SOME FREE ALGEBRAS OF AUTOMORPHIC FORMS ON SYMMETRIC DOMAINS OF TYPE IV

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Abstract. Some arithmetic quotients of symmetric domains of type IV can be interpreted as moduli varieties of multipolarized $K3$ surfaces. Making use of this interpretation, we prove that, for $n = 3, 4, 5, 6, 7$, some natural algebras of automorphic forms on the n-dimensional symmetric domains of type IV are free, and find the degrees of their generators. This implies that the corresponding arithmetic groups are generated by complex reflections.

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1. Introduction

Let $\mathbb{R}^{2,n}$ denote the pseudo-Euclidean vector space of signature $(2, n)$ with inner product

$$(x, y) = x_1y_1 + x_2y_2 - x_3y_3 - \cdots - x_{n+2}y_{n+2}.$$

Set $\mathbb{C}^{2,n} = \mathbb{R}^{2,n} \otimes \mathbb{C}$ and consider the cone

$$\tilde{\mathcal{L}}_n = \{z \in \mathbb{C}^{2,n} : (z, z) = 0,\ (z, \bar{z}) > 0\}.$$

Its elements can be interpreted as pairs $x, y \in R^{2,n}$ with

$$(x, x) = (y, y) > 0,\ (x, y) = 0.$$

The cone $\tilde{\mathcal{L}}_n$ has two complex conjugate connected components. Choose one of them and denote it by $\mathcal{L}_n$. The projectivization of $\mathcal{L}_n$ is a model of the n-dimensional symmetric domain of type IV; we shall denote it by $\mathcal{D}_n$.

The pseudoorthogonal group $O_{2,n}$ acting on $\mathbb{C}^{2,n}$ leaves the cone $\tilde{\mathcal{L}}_n$ invariant. Let $O^+_{2,n} \subset O_{2,n}$ denote the subgroup of index 2 leaving invariant each of the connected components of $\tilde{\mathcal{L}}_n$. It acts on $\mathcal{D}_n$ transitively, and via this action $\mathcal{D}_n$ is identified with the Hermitian symmetric space $O^+_{2,n}/(SO_2 \times O_n)$. The natural holomorphic $\mathbb{C}^*$-bundle

$$\pi : \mathcal{L}_n \to \mathcal{D}_n.$$

is obviously $O^+_{2,n}$-equivariant. Denote by $\bar{\pi}$ the line bundle obtained from π by filling in the zero section. Note that the n-th power of $\bar{\pi}$ is the canonical bundle of $\mathcal{D}_n$ (as an $O^+_{2,n}$-equivariant line bundle).

Let Γ be a lattice (a discrete subgroup of finite covolume) in $O^+_{2,n}$. For any non-negative integer k, consider Γ-invariant holomorphic sections of the k-th power of the above line bundle $\bar{\pi}$ or, equivalently, Γ-invariant holomorphic functions on

$\mathcal{L}_n$ that are homogeneous of degree $-k$ on each fiber. For $n \geq 3$, they constitute a finite-dimensional vector space $A(\mathcal{D}_n, \Gamma)_k$. The algebra

$$A(\mathcal{D}_n, \Gamma) = \bigoplus_{k=0}^{\infty} A(\mathcal{D}_n, \Gamma)_k$$

is called the *natural algebra of automorphic forms* on $\mathcal{D}_n$ with respect to Γ, and the elements of $A(\mathcal{D}_n, \Gamma)_k$ are called *automorphic forms of weight* k. (For $n = 1, 2$ the definition is similar, but one should require some good behaviour of automorphic forms near the boundary of $\mathcal{L}_n$.)

It is known that $A(\mathcal{D}_n, \Gamma)$ is a normal finitely generated graded algebra (with $A(\mathcal{D}_n, \Gamma)_0 = \mathbb{C}$). The affine variety $\operatorname{Spec} A(\mathcal{D}_n, \Gamma)$ contains the analytic quotient $\mathcal{L}_n/\Gamma$ as a Zariski open subset with at most 2-dimensional boundary. The projective variety $\operatorname{Proj} A(\mathcal{D}_n, \Gamma)$ is the Satake–Baily–Borel compactification of $\mathcal{D}_n/\Gamma$ [2].

Let now $\Gamma_n = O^+_{2,n}(\mathbb{Z})$ be the lattice in $O^+_{2,n}$ consisting of all matrices with integer entries. In 1962 J.-I. Igusa [7] proved that, for a certain subgroup $\Gamma \subset O^+_{2,3}$ commensurable with Γ_3, the algebra $A(\mathcal{D}_3, \Gamma)$ is freely generated by forms of weights $4, 6, 10, 12$. The main result of the present paper is the following theorem.

Theorem 1. *For $n \in \{4, 5, 6, 7\}$, the algebra $A(\mathcal{D}_n, \Gamma_n)$ is freely generated by forms of the weights indicated in the following table:*

n	Weights
4	$4, 6, 8, 10, 12$
5	$4, 6, 8, 10, 12, 18$
6	$4, 6, 8, 10, 12, 16, 18$
7	$4, 6, 8, 10, 12, 14, 16, 18$

Corollary. *For $n \in \{4, 5, 6, 7\}$, the quotient $\mathcal{L}_n/\Gamma_n$ is a simply connected complex manifold.*

This means that the group Γ_n acting on $\mathcal{L}_n$, as well as the stabiilizer of each point of $\mathcal{L}_n$, is generated by (complex) reflections. The corresponding group of holomorphic transformations of $\mathcal{D}_n$ is, of course, also generated by reflections. However, the stabilizer of a point of $\mathcal{D}_n$ does not need to be generated by reflections (but it always contains a normal reflection subgroup with a cyclic factorgroup).

In fact we also describe the structure of the algebras $A(\mathcal{D}_n, \Gamma)$ for some groups Γ commensurable with Γ_n (see Theorems 3, 4). Besides, we prove that the algebra $A(\mathcal{D}_3, \Gamma_3)$ is not free but, for a certain index 2 subgroup $\Gamma_3^r \subset \Gamma_3$, the algebra $A(\mathcal{D}_3, \Gamma_3^r)$ is free, with generators of weights $4, 4, 6, 6$ (see Theorems 5, 6).

For any $n \geq 3$, the natural embeddings $\mathcal{L}_n \hookrightarrow \mathcal{L}_{n+1}$ and $\Gamma_n \hookrightarrow \Gamma_{n+1}$ induce a closed embedding

$$\mathcal{D}_n/\Gamma_n \hookrightarrow \mathcal{D}_{n+1}/\Gamma_{n+1}$$

and thereby an epimorphism of graded algebras

$$A(\mathcal{D}_{n+1}, \Gamma_{n+1}) \to A(\mathcal{D}_n, \Gamma_n)$$

(see Theorem 2). It follows from Theorem 1 that, for $n = 4, 5, 6$, the kernel of the latter epimorphism is the principal ideal in $A(\mathcal{D}_{n+1}, \Gamma_{n+1})$ generated by an algebra generator of weight $18, 16, 14$, resp.

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Some notation

We denote by $\mathbb{R}^{k,l}$ the pseudo-Euclidean vector space of signature (k,l) with inner product

$$(x,y) = x_1y_1 + \cdots + x_ky_k - x_{k+1}y_{k+1} - \cdots - x_{k+l}y_{k+l}$$

and by $O_{k,l}$ the group of (pseudo)orthogonal transformations of $\mathbb{R}^{k,l}$. (In particular, $O_{k,0} = O_{0,k} = O_k$.) Further, we denote by $O^+_{k,l}$ the subgroup of $O_{k,l}$ formed by the orthogonal transformations, whose matrix has a positive upper left minor of order k. If $k > 0$, it is a subgroup of index 2 (in particular, $O^+_{k,0} = SO_k$). For any subgroup $\Gamma \subset O_{k,l}$, we denote by Γ^+ its intersection with $O^+_{k,l}$.

We denote by $O_{k,l}(\mathbb{Z})$ the subgroup of $O_{k,l}$ formed by the matrices with integer entries, and set

$$O'_{k,l}(\mathbb{Z}) = \{\gamma = (c_{ij}) \in O^+_{k,l}(\mathbb{Z}) : \sum_{i,j} c_{ij} \equiv k + l \pmod 4\}.$$

This is a subgroup of index 2 in $O^+_{k,l}(\mathbb{Z})$ (except for the case $k = 1$, $l = 0$): see Subsection 2.2.

For $k - l \equiv 4 \pmod 8$ and $k, l > 0$, we denote by $O^{ext}_{k,l}(\mathbb{Z})$ a certain extension of index 3 of $O^+_{k,l}(\mathbb{Z})$, described in Subsection 2.2.

We also set for brevity

$$\Gamma_n = O^+_{2,n}(\mathbb{Z}), \quad \Gamma'_n = O'_{2,n}(\mathbb{Z}),$$

and, for $n \equiv 6 \pmod 8$,

$$\Gamma^{ext}_n = O^{ext}_{2,n}(\mathbb{Z}).$$

Some other notations:

$Q = S^4(\mathbb{C}^4)^*$: the vector space of quartic forms in 4 variables x_0, x_1, x_2, x_3,

$Q^\circ \subset Q$: the (open) subset of forms F defining irreducible surfaces with at most simple singularities,

$SL_4 = SL_4(\mathbb{C})$.

S^g: the subset of fixed elements of a transformation g of a set S.

2. Some quadratic lattices and groups

In this section we collect some definitions and facts related to quadratic lattices, that we need for the proof of our main results. In particular, we reproduce the simple roots of some unimodular hyperbolic quadratic lattices. The reader is recommended just to look through this section and then return to it when it is necessary.

2.1. **Basic definitions.** A *quadratic lattice* is a free abelian group L of finite rank equipped with an inner product $(.,.)$, a non-degenerate symmetric bilinear map $L \times L \to \mathbb{Z}$. Its *discriminant* $d(L)$ is the determinant of the Gram matrix of (any) basis of L. The lattice L is called *unimodular* if $d(L) = \pm 1$. The *signature* (k,l) of L is by definition the signature of the pseudo-Euclidean vector space $L \otimes \mathbb{R}$. If $k, l > 0$, the lattice is called *indefinite*, otherwise it is called (*positive* or *negative*) *definite*. If $k = 1$, $l > 0$, the lattice is called *hyperbolic*. For a non-zero integer c, the symbol $[c]L$ denotes the quadratic lattice obtained from L when multiplying all inner products by c.

A quadratic lattice L is called *even* if $(x,x) \in 2\mathbb{Z}$ for every $x \in L$. Otherwise it is called *odd*. If L is an odd quadratic lattice, then its elements with even squares form an index 2 sublattice L^{ev} called the *even sublattice* of L.

We denote with A_n, D_n, E_6, E_7, E_8 the *root lattices*, the (even positive definite) quadratic lattices generated by the root systems of the corresponding types (assuming the squares of the roots being equal to 2). Their discriminants are $n+1$, 4, 3, 2, 1, respectively.

The *dual lattice* L^* is defined by

$$L^* = \{x \in L \otimes \mathbb{Q} : (x,y) \in \mathbb{Z} \quad \text{for all} \quad y \in L\}.$$

(Note that L^* is not a quadratic lattice in the above sense, unless L is unimodular.) The *discriminant group* $\mathfrak{D}(L) := L^*/L$ is an abelian group of order $|d(L)|$. We shall denote the image of an element $x \in L^*$ in $\mathfrak{D}(L)$ with $[x]$. The inner product in $L \otimes \mathbb{Q}$ induces a non-degenerate symmetric bilinear form on L^* with values in $\mathbb{Q}/\mathbb{Z}$. We will call it the *inner product* in $\mathfrak{D}(L)$ and denote with the same symbol $(.,.)$ as the inner product in L.

For an even quadratic lattice L, the discriminant group $\mathfrak{D}(L)$ is equipped with a quadratic form q with values in $\mathbb{Q}/\mathbb{Z}$ by the formula

$$q([x]) = \frac{1}{2}(x,x) + \mathbb{Z}.$$

We will call q the *norm* in $\mathfrak{D}(L)$. The inner product in $\mathfrak{D}(L)$ is determined by the norm as

$$(\xi, \eta) = q(\xi + \eta) - q(\xi) - q(\eta).$$

An *automorphism* of a quadratic lattice L is an automorphism of the abelian group L preserving the inner product. We will denote the group of automorphisms of L with $O(L)$. If L is even, then clearly $O(L) \subset O(L^{ev})$.

Let $O(\mathfrak{D}(L))$ denote the group of automorphisms of $\mathfrak{D}(L)$ preserving the inner product and (if L is even) the norm. There is a natural homomorphism

$$\mathfrak{d} : O(L) \to O(\mathfrak{D}(L)).$$

2.2. **Some standard lattices.** Let us now introduce some standard quadratic lattices. First, let $I_{k,l}$ be the quadratic lattice of rank $n = k + l$ with an orthogonal basis $\{e_1, \dots, e_n\}$ such that

$$(e_1, e_1) = \cdots = (e_k, e_k) = 1, \; (e_{k+1}, e_{k+1}) = \cdots = (e_n, e_n) = -1.$$

Obviously, it is unimodular and odd. If $k, l > 0$ or $k + l < 12$, any odd unimodular quadratic lattice of signature (k,l) is isomorphic to $I_{k,l}$. (See, e.g., [16], [20].)

Let $D_{k,l} = I_{k,l}^{ev}$, the even sublattice of $I_{k,l}$ (consisting of the vectors whose sum of coordinates is even). It is an even quadratic lattice of discriminant ± 4 and signature (k,l). In particular, $D_{n,0}$ is the root lattice of type D_n.

If $k \equiv l \pmod 8$, then the group generated by $D_{k,l}$ and the vector

$$\sigma = \frac{1}{2}(e_1 + \cdots + e_n) \in I_{k,l} \otimes \mathbb{Q}$$

is an even unimodular quadratic lattice. Denote it by $J_{k,l}$. If $k,l > 0$ or $k+l < 16$, any even unimodular quadratic lattice of signature (k,l) is isomorphic to $J_{k,l}$ [16], [20].

In any case, representatives of cosets of $D_{k,l}$ in $D_{k,l}^*$ are

$$0,\, e_1,\, \sigma,\, \sigma' = \sigma - e_1.$$

The initial lattice $I_{k,l}$ is generated by $D_{k,l}$ and e_1. The discriminant group $\mathfrak{D}(D_{k,l})$ is cyclic (with a generator $[\sigma]$) if n is odd, and a four-group if n is even. The norm q in $\mathfrak{D}(D_{k,l})$ is given by

$$q([e_1]) = \frac{1}{2} + \mathbb{Z}, \quad q([\sigma]) = q([\sigma']) = \frac{k-l}{8} + \mathbb{Z}.$$

Proposition 1. *Let L be an even quadratic lattice of signature (k,l) and discriminant ± 4. Suppose that $k,l > 0$ or $k+l < 12$ and the norm in $\mathfrak{D}(L)$ represents $\frac{1}{2} + \mathbb{Z}$. Then L is isomorphic to $D_{k,l}$.*

Proof. Let $x \in L^*$ be such that $q([x]) = \frac{1}{2} + \mathbb{Z}$. Then (x,x) is an odd integer. It follows that the subgroup generated by L and x, is an odd unimodular quadratic lattice isomorphic to $I_{k,l}$ and L is its even sublattice. □

In particular, if $l > 0$, the lattice $D_{k,l} \oplus E_8$ is isomorphic to $D_{k+8,l}$. Similarly, if $k > 0$, the lattice $D_{k,l} \oplus [-1]E_8$ is isomorphic to $D_{k,l+8}$.

Proposition 2. *If $k,l > 0$ or $k+l < 12$, then the homomorphism $\mathfrak{d} : O(D_{k,l}) \to O(\mathfrak{D}(D_{k,l}))$ is surjective.*

Proof. If $k - l \not\equiv 4 \pmod 8$, then the only non-trivial automorphism of $\mathfrak{D}(D_{k,l})$ preserving the norm permutes σ and σ'. It is induced, for instance, by changing the sign of e_1.

