Jensen measures in potential theory

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Abstract

It is shown that, for open sets in classical potential theory and – more generally – for elliptic harmonic spaces Y, the set $J_x(Y)$ of Jensen measures (representing measures with respect to superharmonic functions on Y) for a point $x \in Y$ is a simple union of closed faces of the compact convex set $M_x(\mathcal{P}(Y))$ of representing measures with respect to potentials on Y, a set which has been thoroughly studied a long time ago. In particular, the set of extreme Jensen measures can be immediately identified. The results hold even without ellipticity (thus capturing also many examples for the heat equation) provided a rather weak approximation property for superharmonic functions holds.

Equally sufficient are a certain transience property and a weak regularity property. More important, each of these properties turns out to be necessary and sufficient for obtaining (in the classical case) that $J_x(Y)$ coincides with the set of all compactly supported probability measures in $M_x(\mathcal{P}(Y))$.

Keywords: Jensen measure, representing measure, harmonic measure, superharmonic function, potential, harmonic space, balayage, fine topology, face of a convex set, extreme point

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1 Introduction

Starting with [5, 6] several papers have been written in the last years, where Jensen measures play a central role [16, 14, 10, 1, 13, 15]. One purpose of this note is to stress that (at least in classical potential theory and – more generally – for elliptic $\mathcal{P}$-harmonic spaces Y) the set of Jensen measures (for a point x with respect to Y) is a simple union of closed faces of the thoroughly studied compact convex set of representing measures (for x with respect to the cone of potentials on Y). In particular, the set of extreme Jensen measures can be immediately identified using results which are known even for general balayage spaces since 25 years (Theorem 1.2 and its consequences Corollary 2.2 and Theorem 3.3).

The statements hold even without ellipticity (thus capturing also many examples for the heat equation) provided Y satisfies some (rather weak) approximation property (AP) for superharmonic functions (Section 1) or a certain h_0-transience for some harmonic function $h_0 > 0$ on Y (Section 2). In classical potential theory,

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1-transience is not only a sufficient, but also a necessary condition for the property that every probability measure, which has compact support in Y and is a representing measure for x with respect to potentials on Y, is a Jensen measure for x (Theorem 3.4). Further, for *bounded* open sets Y in $\mathbb{R}^d$, this property holds if and only if every closed-open set in the boundary of Y intersects the set $\partial_{\mathrm{reg}} Y$ of regular boundary points for Y (Corollary 4.4; we recall that the set $\partial_{\mathrm{irr}} Y := \partial Y \setminus \partial_{\mathrm{reg}} Y$ of irregular boundary points is a polar K_σ-set).

For Jensen measures with respect to compact sets see the equalities (5.1) and (5.2) (Section 5).

To work in reasonable generality, let us assume that $(Y, \mathcal{H})$ is a harmonic space (see [4, p. 129]). The reader, who is not familiar with general potential theory or who wants to restrict his attention to the classical case, may assume right away that Y is a Greenian open set in $\mathbb{R}^d$, $d \geq 1$, that is, Y is an open set in $\mathbb{R}^d$ admitting a strictly positive potential (which only excludes that $Y = \mathbb{R}^d$ in the case $d = 1$ and $\mathbb{R}^2 \setminus Y$ is polar in the case $d = 2$, see [2]), and that, for every open U in Y, the set $\mathcal{H}(U)$ of harmonic functions is the set of all real C^2-functions on U such that $\Delta h = 0$.

Let $\mathcal{P}(Y)$ be the set of all continuous real potentials on Y and, for every open set U in Y, let $\mathcal{S}(U)$ denote the set of all superharmonic functions on U. We recall that $\mathcal{S}^+(Y)$ is the set of all superharmonic limits of increasing sequences in $\mathcal{P}(Y)$.

Given a Borel set E in Y, let $\mathcal{C}(E)$ be the set of all continuous real functions on E. For every convex cone $\mathcal{F}$ of lower semicontinuous functions $f\colon E \to (-\infty, \infty]$ and $x \in Y$, let $M_x(\mathcal{F})$ denote the set of all positive Radon measures μ on Y which are supported by E such that, for every $f \in \mathcal{F}$, $\mu(f^-) < \infty$ and $\mu(f) \leq f(x)$[1], and let $\operatorname{ext} M_x(\mathcal{F})$ denote the set of all extreme points of the convex set $M_x(\mathcal{F})$. We recall that

$$\operatorname{ext} M_x(\mathcal{P}(Y)) = \{\varepsilon_x^A \colon A \subset Y\} = \{\varepsilon_x\} \cup \{\varepsilon_x^A \colon A \text{ Borel set},\ x \notin A\}$$

