BV functions in a Gelfand triple and the
stochastic reflection problem on a convex set of a
Hilbert space

Fonctions BV dans triplet de Gelfand et le
probleme de reflexion sur un ensemble convexe
d’un espace de Hilbert

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Abstract

In this note we introduce BV functions in a Gelfand triple, which is an extension of BV functions in [1], by using Dirichlet form theory. By this definition, we can consider the stochastic reflection problem associated with a self-adjoint operator $A$ and a cylindrical Wiener process on a convex set $\Gamma$. We prove the existence and uniqueness of a strong solution of this problem when $\Gamma$ is a regular convex set. The result is also extended to the non-symmetric case. Finally, we extend our results to the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$

Résumé.

Dans ce papier, on introduit des fonctions BV dans un triplet de Gelfand qui est une extension de fonctions BV dans [1] en utilisant la forme de Dirichlet. Par cette définition, on peut considérer le problème de réflexion stochastique associé à un opérateur auto-adjoint $A$ et un processus de Wiener cylindrique sur un ensemble convexe $\Gamma$. Nous démontrons l’existence et l’unicité d’une solution forte de ce problème si $\Gamma$ et un ensemble convexe régulier. Le résultat est aussi étendu au cas non-symétrique. Finalement, nous utilisons les fonctions BV dans le cas $\Gamma = K_\alpha$, où $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$.

1. Dirichlet form and BV functions——Given a real separable Hilbert space $H$(with scalar product $\langle \cdot, \cdot \rangle$ and norm denoted by $|\cdot|$), assume that:

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Theorem 1.2. Let \( \rho \in QR(H) \). Then \((E^\rho, F^\rho)\) is a quasi-regular local Dirichlet form on \( L^2(F; \rho \cdot \mu) \) in the sense of [6, IV Definition 3.1].

By virtue of Theorem 1.2 and [6], there exists a diffusion process \( M^\rho = (X_t, P_t) \) on \( F \) associated with the Dirichlet form \((E^\rho, F^\rho)\). \( M^\rho \) will be called distorted OU process on \( F \). Since constant functions are in \( F^\rho \) and \( E^\rho(1,1) = 0 \), \( M^\rho \) is recurrent and conservative. Let \( A_{1/2} (x) := \int_{\mathbb{R}} (\log(1+s))^{1/2} dx, 0 \geq s \) and let \( \psi \) be its complementary function, namely, \( \psi(y) := \int_{0}^{y} (A_{1/2}^{-1}(t)dt = \int_{0}^{y} \exp(t^2) - 1)dt \). Define \( L(\log L)^{1/2} := \{ f | A_{1/2} (f) \in L^1 \} \), \( L^\rho := \{ g | \psi(g) \in L^1 \} \) for some \( c > 0 \) (cf.[7]). Let \( c_j \), \( j \in \mathbb{N} \), be a sequence in \([1, \infty)\). Define \( H_1 := \{ x \in H | \int_{\mathbb{R}} c_j (x, e_j) e_j < \infty \} \), equipped with the inner product \( (x, y)_{H_1} := \sum_{j=1}^{\infty} c_j (x, e_j) (y, e_j) \). Then clearly \((H_1, \langle \cdot, \cdot \rangle_{H_1})\) is a Hilbert space such that \( H_1 \subset H \) continuously and densely. Identifying \( H \) with its dual we obtain the continuous and dense embeddings \( H_1 \subset H(\equiv H^* \subset H_1^* \). It follows that \( H_1, H, H_1^* \) is a Gelfand triple. We also introduce a family of \( H \)-valued function on \( H \) by

\[
(C^1_0)_{D(A) \cap H_1} = \{ G : G(z) = \sum_{j=1}^{m} g_j(z) e_j | g_j \in C^1_0(H), e_j \in D(A) \cap H_1 \}
\]

Denote by \( D^* \) the adjoint of \( D : C^1_0(H) \subset L^2(H, \mu) \to L^2(H, \mu; H) \). For \( \rho \in L(\log L)^{1/2}(H, \mu) \), we put \( V(\rho) := \sup_{G \in (C^1_0)_{D(A) \cap H_1}, \|G\|_{H_1} \leq 1} \int_H D^* G(z) \rho(z) \mu(dz) \). A function \( \rho \) on \( H \) is called a BV function in the Gelfand triple \((H_1, H, H_1^*)\) (denoted \( \rho \in BV(H, H_1) \) in notation), if \( \rho \in L(\log L)^{1/2}(H, \mu) \) and \( V(\rho) \) is finite. When \( H_1 = H = H_1^* \), this coincides with the definition of BV functions defined in [1] and clearly \( BV(H, H) \subset BV(H, H_1) \). This definition is a modification of BV function in abstract Wiener space introduced in [3] and [4].