Let now $k - l \equiv 4 \pmod 8$. Then any automorphism of the group $\mathfrak{D}(D_{k,l})$ (which is a four-group) preserves the norm. One may assume that $k > l$. Then $D_{k,l} \simeq D_4 \oplus M$, where M is a direct sum of several copies of E_8 and $[-1]E_8$, so the proof reduces to the case of D_4. But in this case e_1, σ, σ' are just the fundamental weights of the root system of type D_4. They are permuted by the automorphisms of the Dynkin diagram, which yield automorphisms of the lattice D_4. □

Remark 1. *The assertion of the proposition is not valid for the lattices D_{8m+4} with $m > 0$.*

Proposition 3. *If $k - l \not\equiv 4 \pmod 8$, then $O(D_{k,l}) = O(I_{k,l})(= O_{k,l}(\mathbb{Z}))$. If $k - l \equiv 4 \pmod 8$ and $k,l > 0$, then $O(D_{k,l})$ is an extension of index 3 of $O_{k,l}(\mathbb{Z})$.*

Proof. If $k - l \not\equiv 4 \pmod 8$, then any automorphism of $D_{k,l}$ acting on the discriminant group preserves $[e_1]$ (which is the only element with norm $\frac{1}{2} + \mathbb{Z}$) and, hence, it preserves the lattice $I_{k,l}$, generated by $D_{k,l}$ and e_1. Thus, in this case

$O(D_{k,l}) = O_{k,l}(\mathbb{Z})$. If $k - l \equiv 4 \pmod 8$ and $k, l > 0$, then $O(D_{k,l})$ acts transitively on the set of non-zero elements of the discriminant group, so the subgroup preserving $[e_1]$ (and thereby $I_{k,l}$) is of index 3 in $O(D_{k,l})$. $\square$

If $k - l \equiv 4 \pmod 8$ and $k, l > 0$, then clearly also $O^+(D_{k,l})$ is an extension of index 3 of $O^+_{k,l}(\mathbb{Z})$. In this case, we set

$$O^{ext}_{k,l}(\mathbb{Z}) = O^+(D_{k,l}).$$

It is not difficult to describe this group explicitly. It is the union of $O^+_{k,l}(\mathbb{Z})$ and the set of matrices of $O^+_{k,l}$ with entries in $\frac{1}{2} + \mathbb{Z}$, for which the sums of the entries of the columns have the same parity.

Let $O'_{k,l}(\mathbb{Z})$ denote the kernel of the natural homomorphism $O^+_{k,l}(\mathbb{Z}) \to O(\mathfrak{D}(D_{k,l}))$. This is a subgroup of index 2 in $O^+_{k,l}(\mathbb{Z})$. Let us describe it explicitly. Clearly, $\gamma = (c_{ij}) \in O_{k,l}(\mathbb{Z})$ acts trivially on $\mathfrak{D}(D_{k,l})$ if and only if $\gamma\sigma - \sigma \in D_{k,l}$. A straightforward calculation shows that the latter takes place if and only if $\sum_{i,j} c_{ij} \equiv k + l \pmod 4$.

2.3. **Orthogonal complements in unimodular lattices.** A sublattice L of a lattice J is called *primitive*, if the group J/L has no torsion. The orthogonal complement $L^\perp$ of any sublattice L in a quadratic lattice J is clearly primitive. If L is primitive, then the orthogonal projection of J^* to $L \otimes \mathbb{Q}$ coincides with L^* .

Let now J be a unimodular quadratic lattice, and L be a primitive sublattice of J. Set $M = L^\perp$. The orthogonal projections of $J^* = J$ to L^* and M^* give rise to the following commutative diagram of group isomorphisms:

$$\begin{array}{ccc} & J/(L \oplus M) & \\ \swarrow & & \searrow \\ \mathfrak{D}(L) & \xrightarrow[\tau]{\sim} & \mathfrak{D}(M) \end{array}$$

It is immediate that

$$J = \{(x, y) \in L^* \oplus M^* : y + M = \tau(x + L)\}, \tag{1}$$

$$(\xi, \eta) + (\tau(\xi), \tau(\eta)) = 0 \quad \text{for all} \quad \xi, \eta \in \mathfrak{D}(L), \tag{2}$$

and, if J (and, hence, L and M) is even,

$$q(\xi) + q(\tau(\xi)) = 0 \quad \text{for all} \quad \xi \in \mathfrak{D}(L). \tag{3}$$

Conversely, if L and M are two quadratic lattices and $\tau : \mathfrak{D}(L) \to \mathfrak{D}(M)$ is a group isomorphism satisfying (2), then (1) defines a unimodular quadratic lattice J. Moreover, if the lattices L and M are even and τ satisfies (3), then the lattice J is even.

The following proposition is obvious.

Proposition 4. *Under the above assumptions, let $\varphi \in O(L)$ and $\psi \in O(M)$. Then $\varphi \oplus \psi$ extends to an automorphism of J if and only if $\mathfrak{d}(\varphi)$ and $\mathfrak{d}(\psi)$ are consistent in the sense that $\tau\mathfrak{d}(\varphi) = \mathfrak{d}(\psi)\tau$.*

2.4. **Embeddings of $D_{k,l}$ in $J_{p,q}$.** Let $p \equiv q \pmod 8$ and $k \le p$, $l \le q$, $k+l < p+q$. Up to a permutation of basis vectors, we have

$$I_{k,l} \oplus I_{p-k,p-l} = I_{p,q},$$

which yields

$$D_{k,l} \oplus D_{p-k,p-l} \subset D_{p,q} \subset J_{p,q}.$$

Thus obtained primitive embedding $D_{k,l} \hookrightarrow J_{p,q}$ will be called *standard.*

The following proposition is a version of the Witt theorem (cf. [11, Theorem 1.14.4]).

Proposition 5. *If $k < p$, $l < q$ or $(p+q)-(k+l) < 12$, then any isomorphism of primitive sublattices isomorphic to $D_{k,l}$ extends to an automorphism of the lattice $J_{p,q}$.*

Proof. Let L be a primitive sublattice of $J = J_{p,q}$ isomorphic to $D_{k,l}$, and let $M = L^\perp$. Then (1) and (3) hold. It follows that the norm in $\mathfrak{D}(M)$ represents $\frac{1}{2}+\mathbb{Z}$, and by Proposition 2 the lattice M is isomorphic to $D_{p-k,q-l}$.

Let now $\tilde{L} \subset J_{p,q}$ be another primitive sublattice isomorphic to $D_{k,l}$, and $\varphi : L \to \tilde{L}$ be an isomorphism. Set $\tilde{M} = \tilde{L}^\perp$. Under our assumptions on k, l, Proposition 2 implies that there exists an isomorphism $\psi : M \to \tilde{M}$ such that the map

$$(x, y) \mapsto (\varphi(x), \psi(y)) \quad (x \in L^*, y \in M^*, x+y \in J)$$

is an automorphism of J. □

Corollary. *If $q > 0$, the elements $h \in J_{p,q}$ with $(h,h) = 4$ constitute one $O(J_{p,q})$-orbit.*

Proof. Indeed, such elements are just generators of (primitive) sublattices isomorphic to $D_{1,0}$. □

2.5. **Roots.** A primitive element α of a quadratic lattice L is called a *root* or, more precisely, a *k-root*, if $(\alpha,\alpha) = -k < 0$ and the reflection

$$R_\alpha : x \mapsto x + \frac{2(\alpha, x)}{k}\alpha$$

leaves L invariant (which automatically holds if $k = 1$ or 2). Such a reflection is called a *k-reflection.* All the roots of a unimodular quadratic lattice are 2-roots or 1-roots, but in general there may be other roots.

It is easy to see that, if L is indefinite, any reflection belongs to $O^+(L)$. If L is odd, then any k-reflection of L with k odd is a reflection of L^{ev} but the corresponding root of L^{ev} is twice the root of L. Note also that any 1- or 2-reflection acts trivially on $\mathfrak{D}(L)$.

Let $W_2(L)$ (resp. $W(L)$) denote the group generated by all 2-reflections (resp. by all reflections); we shall call it the *small Weyl group* (resp. the *Weyl group*) of L. Clearly, $W_2(L)$ and $W(L)$ are normal subgroups of $O(L)$.

Let now L be a quadratic lattice of signature $(1,n)$. Set

$$\mathbb{R}^{1,n} = L \otimes \mathbb{R}, \quad \tilde{C}_n = \{x \in \mathbb{R}^{1,n} : (x,x) > 0\}.$$

Let C_n be one of the two (opposite) connected components of $\tilde{C}_n$. The group $O^+(L) = O(L) \cap O^+_{1,n}$ acts discretely on the cone C_n and on its projectivization, which is a model of the n-dimensional hyperbolic space H^n. Moreover, the fundamental domain for the action of $O^+(L)$ on H^n is of finite volume.

The groups $W_2(L)$ and $W(L)$ acting on H^n are generated by reflections in the sense of hyperbolic geometry. Choose a fundamental polyhedron $P_2(L)$ for the action of $W_2(L)$ on H^n. The cone over $P_2(L)$ is a fundamental cone for the action of $W_2(L)$ on C_n. Let $A_2(L)$ denote its closure. The roots orthogonal to walls of $A_2(L)$ and looking outside are called the *simple 2-roots* of L, and the corresponding reflections are called the *simple 2-reflections.* The set of simple 2-roots will be denoted by $\Pi_2(L)$.

It is easy to see (cf. [19, Proposition 3]) that

$$W(L) = W_2(L) \leftthreetimes \overline{W}(L),$$

where

$$\overline{W}(L) = \{w \in W(L) : wA_2(L) = A_2(L)\}.$$

Any fundamental cone for the action of $\overline{W}(L)$ on $A_2(L) \cap C_n$ is a fundamental cone for the action of $W(L)$ on C_n. Let $A(L)$ be the closure of such a cone. The roots orthogonal to its walls and looking outside are called the *simple roots* of L, and the corresponding reflections are called the *simple reflections.* The set of simple roots will be denoted by $\Pi(L)$.

Note that every simple root α with $(\alpha, \alpha) = -2$ is a simple 2-root but not every simple 2-root is a simple root, unless all roots are 2-roots.

The group $\overline{W}(L)$ is generated by the reflections R_α with $\alpha \in \Pi(L) \setminus \Pi_2(L)$. Moreover,

$$\Pi_2(L) = \overline{W}(L)(\Pi_2(L) \cap \Pi(L))$$

(see [21, Subsection 1.6]).

2.6. **Isotropic edges of $A(L)$.** The volume of the fundamental polyhedron $P(L)$ for the action of $W(L)$ on H^n is finite if and only if $P(L)$ is a finite polyhedron (i.e., a polyhedron with finitely many faces) or, equivalently, $A(L)$ is a finite polyhedral cone. In this case, the isotropic edges of $A(L)$ (lying on the boundary of C_n) correspond to parabolic subdiagrams of rank $n-1$ of the Coxeter diagram of $A(L)$. Namely, let Σ be any connected component of such subdiagram. It is the extended Dynkin diagram of some finite root system, say, Δ. The extended system of simple roots of Δ is linear dependent with certain mutually prime positive integer coefficients. The same linear combination of the corresponding simple roots of L is just the primitive vector of the edge of $A(L)$ defined by Σ (and by any other connected component of the same parabolic subdiagram of rank $n-1$): see [21, Subsection 1.9].

The same is applicable to the cone $A_2(L)$, provided it is a finite polyhedral cone. Note that the latter condition holds if and only if it holds for $A(L)$ and the group $\overline{W}(L)$ is finite.

2.7. **Simple roots of $I_{1,n}$.** We need a description of the cones $A(L)$ (or, equivalently, of the polyhedra $P(L)$) in some concrete cases, namely, for $L = I_{1,n}$, $n =$ 12, 13, 14, 15. Their Coxeter diagrams taken from [19] (with the numbering of nodes that we use later) are represented in Table 1, where the white (resp. black) nodes correspond to 2-roots (resp. 1-roots).

The group $O^+_{1,n}(\mathbb{Z})$ coincides with $W(I_{1,n})$ for $n = 12$, 13 and is a semidirect product of $W(I_{1,n})$ and the group of order 2 generated by the diagram automorphism, for $n = 14$, 15 [19]. One can also observe that, for $n = 12$, 13, 15 the group $\overline{W}(I_{1,n})$ is finite of order 2, while for $n = 14$ it is an infinite dihedral group.

In the standard basis $\{e_0, e_1, \dots, e_n\}$ of $I_{1,n}$, the simple roots are:

$$\begin{aligned}
&\alpha_i = -e_i + e_{i+1}\,(i = 1, \dots, n-1),\ \alpha_n = -e_n,\\
&\alpha_{n+1} = e_0 + e_1 + e_2 + e_3,\\
&\alpha_{n+2} = 3e_0 + e_1 + \dots + e_{11},\\
&\alpha_{n+3} = 4e_0 + 2e_1 + e_2 + \dots + e_n \quad \text{for} \quad n = 14, 15.
\end{aligned}$$

(The numbering of simple roots matches with the numbering of nodes of the corresponding diagrams in Table 1.)

Applying $\overline{W}(I_{1,n})$ to the simple roots α_i with $(\alpha_i, \alpha_i) = -2$, one obtains all simple 2-roots, in particular, the following new roots:

$$\begin{aligned}
&\alpha'_{n-1} = R_{\alpha_n}\alpha_{n-1} = -e_{n-1} - e_n,\\
&\alpha'_1 = R_{\alpha_{17}}\alpha_1 = 8e_0 + 3e_1 + 3e_2 + 2e_3 + \dots + 2e_{14} \quad \text{for} \quad n = 14,\\
&\alpha'_{18} = R_{\alpha_{15}}\alpha_{18} = 4e_0 + 2e_1 + e_2 + \dots + e_{14} - e_{15} \quad \text{for} \quad n = 15.
\end{aligned}$$

There are no other simple 2-roots for $n = 12$, 13, 15. For $n = 14$, there are infinitely many of them; however, the roots already found suffice for our purposes. Thus obtained Coxeter diagrams of the cones $A_2(I_{1,12})$, $A_2(I_{1,13})$, $A_2(I_{1,15})$, and a part of the Coxeter diagram of the cone $A_2(I_{1,14})$ are represented in Table 2, where the numbers of the new simple 2-roots are indicated.

The following linear dependences between simple 2-roots are obtained when considering the parabolic subdiagrams of types $\tilde{E}_8 + \tilde{D}_{n-9}$ (see Subsection 2.6):

$$(4)\quad 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 + 3\alpha_{n+1}$$
$$= \begin{cases} \alpha_{10} + \alpha_{11} + \alpha'_{11} + \alpha_{14} & \text{for} \quad n = 12,\\ \alpha_{10} + 2\alpha_{11} + \alpha_{12} + \alpha'_{12} + \alpha_{15} & \text{for} \quad n = 13,\\ \alpha_{10} + 2\alpha_{11} + 2\alpha_{12} + \alpha_{13} + \alpha'_{13} + \alpha_{16} & \text{for} \quad n = 14,\\ \alpha_{10} + 2\alpha_{11} + 2\alpha_{12} + 2\alpha_{13} + \alpha_{14} + \alpha'_{14} + \alpha_{17} & \text{for} \quad n = 15. \end{cases}$$

3. The period map for quartic surfaces

Let us recall some facts about complex algebraic $K3$ surfaces (see, for instance, [8] or [4]).

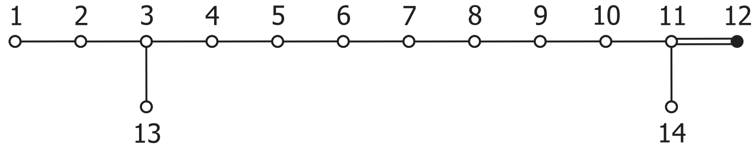
3.1. **Divisors of $K3$ surfaces.** For a smooth $K3$ surface X, the homology group $H_2(X, \mathbb{Z})$ equipped with the intersection form $(.,.)$ is an even unimodular quadratic lattice of signature $(3, 19)$. (Recall that all such quadratic lattices are isomorphic.) The Picard group $S(X)$ (=the Severi group) of X is a primitive sublattice in $H_2(X, \mathbb{Z})$ of signature $(1, n)\,(0 \le n \le 19)$.

Regarding the lattice $S(X)$, we shall use the terminology and notation of Subsection 2.5. However, for brevity we shall write $W(X)$ instead of $W(S(X))$, $A(X)$ instead of $A(S(X))$, etc.

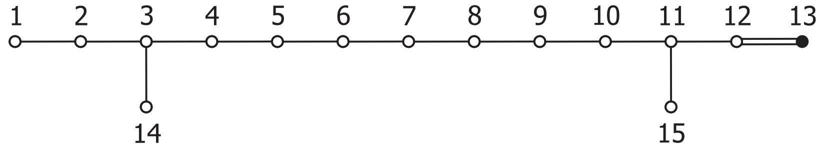
All ample divisor classes of X are contained in one connected component of the cone $\tilde{C}_n$. We will assume that C_n is just this component. Then the closed convex cone generated by the ample classes coincides with the cone $A_2(X)$ under a suitable choice of the fundamental polyhedron $P_2(X)$. We will assume that $P_2(X)$ is chosen in this way. Then the simple 2-roots are just the classes of smooth rational curves on X. Note that any such class contains only one smooth rational curve.

