Note that item (iii) follows by (i) and (ii).  $\square$

**Proposition 2.4.** *Let  $A$  be a Jordan block with an eigenvalue  $|\lambda| = 1$ . Then the following hold.*

- (i)  $J(0) = \{0\}$ .
- (ii) For every linear map  $B : \mathbb{C}^l \rightarrow \mathbb{C}^l$  with eigenvalues of modulus less than 1 it holds that  $J(0) = \{0\}$ .

*Proof.* The proof is trivial and it is omitted.  $\square$

Below we treat the general case.

**Theorem 2.5.** *Let  $A$  be a Jordan block with an eigenvalue  $|\lambda| = 1$ .*

- (i) *If  $l = 2r$  then  $J(x) \neq \emptyset$  if and only if  $x_{r+1} = \dots = x_m = 0$ . In this case  $J(x) = \mathbb{C}^r \times 0$ .*
- (ii) *If  $l = 2r - 1$  then  $J(x) \neq \emptyset$  if and only if  $x_{r+1} = \dots = x_m = 0$ . In this case  $J(x) = \mathbb{C}^{r-1} \times \{Dx_2\} \times 0$ , where  $D$  is the closure of the set  $\{\lambda^n\}$ . In case where  $l = 1$ ,  $J(x) = \{Dx\}$ .*

*Proof.* We only consider the case  $l = 3$ . Let  $x = (x_1, x_2, x_3)$ ,  $z_n = (x_{n1}, x_{n2}, x_{n3}) \rightarrow (x_1, x_2, x_3)$  and  $A^{k_n} z_n \rightarrow y$ . Setting  $y = (y_1, y_2, y_3)$  and  $y_n = (y_{n1}, y_{n2}, y_{n3}) = A^{k_n} z_n$  we get

$$\begin{aligned} y_{n1} &= \lambda^{k_n} x_{n1} + k_n \lambda^{k_n-1} x_{n2} + \frac{k_n(k_n-1)}{2} \lambda^{k_n-2} x_{n3} \\ y_{n2} &= \lambda^{k_n} x_{n2} + k_n \lambda^{k_n-1} x_{n3} \\ y_{n3} &= \lambda^{k_n} x_{n3}. \end{aligned}$$

Using the second equation we conclude that  $x_{n3} \rightarrow 0$  since the sequences  $\{y_{n2}\}$  and  $\{\lambda^{k_n} x_{n2}\}$  are bounded. The last implies that  $x_3 = 0$ . Since

$$\frac{y_{n1}}{k_n} = \frac{\lambda^{k_n} x_{n1}}{k_n} + \lambda^{k_n-1} x_{n2} + \frac{(k_n-1)}{2} \lambda^{k_n-2} x_{n3}$$

and  $\frac{y_{n1}}{k_n} \rightarrow 0$ ,  $\frac{\lambda^{k_n} x_{n1}}{k_n} \rightarrow 0$  we obtain the following

$$\lambda^{k_n-1} x_{n2} + \frac{(k_n-1)}{2} \lambda^{k_n-2} x_{n3} \rightarrow 0.$$

Solving the second equation with respect to  $x_{n3}$  and substitute above, we arrive at

$$2\lambda^{k_n-1} x_{n2} + (k_n-1) \frac{y_{n2} - \lambda^{k_n-2} x_{n2}}{\lambda k_n} \rightarrow 0,$$

or equivalently

$$2\lambda^{k_n-1} x_{n2} + \frac{k_n-1}{\lambda k_n} y_{n2} - \frac{k_n-1}{k_n} \lambda^{k_n-1} x_{n2} \rightarrow 0.$$