(see [12], [4, VI.2.2,VI.12.4-5]). Moreover, if A is a (Borel) set in Y such that $x \notin A$ and $\varepsilon_x^A \neq \varepsilon_x$, then, by [4, VI.2.2, 4.1-4.4], there exists a finely closed G_δ-set F such that

$$A \subset F \subset \overline{A} \setminus \{x\} \quad \text{and} \quad \varepsilon_x^A = \varepsilon_x^F. \tag{1.1}$$

Thus

$$\operatorname{ext} M_x(\mathcal{P}(Y)) = \{\varepsilon_x\} \cup \{\varepsilon_x^F \colon F \text{ finely closed } G_\delta\text{-set},\ x \notin F\}. \tag{1.2}$$

Given $x \in Y$ and an open neighborhood U of x, let $J_x(U)$ denote the set of all *Jensen measures* for x with respect to U, that is,

$$J_x(U) := \{\mu \in M_x(\mathcal{S}(U)) \colon \operatorname{supp} \mu \subset\subset U\}.^2 \tag{1.3}$$

For example, for every open set V such that $x \in V \subset\subset U$, the harmonic measure $\mu_x^V := \varepsilon_x^{V^c}$ is contained in $J_x(U)$. We observe that $\mathcal{S}(U)$ in (1.3) can be replaced

[1] We write $\mu(f)$ instead of $\int f \, d\mu$.

[2] For sets $A \subset B \subset Y$, we write $A \subset\subset B$, if the closure $\overline{A}$ of A is a compact subset of B.

by $\mathcal{S}(U) \cap \mathcal{C}(U)$, since every $u \in \mathcal{S}(U)$ is the limit of an increasing sequence in $\mathcal{S}(U) \cap \mathcal{C}(U)$. If constants are harmonic, then, of course, every $\mu \in J_x(U)$ is a probability measure.

Let us note that our assumptions do not exclude that Y is compact. An example is given by the operator $u'' = u$ on the circle $\{z \in \mathbb{R}^2 \colon |z| = 1\}$, where the derivatives are taken with respect to arc length.

PROPOSITION 1.1. *If Y is compact, then $J_x(Y) = M_x(\mathcal{P}(Y))$ and*

$$\operatorname{ext} J_x(Y) = \{\varepsilon_x\} \cup \{\varepsilon_x^{V^c} \colon x \in V \subset Y,\ V \text{ finely open } K_\sigma\text{-set}\}.$$

Proof. Let Y be compact. Then $\mathcal{S}(U) = \mathcal{S}^+(U)$ by the minimum principle (and the constant function 0 is the only harmonic function on Y). Hence $J_x(Y) = M_x(\mathcal{P}(Y))$, and the proof is finished by (1.2). □

In the following, we assume that Y is not compact. Given an open set $U \subset\subset Y$, let

$$S(U) := \{u \in \mathcal{C}(\overline{U}) \colon u|_U \in \mathcal{S}(U)\}.$$

If W is an open neighborhood of $\overline{U}$, then $(\mathcal{S}(W) \cap \mathcal{C}(W))|_{\overline{U}} \subset S(U)$. So, for $x \in Y$,

$$\bigcup_{x \in U \subset\subset Y} M_x(S(U)) = \bigcup_{x \in U \subset\subset Y} J_x(U) \subset J_x(Y) \subset M_x(\mathcal{P}(Y)). \tag{1.4}$$

THEOREM 1.2. *Let $x \in Y$ and suppose that the union of all $J_x(U)$, $x \in U \subset\subset Y$, is the set $J_x(Y)$. Then* [3]

$$\operatorname{ext} J_x(Y) = \bigcup_{x \in U \subset\subset Y} \operatorname{ext} M_x(S(U)) \tag{1.5}$$

$$= \{\varepsilon_x\} \cup \{\varepsilon_x^{V^c} \colon x \in V \subset\subset Y,\ V \text{ finely open } K_\sigma\text{-set}\}. \tag{1.6}$$

Proof. Let $x \in U \subset\subset Y$. By [4, VII.9.5], $M_x(S(U))$ is a closed face of $M_x(\mathcal{P}(Y))$. Hence (1.4) and our assumption imply (1.5). Moreover, again by [4, VII.9.5],

$$\operatorname{ext} M_x(S(U)) = \{\varepsilon_x\} \cup \{\varepsilon_x^A \colon A \text{ Borel set},\ x \notin A,\ \beta(U^c) \subset A\}, \tag{1.7}$$

where $\beta(U^c)$ is the smallest finely closed subset of U^c such that $U^c \setminus \beta(U^c)$ is semipolar. To prove (1.6) it is sufficient to know that $\beta(U^c)$ is a subset of U^c containing the interior of U^c.