Table 1

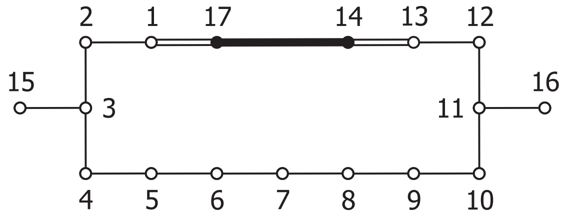
$I_{1,12}$



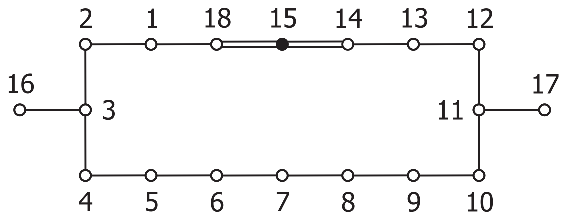
$I_{1,13}$



$I_{1,14}$



$I_{1,15}$



A *d-polarization* of X is a class $h \in S(X) \cap A_2(X)$ with $(h,h) = d > 0$. In this paper, except for Subsection 5.1, we will only consider 4-polarizations. For any such polarization, the linear system $|h|$ defines a morphism

$$\varphi = \varphi_h : X \to \mathbb{C}P^3,$$

which is a birational morphism onto its image if and only if the following condition holds:

(*) There are no isotropic vectors $u \in S(X) \cap A_2(X)$ with $(h,u) = 1$ or 2.

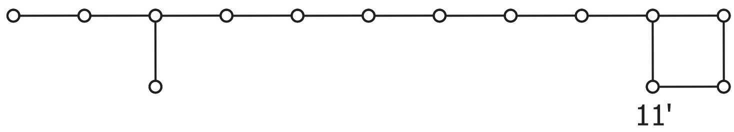
Under this condition, $Y = \varphi(X)$ is a quartic surface with at most simple singularities. (Otherwise Y is a double quadric or a curve.) The morphism φ retracts into singular points the smooth rational curves on X whose classes are orthogonal to h, and it is an isomorphism beyond these curves.

One may assume that $h \in A(X)$. The following lemma (cf. Lemma 1.7 in [21]) is helpful for checking condition (*) in practice.

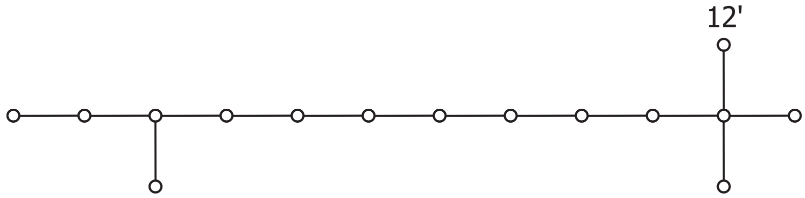
Lemma 1. *If there are no isotropic vectors $u \in S(X) \cap A(X)$ with $(h,u) = 1$ or 2, then there are no such vectors in $S(X)$.*

TABLE 2

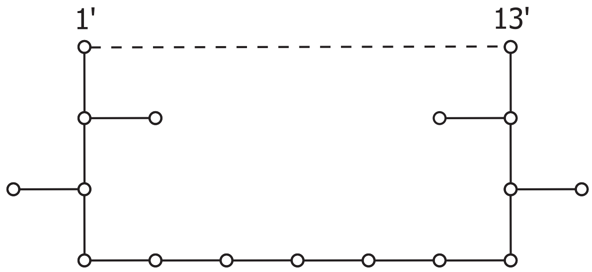
$I_{1,12}$



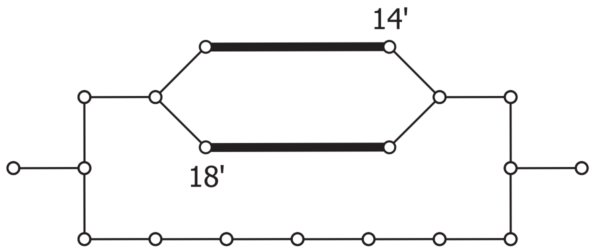
$I_{1,13}$



$I_{1,14}$



$I_{1,15}$



Proof. Note that $(h, u) > 0$ for any $u \in \bar{C}_n$. Let $u \in S(X) \cap \bar{C}_n$ be an isotropic vector with the minimal value of (h, u). If $u \notin A(X)$, then there exists a reflection $R_\alpha \in W(X)$ such that $(\alpha, x) \geq 0$ for all $x \in A(X)$, but $(\alpha, u) < 0$. If $(\alpha, h) > 0$, then

$$(h, R_\alpha u) = (h, u) - \frac{2(\alpha, u)(\alpha, h)}{(\alpha, \alpha)} < (h, \alpha),$$

which contradicts our choice of u. Thus, $(\alpha, h) = 0$ and $(h, R_\alpha u) = (h, u)$ for any such α. Now, applying a suitable element of the stabilizer of h in $W(X)$, one can move u into $A(X)$ without changing (h, u). □

3.2. **Quartic surfaces.** Let $Q = S^4(\mathbb{C}^4)^*$ be the vector space of quartic forms in 4 variables x_0, x_1, x_2, x_3. Each non-zero form $F \in Q$ defines a quartic surface $Y = Y(F) \subset \mathbb{C}P^3$ given in the homogeneous coordinates by the equation $F = 0$. Denote by Q° the set of forms F for which the surface $Y(F)$ is irreducible and has at most simple singularities. It is a Zariski open subset in Q. The boundary $Q \setminus Q^\circ$

is of codimension ≥ 2, since it is a proper subvariety in the irreducible codimension one subvariety of forms F for which the surface $Y(F)$ is singular. In particular, $\mathbb{C}[Q^\circ] = \mathbb{C}[Q]$.

For each $F \in Q^\circ$, the desingularization $X = X(F)$ of $Y(F)$ is a $K3$ surface. The class of a plane section of Y defines a polarization $h = h_F$ of X satisfying condition (*).

On any $K3$ surface, there is a non-vanishing regular differential 2-form defined uniquely up to a constant factor. It is important that, for the surface $X(F)$, such a form can be canonically defined by F. Namely, consider the cone

$$\widehat{Y} = \widehat{Y}(F) = \{v \in \mathbb{C}^4 : F(v) = 0\},$$

whose projectivization is Y. For $v \in \widehat{Y}^{reg}$, $\xi, \eta, \zeta \in T_v(\widehat{Y})$, we have $\det(v, \xi, \eta, \zeta) = 0$, since $v \in T_v(\widehat{Y})$. Hence, for $v \in \widehat{Y}^{reg}$, $\xi, \eta \in T_v(\widehat{Y})$, and any $\zeta \in \mathbb{C}^4$ we have

$$\det(v, \xi, \eta, \zeta) = \hat{\omega}(\xi, \eta) dF(\zeta),$$

where $\hat{\omega} = \hat{\omega}(F)$ is some nowhere vanishing regular differential 2-form on $\widehat{Y}^{reg}$. In coordinates,

$$\hat{\omega} = \frac{-x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 - x_3 dx_1 \wedge dx_2}{\partial F / \partial x_0}.$$

It is easy to see that the form $\hat{\omega}$ is invariant under homotheties, and the radial direction lies in its kernel. It follows that $\hat{\omega}$ defines a regular differential 2-form $\omega = \omega(F)$ on Y^{reg}. It lifts to a regular differential 2-form on X, which we will denote with the same symbol. Clearly,

$$\omega(tF) = t^{-1}\omega(F).$$

The last formula is of crucial significance. As we shall see, it has the consequence that, when passing from the algebraic invariant theory to the “transcendental invariant theory” (the theory of automorphic forms), one should change the signs: invariant homogeneous polynomials considered in the algebraic invariant theory, are of positive degree of homogeneity, while automorphic forms are naturally interpreted as homogeneous holomorphic functions of negative degree of homogeneity.

By the Poincaré duality one can consider the form ω as an element of $H_2(X, \mathbb{C})$ so that, for any 2-dimensional cycle c,

$$\int_c \omega = (\omega, [c]),$$

where $[c]$ stands for the homology class of c. Clearly, ω is orthogonal to all classes of algebraic cycles and, in particular,

$$(\omega, h) = 0. \tag{5}$$

In fact, the classes of algebraic cycles are characterized by this property, i.e.,

$$S(X) = \{a \in H_2(X, \mathbb{Z}) : (\omega, a) = 0\}.$$

Moreover,

$$(\omega, \omega) = \int_X \omega \wedge \omega = 0, \tag{6}$$

$$(\omega, \bar{\omega}) = \int_X \omega \wedge \bar{\omega} > 0. \tag{7}$$

The group $SL_4 = SL_4(\mathbb{C})$ naturally acts on Q. With respect to this action, all points of Q° are stable in the sense of Mumford, which means that their orbits are closed and have the maximal dimension (equal to $\dim SL_4 = 15$). (See Proposition 9.) It follows that the restriction to Q° of the categorical quotient

$$\pi_{SL_4} : Q \to Q /\!\!/ SL_4 := \operatorname{Spec} \mathbb{C}[Q]^{SL_4}$$

is a geometric quotient. We shall denote it as

$$\pi_{SL_4} : Q^\circ \to Q^\circ / SL_4.$$

For each $g \in SL_4$, we have $Y(gF) = gY(F)$, so g induces an isomorphism $X(F) \to X(gF)$ of $K3$ surfaces and thereby an isomorphism $H_2(X(F), \mathbb{Z}) \to H_2(X(gF), \mathbb{Z})$ of quadratic lattices. In this sense, the map $F \mapsto \omega(F)$ is SL_4-equivariant.

3.3. **The period map.** Let $J = J_{3,19}$, and let $h_0 \in J$ be a fixed vector with $(h_0, h_0) = 4$. The orthogonal complement of h_0 in J is $D_{2,19} = I^{ev}_{2,19}$. We set $I_{2,19} \otimes \mathbb{R} = \mathbb{R}^{2,19}$ and use the notation of the introduction, assuming that $I_{2,19}$ is the lattice of integer vectors in $\mathbb{R}^{2,19}$.

Let $X = X(F)$ be as in Subsection 1.3, and let $\varphi : H_2(X, \mathbb{Z}) \to J$ be an isomorphism of quadratic lattices, taking h to h_0. Then $z_F = \varphi(\omega(F))$ belongs to the cone $\tilde{\mathcal{L}}_{19}$ by (5)-(7). Let us consider only those φ, for which $z_F \in \mathcal{L}_{19}$. Then φ is defined up to a left multiplication by an element of the group

$$O^+(J, h_0) = \{\gamma \in O^+(J) : \gamma h_0 = h_0\}.$$

By Proposition 4 the restriction of $O^+(J, h_0)$ to $D_{2,19}$ is the group $O'_{2,19}(\mathbb{Z})$ (see Subsection 2.2 for the notation). For brevity, we will denote it by Γ'_{19}.

Thus, we obtain a map $Q^\circ \to \mathcal{L}_{19}/\Gamma'_{19}$. It factors through a map

$$\hat{p} : Q^\circ / SL_4 \to \mathcal{L}_{19}/\Gamma'_{19}.$$

Clearly, the map $\hat{p}$ is $\mathbb{C}^*$-equivariant, if the action of $t \in \mathbb{C}^*$ on $\mathbb{C}^{2,19}$ is defined as the multiplication by t^{-1}. Passing to the quotients by $\mathbb{C}^*$, we obtain a map

$$p : (Q^\circ / SL_4)/C^* \to \mathcal{D}_{19}/\Gamma'_{19},$$

which is called *the period map* for quartic surfaces. We will also call $\hat{p}$ the period map.

The global Torelli theorem for $K3$ surfaces [12] implies that $\hat{p}$ is a $\mathbb{C}^*$-equivariant analytic isomorphism of Q°/SL_4 onto a dense open subset in $\mathcal{L}_{19}/\Gamma'_{19}$ ([9], Theorem 8.6). The complement of this subset is the union of two irreducible divisors $\mathcal{H}_1$ and $\mathcal{H}_2$, arising from condition (*). Namely, for any isotropic vector $u \in J$ with $(h_0, u) = 1$ or 2, set

$$\mathcal{H}(u) = \{z \in \mathcal{L}_{19} : (z, u) = 0\}.$$

The projectivization of $\mathcal{H}(u)$ is a "hyperplane" in $\mathcal{D}_{19}$, a totally geodesic complex hypersurface. The hyperplanes $\mathcal{H}(u)$ decompose into two Γ'_{19}-orbits depending on whether $(h_0, u) = 1$ or 2, and constitute a discrete arrangement. The image in $\mathcal{L}_{19}/\Gamma'_{19}$ of each $\mathcal{H}(u)$ with $(h_0, u) = 1$ (resp. 2) is the divisor $\mathcal{H}_1$ (resp. $\mathcal{H}_2$).

We denote the complement of the above arrangement by $\mathcal{L}^\circ_{19}$. Then $\hat{p}$ is a $\mathbb{C}^*$-equivariant analytic isomorphism of Q°/SL_4 onto $\mathcal{L}^\circ_{19}/\Gamma'_{19}$, the complement of $\mathcal{H}_1 \cup \mathcal{H}_2$ in $\mathcal{L}_{19}/\Gamma'_{19}$. It induces an isomorphism of graded algebras

$$\hat{p}^* : \tilde{A}_2(D_{19}, \Gamma'_{19}) \xrightarrow{\sim} \mathbb{C}[Q]^{SL_4},$$

where $\tilde{A}_2(D_{19}, \Gamma'_{19})$ is the algebra of meromorphic automorphic forms introduced by Looijenga [9] and constituted by the automorphic forms that are allowed to have poles on the hyperplanes $\mathcal{H}(u)$.

4. Morphisms of arithmetic quotients

The algebra $\mathbb{C}[Q]^{SL_4}$ is enormously complicated. In order to obtain reasonable explicit results, we are going to pass to some subvarieties in $\mathcal{L}_{19}/\Gamma'_{19}$ and, correspondingly, in Q°/SL_4.

Our subvarieties in $\mathcal{L}_{19}/\Gamma'_{19}$ will be the images of suitable subcones $\mathcal{L}_n \subset \mathcal{L}_{19}$. The first step is to study thus obtained morphisms of the relevant arithmetic quotients of $\mathcal{L}_n$ to $\mathcal{L}_{19}/\Gamma'_{19}$. We will do this in a more general situation.

4.1. **The general case.** Let M be a quadratic lattice embedded in $\mathbb{R}^{2,m}$ so that $\mathbb{R}^{2,m} = M \otimes \mathbb{R}$ (as a quadratic vector space). Then $O^+(M)$ is an arithmetic discrete group of holomorphic transformations of $\mathcal{L}_m$ (and of $\mathcal{D}_m$). Let $\Gamma \subset O^+(M)$ be a subgroup of finite index.

Let now $n < m$. The standard embedding $\mathbb{R}^{2,n} \hookrightarrow \mathbb{R}^{2,m}$ induces embeddings

$$\mathcal{L}_n \hookrightarrow \mathcal{L}_m, \quad \mathcal{D}_n \hookrightarrow \mathcal{D}_m,$$

so that $\mathcal{D}_n$ is a totally geodesic complex submanifold of $\mathcal{D}_m$, an n-dimensional "plane". Assume that $\mathbb{R}^{2,n} = N \otimes \mathbb{R}$, where $N = M \cap \mathbb{R}^{2,n}$. Let Δ denote the group formed by the restrictions to $\mathcal{L}_n$ of the transformations $\gamma \in \Gamma$ that leave the subspace $\mathbb{R}^{2,n}$ invariant. (This is a subgroup of finite index in $O^+(N)$.)

We have the following commutative diagrams:

$$\begin{CD} \mathcal{L}_n @>>> \mathcal{L}_m \\ @VVV @VVV \\ \mathcal{L}_n/\Delta @>>{\hat{\nu}}> \mathcal{L}_m/\Gamma \end{CD} \qquad \begin{CD} \mathcal{D}_n @>>> \mathcal{D}_m \\ @VVV @VVV \\ \mathcal{D}_n/\Delta @>>{\nu}> \mathcal{D}_m/\Gamma \end{CD}$$

where the vertical arrows are the factorization morphisms.

Proposition 6. *The morphisms $\hat{\nu}$ and ν are the normalizations over their images, which are closed analytic subspaces (and algebraic subvarieties) of $\mathcal{L}_m/\Gamma$ and $\mathcal{D}_m/\Gamma$, respectively.*

Proof. It suffices to prove that $\hat{\nu}$ and ν have finite fibers, are injective on dense open subsets, and proper.