Theorem 1.3. (i) Suppose \( \rho \in BV(H, H_1) \cap L^1(H, \mu) \), then there exist a positive finite measure \( \|d\rho\| \) on \( H \) and a Borel-measurable map \( \sigma_\rho : H \to H_1^* \) such that \( \|\sigma_\rho(z)\|_{H_1} = 1 \cdot \|d\rho\| - a.e. \), \( V(\rho) = \|d\rho\|(H) \).

\[
\int_H D^* G(z) \rho(z) \mu(dz) = \int_H H_1(G(z), \sigma_\rho(z))_{H_1^*} \|d\rho\|(dz), \forall G \in (C^1_0)_{D(A) \cap H_1}.
\]

Further, if \( \rho \in QR(H) \), \( \|d\rho\| \) is \( E^\rho \)-smooth, also, \( \sigma_\rho \) and \( \|d\rho\| \) are uniquely determined.

(ii) Conversely, if Eq.(1.1) holds for \( \rho \in L(\log L)^{1/2}(H, \mu) \) and for some positive finite measure \( \|d\rho\| \) and a map \( \sigma_\rho \) with the stated properties, then \( \rho \in BV(H, H_1) \) and \( V(\rho) = \|d\rho\|(H) \).

Theorem 1.4 Let \( \rho \in QR(H) \cap BV(H, H_1) \) and consider the measure \( \|d\rho\| \) and \( \sigma_\rho \) from Theorem 1.3(i). Then there is an \( E^\rho \)-exceptional set \( S \subset F \) such that \( \forall z \in F \setminus S \), under \( P_z \) there exists an \( \mathcal{M}_t \) -cylindrical Wiener process \( W^z \), such that the sample paths of the associated distorted
OU-process $M^\rho$ on $F$ satisfy the following: for $l \in D(A) \cap H_1$

$$
\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW^x_s \rangle + \frac{1}{2} \int_0^t H_l \langle l, \sigma(x_s) \rangle H_t dL^\rho_s - \int_0^t \langle A_l, X_s \rangle ds \ \forall t \geq 0 \ \text{P}_z\text{-a.s.}
$$

Here $L^\rho_t$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence.

2. **Reflected OU process**—Consider the situation when $\rho = I_G$, the indicator of a set.

**Remark 2.1** We emphasize that if $\Gamma$ is a convex closed set in $H$, then for each $z, l \in H$ the set \{s $\in \mathbb{R}|z+sl \in \Gamma$\} is a closed interval in $\mathbb{R}$, whose indicator function hence trivially has the Hamza property. Hence, in particular, $I_G \in \mathcal{QR}(H)$.

2.1 **Reflected OU processes on regular convex set**—Denote the corresponding objects $\sigma_\rho, \|dI\|$ in Theorem 1.3(i) by $-n_\Gamma, \|\partial \Gamma\|$, respectively.

**Hypothesis 2.1.1** There exists a convex $C^\infty$ function $g: H \to \mathbb{R}$ with $g(0) = 0, g'(0) = 0$, and $D^2g$ strictly positively definite, that is, $\langle D^2g(x)h, h \rangle \geq \gamma |h|^2, \forall h \in H$ where $\gamma > 0$, such that

$$
\Gamma = \{x \in H : g(x) \leq 1\}, \partial \Gamma = \{x \in H : g(x) = 1\}
$$

Moreover, we also suppose that $D^2g$ is bounded on $\Gamma$. Finally, we also suppose that $g$ and all its derivatives grow at infinity at most polynomially.

By using [2, Lemma 2.1], we have (1.1) for $\rho = I_G$ with $H = H_1$. By the continuity property of surface measure given in [5], we have the following two theorems.

**Theorem 2.1.2** Assume Hypothesis 2.1.1. Then $I_G \in \mathcal{BV}(H, H) \cap \mathcal{QR}(H)$.

**Theorem 2.1.3** Assume Hypothesis 2.1.1. Then there exists an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \setminus S$, under $P_z$ there exists an $\mathcal{M}_\Gamma$ cylindrical Wiener process $W^z$, such that the sample paths of the associated reflected OU-process $M^\rho$ on $F$ with $\rho = I_G$ satisfy the following: for $l \in D(A) \cap H_1$

$$
\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW^x_s \rangle - \frac{1}{2} \int_0^t \langle l, n_\Gamma(X_s) dL^\rho_s \rangle - \int_0^t \langle A_l, X_s \rangle ds \ \forall t \geq 0 \ \text{P}_z\text{-a.e.}
$$

where $n_\Gamma := \frac{\partial \mu_\Gamma}{\partial \mathcal{P}_\Gamma}$ is the exterior normal to $\Gamma$, satisfying $\langle n_\Gamma(x), x - y \rangle \geq 0$, for any $y \in \Gamma, x \in \partial \Gamma$ and $\|\partial \Gamma\| = \mu_\Gamma$, where $\mu_\Gamma$ is the surface measure induced by $\mu$ (c.f [2], [5]).