Indeed, given a finely open set V such that $x \in V \subset\subset Y$, we may choose an open set $U \subset\subset Y$ such that $\overline{V} \subset U$, and then $\varepsilon_x^{V^c} \in \operatorname{ext} M_x(S(U))$, by (1.7). Conversely, let U be open, $x \in U \subset\subset Y$, and $\mu \in \operatorname{ext} M_x(S(U)) \setminus \{\varepsilon_x\}$. Then, by (1.7) and (1.1), there exists a finely open K_σ-set V such that $x \in V \subset \overline{U}$ and $\mu = \varepsilon_x^{V^c}$. □

[3] If Y is a $\mathcal{P}$-harmonic Brelot space satisfying the axiom of domination, then we may replace "V finely open" by "V fine domain" (see [7, Theorem 12.7]).

2 Application based on approximation

For a first application of Theorem 1.2, we introduce the following rather weak approximation property:

(AP) For every compact K in Y, there exists a bounded[4] open neighborhood U of K such that, for all $u \in \mathcal{S}(U) \cap \mathcal{C}(U)$ and $\varepsilon > 0$, there is a function $v \in \mathcal{S}(Y) \cap \mathcal{C}(Y)$ satisfying $|u - v| < \varepsilon$ on K.

If F is a closed set in Y, then the connected components of F^c are open, since Y is locally connected. If K is a compact in Y, then the union $\hat{K}$ of K and all bounded connected components of K^c is compact (see [9, Lemm a 1]; its proof uses only that Y is locally compact and locally connected).

PROPOSITION 2.1. *If Y is elliptic, then* (AP) *holds.*

Proof. Let K be a compact set in Y, let L be a compact neighborhood of $\hat{K}$, and let U be the interior of the compact set $\hat{L}$. Since $Y \setminus \hat{L}$ is the union of all unbounded connected components of L^c, the set $(Y \setminus \hat{L}) \cup \{\infty\}$ is connected in the Aleksandrov compactification $Y_\infty := Y \cup \{\infty\}$, and hence its closure, that is, the set $(Y \setminus U) \cup \{\infty\}$, is connected as well. Therefore every connected component of U^c is unbounded.

By [3, Theorem 6.1 and Remark 6.2.1], we obtain that, given $u \in \mathcal{S}(U) \cap \mathcal{C}(U)$ and $\varepsilon > 0$, there exists $v \in \mathcal{S}(Y) \cap \mathcal{C}(Y)$ such that $|u - v| < \varepsilon$ on $\hat{K}$ (cf. also [8, Theorem 6.9] for the classical case and [9, Theorem 1] for the case of a Brelot space satisfying the axiom of domination). $\square$

Moreover, by [3, Theorem 6.1], it is clear that (AP) holds for many open sets Y in $\mathbb{R}^d \times \mathbb{R}$, $d \geq 1$, with respect to the heat equation. For example, (AP) holds if Y is a convex open set in $\mathbb{R}^d \times \mathbb{R}$.

By Theorem 1.2, we now obtain the following (for a different approach see Theorem 3.3 in connection with Proposition 3.2).

COROLLARY 2.2. *Suppose that Y is elliptic or, more generally, that* (AP) *holds. Then, for every $x \in Y$, $J_x(Y)$ is the union of all $J_x(U)$, $x \in U \subset\subset Y$, and* (1.5), (1.6) *hold.*

Proof. Let $x \in Y$, $\mu \in J_x(Y)$, and $K := \{x\} \cup \operatorname{supp} \mu$. We choose an open set $U \subset\subset Y$ according to (AP). Let $u \in \mathcal{S}(U) \cap \mathcal{C}(U)$ and $\varepsilon > 0$. By (AP), there exists $v \in \mathcal{S}(Y) \cap \mathcal{C}(Y)$ such that $|u - v| < \varepsilon$ on K, and therefore

$$\mu(u) \leq \mu(v) + \varepsilon\mu(K) \leq v(x) + \varepsilon\mu(K) \leq u(x) + \varepsilon(1 + \mu(K)).$$

Hence $\mu(u) \leq u(x)$, $\mu \in J_x(U)$, and the proof is finished by Theorem 1.2. $\square$

For the heat equation on an open set Y in $\mathbb{R}^{d+1}$, the union $\bigcup_{x \in U \subset\subset Y} J_x(U)$ may be a proper subset of $J_x(Y)$.

[4] We say that a subset of Y is bounded, if it is relatively compact (in Y), unbounded, if not.