The n-dimensional planes in $\mathcal{D}_m$ are parameterized by the manifold $\mathcal{G}$ of subspaces of signature $(2, n)$ in $\mathbb{R}^{2,m}$. Associating to every such subspace the exterior product of its basis vectors with a fixed Gram determinant, one can represent $\mathcal{G}$ as a submanifold of the corresponding Grassmann cone. For the subspaces $\gamma\mathbb{R}^{2,n}$ with $\gamma \in \Gamma$ such bases can be chosen in the lattice M, so that their Plücker coordinates (with respect to a basis of M) will be integer. It follows that the planes $\gamma\mathcal{D}_n$, $\gamma \in \Gamma$, constitute a discrete arrangement in $\mathcal{D}_m$. In particular, every point of $\mathcal{D}_n$ belongs only to finitely many of such planes, which means that the fibers of ν are finite.

Further, the intersections of $\mathcal{D}_n$ with other planes of the above arrangement constitute a Δ-invariant discrete arrangement in $\mathcal{D}_n$. Let $\mathcal{A}$ be the union of the planes of this arrangement. Then the morphism ν is injective on $(\mathcal{D}_n \setminus \mathcal{A})/\Delta$.

It follows from the definition of the Satake–Baily–Borel compactification [2] that the morphism ν extends to a morphism of the compactifications, under which the boundary of $\mathcal{D}_n/\Delta$ goes to the boundary of $\mathcal{D}_m/\Gamma$. This implies that the morphism ν is proper.

The same reasoning applied to the cone $\mathcal{L}_m$ shows that $\hat{\nu}$ also has finite fibers and is injective on a dense open subset. Finally, since $\hat{\nu}$ is $\mathbb{C}^*$-equivariant and acts on each $\mathbb{C}^*$-orbit as $z \mapsto z^k$ $(k \in \mathbb{N})$ (in suitable coordinates on the orbit and its image), the properness of ν implies properness of $\hat{\nu}$. □

4.2. **Special cases.** In some special cases, one can prove more about the morphisms ν and $\hat{\nu}$. We will do this for $M = I_{2,m}$, when $\Gamma \subset \Gamma_m(= O^+_{2,m}(\mathbb{Z}))$ and $\Delta \subset \Gamma_n(= O^+_{2,n}(\mathbb{Z}))$. Note that this result is not needed for the proof of our main theorem.

Theorem 2. *In the notation of Subsection 4.1, if $M = I_{2,m}$ and Γ contains all 2-reflections of the lattice M, the morphisms $\hat{\nu}$ and ν are closed embeddings.*

To prove this theorem, we need two lemmas. The first one is a sort of the Chinese remainder theorem.

Lemma 2. *Let V be a complex vector space with a fixed basis and $V_1, \dots, V_s \subset V$ be some subspaces spanned by basis vectors. Let $f_1, \dots, f_s$ be polynomial functions on $V_1, \dots, V_s$, respectively, such that $f_i = f_j$ on $V_i \cap V_j$ for all i, j. Then there is a polynomial function f on V such that $f = f_i$ on V_i for all i.*

Proof. For $i = 1, \dots, s$, let us consider the algebra $\mathbb{C}[V_i]$ as a subalgebra of $\mathbb{C}[V]$ via the projection of V to V_i along basis vectors. The assumption of the lemma means that every monomial has one and the same coefficient in all polynomials f_i where it occurs. For f one can take the sum of all different monomials (with their coefficients) occurring in the polynomials $f_1, \dots, f_s$. □

Lemma 3. *Let $G \subset GL(V)$ be a finite group of linear transformations of a complex vector space V. Let $H \subset G$ be a subgroup and $U \subset V$ be an H-invariant subspace of codimension 1. Suppose that the natural morphism $\eta : U/H \to V/G$ is injective and all the different subspaces of the form gU $(g \in G)$ are transversal (i.e. the codimension of their intersection equals their number). Then η is a closed embedding.*

Proof. Since $V/G = \operatorname{Spec} \mathbb{C}[V]^G$ and $U/H = \operatorname{Spec} \mathbb{C}[U]^H$, we are to prove that the homomorphism $\eta^* : \mathbb{C}[V]^G \to \mathbb{C}[U]^H$ is surjective, i.e., that every H-invariant polynomial function f_0 on U extends to an G-invariant polynomial function on V. Let $V_1 = g_1U, \dots, V_m = g_mU$ be all the different subspaces of the form gU $(g \in G)$. Since they are transversal, they are spanned by basis vectors of some fixed basis of V. Define a polynomial function f_i on V_i by $f_i(g_ix) = f_0(x)$ for $x \in U$. If $g_ix = g_jy$ for $x, y \in U$, then, by our assumption, x and y are H-equivalent and, hence,

$$f_i(g_ix) = f_0(x) = f_0(y) = f_j(g_jy).$$

By Lemma 2 there exists a polynomial function f on V such that $f = f_i$ on V_i for all i. The definition of the functions f_i implies that the restriction of f to $\cup_{i=1}^s V_i$ is G-invariant. Averaging f over the group G, we obtain an G-invariant polynomial function on V, whose restriction to U coincides with f_0. □

Proof of Theorem 2. We shall prove the theorem for $m = n+1$, which will imply the general case.

Let $\{e_1, \ldots, e_{n+3}\}$ be the standard basis of the space $\mathbb{R}^{2,n+1}$ as in Subsection 2.2. Then the subspace $\mathbb{R}^{2,n} \subset \mathbb{R}^{2,n+1}$ is the orthogonal complement of the vector $e = e_{n+3}$ with $(e,e) = -1$.

The submanifolds $\mathcal{D}_n$ and $\gamma\mathcal{D}_n \quad (\gamma \in \Gamma)$ intersect in $\mathcal{D}_{n+1}$ if and only if $|(e, \gamma e)| < 1$, which in fact means that $(e, \gamma e) = 0$. But then $(e - \gamma e, e - \gamma e) = 2$, so the reflection $R_{e-\gamma e}$ belongs to the group Γ. Since it permutes $\mathcal{D}_n$ and $\gamma\mathcal{D}_n$, we obtain that $R_{e-\gamma e}\gamma$ leaves $\mathcal{D}_n$ invariant. If now $p, q \in \mathcal{D}_n$ and $p = \gamma q$, then $p = R_{e-\gamma e}\gamma q$, so p and q are Δ-equivalent. This shows that the map ν is injective.

It remains to prove that ν is an immersion. Let $p \in \mathcal{D}_n$. Denote by V (resp. U) the tangent space of $\mathcal{D}_{n+1}$ (resp. of $\mathcal{D}_n$) at p and by G the stabilizer of p in Γ. Let H be the normalizer of U in G. The morphism ν locally at p looks as the natural morphism $\eta : U/H \to V/G$ at 0. Since ν is injective, η is injective as well. By the above the different images of U under the action of G are mutually perpendicular and, hence, transversal. According to Lemma 3 this implies that η is a closed embedding. Thus, ν is an injective immersion and, hence, a closed embedding.

The same reasoning applied to the cones $\mathcal{L}_{n+1}$ ans $\mathcal{L}_n$ shows that $\hat{\nu}$ is also a closed embedding. □

5. Multipolarized quartic surfaces

Recall that the period map $\hat{p}$ (depending on the choice of the vector $h_0 \in J = J_{3,19}$) defines a $\mathbb{C}^*$-equivariant analytic isomorphism of the quotient Q°/SL_4 onto the complement of two irreducible divisors $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{L}_{19}/\Gamma'_{19}$ (see Subsection 3.3). In this section we show that under a suitable choice of h_0 the image of $\mathcal{L}_7$ in $\mathcal{L}_{19}/\Gamma'_{19}$ does not intersect the divisors $\mathcal{H}_1$ and $\mathcal{H}_2$, and describe the corresponding subvariety of Q°/SL_4.

5.1. **Multipolarizations of $K3$ surfaces.** Let $h_0 \in J$ be a primitive vector with $(h_0, h_0) = d > 0$, and $S_0 \subset J$ be a primitive hyperbolic sublattice containing h_0. Let T_0 be the orthogonal complement of S_0 in J. Set $T_0 \otimes \mathbb{R} = \mathbb{R}^{2,n}$ and use the notation of the introduction.

A *multipolarization* of type (h_0, S_0) of a $K3$ surface X is a vector $h \in S(X) \cap A_2(X)$ and a sublattice $S \subset S(X)$ containing h such that there exists an isomorphism $\varphi : H_2(X, \mathbb{Z}) \to J$ taking h to h_0 and S to S_0. If $\omega \in H_2(X, \mathbb{Z})$ is a non-zero regular differential 2-form on X, then $\varphi(\omega) \in \tilde{\mathcal{L}}_n$. We will require in addition that $\varphi(\omega) \in \mathcal{L}_n$. Then φ is defined up to a left multiplication by an element of $O^+(J)$ leaving h_0 and S_0 (or, equivalently, h_0 and T_0) invariant. Denote with $O^+(T_0, h_0)$ the group formed by the restrictions to T_0 of such elements of $O^+(J)$. It is a subgroup of finite index in $O^+(T_0)$. Via the map $(X, h, S) \mapsto \varphi(\omega)$ the multipolarized $K3$ surfaces of type (h_0, S_0) are parameterized by the variety $\mathcal{D}_n/O^+(T_0, h_0)$.

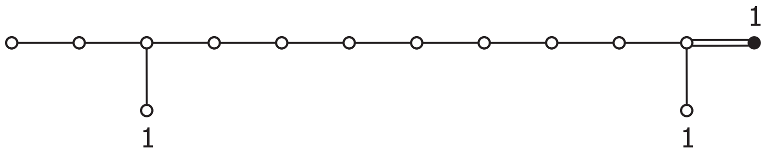
Forgetting S defines a morphism

$$\nu : \mathcal{D}_n/O^+(T_0, h_0) \to \mathcal{D}_{19}/\Gamma'_{19},$$

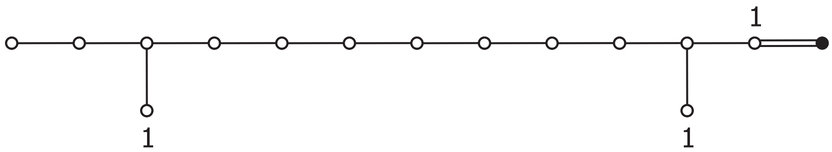
which is the normalization of its (closed) image by Proposition 6. The image of ν parameterizes the d-polarized $K3$ surfaces that admit a multipolarization of type (h_0, S_0).

Table 3

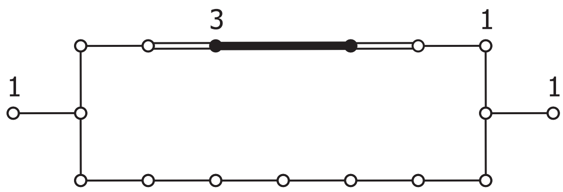
$I_{1,12}$



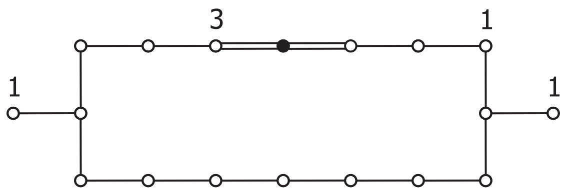
$I_{1,13}$



$I_{1,14}$



$I_{1,15}$



In a sense, the typical situation for a multipolarized $K3$ surface X is that $S(X) = S$. More precisely, this holds beyond the images in $\mathcal{D}_n/O^+(T_0, h_0)$ of countably many hyperplanes of the form

$$\mathcal{H}(a) = \{z \in \mathcal{L}_n : (z, a) = 0\} \quad \text{with} \quad a \in T_0, \quad (a, a) < 0.$$

This is similar to the distribution of irrational and rational numbers on the real line. Following this analogy, we will call a multipolarization (h, S) of a $K3$ surface X (and the corresponding point of $\mathcal{D}_n/O^+(T_0, h_0)$) *irrational*, if $S(X) = S$, and *rational* otherwise.

If one considers $K3$ surfaces X with a fixed 2-form ω, then all written above lifts to the level of the cone $\mathcal{L}_n$. In particular, the multipolarized pairs (X, ω) of type (h_0, S_0) are parameterized by the variety $\mathcal{L}_n/O^+(T_0, h_0)$, while the d-polarized pairs (X, ω) admitting a multipolarization of type (S_0, h_0) are parameterized by the image of the morphism

$$\hat{\nu} : \mathcal{L}_n/O^+(T_0, h_0) \to \mathcal{L}_{19}/\Gamma'_{19}.$$

5.2. **The choice of multipolarization types.** Slightly changing the notation of Subsection 2.2, define $I = I_{3,19}$ as the quadratic lattice with an orthogonal basis $\{e_0, e_1, \dots, e_{19}, e_{20}, e_{21}\}$ such that

$$(e_0, e_0) = (e_{20}, e_{20}) = (e_{21}, e_{21}) = 1,\ (e_1, e_1) = \dots = (e_{19}, e_{19}) = -1,$$

and $J = J_{3,19}$ as the quadratic lattice generated by the sublattice $I^{ev} \subset I$ and the vector

$$\sigma = \frac{1}{2}(e_0 + e_1 + \dots + e_{21}).$$

Let $3 \le n \le 7$ and $m = 19 - n$. Denote by S_n the intersection of the lattice J with the linear span of $e_0, e_1, \dots, e_m$. This is a quadratic lattice of type $D_{1,m}$, the even sublattice of $I_{1,m}$, and its orthogonal complement T_n is a quadratic lattice of type $D_{2,n}$ (see Subsection 2.4). We set

$$h_0 = 4e_0 + e_1 + \dots + e_{12}.$$

Clearly, $h_0 \in S_n$ and $(h_0, h_0) = 4$. In what follows the period map for quartic surfaces is supposed to be defined under this choice of h_0, and we will consider multipolarizations of types (h_0, S_n). Note that T_n lies in the orthogonal complement M of h_0 in J, which is a quadratic lattice of type $D_{2,19}$, in a non-standard way!

In Table 3 (resp. Table 4) the inner products of h_0 with simple roots (resp. simple 2-roots) of the lattices $I_{1,m}$, $m = 12, 13, 14, 15$ are indicated near the corresponding nodes of the Coxeter diagrams. Absence of a label means that the inner product equals 0. (For $m = 14, 15$ not all the simple 2-roots are included in the diagrams of Table 4.)

Proposition 7. *Under the above choice of S_0 and h_0, one has*

$$O^+(T_n, h_0) = \begin{cases} \Gamma'_7 & \textit{for} \quad n = 7, \\ \Gamma_6^{ext} & \textit{for} \quad n = 6, \\ \Gamma_n & \textit{for} \quad n = 5,\ 4,\ 3. \end{cases}$$

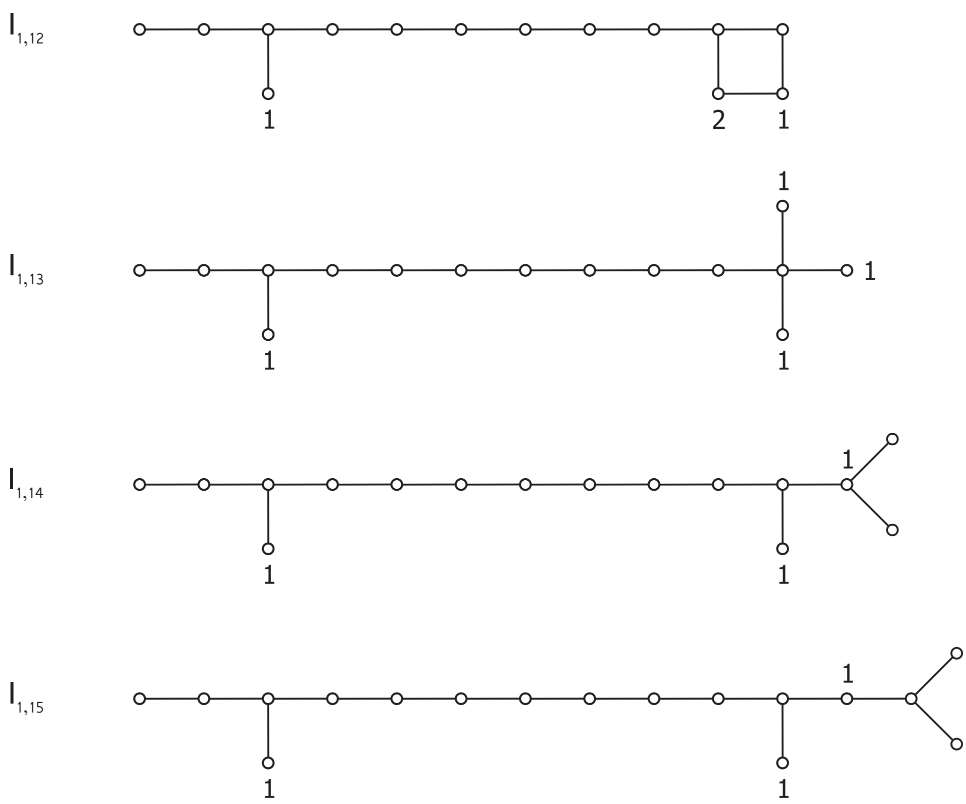
Proof. We are to determine, which elements of $O^+(T_n)$ extend to automorphisms of the lattice J fixing h_0. This can be done with help of Proposition 4. To apply it, we should first to determine, which automorphisms of the group $\mathfrak{D}(S_n)$ are realized by elements of $O(S_n)$ fixing h_0. (Note that such elements automatically belong to $O^+(S_n)$.)

