THE STABLE MONOMORPHISM CATEGORY OF A FROBENIUS CATEGORY

XIAO-WU CHEN

ABSTRACT. For a Frobenius abelian category $\mathcal{A}$, we show that the category $\mathrm{Mon}(\mathcal{A})$ of monomorphisms in $\mathcal{A}$ is a Frobenius exact category; the associated stable category $\underline{\mathrm{Mon}}(\mathcal{A})$ modulo projective objects is called the stable monomorphism category of $\mathcal{A}$. We show that a tilting object in the stable category $\underline{\mathcal{A}}$ of $\mathcal{A}$ modulo projective objects induces naturally a tilting object in $\underline{\mathrm{Mon}}(\mathcal{A})$. We show that if $\mathcal{A}$ is the category of (graded) modules over a (graded) self-injective algebra A, then the stable monomorphism category is triangle equivalent to the (graded) singularity category of the (graded) 2×2 upper triangular matrix algebra $T_2(A)$. As an application, we give two characterizations to the stable category of Ringel-Schmidmeier ([35]).

1. INTRODUCTION

Let $\mathcal{A}$ be an abelian category. Denote by $\mathrm{Mor}(\mathcal{A})$ the category of morphisms in $\mathcal{A}$ ([3, p.101]): the objects are morphisms in $\mathcal{A}$ and the morphisms are given by commutative squares in $\mathcal{A}$. It is an abelian category ([16, Proposition 1.1]). We are mainly concerned with the full subcategory $\mathrm{Mon}(\mathcal{A})$ of $\mathrm{Mor}(\mathcal{A})$ consisting of monomorphisms in $\mathcal{A}$, which is called the *monomorphism category* of $\mathcal{A}$. It is an additive subcategory of $\mathrm{Mor}(\mathcal{A})$ which is closed under extensions, thus it becomes an exact category in the sense of Quillen ([21, Appendix A]).

In the case that the abelian category $\mathcal{A}$ is the module category over a ring, the monomorphism category $\mathrm{Mon}(\mathcal{A})$ is known as the *submodule category*. Recently it is studied intensively by Ringel and Schmidmeier ([33, 34, 35]). If the ring is the truncated polynomial ring $\mathbb{Z}[t]/(t^p)$ with $p \geq 2$ and t an indeterminant, the study of the submodule category goes back to Birkhoff ([7; see also [1]). The case that the ring is $k[t]/(t^p)$ with k a field is studied by Simson ([36]) and also by Beligiannis ([6]). In this case, the study of indecomposable objects in $\mathrm{Mon}(\mathcal{A})$ shows an example of the typical trichotomy phenomenon "finite/tame/wild" in the representation theory of finite dimensional algebras, where the trichotomy depends on the parameter p; see [35, section 6]. Moreover, the case where the abelian category $\mathcal{A}$ is given by the graded module category over the graded algebra $k[t]/(t^p)$ with deg $t = 1$ plays an important role in [35]; in this case, the monomorphism category $\mathrm{Mon}(\mathcal{A})$ is denoted by $\mathcal{S}(\widetilde{p})$. It is a Frobenius exact category ([26]; also see Lemma 2.1 and compare [21, section 5]). Then by [17, Chapter I, Theorem 2.6] its stable category $\underline{\mathcal{S}}(\widetilde{p})$ modulo projective objects is triangulated. A very recent and remarkable result

due to Kussin, Lenzing and Meltzer claims that the stable category $\underline{\mathcal{S}}(\widetilde{p})$ is triangle equivalent to the stable category of vector bundles on the weighted projective lines of type $(2, 3, p)$; see [26]. Note that a similar trichotomy phenomenon "domestic/tubular/wild" occurs in the classification of indecomposable vector bundles on the weighted projective lines of type $(2, 3, p)$, while the trichotomy again depends on the parameter p; see [28, 25]. In this paper, we will call the triangulated category $\underline{\mathcal{S}}(\widetilde{p})$ the *stable category of Ringel-Schmidmeier*.