EXAMPLE 2.3. *Let $Y := \{(y', t) \in \mathbb{R}^2 \colon t < 1+\cos \pi y'\}$ be equipped with the sheaf of solutions to the heat equation $\partial^2 u/\partial (y')^2 = \partial u/\partial t$ and let*

$$V := (-2,2) \times (-1,0), \quad x := (0,0), \quad \mu := {}^{Y}\varepsilon_x^{V^c}.$$

Then

$$\mu \in J_x(Y) \setminus \bigcup_{x \in U \subset\subset Y} J_x(U).$$

Proof. Indeed, $\operatorname{supp} \mu$ is the compact set $\partial V \setminus \big((-2,2) \times \{0\}\big)$ in Y and $\mu(1) = 1$. For every $n \in \mathbb{N}$, let

$$V_n := \{(y', t) \in V \colon t < -1/n\} \quad \text{and} \quad x_n := (0, -2/n).$$

Then, for every $u \in \mathcal{S}(Y) \cap \mathcal{C}(Y)$,

$$\mu(u) = \lim_{n \to \infty} \mu_{x_n}^{V_n}(u) \le \lim_{n \to \infty} u(x_n) = u(x).$$

Therefore $\mu \in J_x(Y)$. Now let U be an open set such that $x \in U \subset\subset Y$ and $\operatorname{supp} \mu \subset U$. There exists $\eta > 0$ such that the line segment $\{1\} \times (-\eta, 0)$ does not intersect U. For $y = (y', t) \in U$, let $u(y) := 0$, if $t \le -\eta$ or $y' < 1$, and $u(y) := t + \eta$, if $t > -\eta$ and $y' > 1$. Then $u \in \mathcal{S}(U)$ and $\mu(u) > 0 = u(x)$. Hence $\mu \notin J_x(U)$. □

3 Application based on h_0-transience

For a second application of Theorem 1.2, which yields an even stronger result, we introduce a general transience property.

DEFINITION 3.1. *Given $h_0 \in \mathcal{H}(Y)$, $h_0 > 0$, we shall say that Y is* h_0-transient, *if, for every compact K in Y, the set $\{R_{h_0}^K = h_0\}$ is compact.*

For the moment, let us fix $h_0 \in \mathcal{H}(Y)$, $h_0 > 0$. Then the function $R_{h_0}^K$ is harmonic on K^c, and $R_{h_0}^K = h_0$ on $\hat{K}$, by the minimum principle. Since $\hat{K}$ is compact, we see that the set $\{R_{h_0}^K = h_0\}$ is compact, if $R_{h_0}^K < h_0$ on every unbounded connected component W of K^c. In particular, Y is h_0-transient, if Y is elliptic and if, for every compact K in Y, there is only one unbounded connected component W of K^c. Indeed, otherwise $R_{h_0}^K = h_0$ on W, and hence $R_{h_0}^K = h_0$ on Y, which is impossible, since $R_{h_0}^K$ is bounded by a potential.

Moreover, in the classical case, every bounded regular set Y in $\mathbb{R}^d$, $d \ge 1$, is 1-transient, since the functions R_1^K, K compact in Y, vanish at ∂Y. In particular, finite intervals in $\mathbb{R}$ are 1-transient, whereas half-lines in $\mathbb{R}$ are not 1-transient (if, for example, $Y = (a, \infty)$, $a \in \mathbb{R}$, then $R_1^{\{a+1\}}(y) = 1$, for every $y \ge a+1$). Further, the punctured unit ball $Y := \{y \in \mathbb{R}^d \colon 0 < |y| < 1\}$, $d \ge 2$, is not 1-transient (since, taking $U := \{y \in \mathbb{R}^d \colon |y| < 1/2\}$ and $K = \partial U$, $R_1^K(y) = 1$ on $U \setminus \{0\}$). However, it is easily seen that the half-line $(0, \infty)$ is $(1+y)$-transient, the punctured disk in $\mathbb{R}^2$ is $(1 + \ln |y|^{-1})$-transient, and the punctured unit ball in $\mathbb{R}^d$, $d \ge 3$, is $|y|^{2-d}$-transient (for a general result see Proposition 3.2).

For $y, z \in \mathbb{R}^d$, let

$$u_z(y) := \begin{cases} \ln 1/|y-z|, & \text{if } d \ge 2, \\ |y-z|^{2-d}, & \text{if } d \ge 3. \end{cases}$$

PROPOSITION 3.2. *For every Greenian open set Y in $\mathbb{R}^d$, $d \geq 1$, there exists $h_0 \in \mathcal{H}(Y)$, $h_0 > 0$, such that Y is h_0-transient.*

Proof. We may assume without loss of generality that Y is connected.

If $d = 1$, then Y is a finite interval or a half-line. Both cases were settled above.