Since  $\frac{k_n-1}{\lambda k_n} y_{n2} \rightarrow \frac{1}{\lambda} y_2$ , without loss of generality we may assume that  $\lambda^{k_n-1} \rightarrow \mu$  for some  $\mu$  of modulus 1. Then every  $y = (y_1, y_2, y_3) \in J(x)$  satisfies the following

$$y_2 + \lambda \mu x_2 = 0.$$

That is  $y_2 \in \{Dx_2\}$ . Hence,  $J(x) \subset \mathbb{C} \times \{Dx_2\} \times 0$ . Next we show the inverse inclusion. Let  $y = (y_1, y_2, 0)$  where  $y_2 \in \{Dx_2\}$ . Hence there exists a sequence of the form  $\{\lambda^{k_n}\}$  such that  $\lambda^{k_n-1} \rightarrow -\mu$  for some  $\mu$  of modulus 1 and  $\lambda^{k_n} x_2 \rightarrow y_2$ . Hence,  $y_2 + \lambda \mu x_2 = 0$ . Setting  $y_{n1} = y_1$ ,

$x_{n1} = x_1$ ,  $x_{n2} = x_2$  we solve again the corresponding system of the linear equations. Therefore

$$\begin{aligned} x_{n3} &= (y_1 - \lambda^{k_n} x_1 - k_n \lambda^{k_n-1} x_2) \frac{2}{k_n(k_n-1)\lambda^{k_n-2}} \rightarrow 0 \\ y_{n2} &= \lambda^{k_n} x_2 + \frac{2\lambda}{k_n-1} (y_1 - \lambda^{k_n} x_1) - \frac{2\lambda k_n}{k_n-1} \lambda^{k_n-1} x_2 \rightarrow -\lambda \mu x_2 = y_2 \\ y_{n3} &= \lambda^{k_n} x_{n3} \rightarrow 0 \end{aligned}$$

since  $x_{n3} \rightarrow x_3 = 0$  and this finishes the proof of the theorem.  $\square$

**Proposition 2.6.** *Let  $A$  be a Jordan block with an eigenvalue  $|\lambda| > 1$ .*

- (i) *If  $x \neq 0$  then  $J(x) = \emptyset$ .*
- (ii) *If  $x = 0$  then  $J(0) = \mathbb{C}^l$ .*

*Proof.* We give the proof for the case  $l = 3$ . Since  $y_{n3} = \lambda^{k_n} x_{n3} \rightarrow y_3$  then  $k_n(k_n - 1)x_{n3} \rightarrow 0$ . From  $y_{n2} \rightarrow y_2$  we get

$$\frac{y_{n2}}{k_n} = \frac{\lambda^{k_n}}{k_n^2} k_n x_{n2} + \lambda^{k_n-1} x_{n3} \rightarrow 0.$$

Using the fact that  $\lambda^{k_n} x_{n3} \rightarrow y_3$  it follows that the sequence  $\{\frac{\lambda^{k_n}}{k_n^2} k_n x_{n2}\}$  converges to a finite complex number, hence  $k_n x_{n2} \rightarrow 0$ . The last implies  $x_{n2} \rightarrow 0$ , therefore  $x_2 = 0$ . We have

$$x_{n1} = \frac{y_{n1}}{\lambda^{k_n}} - \frac{1}{\lambda} k_n x_{n2} - \frac{1}{2} \lambda^2 k_n (k_n - 1) x_{n3}.$$

Observing that each one term on the right hand side in the previous equality goes to 0 ( $y_{n3} \rightarrow y_3$ ) we arrive at  $x_1 = 0$ . Therefore  $x = 0$ . Hence by Theorem 2.3,  $J(0) = \mathbb{C}^3$ .  $\square$

**Proposition 2.7.** *If  $A$  is a Jordan block with an eigenvalue  $|\lambda| < 1$  then  $J(x) = \{0\}$  for every  $x \in \mathbb{C}^l$ .*

*Proof.* This is clear from the form of the equations

$$\begin{aligned} y_{n1} &= \lambda^{k_n} x_{n1} + k_n \lambda^{k_n-1} x_{n2} + \frac{k_n(k_n-1)}{2} \lambda^{k_n-2} x_{n3} \\ y_{n2} &= \lambda^{k_n} x_{n2} + k_n \lambda^{k_n-1} x_{n3} \\ y_{n3} &= \lambda^{k_n} x_{n3}. \end{aligned}$$

$\square$

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