By Proposition 3, if $n \ne 6$, then $O^+(S_n) = O^+_{1,m}(\mathbb{Z})$ and $O^+(T_n) = O^+_{2,n}(\mathbb{Z}) = \Gamma_n$, while $O^+(S_6) = O^{ext}_{1,13}(\mathbb{Z})$ (resp. $O^+(T_6) = O^{ext}_{2,6}(\mathbb{Z}) = \Gamma_6^{ext}$), an extension of index 3 of $O^+_{1,13}(\mathbb{Z})$ (resp. of Γ_6).

For $n \ne 7$, we have $(h_0, \alpha_m) = 0$, so the reflection R_{α_m} fixes h_0. Clearly, it acts non-trivially on $\mathfrak{D}(S_n)$. Further, the automorphisms of the system of simple 2-roots of $I_{1,13}$ permuting α_{12}, $\alpha'_{12}, \alpha_{15}$ (see Table 4) extend to automorphisms of S_n fixing h_0 and accordingly permuting the elements of the four-group $\mathfrak{D}(S_6)$ (cf. the proof of Proposition 2). It follows that, for $n \ne 7$, the stabilizer of h_0 in $O(S_n)$ maps onto $\mathfrak{D}(S_n)$. As for the stabilizer of h_0 in $O(S_7)$, it is generated by the simple reflections R_{α_k}, $k \le 11$, which are 2-reflections, and therefore acts trivially on $\mathfrak{D}(S_7)$.

Thus, for $n \ne 7$, we have $O^+(T_n, h_0) = O^+(T_n)$, whence the assertion of the proposition follows. As for the group $O^+(T_7, h_0)$, it coincides with $O'(T_7) = O'_{2,7}(\mathbb{Z}) = \Gamma'_7$. □

Table 4



The vector h_0 is linearly expressed in terms of simple 2-roots as follows:

(8) $h_0 = 3\alpha_1 + 6\alpha_2 + 9\alpha_3 + 8\alpha_4 + 7\alpha_5 + 6\alpha_6 + 5\alpha_7 + 4\alpha_8 + 3\alpha_9 + 2\alpha_{10} + \alpha_{11} + 4\alpha_{n+1}.$

Another expression can be obtained using the linear dependence (4).

5.3. **Verification of condition (*).** The following proposition plays a key role in this work.

Proposition 8. *There are no isotropic vectors $u \in J$ with $(h_0, u) = 1$ or 2 such that the subspace of $\mathbb{R}^{3,19}$ spanned by $e_0, \dots, e_{12}$ and u is hyperbolic.*

Proof. First, we shall show that there are no such vectors in S_7. According to Proposition 30, it suffices to test the isotropic vectors from $S_7 \cap A(S_7)$. Since $O^+(S_7) = O^+_{1,12}(\mathbb{Z})$, we have $A(S_7) = A(I_{1,12})$. Note also that the isotropic vectors of S_7 and $I_{1,12}$ are the same.

Using the algorithm described in Subsection 2.6, we see that the cone $A(I_{1,12})$ has exactly two isotropic edges, corresponding to parabolic subdiagrams of types $\tilde{E}_8 + \tilde{B}_3$ and $\tilde{B}_{11}$, and if u_1 and u_2 are the primitive vectors of these edges, then $(h_0, u_1) = (h_0, u_2) = 3$.

Suppose now that there is an isotropic vector $u \in J \setminus S_7$ satisfying the conditions of the proposition. We have $u = u' + u''$ with $u' \in S_7^*$, $u'' \in T_7^*$ (see Subsection 2.3). By our assumption the subspace spanned by $e_0, e_1, \dots, e_{12}$ and u'' is hyperbolic.

This means that $(u'', u'') < 0$, whence $(u', u') > 0$. Moreover, since $2u' \in I_{1,12}$, we have $4(u', u') \in \mathbb{Z}$. If $u' \in I_{1,12}$, then $(u', u') \in \mathbb{Z}$. Otherwise all the coordinates of u' are half-integers, whence $4(u', u') \equiv 5 \pmod 8$. Thus, in any case $(u', u') \geq 1$.

Taking into account that the Gram determinant of h_0 and u' cannot be positive, we obtain that $(u', u') = 1$, $(h_0, u') = 2$, and the Gram determinant equals 0. This means that $u' = \frac{1}{2}h_0$. But it is easy to see that $\frac{1}{2}h_0 \notin S_7^*$, a contradiction. □

5.4. **Strategy of the proof of Theorem 1.** Let $3 \leq n \leq 7$. For a form $F \in Q^\circ$ let $[F]$ denote its image in Q°/SL_4. Let Q_n° denote the set of those $F \in Q^\circ$, for which $\hat{p}([F])$ lies in the image of $\hat{\nu}$ or, in other words, $X(F)$ admits a multipolarization of type (h_0, S_n). Since the image of $\hat{\nu}$ is an $(n+1)$-dimensional closed subvariety of $\mathcal{L}_{19}^\circ/\Gamma_{19}'$, the subset Q_n°/SL_4 is an $(n+1)$-dimensional closed subvariety of Q°/SL_4.

Let T be the maximal torus of SL_4 consisting of the diagonal matrices. We will construct a T-invariant $(n+4)$-dimensional subvariety (in fact, subspace, unless $n = 3$) $R_n \subset Q$ with the following properties:

(P1) Every form $F \in Q^\circ$, for which $X(F)$ admits an irrational multipolarization of type (h_0, S_n), is SL_4-equivalent to a form of R_n, which is uniquely defined up to the action of T.

(P2) The natural morphism $\hat{\mu} : R_n/\!\!/T \to Q/\!\!/SL_4$ is finite.

(P3) If $R_n^\circ = R_n \cap Q^\circ$, then the complement of R_n°/T in $R_n/\!\!/T$ is of codimension ≥ 2.

Since the set of irrational points is dense in $\mathcal{L}_n/O^+(T_n, h_0)$, property (P1) will imply that the subvariety Q_n°/SL_4 lies in the closure of $\hat{\mu}(R_n^\circ/T)$ in Q°/SL_4; but for the dimension reason it must coincide with this closure.

Further, (P1) and (P2) will imply that $\hat{\mu}(R_n/\!\!/T)$ is closed in $Q/\!\!/SL_4$ and the morphism $\hat{\mu} : R_n/\!\!/T \to Q/\!\!/SL_4$ is the normalization over its image. It will follow that $\hat{\mu}(R_n^\circ/T)$ is closed in Q°/SL_4. Therefore,

$$Q_n^\circ/SL_4 = \hat{\mu}(R_n^\circ/T),$$

and R_n°/T is the normalization of Q_n°/SL_4.

Lifting the isomorphism

$$\hat{p} : Q_n^\circ/SL_4 \xrightarrow{\sim} \hat{\nu}(\mathcal{L}_n/O^+(T_n, h_0)),$$

to an isomorphism of the normalizations

$$\hat{q} : R_n^\circ/T \xrightarrow{\sim} \mathcal{L}_n/O^+(T_n, h_0),$$

will then yield an isomorphism of graded algebras

$$\hat{q}^* : A(\mathcal{D}_n, O^+(T_n, h_0)) \xrightarrow{\sim} \mathbb{C}[R_n^\circ/T].$$

Finally, (P3) will imply that the algebra $\mathbb{C}[R_n^\circ/T]$ coincides with the algebra $\mathbb{C}[R_n/\!\!/T] = \mathbb{C}[R_n]^T$, which we will easily calculate.

Some extra efforts will be needed for $n = 6$ and 7 in order to pass from the group $O^+(T_n, h_0)$ to the group Γ_n.

The following commutative diagram demonstrates the strategy of our proof:

$$\begin{array}{ccccc} R_n/\!\!/T & \longleftarrow & R_n^\circ/T & \xrightarrow[\sim]{\hat{q}} & \mathcal{L}_n/O^+(T_n, h_0) \\ \downarrow{\scriptstyle \hat{\mu}} & & \downarrow & & \downarrow{\scriptstyle \hat{\nu}} \\ Q/\!\!/SL_4 & \longleftarrow & Q^\circ/SL_4 & \xrightarrow[\hat{p}]{\sim} & \mathcal{L}_{19}^\circ/\Gamma_{19}' \end{array}$$

Here the non-marked horizontal arrows are open embeddings; all the vertical arrows are finite morphisms, the normalizations of their images.

5.5. **Canonical equations for the multipolarized quartics.** Table 4 and formulas (8), (4) provide enough information in order to explicitly determine equations of quartics admitting multipolarizations of types (h_0, S_n) $(n = 3, 4, 5, 6, 7)$. Let us first formulate the facts we are going to use.

Let $Y = Y(F) \subset \mathbb{C}P^3$ be a quartic with at most simple singularities, and $X = X(F)$ be the corresponding $K3$ surface, with the desingularization morphism $\varphi : X \to Y$ and the polarization $h \in S(X)$. Let $\alpha_1, \alpha_2, \dots$ be the simple 2-roots of the lattice $S(X)$. Each of them is a class of a (uniquely defined) smooth rational curve on X, which we shall denote by the same symbol. If $(h, \alpha_i) = 0$, then φ retracts the curve α_i into a (singular) point. The configuration of such curves defines the types of singular points of the quartic Y. If $(h, \alpha_i) = 1$ (resp. $(h, \alpha_i) = 2$), then $\varphi(\alpha_i)$ is a line (resp. a conic) on Y. If $h = \sum_i k_i\alpha_i$, then the divisor $\sum_{i:(h,\alpha_i)\neq 0} k_i\varphi(\alpha_i)$ is a plane section of Y.

Assume now that X admits an irrational multipolarization of type (h_0, S_n). Then $S(X) \simeq S_n$, and the diagrams of Table 4 show that the quartic Y has a singular point o of type A_{11} and contains two lines

$$l_1 = \varphi(\alpha_{m+1}), \quad l_2 = \varphi(\alpha_{m+2}) \quad (m = 19 - n),$$

passing through o. Denote by $\bar{S}$ the subgroup of S generated by the simple 2-roots orthogonal to h. It follows from (8) that

$$h \equiv 4\alpha_{m+1} \pmod{\bar{S}}.$$

This means that there is a plane $P_1 \subset \mathbb{C}P^3$ such that

$$P_1 \cap Y = 4l_1.$$

Thus, the flag (o, l_1, P_1) is canonically associated with F. By the action of the group SL_4 it can be moved to the standard co-ordinate flag, so we may (and will) assume that

$$o = (1 : 0 : 0 : 0), \quad l_1 : x_2 = x_3 = 0, \quad P_1 : x_3 = 0. \tag{9}$$

Consider now the cases $n = 7$, 6, 5, 4 separately. (The case $n = 3$ will be treated in Section 8.)

<u>Case $n = 7$.</u> Apart from the lines l_1, l_2, the surface Y contains a conic

$$q = \varphi(\alpha'_{11}),$$

also passing through o. Besides, it follows from (8) and (4) that

$$h \equiv \alpha_{13} + \alpha_{14} + \alpha'_{11} \pmod{\bar{S}},$$

which means that there is a plane $P_2 \subset \mathbb{C}P^3$ such that

$$P_2 \cap Y = l_1 + l_2 + q.$$

In addition to our assumptions (9), we may assume that

$$P_2 : x_2 = 0, \quad l_2 : x_1 = x_2 = 0. \tag{10}$$

Note that the conic q intersect the line l_1 only at the singular point o, because the curves α_{13} and α'_{11} do not intersect on X. This means that the line l_1 is tangent to q at o. (See Fig. 1.)

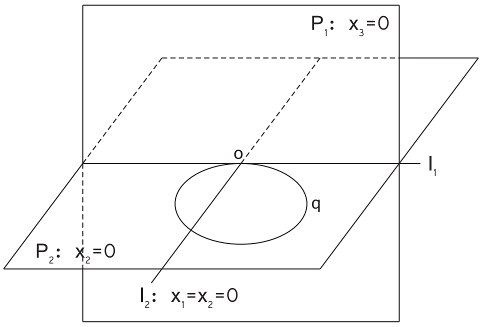


FIGURE 1

Taking all this into account, we see that, under our assumptions,

$$F = x_0x_2x_3(Ax_0 + f_1(x_1, x_2, x_3)) + x_1x_3(Bx_1^2 + x_3g_1(x_0, x_1, x_3)) \\ + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3) \quad (A, B, C \neq 0),$$

where f_1, g_1 are linear forms, and f_2 is a quadratic form.

By means of a linear substitution

$$x_0 \mapsto x_0 + a_1x_1 + a_2x_2 + a_3x_3$$

with some uniquely defined coefficients a_1, a_2, a_3, one can kill f_1, and by means of a linear substitution

$$x_1 \mapsto x_1 + bx_2$$

with some uniquely defined coefficient b, one can kill the term x_1^2 in f_2 (without violating (9) and (10)). Thus, we come to the form

$$F = Ax_0^2x_2x_3 + x_1x_3(Bx_1^2 + x_3g_1(x_0, x_1, x_3)) + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3), \tag{11}$$

where f_2 does not contain x_1^2 (and $A, B, C \neq 0$). It is important that this form is uniquely defined up to a (unimodular) diagonal linear substitution.

The space of forms (11), where f_2 does not contain x_1^2 (but with arbitrary A, B, C) will be denoted by R_7.

Case $n = 6$. As the second diagram of Table 4 shows, here the conic q degenerates into two lines

$$l_3 = \varphi(\alpha_{12}), \quad l_4 = \varphi(\alpha'_{12})$$

passing through o. We also have

$$h \equiv \alpha_{14} + \alpha_{15} + \alpha_{12} + \alpha'_{12} \pmod{\bar{S}},$$

so the lines l_1, l_2, l_3, l_4 lie on one plane, say, P_2. We may (and will) assume that P_2 is the plane $x_2 = 0$.

The lines l_2, l_3, l_4 are on equal foot. If we distinguish one of them, say, l_2, we may assume that it is given by the equations $x_1 = x_2 = 0$ (see Fig. 2) and can do as for $n = 7$, but then we should reduce the group Γ_6^{ext} acting on $\mathcal{L}_6$ to the group Γ_6, which is exactly what we need for the proof of Theorem 1. Doing in this way, we shall come to the canonical form

$$F = Ax_0^2x_2x_3 + x_1x_3(Bx_1^2 + x_3g_1(x_1, x_3)) + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3), \tag{12}$$

where f_2 does not contain x_1^2, This differs from (11) only by the absence of the term x_0 in g_1.

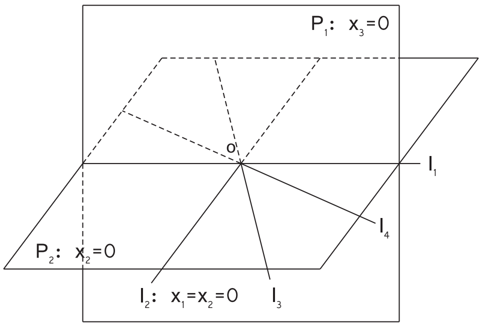


FIGURE 2

The space of forms (12), where f_2 does not contain x_1^2 (but with arbitrary A, B, C) will be denoted by R_6'.