The present paper studies the monomorphism category $\mathrm{Mon}(\mathcal{A})$ of a Frobenius abelian category $\mathcal{A}$, in particular, the stable category $\underline{\mathcal{A}}$ modulo projective objects is triangulated. We show that $\mathrm{Mon}(\mathcal{A})$ is a Frobenius exact category and then the stable category $\underline{\mathrm{Mon}}(\mathcal{A})$ modulo projective objects is triangulated; it is called the *stable monomorphism category* of $\mathcal{A}$. Recently this category is also studied by Iyama, Kato and Miyachi ([20]). Note that the triangulated categories above are algebraical in the sense of Keller. We have a well-behaved notion of *tilting object* for an algebraical triangulated category ([23]). We prove that a tilting object in $\underline{\mathcal{A}}$ induces naturally a tilting object in $\underline{\mathrm{Mon}}(\mathcal{A})$; see Theorem 3.2. Moreover, if the category $\mathcal{A}$ is the (graded) module category over a (graded) self-injective algebra A, we relate the category $\underline{\mathrm{Mon}}(\mathcal{A})$ to the category of (graded) Gorenstein-projective modules and then to the (graded) singularity category of the 2×2 upper triangular matrix algebra $T_2(A)$ of A (for $T_2(A)$, see [16, p.115] and [3, Chapter III, section 2]); see Theorem 4.1. We are inspired by a computational result by Li and Zhang on Gorenstein-projective modules ([29]; compare [6, 20]). Here the Gorenstein-projective module is in the sense of Enochs and Jenda ([15, Chapter 10]), and the singularity category is in the sense of Orlov ([31, 32]; compare [9, 18]).

Combining all these together, we give two characterizations to the stable category of Ringel-Schmidmeier in Theorem 5.1: we characterize the stable category $\underline{\mathcal{S}}(\widetilde{p})$ as the bounded derived category of $T_2(k\mathbb{A}_{p-1})$, where $\mathbb{A}_{p-1}$ is the linear quiver with $p-1$ vertices and linear orientation, and $k\mathbb{A}_{p-1}$ is the path algebra; we characterize the stable category $\underline{\mathcal{S}}(\widetilde{p})$ as the graded singularity category of $T_2(k)[t]/(t^p)$, where the algebra $T_2(k)[t]/(t^p)$ is graded such that $\deg T_2(k) = 0$ and $\deg t = 1$.

For the convention, throughout we fix a commutative artinian ring R. All artin algebras are artin R-algebras, and all categories and functors are R-linear. For an artin algebra A, denote by $\mathrm{mod}\, A$ the category of finitely generated right A-modules and by $\mathrm{proj}\, A$ the full subcategory consisting of projective modules. We denote by A_A and $_A A$ the right and left regular modules of the artin algebra A, respectively. For triangulated categories and derived categories, we refer to [19, 17, 22, 23].

2. Monomorphism Category

Let $\mathcal{A}$ be a Frobenius abelian category. Thus $\mathcal{A}$ has enough projective objects and enough injective objects, and the class of projective objects coincides with the class of injective objects. Denote by $\mathcal{P}$ the full subcategory of $\mathcal{A}$ consisting of projective objects. Denote by $\underline{\mathcal{A}}$ the *stable category* of $\mathcal{A}$ modulo $\mathcal{P}$: the objects are the same as $\mathcal{A}$, and the morphism spaces are factors of the morphism spaces in $\mathcal{A}$ modulo those factoring through projective objects ([3, p.101]). The stable category $\underline{\mathcal{A}}$ is a triangulated category such that its shift functor is given by the inverse of the syzygy functor on $\underline{\mathcal{A}}$ and triangles are induced by short exact sequences in $\mathcal{A}$; for details, see [17, Chapter I, section 2].

Recall that $\mathrm{Mor}(\mathcal{A})$ is the category of morphisms in $\mathcal{A}$: the objects are morphisms $\alpha \colon A \to B$ in $\mathcal{A}$ and the morphisms are commutative squares in $\mathcal{A}$, that is, of the form $(f, g) \colon \alpha \to \alpha'$ where $f \colon A \to A'$ and $g \colon B \to B'$ are morphisms in $\mathcal{A}$ such that

$\alpha' \circ f = g \circ \alpha$ ([3, p.101]). For an object $\alpha\colon A \to B$ in $\mathrm{Mor}(\mathcal{A})$, we write $s(\alpha) = A$ and $t(\alpha) = B$, which are called the *source* and *target* of α, respectively. Note that $\mathrm{Mor}(\mathcal{A})$ is an abelian category such that a sequence $\alpha' \to \alpha \to \alpha''$ is exact if and only if the induced sequences of sources and targets are exact in $\mathcal{A}$ ([16, Corollary 1.2]).