Let $G$ satisfy Hypothesis 2.1.1 and $A$ satisfy Hypothesis 1.1. Consider the following stochastic differential inclusion in the Hilbert space $H$,

$$
\begin{cases}
  dX(t) + (AX(t) + N_G(X(t)))dt \ni dW(t),
  X(0) = x
\end{cases}
$$

(2.1)

where $W(t)$ is a cylindrical Wiener process in $H$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_G(x)$ is the normal cone to $\Gamma$ at $x$.

**Definition 2.1.4** A pair of continuous $H \times \mathbb{R}$ valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$, is called a solution of (2.1) if the following conditions hold:

(i) $X(t) \in \Gamma, P - a.s$, for all $t \in [0, T]$, 

(ii) $L$ is an increasing process with the property $\int_0^t I_{\partial \Gamma}(X_s(\omega)) dL_s(\omega) = L_t(\omega), t \geq 0$ and we have for any $l \in D(A), \langle l, X_t(\omega) - x \rangle = \langle l, W_t(\omega) - \int_0^t n_\Gamma(X_s(\omega)) dL_s(\omega) \rangle - \langle A_l, \int_0^t X_s(\omega) ds \rangle$ where $n_\Gamma$ is the exterior normal to $\Gamma$, satisfying $\langle n_\Gamma(x), x - y \rangle \geq 0, \forall y \in \Gamma, x \in \partial \Gamma$. 

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Theorem 2.1.5 If $\Gamma$ satisfies Hypothesis 2.1.1, then there exists $M, I_{\Gamma} \cdot \mu(M) = 1$, such that for every $x \in M$, (2.1) has a pathwise unique continuous strong solution in the sense of Definition 2.1.4, such that $X(t) \in M$ for all $t \geq 0$ $P_{\sigma}$-a.s.

Remark 2.1.6 We can extend all these results to non-symmetric Dirichlet forms obtained by first order perturbation of the above Dirichlet form.

2.2 Reflection OU processes on a class of convex sets——Now we consider the case when $H = L^2(0,1), \rho = I_{K_\alpha}$, where $K_\alpha = \{ f \in H | f \geq -\alpha \}, \alpha \geq 0$ and $A = -\frac{1}{2} \partial^2$ with Dirichlet boundary condition on $[0,1]$. Take $c_j = (j\pi)^{\frac{1}{2}+\varepsilon}$ if $\alpha > 0$, $c_j = (j\pi)^{\beta}$ if $\alpha = 0$, where $\varepsilon \in (0, \frac{3}{2}]$ and $\beta \in (\frac{1}{2}, 2]$ respectively. By using [8, (1) (2), Theorem 5 ], we can prove the following theorem.

Theorem 2.2.1 $I_{K_\alpha} \in BV(H, H_1) \cap QR(H)$.

Remark 2.2.2 It has been proved by Guan Qingyang that $I_{K_\alpha}$ is not in $BV(H, H)$. Since we have Theorem 2.2.1, we denote the corresponding objects $\sigma_{\alpha}, ||dI_{K_\alpha}||$ in Theorem 1.3 (i) by $n_{\alpha}, |\sigma_{\alpha}|$, respectively.

Theorem 2.2.3 Let $\rho = I_{K_\alpha}$. Then there is an $E^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \setminus S$, under $P_z$ there exists an $\mathcal{M}_\varepsilon$-cylindrical Wiener process $W^z$, such that the sample paths of the associated distorted OU-process $M^\rho$ on $F$ satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s \rangle + \frac{1}{2} \int_0^t \langle l, n_\alpha(X_s) \rangle_{H_1} dL_t^{\alpha, \rho} - \int_0^t \langle Al, X_s \rangle ds P_z - a.e.$$

Here, $L_t^{\alpha, \rho}(\omega)$ is a real valued PCAF associated with $|\sigma_\alpha|$ by the Revuz correspondence, satisfying

$I_{(X_t, \alpha=0)} dL_t^{\alpha, \rho} = 0$, and for every $z \in F, P_z[ X_t \in C_0[0,1] ] = 0$ for a.e. $t \in [0, \infty]$ = 1

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References

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