If we consider the lines l_2, l_3, l_4 as equally standing, then we obtain the following intermediate form

$$F = x_0x_2x_3(Ax_0 + f_1(x_1, x_2, x_3)) + x_3(Bx_1^3 + x_3g_2(x_1, x_3)) \\ + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3) \quad (A, B, C \neq 0),$$

where f_1 is a linear form, and f_2, g_2 are quadratic forms. By means of a linear substitution

$$x_0 \mapsto x_0 + a_1x_1 + a_2x_2 + a_3x_3,$$

one can kill f_1, and by means of a linear substitution

$$x_1 \mapsto x_1 + b_2x_2 + b_3x_3,$$

one can kill the terms with x_1^2 in f_2 and g_2. The coefficients of these substitutions are uniquely defined. As a result, we obtain the form

$$F = Ax_0^2x_2x_3 + x_3(Bx_1^3 + x_3^2g_1(x_1, x_3)) + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3), \tag{13}$$

where f_2 does not contain x_1^2 (and $A, B, C \neq 0$). It is uniquely defined up to a diagonal linear substitution.

The space of forms (13), where f_2 does not contain x_1^2 (but with arbitrary A, B, C) will be denoted by R_6.

Case $n = 5$. Here the lines l_3 and l_4 stick together. Assuming that this double line is the line $x_1 = x_2 = 0$ (see Fig. 3) and proceeding as for $n = 7$, we obtain the canonical form

$$F = Ax_0^2x_2x_3 + x_1^2x_3(Bx_1 + B'x_3) + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3), \tag{14}$$

where f_2 does not contain x_1^2 (and $A, B, B', C \neq 0$).

The space of forms (14), where f_2 does not contain x_1^2 (but with arbitrary A, B, B', C) will be denoted by R_5.

Case $n = 4$. Here two singular points of type A_1 lying on the double line $x_1 = x_2 = 0$ stick together into a singular point of type A_3. It is easy to see that the co-ordinates of these points are defined by the equation

$$Ax_0^2 + f_{33}x_3^2 = 0,$$

where f_{33} is the coefficient of x_3^2 in f_2. These points stick together (into the point $p = (0 : 0 : 0 : 1)$), when $f_{33} = 0$.

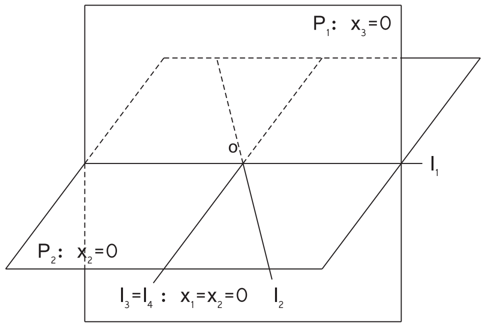


FIGURE 3

The space of forms (14), where f_2 does not contain x_1^2 and x_3^2 (but with arbitrary A, B, B', C) will be denoted by R_4.

In all the cases $\dim R_n = n + 4$ (and $\dim R'_6 = 10$), and property (P1) holds by the construction. It remains to prove properties (P2) and (P3).

6. Some geometric invariant theory for quartic forms

We turn now to the study of the canonical quartic forms introduced in the previous section, from the point of view of geometric invariant theory, i.e., to the study of their orbits and invariants.

6.1. **The weight decomposition of Q.** Let, as above, T be the maximal torus of SL_4 consisting of the diagonal matrices. Let ϵ_0, ϵ_1, ϵ_2, ϵ_3 be the diagonal entries of the matrix of T considered as elements of the character group of T in the additive notation (so $\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 = 0$). Then the roots of SL_4 are $\epsilon_i - \epsilon_j$ ($i \neq j$), and the simple roots are

$$\alpha_1 = \epsilon_0 - \epsilon_1,\ \alpha_2 = \epsilon_1 - \epsilon_2,\ \alpha_3 = \epsilon_2 - \epsilon_3.$$

For any root α, we denote by e_α the corresponding root vector (a matrix unit). We denote by B^+ (resp. B^-) the subgroup of upper (resp. lower) triangular matrices of SL_4.

The weight vectors for the representation of SL_4 in the space Q of quartic forms are the monomials

$$x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3}\ (k_0 + k_1 + k_2 + k_3 = 4),$$

and the corresponding weights are

$$k_0\epsilon_0 + k_1\epsilon_1 + k_2\epsilon_2 + k_3\epsilon_3.$$

They are the integer points of a regular tetrahedron: see Fig. 4.

6.2. **Stability of quartics with simple singularities.** Recall that a vector v in the representation space V of a reductive algebraic group G is called *stable* (in the sense of Mumford), if its stabilizer is finite and its orbit is closed. According to Mumford ([10, Theorem 2.1]; see also [13, Subsection 6.13]), this is not the case if and only if the representation weights involving in the weight decomposition of some vector of the orbit of v lie in one closed half-space (with boundary passing through 0) in the character space of a fixed maximal torus T of G.

The connection between the geometry of the SL_4-orbits of quartic forms in 4 variables and the singularities of the corresponding quartic surfaces was studied in details by J. Shah [17]. We need, however, only a small part of the information contained in [17], namely, the following

Proposition 9. *If the surface $Y(F) \subset \mathbb{C}P^3$ $(F \in Q)$ has at most simple singularities, then the form F is stable in the sense of Mumford.*

This follows from Theorem 2.4 of [17]. For convenience of the reader, we give here a sketch of the proof. All the facts on singularities of hypersurfaces that we use in this and subsequent proofs can be found in [1, §16].

Proof. We are to prove that if the weights of all non-zero terms of a form $F \in Q$ are contained in a closed half-space, then the surface $Y = Y(F)$ has non-simple singularities.

It is not hard to find all maximal sets of weights contained in a closed half-space. Up to permutations of x_0, x_1, x_2, x_3, they are defined by the following inequalities:
(H1) $3k_0 \leq k_1 + k_2 + k_3$ (or,equivalently, $k_0 \leq 1$);
(H2) $k_0 \leq k_1$;
(H3) $3k_0 + k_3 \leq 3k_1 + k_2$;
(H4) $k_0 + k_1 \leq k_2 + k_3$ (or,equivalently, $k_0 + k_1 \leq 2$);
(H5) $3k_0 \geq k_1 + k_2 + k_3$ (or, equivalently, $k_0 \geq 1$).
Suppose that the weights of all non-zero terms of a form $F \in Q$ are contained in one of these sets, and consider the five cases separately.

Case (H1). Consider the point $o = (1 : 0 : 0 : 0) \in Y$. Set $f(x_1, x_2, x_3) = F(1, x_1, x_2, x_3)$. Then

$$d_o f = 0, \quad d_o^2 f = 0,$$

so o is a non-simple singular point of Y.

Case (H2). Here we have

$$d_o f = 0, \quad d_o^2 f = a x_1^2, \quad d_o^3 f = x_1 g_2(x_1, x_2, x_3),$$

where g_2 is a quadratic form. (For simplicity, we write x_i instead of dx_i, considering the multiple differentials of f as homogeneous polynomials in x_1, x_2, x_3.) One may assume that $a \neq 0$, since otherwise (H1) holds. Then f is formally equivalent to a function $\tilde{f}$ such that

$$d_o \tilde{f} = 0, \quad d_o^2 \tilde{f} = x_1^2, \quad d_o^3 \tilde{f} = 0,$$

so o is a non-simple singular point.

Case (H3). We have

$$d_o f = 0, \quad d_o^2 f = a x_1^2, \quad d_o^3 f = b x_2^3 + x_1 g_2(x_1, x_2, x_3),$$

where g_2 is a quadratic form not containing x_3^2. One may assume that $a, b \neq 0$, since otherwise (H1) of (H2) holds. Then f is formally equivalent to a function $\tilde{f}$ such that

$$d_o \tilde{f} = 0, \quad d_o^2 \tilde{f} = x_1^2, \quad d_o^3 \tilde{f} = x_2^3, \quad d_0^4 \tilde{f} = 0, \quad d_0^5 \tilde{f} = 0,$$

so o is again a non-simple singular point.

Case (H4). Here all the points of the line $x_2 = x_3 = 0$ are singular, so the singularities of Y are not isolated.

Case (H5). In this case the surface Y decomposes into the plane $x_0 = 0$ and a cubic surface (so its singularities are not isolated). □

6.3. **A general finiteness criterion.** Let $G \subset GL(V)$ be a reductive algebraic linear group, and $\pi_G : V \to V/\!\!/G$ be the factorization morphism. The cone $\mathfrak{N}_G(V) := \pi_G^{-1}(\pi_G(0))$ is called the *null-cone* for the action of G on V. Let T be a maximal torus of G. According to Hilbert–Mumford ([10, Theorem 2.1]; see also [13, Subsection 5.4]), a vector $v \in V$ belongs to $\mathfrak{N}_G(V)$ if and only if $gv \in \mathfrak{N}_T(V)$ for some $g \in G$. The latter takes place if and only if the representation weights involving in the weight decomposition of gv lie in one open half-space (with boundary passing through 0) in the character space of a fixed maximal torus T of G.

Let now $H \subset G$ be a reductive subgroup, and $U \subset V$ be an H-invariant subspace. Then the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle \pi_H}\downarrow & & \downarrow{\scriptstyle \pi_G} \\ U/\!\!/H & \underset{\mu}{\longrightarrow} & V/\!\!/G \end{array}$$

defines a natural morphism $\mu : U/\!\!/H \to V/\!\!/G$.

Proposition 10. *The morphism μ is finite if and only if*

$$\mathfrak{N}_H(U) = \mathfrak{N}_G(V) \cap U. \tag{15}$$

Proof. Set $A = \mathbb{C}[V]^G$, $B = \mathbb{C}[U]^H$. Finiteness of μ means that the algebra B is a finite extension of the subalgebra $\mu^* A$ formed by the restrictions to U of G-invariant polynomial functions on V. Let A_+ be the maximal ideal of A, constituted by the polynomials without constant term. Then a minimal system of homogeneous generators of the A-module B is a basis of a complementary subspace of the ideal $I = B\mu^* A_+$ in B. Hence, B is a finitely generated A-module (i.e., a finite extension of $\mu^* A$) if and only if the ideal I has a finite codimension in B. Geometrically, this means that $\mu^{-1}(\pi_G(0)) = \pi_H(0)$, which is equivalent to (15). □

6.4. **The space R.** All the subspaces R_n, $n = 4, 5, 6, 7$, and R_6' introduced in Subsection 5.5, are contained in the B^--invariant subspace $R \subset Q$ generated by the monomials

$$x_0^2 x_2 x_3, \quad x_1^3 x_3, \quad x_2^4, \tag{16}$$

whose weights are

$$\alpha = 2\epsilon_0 + \epsilon_2 + \epsilon_3, \quad \beta = 3\epsilon_1 + \epsilon_3, \quad \gamma = 4\epsilon_2. \tag{17}$$

It is remarkable that all the other weights of R have the form

$$-p\alpha - q\beta - r\gamma \tag{18}$$

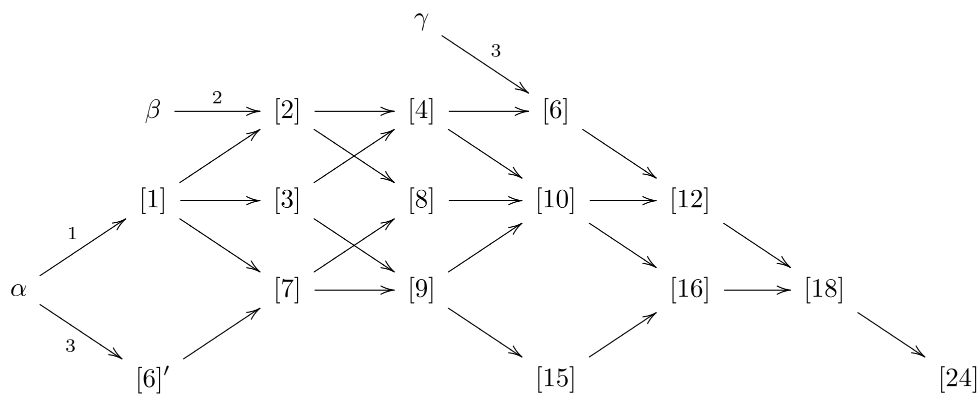
with non-negative integer coefficients p, q, r. We denote the set of weights of R by Λ. They are shown as punctured and black nodes on Fig. 4 representing all the weights of Q.

The set Λ also represented in the following commutative diagram, where the weight (18) is denoted by the symbol $[p + q + r + 1]$ (whose sense will be revealed below). There is only one case, when for different monomials of R the sums $p + q + r + 1$ are equal, namely, there are two monomials with $p + q + r + 1 = 6$; in this case we denote the corresponding weights by $[6]$ and $[6]'$. The arrow going from a weight λ to a weight μ, means that μ is obtained from λ by subtracting a simple root; the

TABLE 5

Weight	k_0, k_1, k_2, k_3	p, q, r
[1]	1 1 1 1	0 0 0
[2]	0 2 1 1	1 0 0
[3]	1 0 2 1	1 1 0
[4]	0 1 2 1	2 1 0
[6]	0 0 3 1	3 2 0
[6]′	2 0 0 2	2 2 1
[7]	1 1 0 2	3 2 1
[8]	0 2 0 2	4 2 1
[9]	1 0 1 2	4 3 1
[10]	0 1 1 2	5 3 1
[12]	0 0 2 2	6 4 1
[15]	1 0 0 3	7 5 2
[16]	0 1 0 3	8 5 2
[18]	0 0 1 3	9 6 2
[24]	0 0 0 4	12 8 3

number of this simple root is defined by the direction of the arrow as indicated on the arrows going from α, β, γ.



All the monomials $x_0^{k_0}x_1^{k_1}x_2^{k_2}x_3^{k_3} \in R$, distinct from the monomials (16), together with their weights and the coefficients p, q, r, are listed in Table 5.

It follows from the structure of Λ that the algebra $\mathbb{C}[R]^T$ is freely generated by monomials (in the basis of R formed by weight vectors). More precisely, write any $F \in R$ as

$$F = Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4 + \sum_{p,q,r} f_{pqr}x^{[pqr]} \quad (f_{pqr} \in \mathbb{C}),$$

where $x^{[pqr]} = x_0^{k_0}x_1^{k_1}x_2^{k_2}x_3^{k_3}$ is the monomial of weight $-p\alpha - q\beta - r\gamma$. Then

$$I_{p+q+r+1} = A^pB^qC^rf_{pqr} \tag{19}$$

is a T-invariant polynomial on R of degree $p+q+r+1$. Clearly, thus defined invariants are algebraically independent and generate the algebra $\mathbb{C}[R]^T$. In the only case, when for two different monomials the sums $p+q+r+1$ are the same (and are equal to 6), we shall denote the corresponding invariants by I_6 and I_6'.

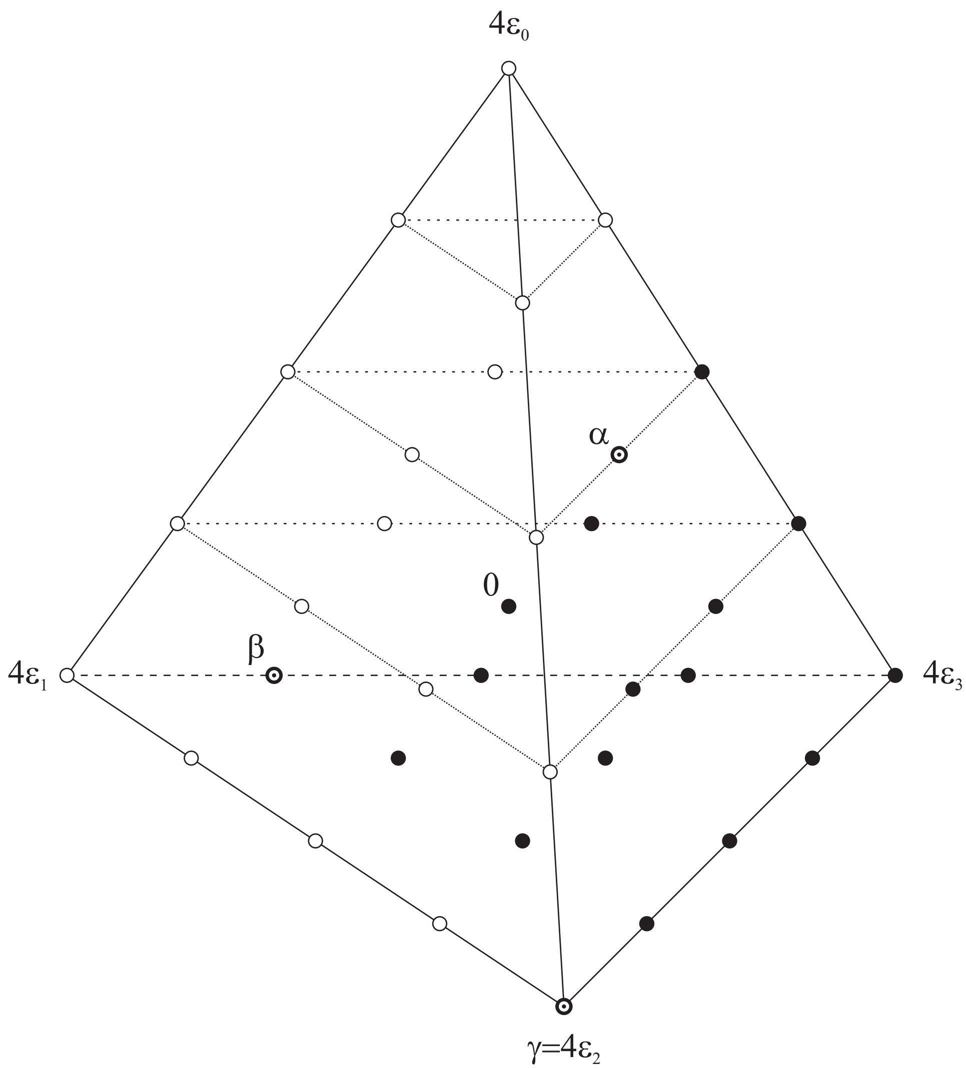


FIGURE 4

6.5. **Finiteness of the morphisms $R_n/\!\!/T \to Q/\!\!/SL_4$.** Let $R_0 \subset R$ be a T-invariant subspace containing the monomials (16), and let $\Lambda_0 \subset \Lambda$ be the set of weights of R_0. It is obvious that the algebra $\mathbb{C}[R_0]^T$ is freely generated by the invariants (19) corresponding to the weights of Λ_0.