Let us next suppose that $d \geq 3$. There exists a finite measure μ on $P := \partial_{\rm irr} Y$ such that $p := \int u_z\, d\mu(z) = \infty$ on P (see [11, Lemma 13.22]; if Y is regular, we take $\mu = 0$). We note that $h := p|_Y \in \mathcal{H}^+(Y)$ and define

$$h_0 := 1 + h.$$

Let K be a compact in Y. Then $R_{h_0}^K = R_1^K + R_h^K$, where $R_1^K \leq 1$ and $R_h^K \leq h$.

Since $R_1^K \leq {}^{\mathbb{R}^d} R_1^K$ and $\lim_{|x|\to\infty} {}^{\mathbb{R}^d} R_1^K(x) = 0$, there exists a closed ball B in $\mathbb{R}^d$ such that $R_1^K < 1$ on $Y \setminus B$ (of course, $K \subset B$). If $z \in \partial_{\rm reg} Y$, then $\lim_{y\to z} R_1^K(y) = 0$, and hence there exists an open neighborhood V_z of z in $\mathbb{R}^d$ such that $R_1^K < 1$ on $V_z \cap Y$. Of course, $a := \sup h_0(K) < \infty$ and $R_{h_0}^K \leq a$. If $z \in \partial_{\rm irr} Y$, then $\lim_{y\to z} h(y) = \infty$, and hence there exists an open neighborhood V_z of z in $\mathbb{R}^d$ such that $a < h_0$ on $V_z \cap Y$. The compact $\partial Y \cap B$ can be covered by a finite union V of sets V_z, $z \in \partial Y \cap B$ (we take $V = \emptyset$, if $\partial Y \cap B = \emptyset$). Then $L := (Y \cap B) \setminus V$ is compact in Y and $R_{h_0}^K < h_0$ on $Y \setminus L$.

The case $d = 2$ is treated similarly using $h_0 \in \mathcal{H}^+(Y)$ with $\lim_{y\to z} h_0(y) = \infty$, for every $z \in \partial_{\rm irr} Y$, and if Y is unbounded, $\lim_{|y|\to\infty} h_0(y) = \infty$. To see that such a function exists let us choose $m_0 \in \mathbb{N}$, such that $B(0, m_0) \setminus Y$ is non-polar. Then there exists a probability measure ν on $B(0, m_0) \setminus Y$ such that the function $q := \int u_z\, d\nu(z)$ is continuous and real on $\mathbb{R}^2$. For the moment, let us fix $m \geq m_0$. Again by [11, Lemma 13.22], there exists a measure μ_m on $P_m := B(0, m) \cap \partial_{\rm irr} Y$ such that $\|\mu_m\| < 1$ and $p_m(x) := \int u_z\, d\mu_m(z)$ satisfies $p_m = \infty$ on P_m (if $P_m = \emptyset$, we take $\mu_m = 0$). Clearly, $\lim_{|x|\to\infty}(p_m - q)(x) = \infty$. So there exists $a_m > 0$ such that the harmonic function

$$h_m := a_m + (p_m - q)|_Y$$

is positive on Y and satisfies $\lim_{|y|\to z} h_m(y) = \infty$ for every $z \in P_m$ and, if Y is unbounded, $\lim_{|y|\to\infty} h_m(y) = \infty$. To finish the proof it now suffices to choose $x_0 \in Y$ and to define $h_0 := 1 + \sum_{m=m_0}^{\infty} 2^{-m} h_m / h_m(x_0)$. $\square$

For the next result, we may now return to the case of a general harmonic space $(Y, \mathcal{H})$.

THEOREM 3.3. *Suppose that $h_0 \in \mathcal{H}(Y)$, $h_0 > 0$, such that Y is h_0-transient. Then, for every $x \in Y$,*

$$(3.1) \qquad \bigcup_{x \in U \subset\subset Y} J_x(U) = J_x(Y) = \{\mu \in M_x(\mathcal{P}(Y))\colon\ \operatorname{supp}\mu \subset\subset Y,\ \mu(h_0) = h_0(x)\}.$$

Moreover, (1.5) *and* (1.6) *hold.*

Proof. We fix a point $x \in Y$, a compact K in Y, and define

$$N := \{\mu \in \mathcal{M}_x(\mathcal{P}(Y))\colon\ \operatorname{supp}\mu \subset K,\ \mu(h_0) = h_0(x)\}.$$

By assumption, there exists a bounded open neighborhood W of $K \cup \{x\}$ such that $R^K_{h_0} < h_0$ on W^c. Let U be a bounded open neighborhood of $\overline{W}$. If we show that

$$N \subset M_x(S(U)), \tag{3.2}$$

then we obtain, by (1.4), that (3.1) holds, and the proof is finished by Theorem 1.2.

