

Infima of superharmonic functions

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Abstract

Let Ω be a Greenian domain in \mathbb{R}^d , $d \geq 1$, or – more generally – let Ω be a connected \mathcal{P} -Brelot space satisfying axiom D, and let u be a numerical function on Ω , $u \not\equiv \infty$, which is locally bounded from below. A short proof yields the following result: The function u is the infimum of its superharmonic majorants if and only if each set $\{u > t\}$, $t \in \mathbb{R}$, differs from an analytic set only by a polar set and $\int u d\mu_x^V \leq u(x)$, whenever V is a relatively compact open set in Ω and $x \in V$.

The purpose of this paper is to provide a *short* proof for a *full* characterization of infima of superharmonic functions in terms of mean value inequalities and a weak measurability property (Theorem 2). Its background is the publication [6], where the main results Theorem 1.1 and Theorem 1.5 can be stated as follows.

Theorem. *Let Ω be a Greenian domain in \mathbb{R}^d , $d \geq 2$, and $u: \Omega \rightarrow (-\infty, \infty]$, $u \not\equiv \infty$, such that u is locally bounded from below and, for every $t \in \mathbb{R}$, the set $\{u > t\}$ is analytic. Then the following statements are equivalent:*

- (i) *The greatest lower semicontinuous minorant \hat{u} of u is superharmonic, and the set $\{\hat{u} < u\}$ is polar.*
- (ii) *u is the infimum of its superharmonic majorants.*
- (iii) *For all $x \in \Omega$ and Jensen measures μ for x , $\int u d\mu \leq u(x)$.*

Let us recall that a Jensen measure for x with respect to Ω is a (Radon) measure μ , supported on a compact set in Ω , such that every superharmonic function u on Ω satisfies $\int u d\mu \leq u(x)$. In particular, for every relatively compact open V in Ω and every $x \in V$, the harmonic measure μ_x^V is a Jensen measure for x .

The proof for (i) \Rightarrow (ii) is short and straightforward (it can be reduced to four lines; see the proof of Theorem 2) and the implication (ii) \Rightarrow (iii) is almost trivial. However, the proof for the implication (iii) \Rightarrow (i) given in [6] is quite involved and uses a duality theorem for Jensen measures which is based on an abstract duality theorem by D.A. Edwards.

Clearly, (iii) implies that u is nearly hyperharmonic and hence \hat{u} is superharmonic ([2, 4]). Moreover, it is well known that (ii) implies the polarity of the set $\{\hat{u} < u\}$ (see [5] or, for example, [1, Theorem 5.7.1]). So the (only) interesting part in the Theorem above is the fact that already (iii) implies that the set $\{\hat{u} < u\}$ is polar.

We shall present a short proof for this implication (using harmonic measures only so that we might dispense with general Jensen measures) and, at the same time, remove the a priori measurability assumption on u . The latter is possible, since

both (i) and (ii) already provide some regularity of the function u (it differs from a lower semicontinuous function only on a polar set, it is finely upper semicontinuous, respectively).

In the sequel let Ω be as in the Theorem above or, more generally, let Ω be a connected (metrizable) \mathcal{P} -Brelot space satisfying the axiom of domination. We introduce the following (weak) measurability property for numerical functions u on Ω .

(QA) For every $t \in \mathbb{R}$, the set $E := \{u > t\}$ is *quasi-analytic*, that is, there exists an analytic set A in Ω such that the symmetric difference $E \Delta A$ is polar.

If u is a numerical function on Ω having property (QA) and μ is a Jensen measure for $x \in \Omega$, then u is μ -measurable, since μ does not charge polar sets in $\Omega \setminus \{x\}$.

LEMMA 1. *If u, v are numerical functions on Ω such that $u = v$ quasi-everywhere and v is finely lower or finely upper semicontinuous, then u satisfies (QA) (even with $G_{\delta\sigma}$ -sets A).*

Proof. Obviously, it suffices to show that v satisfies (QA).

We first suppose that v is finely lower semicontinuous, and fix $t \in \mathbb{R}$. Then $V := \{v > t\}$ is finely open. Hence the quasi-Lindelöf property implies that there exists a countable union A of compact sets and a polar set P such that $V = A \cup P$ (see [7, 1.XI.11], [3, II.4.1, VI.5.14]).

Next, let us assume that v is finely upper semicontinuous, and hence $-v$ is finely lower semicontinuous. So, for each $n \in \mathbb{N}$, there exists a countable union A_n of compact sets and a polar set P_n such that $\{-v > -t - 1/n\} = A_n \cup P_n$. Then $A := \bigcup_{n=1}^{\infty} A_n^c$ is a $G_{\delta\sigma}$ -set, $\{v > t\} = \bigcup_{n=1}^{\infty} (A_n \cup P_n)^c \subset A$, and the set $A \setminus \{v > t\}$ is polar, since it is contained in the polar set $\bigcup_{n=1}^{\infty} P_n$. \square

THEOREM 2. *Let $u: \Omega \rightarrow (-\infty, \infty]$ such that $u \not\equiv \infty$ and u is locally lower bounded. Then the following statements are equivalent.*

- (1) \hat{u} is superharmonic and $\hat{u} = u$ quasi-everywhere.
- (2) u is the infimum of its superharmonic majorants.
- (3) (QA) holds and $\int u d\mu \leq u(x)$, for all $x \in \Omega$ and Jensen measures μ for x .
- (4) (QA) holds and $\int u d\mu_x^V \leq u(x)$, for all $x \in V \subset \Omega$, V relatively compact open.