Suppose that

(R1) $\Lambda_0 \not\ni [1], [2], [3], [6]', [9]$;

(R2) $\Lambda_0 \not\supset \{[8], [16], [24]\}$.

Proposition 11. *Under the above assumptions, the natural morphism*

$$\hat{\mu} : R_0/\!\!/T \to Q/\!\!/SL_4$$

is finite.

This is applicable to the subspaces R_n, $n = 4, 5, 6, 7$, and R_6', and thereby proves property (P2) of Subsection 5.4.

Proof. In view of Proposition 10, we are to prove that if a form $F \in R_0$ does not lie in $\mathfrak{N}_T(R_0)$, then it does not lie in $\mathfrak{N}_{SL_4}(Q)$. Clearly,

$$\mathfrak{N}_T(R_0) = \mathfrak{N}_T(R) \cap R_0.$$

The cone $\mathfrak{N}_T(R)$ is the union of the maximal T-invariant subspaces of R, whose weights lie in an open half-space. It is easy to see that there are 4 such subspaces N_1, N_2, N_3, N_4, whose sets of weights are

$$\begin{aligned}
\Lambda_1 &= \Lambda \setminus \{\alpha,\, [1]\},\\
\Lambda_2 &= \Lambda \setminus \{\beta,\, [1],\, [2]\},\\
\Lambda_3 &= \Lambda \setminus \{\gamma,\, [1],\, [2],\, [3],\, [4],\, [6]\},\\
\Lambda_4 &= \{\alpha,\, \beta,\, \gamma\}.
\end{aligned}$$

Suppose that $F \in R_0 \setminus \mathfrak{N}_T(R)$ but $F \in \mathfrak{N}_{SL_4}(Q)$. Then $gF \in \mathfrak{N}_T(R)$ for some $g \in SL_4$. One may assume that the weights of non-zero terms of gF lie in a B^--invariant open half-space. Let U^- denote the unipotent radical of B^-. Using the Bruhat decomposition, write $g = bwu$, where $b \in B^-$, $u \in U^-$, and w is an element of the normalizer of T. Then clearly the weights of non-zero terms of $uF \in R$ also lie in an open half-space, that is, uF belongs to one of the subspaces N_1, N_2, N_3, N_4 (but F does not belong to any of these subspaces).

Consider now the four cases separately.

Case 1: $uF \in N_1$. Since the subspace N_1 is U^--invariant (see the diagram of Λ in Subsection 6.4), we immediately come to a contradiction with the assumption $F \notin N_1$.

Case 2: $uF \in N_2$ but $uF \notin N_1$. The subspace N_2 is invariant under the regular subgroup of U^- generated by all negative root vectors but $e_{-\alpha_1}$. Therefore, one may assume that $u = \exp te_{-\alpha_1}$ $\quad (t \in \mathbb{C})$. Since the form uF does not lie in N_1, it contains a term of weight α. Hence, if $t \neq 0$, then $F = u^{-1}uF$ contains a term of weight [1], which contradicts (R1). Thus, $t = 0$ and $F = uF$, which contradicts the assumption $F \notin N_2$.

Case 3: $uF \in N_3$ but $uF \notin N_1 \cup N_2$. The subspace N_3 is invariant under the regular subgroup of U^- generated by all negative root vectors but $-\alpha_1$, $-\alpha_2$, $-\alpha_1-\alpha_2$. Therefore, one may assume that $u = \exp t_1e_{-\alpha_1}{\cdot}\exp t_2e_{-\alpha_2}{\cdot}\exp t_3e_{-\alpha_1-\alpha_2}$ $\quad (t_1, t_2, t_3 \in \mathbb{C})$. Since the form uF does not lie in $N_1 \cup N_2$, it contains terms of weights α and β. Hence, if $t_1 \neq 0$, then $F = u^{-1}uF$ contains a term of weight [1]; if $t_2 \neq 0$, then F contains a term of weight [2]; if $t_3 \neq 0$, then F contains a term of weight [3]. In all these cases we come to a contradiction with (R1). Thus, $t_1 = t_2 = t_3 = 0$ and $F = uF$, which contradicts the assumption $F \notin N_3$.

Case 4: $uF \in N_4$ but $uF \notin N_1 \cup N_2 \cup N_3$. In this case

$$uF = Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4$$

with $A, B, C \neq 0$. Let

$$u = \exp t_1e_{-\alpha_1}{\cdot}\exp t_2e_{-\alpha_2}{\cdot}\exp t_3e_{-\alpha_1-\alpha_2}{\cdot}\exp t_4e_{-\alpha_3}{\cdot}\exp t_5e_{-\alpha_2-\alpha_3}{\cdot}\exp t_6e_{-\alpha_1-\alpha_2-\alpha_3}.$$

Considering consequently the coefficients of the monomials of weights [1], [2], [3], [6]′, [9] in $F = u^{-1}uF$ and using (R1), we prove that $t_1 = t_2 = t_3 = t_4 = t_6 = 0$, so $u = \exp t_5 e_{-\alpha_2-\alpha_3}$. Now, if $t_5 \neq 0$, then all the coefficients of the monomials of weights [8], [16], [24] in F are non-zero, which contradicts (R2). Thus, $F = uF$, which contradicts the assumption $F \notin N_4$. □

7. The singularities of canonical quartics

In this section we prove property (P3) and complete the proof of the main theorem.

7.1. **Singularities of quartic curves.** We need the following auxiliary result, which must be known to specialists.

Proposition 12. *Any singularity of an irreducible quartic curve in* $\mathbb{C}P^2$ *is simple.*

Proof. Let the curve is given by the equation $f(x,y) = 0$ on an affine chart, where the point $o = (0,0)$ is singular. If $d_o^2 f \neq 0$, then o is a singular point of type A. If $d_o^2 f = 0$, then the cubic form $d_o^3 f$ must be non-zero, since otherwise f is a quartic form and, hence, the curve decomposes into four lines. If $d_o^2 f = 0$ but the form $d_o^3 f$ has at least two different roots, then o is a singular point of type D. Thus, we may assume that

$$f(x,y) = x^3 + h(x,y),$$

where h is a quartic form. The form h must contain the term y^4, since otherwise f is divisible by y; but then o is a singular point of type E_6. □

7.2. **Some family of quartic surfaces with simple singularities.** Consider the subspace $\bar{R} \subset R$ consisting of the forms

$$F = Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4 + (Dx_1^2 + Ex_1x_2 + Fx_2^2)x_3^2. \tag{20}$$

It is contained in all the subspaces R_7, R_6, R_6', R_5, R_4 introduced in Subsection 5.5. The algebra $\mathbb{C}[\bar{R}]^T$ is freely generated by the invariants I_8, I_{10}, I_{12}.

Proposition 13. *If* $A, B, C \neq 0$ *and not all the coefficients* D, E, F *are equal to* 0*, then the quartic surface* $Y = Y(F)$ *is irreducible and has only simple singularities.*

Proof. To prove the irreducibility of F, assign the weights 10, 9, 7, 1 to the variables x_0, x_1, x_2, x_3. Then the highest weight component of F will be $Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4$ (of weight 28), which is obviously an irreducible polynomial. It follows that the very polynomial F is also irreducible.

Consider the projection of Y to the plane $x_0 = 0$. In the domain $x_2, x_3 \neq 0$ it is a two-sheeted covering, ramified over the quartic curve

$$Z: \quad Bx_1^3x_3 + Cx_2^4 + (Dx_1^2 + Ex_1x_2 + Fx_2^2)x_3^2 = 0.$$

Using the same weights of the variables as above, one can easily prove that this curve is irreducible. The singularities of Y in the domain $x_2, x_3 \neq 0$ are nothing but the singularities of the curve Z, and their stable types are the same. By Proposition 12 they are simple.

Further, it is easy to see that the only singular point $(x_0 : x_1 : x_2 : x_3)$ of Y with $x_2 = 0$, $x_3 \neq 0$ is the point $p = (0 : 0 : 0 : 1)$. The singularity of Y at p is the singularity of the function

$$g(x_0, x_1, x_2) = F(x_0, x_1, x_2, 1) \\ = (Dx_1^2 + Ex_1x_2 + Fx_2^2) + (Ax_0^2x_2 + Bx_1^3) + Cx_2^4$$

at the origin. If the quadratic form $Dx_1^2 + Ex_1x_2 + Fx_2^2$ is non-degenerate, then the singularity is of type A. Otherwise, this quadratic form is the square of a non-zero linear form $l(x_1, x_2)$. If both coefficients of l are non-zero, then the 3-jet of g is equivalent to $x_2^2 + (ax_0^2x_1 + bx_1^3)$ with $a, b \neq 0$, and the singularity is of type D_4. If l is proportional to x_1, then g is formally equivalent to $x_1^2 + x_0^2x_2 + x_2^4$, and the singularity is of type D_5. If l is proportional to x_2, then g is formally equivalent to $x_2^2 + x_1^3 + x_0^4$, and the singularity is of type E_6.

Finally, the only singular point $(x_0 : x_1 : x_2 : x_3)$ of Y with $x_2 = x_3 = 0$ is the point $o = (1 : 0 : 0 : 0)$. The singularity of Y at o is the singularity of the function

$$f(x_1, x_2, x_3) = F(1, x_1, x_2, x_3) \\ = Ax_2x_3 + Bx_1^3x_3 + Cx_2^4 + (Dx_1^2 + Ex_1x_2 + Fx_2^2)x_3^2,$$

and is of type A. □

7.3. **Completing the proof of the main theorem.** Let $R_0 \subset R$ be any T-invariant subspace containing $\bar{R}$. Under the factorization morphism $R_0 \to R_0/\!\!/T$ the subspace $\bar{R}$ goes to the 3-dimensional subspace $\bar{R}/\!\!/T \subset R_0/\!\!/T$, defining by the equations $I_d = 0$ with $d \neq 8$, 10, 12. One can observe that all the forms (20) that do not satisfy the conditions of Proposition 13 goes to 0 under this morphism.

Set $R_0^\circ = R_0 \cap Q^\circ$. Then by Proposition 13 R_0°/T contains $\bar{R}/\!\!/T \setminus \{0\}$. This implies that the complement of R_0°/T in $R_0/\!\!/T$ cannot contain divisors, since any such divisor would intersect the subspace $\bar{R}/\!\!/T$ non-trivially.

This is applicable to all the subspaces R_7, R_6, R_6', R_5, R_4 introduced in Subsection 5.5. Thus, we have proved properties (P1), (P2), (P3) of Subsection 5.4 for these subspaces and thereby, taking into account Proposition 7, have proved the isomorphisms

$$A(\mathcal{D}_n, \Gamma_7') \simeq \mathbb{C}[R_7]^T. \tag{21}$$

$$A(\mathcal{D}_n, \Gamma_6^{ext}) \simeq \mathbb{C}[R_6]^T. \tag{22}$$

$$A(\mathcal{D}_n, \Gamma_5) \simeq \mathbb{C}[R_5]^T. \tag{23}$$

$$A(\mathcal{D}_n, \Gamma_4) \simeq \mathbb{C}[R_4]^T. \tag{24}$$

We can also state (see Case $n = 6$ in Subsection 5.5) that

$$A(\mathcal{D}_n, \Gamma_6) \simeq \mathbb{C}[R_6']^T. \tag{25}$$

Recall that the algebras $\mathbb{C}[R_0]^T$ for T-invariant subspaces $R_0 \subset R$ containing the monomials (16) were described in Subsection 6.4. Being combined with that description, the isomorphisms (21) and (22) yield the following theorems.

Theorem 3. *The algebra $A(\mathcal{D}_n, \Gamma_7')$ is freely generated by forms of weights* 4, 6, 7, 8, 10, 12, 16, 18.

Theorem 4. *The algebra $A(\mathcal{D}_6, \Gamma_6^{ext})$ is freely generated by forms of weights* 4, 6, 10, 12, 16, 18, 24.

Since the group Γ_7 is generated by Γ_7' and the operator $-1 \in O_{2,7}^+$, we obtain the assertion of Theorem 1 for $n = 7$. The cases $n = 6$, 5, 4 follow from (25), (23), (24), resp.

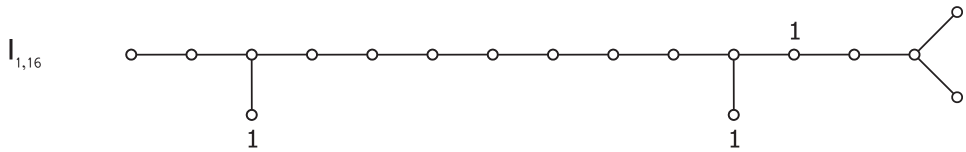
8. The case $n = 3$

In this section we apply the above method to the case $n = 3$. Unlike the cases $n = 7$, 6, 5, 4, the algebra $A(\mathcal{D}_3, \Gamma_3)$ turns to be non-free, but its generators are subject just to one quadratic relation. Moreover, the subgroup Γ_3^r generated by all reflections contained in Γ_3 is of index 2, and the algebra $A(\mathcal{D}_3, \Gamma_3^r)$ is free.

8.1. **The algebra** $A(\mathcal{D}_3, \Gamma_3)$**.** The simple 2-roots of the lattice $I_{1,16}$ include the roots

$$\alpha_i = -e_i + e_{i+1} \quad (i = 1, \dots, 15), \quad \alpha_{15}' = -e_{15} - e_{16},$$
$$\alpha_{17} = e_0 + e_1 + e_2 + e_3,$$
$$\alpha_{18} = 3e_0 + e_1 + \dots + e_{11}.$$

Their Coxeter diagram, with the inner products of the roots with h_0 indicated near the nodes, looks as follows:



As in the case $n = 4$, every form $F \in Q^\circ$, for which $X(F)$ admits an irrational multipolarization of type (h_0, S_3), is SL_4-equivalent to a uniquely defined, up to the action of T, form (14), with f_2 not containing x_1^2 and x_3^2, and $A, B, B', C \neq 0$. The difference is that the singular point $p = (0 : 0 : 0 : 1)$ is now of type D_4 rather than A_3. This means that the rank of the second differential of the function $f(x_0, x_1, x_2) = F(x_0, x_1, x_2, 1)$ at the origin reduces to 1. If

$$f_2 = 2f_{12}x_1x_2 + f_{22}x_2^2 + 2f_{13}x_1x_3 + 2f_{23}x_2x_3,$$

then

$$f = (B'x_1^2 + 2f_{13}x_1x_2 + 2f_{23}x_2^2) + (\text{members of degree 3 and 4}).$$

Thus, there must be

$$2B'f_{23} = f_{13}^2. \tag{26}$$

Let R_3 denote the T-invariant quadratic cone in R_4 defined by the equation (26). Then property (P1) holds for R_3 by the construction. Property (P2) holds, since it holds for R_4 and, hence, for any closed T-invariant subvariety of R_4. The algebra $\mathbb{C}[R_3]^T = \mathbb{C}[R_3/\!\!/T]$ is generated by the restrictions J_4, J_6, J_8, J_{10}, J_{12} of the invariants I_4, I_6, I_8, I_{10}, I_{12} to R_3, which are subject to the only defining relation

$$2J_8J_{12} = J_{10}^2,$$

coming from (26).