To prove (3.2) we observe that N is a closed face of $M_x(\mathcal{P}(Y))$. Hence it suffices to show that

$$N \cap \operatorname{ext} M_x(\mathcal{P}(Y)) \subset M_x(S(U)). \tag{3.3}$$

So let us fix $\mu \in \operatorname{ext} M_x(\mathcal{P}(Y))$, $\mu \neq \varepsilon_x$, such that $\operatorname{supp} \mu \subset K$ and $\mu(h_0) = h_0(x)$. By (1.2), there is a finely closed G_δ-set F such that $x \notin F$ and $\mu = \varepsilon^F_x$. By [4, VI.9.4],

$$\varepsilon^{F\cap K}_x = \varepsilon^F_x|_K + (\varepsilon^F_x|_{K^c})^{F\cap K} = \varepsilon^F_x.$$

So we may assume that $F \subset K$. The set $V := W \setminus F$ is a finely open K_σ-set, $x \in V$, and $\overline{V} \subset U$. We define $\sigma := 1_F \varepsilon^{V^c}_x$, $\tau := 1_{W^c}\varepsilon^{V^c}_x$, and note that

$$\varepsilon^{V^c}_x = \sigma + \tau \quad \text{and} \quad \mu = \varepsilon^F_x = \sigma + \tau^F,$$

where the last equality follows by [4, VI.9.4]. If $\tau \neq 0$, then

$$\tau^F(h_0) = \tau(R^F_{h_0}) \le \tau(R^K_{h_0}) < \tau(h_0),$$

and therefore

$$h_0(x) = \mu(h_0) = (\sigma + \tau^F)(h_0) < (\sigma + \tau)(h_0) = \varepsilon^{V^c}_x(h_0),$$

a contradiction. Thus $\tau = 0$ and $\mu = \sigma = \varepsilon^{V^c}_x \in \mathcal{M}_x(S(U))$. □

For classical potential theory and $h_0 = 1$, we have the following strong converse to Theorem 3.3.

THEOREM 3.4. *Let Y be a Greenian open set in $\mathbb{R}^d$, $d \ge 1$. Then Y is 1-transient, if*

$$J_x(Y) = \{\mu \in M_x(\mathcal{P}(Y))\colon \operatorname{supp}\mu \subset\subset Y,\ \mu(1) = 1\} \qquad (x \in Y). \tag{3.4}$$

Proof. We may assume without loss of generality that Y is connected.

If $d = 1$, then Y is a finite interval or a half-line. In the first case, Y is 1-transient. If $Y = (a, \infty)$, $a \in \mathbb{R}$, every $p \in \mathcal{P}(Y)$ is increasing, whereas the function $z \mapsto -z$ is harmonic on Y, and therefore $\varepsilon_y \in M_x(\mathcal{P}(Y)) \setminus J_x(Y)$, whenever $a < y < x < \infty$. Similarly, if $Y = (-\infty, a)$.

So let $d \ge 2$ and suppose that Y is not 1-transient. Then $Y \neq \mathbb{R}^d$, and there exist a compact set K in Y and $x_n \in Y \setminus K$ such that $R^K_1(x_n) = 1$, for every $n \in \mathbb{N}$, and the sequence (x_n) is unbounded in Y. So, for every $n \in \mathbb{N}$,

$$\mu_n := \varepsilon^K_{x_n} \in M_{x_n}(\mathcal{P}(Y)), \quad \operatorname{supp}\mu_n \subset K, \quad \mu_n(1) = 1.$$

Let us assume first that $\sup_{n\in\mathbb{N}} |x_n| = \infty$. Fixing any $z \in \mathbb{R}^d \setminus Y$, we may choose $n \in \mathbb{N}$ such that $u_z(x_n) < u_z$ on K. Then $u_z(x_n) < \mu_n(u_z)$, and hence $\mu_n \notin J_{x_n}(Y)$. Next, we suppose that $\sup_{n\in\mathbb{N}} |x_n| < \infty$. Then (x_n) has a limit point $z \in \partial Y$, and we may choose $n \in \mathbb{N}$ such that $u_z(x_n) > u_z$ on K. Now $-u_z(x_n) < \mu_n(-u_z)$, and again $\mu_n \notin J_{x_n}(Y)$.

Thus in both cases (3.4) does not hold. This finishes the proof. □

4 A topological characterization

To obtain a topological characterization for (3.4) we introduce the following weak regularity property for open sets Y in $\mathbb{R}^d$, $d \geq 1$.

(WRP) Every non-empty closed-open set in ∂Y intersects the set $\partial_{\text{reg}} Y$ of regular boundary points for Y.