Proof. (1) \Rightarrow (2): Let $x \in \Omega$ such that $u(x) < \infty$, and let $\varepsilon > 0$. There exists a superharmonic function w on Ω such that $w \geq 0$, $w(x) = u(x) - \hat{u}(x) + \varepsilon$, and $w(y) = \infty$, whenever $y \in \Omega$, $y \neq x$, and $\hat{u}(y) < u(y)$. Let $v = \hat{u} + w$. Then v is superharmonic on Ω , $v \geq u$, and $v(x) = u(x) + \varepsilon$.

(2) \Rightarrow (3): Clearly, u is finely upper semicontinuous. So (QA) holds, by Lemma 1. Let μ be a Jensen measure for $x \in \Omega$. Then $\int u d\mu \leq \int v d\mu \leq v(x)$, whenever v is a superharmonic majorant of u . Thus $\int u d\mu \leq u(x)$.

(3) \Rightarrow (4): Trivial.

(4) \Rightarrow (1): Let w_0 be a continuous superharmonic function on Ω , $0 < w_0 < \infty$. Let us suppose first that $u \leq Mw_0$ for some $M > 0$. Then \hat{u} is superharmonic, since the function u is nearly hyperharmonic.

The set $\{\hat{u} < u\}$ is the union of the sets $\{\hat{u} \leq s\} \cap \{s < u\}$, s rational, where the sets $\{\hat{u} \leq s\}$ are closed and the sets $\{s < u\}$ are quasi-analytic. Hence there exists an analytic set A and a polar set P in Ω such that

$$\{\hat{u} < u\} = A \triangle P.$$

To prove that $\hat{u} = u$ quasi-everywhere, it clearly suffices to show that all compact sets in A are polar.

So let us fix a compact set K in A . Let U be a relatively compact open neighborhood of K in Ω and $V := U \setminus K$. The functions $g: x \mapsto \int \hat{u} d\mu_x^V$ and $h: x \mapsto \int u d\mu_x^V$ are harmonic on V , $g \leq h$. Moreover, $h \leq u$, by assumption, and hence $h \leq \hat{u}$, by the continuity of h . By the axiom of domination, g is the the greatest harmonic minorant of \hat{u} on V . So $g = h$. Fixing $x \in V$, we conclude that $\mu_x^V(K \setminus P) = 0$, since $\hat{u} \leq u$ on Ω and $\hat{u} < u$ on $K \setminus P$. Knowing that μ_x^V does not charge polar sets, we see that $\mu_x^V(K) = 0$, and hence

$$\inf\{s(x): s \text{ superharmonic on } U, s \geq 1_K w_0\} =: {}^U R_{w_0}^K(x) = \int_K w_0 d\mu_x^V = 0$$

(see, for example, [3, VI.2.9]). So the lower semicontinuous regularization w of ${}^U R_{w_0}^K$ is a positive superharmonic function on U which vanishes on $U \setminus K$. By ellipticity, $w = 0$ on U . Thus K is polar.

In the general case, we apply the preceding result to $u_m := \inf\{u, mw_0\}$, $m \in \mathbb{N}$. For every $m \in \mathbb{N}$, the function u_m satisfies (4), and $\hat{u}_m = \inf\{\hat{u}, mw_0\}$. In particular, $\hat{u}_m \uparrow \hat{u}$ and $\{\hat{u}_m < u_m\} \uparrow \{\hat{u} < u\}$ as $m \rightarrow \infty$. Thus (1) holds. \square

REMARKS 3. a) If there exists a nonpolar set A in Ω which is inner polar and measurable with respect to all harmonic measures, then the assumption (QA) in (4) cannot be omitted. To see this it suffices to consider the function $u := 1_A$. Indeed, such a set A has no interior points, hence $\hat{u} = 0$, and the set $\{\hat{u} < u\} = A$ is nonpolar. But, for all relatively compact open sets V in Ω and $x \in V$,

$$\int u d\mu_x^V = \mu_x^V(A) = \sup\{\mu_x^V(K): K \text{ compact in } A\} = 0 \leq u(x).$$

b) It is easily seen (directly or noting that (1) is a local statement) that we may replace (4) by

(4') (QA) holds and there exists a covering of Ω by open sets Ω_i , $i \in I$, such that $\int u d\mu_x^V \leq u(x)$, for all $x \in V \subset \Omega$, V relatively compact open and $\bar{V} \subset \Omega_i$ for some $i \in I$.

However, it is certainly not sufficient to consider sets V from an arbitrary base \mathcal{V} of open sets in Ω . Indeed, let $\Omega := \mathbb{R}^d$, $d \geq 3$. For $1 \leq j \leq d$, let A_j be the set of all $x = (x_1, \dots, x_d)$ such that x_j is rational, and let $A := \bigcup_{j=1}^d A_j$, $u := 1_{A^c}$, $\mathcal{V} := \{\prod_{j=1}^d (\alpha_j, \beta_j): \alpha_j, \beta_j \text{ rational}\}$. For every $V \in \mathcal{V}$, $u = 0$ on the boundary of V , and hence $\int u d\mu_x^V = 0 \leq u(x)$, $x \in V$. Of course, $\hat{u} = 0$ and $\{\hat{u} = u\}$ is the Lebesgue null set A so that $\{\hat{u} < u\} = A^c$ is far from being polar.

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