Let us now prove property (P3) for R_3. Let $\bar{R} \subset R_4$ be the subspace of forms (20). Then $\bar{R}/\!\!/T$ is a 3-dimensional subspace of $R_4/\!\!/T$, and by Proposition 13

$$(\bar{R}/\!\!/T) \setminus 0 \subset R_4^\circ/T.$$

It follows that $(\bar{R} \cap R_3)/\!\!/T$ is a 2-dimensional subcone of the cone $R_3/\!\!/T$, and, except for its vertex, it lies in R_3°/T. This implies that the complement of R_3°/T in $R_3/\!\!/T$ contains no divisors, since any such divisor would intersect the cone $(\bar{R} \cap R_3)/\!\!/T$ along a line.

Thus, all the properties (P1)—(P3) hold for R_3 and, hence, as was explained in Subsection 5.4, the period map induces a $\mathbb{C}^*$-equivariant isomorphism of algebraic varieties

$$R_3^\circ/T \xrightarrow{\sim} \mathcal{L}_3/\Gamma_3 \tag{27}$$

and thereby an isomorphism of graded algebras

$$A(\mathcal{D}_3, \Gamma_3) \xrightarrow{\sim} \mathbb{C}[R_3]^T. \tag{28}$$

This yields

Theorem 5. *The algebra $A(\mathcal{D}_3, \Gamma_3)$ is generated by forms Φ_4, Φ_6, Φ_8, Φ_{10}, Φ_{12} of weights 4, 6, 8, 10, 12, which are subject to the only defining relation*

$$\Phi_8\Phi_{12} = \Phi_{10}^2$$

(under a suitable normalization of these forms).

8.2. **The essential fundamental group.** Let $\mathcal{X}$ be a normal irreducible analytic space. Following O. V. Shvartsman [18], we shall call the group $\pi_1(\mathcal{X}^{\mathrm{reg}})$ the *essential fundamental group* of $\mathcal{X}$ and denote it by $\pi_1^e(\mathcal{X})$. Clearly, it does not change when deleting any subvariety of codimension > 1. The variety $\mathcal{X}$ is called *strongly simply connected*, if $\pi_1^e(\mathcal{X}) = 0$.

An automorphism γ of $\mathcal{X}$ is called a *(complex) reflection* if the subspace $\mathcal{X}^\gamma$ of its fixed points is of codimension 1.

Let Γ be a discrete group of holomorphic automorphisms of $\mathcal{X}$. If it is generated by reflections, then the natural homomorphism

$$\varphi_\Gamma : \pi_1^e(\mathcal{X}) \to \pi_1^e(\mathcal{X}/\Gamma)$$

is surjective [18, Theorem 2.2].

On the contrary, if Γ does not contain reflections, then φ_Γ is injective and the factorgroup $\pi_1^e(\mathcal{X}/\Gamma)/\varphi_\Gamma(\pi_1^e(\mathcal{X}))$ is naturally isomorphic to Γ. More precisely, let $\mathcal{X}' \subset \mathcal{X}$ be the open subset obtained by deleting the singular locus and the subvarieties $\mathcal{X}^\gamma$ for all non-identity $\gamma \in \Gamma$. Then the factorization map $\mathcal{X}' \to \mathcal{X}'/\Gamma$ is a non-ramified topological covering, whence the assertion follows, since $\pi_1^e(\mathcal{X}') = \pi_1^e(\mathcal{X})$ and $\pi_1^e(\mathcal{X}'/\Gamma) = \pi_1^e(\mathcal{X}/\Gamma)$.

We are going to apply this ideology to the affine algebraic variety

$$\mathcal{X} = \operatorname{Spec} \mathbb{C}[R_3]^T = R_3/\!\!/T.$$

First of all, we shall prove that $\pi_1^e(\mathcal{X}) = \mathbb{Z}_2$ by constructing a 2-sheeted covering $\mathbb{C}^4 \to \mathcal{X}$.

Set for brevity

$$A = \mathbb{C}[R_3]^T = \mathbb{C}[J_4,\, J_6,\, J_8,\, J_{10},\, J_{12}]$$

and consider the algebra

$$A_1 = A[\sqrt{J_8}, \sqrt{J_{12}}],$$

having in mind that $\sqrt{J_8}\sqrt{J_{12}} = J_{10}$. It has a natural grading extending the grading of A, and is freely generated by the homogeneous elements J_4, $\sqrt{J_8}$, J_6, $\sqrt{J_{12}}$ of degrees 4, 4, 6, 6 respectively. Set $\mathcal{X}_1 = \operatorname{Spec} A_1 \simeq \mathbb{C}^4$, and let

$$\varphi_1 : \mathcal{X}_1 \to \mathcal{X}$$

be the morphism defined by the embedding $A \hookrightarrow A_1$.

We have $A = A_1^{\theta_1}$, where θ_1 is the involution multiplying $\sqrt{J_8}$ and $\sqrt{J_{12}}$ by -1 and fixing J_4 and J_6. Denote by the same letter the involutory automorphism of the variety $\mathcal{X}_1$ defined by θ_1. Then φ_1 can be viewed as the factorization by the group $\langle\theta_1\rangle$. It is important that θ_1 is not a reflection, since $\dim \mathcal{X}_1^{\theta_1} = 2$. This implies that

$$\pi_1^e(\mathcal{X}) = \mathbb{Z}_2.$$

8.3. **The reflection subgroup of Γ_3.** Set $\mathcal{X}^\circ = R_3^\circ/T \subset \mathcal{X}$. This is a Zariski open subset of $\mathcal{X}$, whose complement does not contain divisors. Via the period map, it is identified with $\mathcal{L}_3/\Gamma_3$.

Let Γ_3^r be the subgroup generated by all reflections contained in Γ_3. Set

$$\mathcal{X}_2^\circ = \mathcal{L}_3/\Gamma_3^r.$$

Then

$$\mathcal{X}^\circ = \mathcal{X}_2^\circ/(\Gamma_3/\Gamma_3^r).$$

Since the group Γ_3/Γ_3^r, acting on $\mathcal{X}_2^\circ$, does not contain reflections, we have the following exact sequence:

$$0 \longrightarrow \pi_1^e(\mathcal{X}_2^\circ) \longrightarrow \pi_1^e(\mathcal{X}^\circ) \longrightarrow \Gamma_3/\Gamma_3^r \longrightarrow 0. \tag{29}$$

Since the middle term is $\mathbb{Z}_2$, there are only two possibilities for all these groups. In fact we shall prove below that $\Gamma_3^r \neq \Gamma_3$. It will then follow that $\pi_1^e(\mathcal{X}_2^\circ) = 0$, i.e. the variety $\mathcal{X}_2^\circ$ is strongly simply connected.

Proposition 14. *The group $\Gamma_3 = O_{2,3}^+(\mathbb{Z})$ is not generated by reflections.*

Proof. Consider the group $\mathbb{Z}_2^{2,3} = D_{2,3}/2D_{2,3}$, where $D_{2,3} = I_{2,3}^{ev}$ (see the notation in Section 2). It can be viewed as a 5-dimensional vector space over $\mathbb{Z}_2$. The inner product in $D_{2,3}$ induces an alternating form in $\mathbb{Z}_2^{2,3}$, which we will denote with the same symbol $(.,.)$. Moreover, one can define a quadratic form q in $\mathbb{Z}_2^{2,3}$ by

$$q([x]) = \frac{1}{2}(x,x) + 2\mathbb{Z} \quad \text{for} \quad x \in D_{2,3},$$

where $[x]$ stands for the image of x in $\mathbb{Z}_2^{2,3}$, so that

$$(\xi, \eta) = q(\xi + \eta) - q(\xi) - q(\eta).$$

There is a basis $\{f_0, f_1, f_2, f_3, f_4\}$ of the lattice $I_{2,3}$ such that

$$(f_0, f_0) = (f_1, f_2) = (f_2, f_1) = (f_3, f_4) = (f_4, f_3) = 1,$$

and all the other inner products of the basis vectors equal 0. Then $\{2f_0, f_1, f_2, f_3, f_4\}$ is a basis of $D_{2,3}$. The kernel of the inner product in $\mathbb{Z}_2^{2,3}$ is spanned by $[f_0]$, and $q = 0$ on it. Hence, the form q induces a quadratic form $\bar{q}$ on $\mathbb{Z}^{2,2} = \mathbb{Z}^{2,3}/\langle f_0\rangle$. In the basis constituted by the images of $[f_1]$, $[f_2]$, $[f_3]$, $[f_4]$, the form $\bar{q}$ is written as $x_1x_2 + x_3x_4$. The space $\mathbb{Z}^{2,2}$ can be modeled as the space of (2×2)-matrices over $\mathbb{Z}_2$ so that the form $\bar{q}$ becomes the determinant. It is known (see [3, Ch. II, §10, 10)]) that in this model the orthogonal group $O(\bar{q})$ is generated by the left and right multiplications by matrices of $SL_2(\mathbb{Z}_2)$ and taking the transposed matrix. The orthogonal transvections generate a subgroup of index 2 in $O(\bar{q})$. It does not contain, for example, the orthogonal transformation

$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_4, x_2, x_3 + x_2, x_4). \tag{30}$$

The group Γ_3 naturally acts on $\mathbb{Z}_2^{2,3}$ preserving the inner product and the quadratic form q, and thereby it acts on $\mathbb{Z}_2^{2,2}$ preserving the form $\bar{q}$. There are only 1- and 2-reflections in Γ_3. It is easy to see that 1-reflections act trivially on $\mathbb{Z}_2^{2,2}$, while 2-reflections act as orthogonal transvections. To prove the proposition, it suffices to present an element $\gamma \in \Gamma_3$ that induces the transformation (30) on $\mathbb{Z}_2^{2,2}$. Such an element can be given by

$$\gamma(f_0) = f_0,\ \gamma(f_1) = f_1,\ \gamma(f_2) = f_2 + f_3,\ \gamma(f_3) = f_3,\ \gamma(f_4) = f_4 - f_1.$$

□

Set now $\mathcal{X}_2 = \operatorname{Spec} A(\mathcal{D}_3, \Gamma_3^r)$, and let

$$\varphi_2 : \mathcal{X}_2 \to \mathcal{X}$$

be the morphism defined by the embedding $A(\mathcal{D}_3, \Gamma_3) \hookrightarrow A(\mathcal{D}_3, \Gamma_3^r)$. The morphism φ_2 is the factorization by the group Γ_3/Γ_3^r (of order 2), and $\mathcal{X}_2^\circ$ is a Zariski open subset of $\mathcal{X}_2$, whose complement does not contain divisors.

8.4. **Finite coverings of algebraic varieties.** In the previous two subsections we constructed two $\mathbb{C}^*$-equivariant finite morphisms of degree 2:

$$\varphi_1 : \mathcal{X}_1 \to \mathcal{X}, \quad \varphi_2 : \mathcal{X}_2 \to \mathcal{X},$$

with strongly simply connected normal affine $\mathbb{C}^*$-varieties $\mathcal{X}_1$ and $\mathcal{X}_2$. We are now going to prove that $\mathcal{X}_1 \simeq \mathcal{X}_2$.

Set

$$\mathcal{X}' = \mathcal{X}^{reg} \setminus (\varphi_1(\mathcal{X}_1^{\theta_1}) \cup \varphi_2(\mathcal{X}_2^{\theta_2}),$$

where θ_2 is the generator of Γ_3/Γ_3^r, and

$$\mathcal{X}_1' = \varphi_1^{-1}(\mathcal{X}'), \quad \mathcal{X}_2' = \varphi_2^{-1}(\mathcal{X}').$$

Then $\mathcal{X}_1'$ and $\mathcal{X}_2'$ are simply connected analytic manifolds and φ_1 and φ_2 are non-ramified analytic coverings (of degree 2). It follows that there is an analytic isomorphism $\psi : \mathcal{X}_1' \to \mathcal{X}_2'$ such that the diagram

$$\begin{array}{ccc} \mathcal{X}_1' & \xrightarrow[\sim]{\psi} & \mathcal{X}_2' \\ & {\scriptstyle\varphi_1}\searrow \quad \swarrow{\scriptstyle\varphi_2} & \\ & \mathcal{X}' & \end{array}$$

is commutative.

Now the desired result follows from

Proposition 15. *Let $\mathcal{X}$, $\mathcal{X}_1$, $\mathcal{X}_2$ be normal irreducible affine algebraic varieties, and $\varphi_1 : \mathcal{X}_1 \to \mathcal{X}$, $\varphi_2 : \mathcal{X}_2 \to \mathcal{X}$ be finite morphisms. Let $\mathcal{X}' \subset \mathcal{X}$ be a non-empty Zariski open subset and $\mathcal{X}_1' = \varphi_1^{-1}(\mathcal{X}')$, $\mathcal{X}_2' = \varphi_2^{-1}(\mathcal{X}')$. Let $\psi : \mathcal{X}_1' \to \mathcal{X}_2'$ be an analytic isomorphism such that the above diagram is commutative. Then ψ extends to an algebraic isomorphism $\mathcal{X}_1 \to \mathcal{X}_2$.*

This is a part of a more general result of H. Grauert and R. Remmert [5] (see also [6, Appendix B, Proposition 2.2]) but we give here a proof for convenience of the reader.

Lemma 4. *Let $\mathcal{X}$ be a normal irreducible affine algebraic variety. If an analytic function f on some non-empty Zariski open subset $\mathcal{X}' \subset \mathcal{X}$ is integral over $\mathbb{C}[\mathcal{X}]$, then it is the restriction to $\mathcal{X}'$ of some polynomial function on $\mathcal{X}$.*

Proof. Consider the variety $\tilde{\mathcal{X}} = \operatorname{Spec} \mathbb{C}[\mathcal{X}][f]$. Let $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ be the morphism defined by the embedding $\mathbb{C}[\mathcal{X}] \hookrightarrow \mathbb{C}[\mathcal{X}][f]$. Let $\Delta \in \mathbb{C}[\mathcal{X}]$ be the discriminant of the minimal polynomial of f over $\mathbb{C}[\mathcal{X}]$. Reducing $\mathcal{X}'$, one may assume that Δ nowhere vanishes in $\mathcal{X}'$. Let $\tilde{\mathcal{X}}' = \rho^{-1}(\mathcal{X}')$. Then the restriction of ρ to $\tilde{\mathcal{X}}'$ is a non-ramified analytic covering, and the set

$$\{p \in \tilde{\mathcal{X}}' : f(p) = f(\rho(p)\}$$

is open and closed (in the real topology) in $\tilde{\mathcal{X}}'$, hence coincides with $\tilde{\mathcal{X}}'$. Therefore, ρ is injective on $\tilde{\mathcal{X}}'$. Since $\mathcal{X}$ is normal, this implies that ρ is an isomorphism, i.e. $f \in \mathbb{C}[\mathcal{X}]$. □

Proof of Proposition 15. Take any function $f \in \mathbb{C}[\mathcal{X}_2]$. It is integral over $\mathbb{C}[\mathcal{X}]$. Therefore, the function $\psi^* f$ is also integral over $\mathbb{C}[\mathcal{X}]$ and, hence, over $\mathbb{C}[\mathcal{X}_1]$. At the same time it is analytic on $\mathcal{X}_1'$. By Lemma 4, this implies that $\psi^* f \in \mathbb{C}[\mathcal{X}_1]$. This means that ψ is (the restriction of) a morphism of affine varieties. In the same way, replacing ψ with ψ^{-1}, one can prove that ψ^{-1} is a morphism of affine varieties. □

Coming back to the situation of the beginning of this subsection, we see that there exists an isomorphism $\psi : \mathcal{X}_1 \to \mathcal{X}_2$ (of algebraic varieties) such that the diagram

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow[\sim]{\psi} & \mathcal{X}_2 \\ & {\scriptstyle\varphi_1}\searrow \quad \swarrow{\scriptstyle\varphi_2} & \\ & \mathcal{X} & \end{array}$$

is commutative. Note that it is automatically $\mathbb{C}^*$-equivariant. Indeed, for any $t \in \mathbb{C}^*$ and $x_1 \in \mathcal{X}_1$, the points $\psi(tx_1)$ and $t\psi(x_1)$ have the same image in $\mathcal{X}$ and, hence, may differ only by θ_2; but for the continuity reason they must coincide. Thus, we have proved

Theorem 6. *The algebra $A(\mathcal{D}_3, \Gamma_3^r)$ is freely generated by forms of weights* 4, 4, 6, 6.

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