Of course, (WRP) holds if $\partial_{\text{reg}} Y$ is dense in ∂Y. So it trivially holds, if $d = 1$.

Suppose next that $d = 2$. Then, for all $z \in \partial_{\text{irr}} Y$ and $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that $\partial B(z, \delta) \cap \partial Y = \emptyset$ (see [2, Theorem 7.3.9]), and hence $B(z, \delta) \cap \partial Y$ is closed and open in ∂Y. So (WRP) holds if and only if $\partial_{\text{reg}} Y$ is dense in ∂Y.

Finally, let Y_0 denote the open unit ball in $\mathbb{R}^d$, $d \geq 3$. If P is a connected compact polar set, $P \subset \overline{Y}_0$, then $Y := Y_0 \setminus P$ satisfies (WRP) if and only if $P \cap \partial Y_0 \neq \emptyset$. Moreover, the following example may be instructive. Let I be a closed line segment, $0 \in I \subset Y_0$ and, for every $n \in \mathbb{N}$, let B_n be a closed ball in $B(0, 1/n) \setminus I$. Then $Y := Y_0 \setminus (I \cup B_1 \cup B_2 \cup \dots)$ satisfies (WRP).

PROPOSITION 4.1. *Let Y be an open set in $\mathbb{R}^d$, $d \geq 2$, bounded if $d = 2$, such that* (WRP) *is satisfied. Then Y is* 1*-transient.*

Proof. If $d \geq 3$, let $X := \mathbb{R}^d$ and $p := |\cdot|^{2-d}$. If $d = 2$, we take $R > 0$ such that $\overline{Y}$ is contained in the open disc $X := B(0, R)$, and define $p := \ln(R/|\cdot|)$.

We fix a compact K in Y and claim that $R_1^K < 1$ outside the compact set $\hat{K}$. So let W be a connected component of $Y \setminus K$ which is unbounded in Y. By ellipticity, it suffices to show that $R_1^K(y) < 1$ for some point $y \in W$. Clearly, this is true, if W is unbounded in X, since R_1^K is bounded by a multiple of p.

So let us suppose that W is bounded in X. Then $A := \partial W \setminus K$ is a non-empty closed set in ∂Y. We claim that A intersects the closure F of $\partial_{\text{reg}} Y$. Indeed, assume that $A \cap F = \emptyset$. Let $z \in A$ and let B be an open ball in $X \setminus (K \cup F)$ containing z. Then the set $B \cap \partial Y$ is polar, and hence $B \setminus \partial Y$ is connected (see [2, Corollary 5.1.5]). This implies that $B \setminus \partial Y \subset W$ and $B \cap \partial Y \subset A$. So A is open in ∂Y. By (WRP), $A \cap \partial_{\text{reg}} Y \neq \emptyset$, contradicting our assumption. This proves our claim.

So there exists a point $z \in A \cap F$. Let B be an open ball, $z \in B \subset\subset X \setminus K$. Then $B \cap \partial_{\text{reg}} Y \neq \emptyset$, and therefore $B \setminus Y$ is nonpolar. We fix $y \in B \cap W$. By [4, VI.9.4] or by the minimum principle,

$$\mu_y^{Y \setminus K}(B) \geq \mu_y^{Y \cap B}(B) > 0.$$

(Since $\mu_y^W = \mu_y^{Y \setminus K}$, this implies that $A \cap B$ is nonpolar, and hence $A \cap \partial_{\text{reg}} Y \neq \emptyset$.) By [4, VI.2.9], we finally conclude that

$$R_1^K(y) = {}^Y\varepsilon_y^K(K) = \mu_y^{Y \setminus K}(K) \leq 1 - \mu_y^{Y \setminus K}(B) < 1.$$

□

REMARK 4.2. If $d = 2$, the boundedness of Y in Proposition 4.1 may not be dropped. Indeed, let $Y := \{y \in \mathbb{R}^2 \colon |y| > 1\}$. Then $\partial_{\text{reg}} Y = \partial Y$, and hence (WRP) holds. Choosing $K := \{y \in \mathbb{R}^2 \colon |y| = 2\}$, the set $\{R_1^K = 1\} = \{y \in \mathbb{R}^2 \colon |y| \geq 2\}$ is not compact (and hence Y is not 1-transient). Moreover, taking $x := (3, 0)$ and $\mu := {}^Y\varepsilon_x^K$, we have $\mu \in M_x(\mathcal{P}(Y))$, $\operatorname{supp} \mu = K$, and $\mu(1) = 1$, but $\mu \notin J_x(Y)$, since $\mu(u_0) > u_0(x)$. So even (3.4) does not hold.

PROPOSITION 4.3. *Let Y be a bounded open set in $\mathbb{R}^d$, $d \geq 2$, such that* (WRP) *does not hold. Then* (3.4) *does not hold.*

Proof. By assumption, there exists a compact $A \neq \emptyset$ in ∂Y such that $F := \partial Y \setminus A$ is closed and $\partial_{\text{reg}} Y \subset F$. For the moment, let us fix $z \in A$. There exists a connected open neighborhood V_z of z such that $\overline{V}_z \cap F = \emptyset$. Then the set $V_z \cap \partial Y$ is polar, and hence $V_z \setminus \partial Y$ is connected (see [2, Corollary 5.1.5]). This implies that

$$V_z \setminus \partial Y = V_z \cap Y. \tag{4.1}$$

Since A is compact, we may choose $z_1, \dots, z_m \in A$ such that A is covered by the union V of $V_{z_1}, \dots, V_{z_m}$. Then $\overline{V} \cap F = \emptyset$ and $\partial V \cap A = \emptyset$. Since $A \cup F = \partial Y$, we hence see that $\partial V \cap \partial Y = \emptyset$. Moreover, by (4.1), $V \setminus \partial Y = V \cap Y$. So $V \setminus Y$ is the polar set A and $\overline{V} \subset \overline{Y}$, $\partial V \subset Y$.

We fix $z \in A$ and $x \in V \cap Y$ such that $|x - z| < \operatorname{dist}(z, \partial V)$, and define $\nu := \mu_x^V$. Then $\operatorname{supp} \nu \subset \partial V \subset Y$, and $\nu \in M_x(\mathcal{P}(Y))$, since polar sets are removable singularities for functions in $\mathcal{P}(Y)$. However, $-u_z > -u_z(x)$ on ∂V, and hence $\nu(-u_z) > -u_z(x)$, $\nu \notin J_x(Y)$. □

Combining Theorem 3.3 with Propositions 4.1 and 4.3 we obtain the following result.

COROLLARY 4.4. *For every bounded open set Y in $\mathbb{R}^d$, $d \geq 2$, the following statements are equivalent:*

- *Every non-empty closed-open set in ∂Y intersects $\partial_{\text{reg}} Y$.*
- *Y is* 1*-transient.*
- *For every $x \in Y$,*

$$\bigcup_{x \in U \subset\subset Y} J_x(U) = J_x(Y) = \{\mu \in M_x(\mathcal{P}(Y))\colon\ \operatorname{supp} \mu \subset\subset Y,\ \mu(1) = 1\}.$$

- *For every $x \in Y$,*

$$J_x(Y) = \{\mu \in M_x(\mathcal{P}(Y))\colon\ \operatorname{supp} \mu \subset\subset Y,\ \mu(1) = 1\}.$$

REMARK 4.5. Again, the boundedness of Y cannot be dropped. For $d = 2$ see the example discussed in Remark 4.2, where only (WRP) holds. Suppose now that $d \geq 3$, let $z_0 := (1, 0, \dots, 0)$, and $Y := \mathbb{R}^d \setminus \mathbb{R} z_0$. Then $\partial Y = \mathbb{R} z_0$ and $\partial_{\text{reg}} Y = \emptyset$ so that (WRP) is not satisfied. But Y is 1-transient (and the other statements hold), since $y \mapsto |y|^{2-d}$ is a potential on Y. However, *all* properties are satisfied, if $Y := B(0,1) \setminus \mathbb{R} z_0$.

5 Jensen measures for compact sets

Finally, let K be a compact set in (a general harmonic space) Y and $x \in K$. Then it is, of course, possible to define

$$J_x(K) := \bigcap_{K \subset U} J_x(U)$$

(see [13]). However, this does not yield anything new (see [4, Sections VII.8 and VII.9]). Indeed, clearly

$$S_0(K) := \bigcup_{K \subset U} S(U)|_K = \bigcup_{K \subset U} (\mathcal{S}(U) \cap \mathcal{C}(U))|_K.$$

Let G be the fine interior of K. Then, by [4, VII.9.2], the uniform closure of $S_0(K)$ is the set $S(K,G)$ of all continuous real functions u on K such that $\varepsilon_y^{V^c}(u) \le u(y)$, for all finely open $V \subset\subset G$ and all $y \in V$. Thus

$$J_x(K) = M_x(S_0(K)) = M_x(S(K,G)). \tag{5.1}$$

By [4, VII.9.5], $M_x(S(K,G))$ is a closed face of $M_x(\mathcal{P}(Y))$ and

$$\operatorname{ext} M_x(S(K,G)) = \{\varepsilon_x\} \cup \{\varepsilon_x^{B^c} : B \text{ Borel}, \ x \in B \subset K\}. \tag{5.2}$$

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