Recall that an exact category in the sense of Quillen is an additive category together with an *exact structure*, that is, a distinguished class of ker-coker sequences, which are called *conflations*, subject to certain axioms. Note that a full additive subcategory of an abelian category which is closed under extensions has a natural exact structure such that conflations are just short exact sequences with terms in the subcategory ([21, Appendix A] and [22, section 4]). Moreover, there is a notion of Frobenius exact category and the associated stable category modulo projective objects is still triangulated; compare [17, p.10-11], [21, subsection 1.2 b)] and [22, section 6].

Recall that our main concern is the *monomorphism category* $\mathrm{Mon}(\mathcal{A})$, which is the full subcategory of $\mathrm{Mor}(\mathcal{A})$ consisting of monomorphisms in $\mathcal{A}$. We will consider the following two functors: the first functor $i_1\colon \mathcal{A} \to \mathrm{Mon}(\mathcal{A})$ is defined such that $i_1(A) = 0 \to A$ and $i_1(f) = (0, f)$ where A is an object and f is a morphism in $\mathcal{A}$; the second $i_2\colon \mathcal{A} \to \mathrm{Mon}(\mathcal{A})$ is defined such that $i_2(A) = A \overset{\mathrm{Id}_A}{\to} A$ and $i_2(f) = (f, f)$. Note that both functors are exact and fully faithful.

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category. Then the monomorphism category $\mathrm{Mon}(\mathcal{A})$ is an exact category such that conflations are given by sequences $\alpha' \to \alpha \to \alpha''$ with the induced sequences of sources and targets short exact in $\mathcal{A}$.*

Assume further that $\mathcal{A}$ is Frobenius. Then the exact category $\mathrm{Mon}(\mathcal{A})$ is Frobenius such that its projective objects are equal to direct summands of objects of the form $i_1(P) \oplus i_2(P)$ where P is a projective object in $\mathcal{A}$.

Proof. Note that $\mathrm{Mon}(\mathcal{A})$ is an additive subcategory of the abelian category $\mathrm{Mor}(\mathcal{A})$ which is closed under extensions by Snake Lemma. Then it is an exact category with conflations induced by short exact sequences in $\mathrm{Mor}(\mathcal{A})$; see Example 4.1 in [22].

Assume now that the abelian category $\mathcal{A}$ is Frobenius. We will show first that objects of the form $i_1(P)$ and $i_2(P)$ are projective and injective. Recall that for an object α in $\mathrm{Mon}(\mathcal{A})$ we denote by $s(\alpha)$ and $t(\alpha)$ the source and target of α, respectively. Note that

$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(i_1(P), \alpha) \simeq \mathrm{Hom}_{\mathcal{A}}(P, t(\alpha))$ and $\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(i_2(P), \alpha) \simeq \mathrm{Hom}_{\mathcal{A}}(P, s(\alpha))$.

These isomorphisms show that the objects are projective. Note that

$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(\alpha, i_1(P)) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{Cok}\, \alpha, P)$ and $\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(\alpha, i_2(P)) \simeq \mathrm{Hom}_{\mathcal{A}}(t(\alpha), P)$.

These isomorphisms show that the objects are injective (note that the functor Cok of taking the cokernels is exact on $\mathrm{Mon}(\mathcal{A})$ by Snake Lemma).

Let α be an object in $\mathrm{Mon}(\mathcal{A})$. Take epimorphisms $P \to s(\alpha)$ and $P \to t(\alpha)$ with P projective in $\mathcal{A}$. Then we have an epimorphism $i_1(P) \oplus i_2(P) \to \alpha$ whose kernel lies in $\mathrm{Mon}(\mathcal{A})$. This shows that the exact category $\mathrm{Mon}(\mathcal{A})$ has enough projective objects. On the other hand, for the object α, take monomorphisms $a\colon t(\alpha) \to P$ and $b'\colon \mathrm{Cok}\, \alpha \to P$ with P projective in $\mathcal{A}$. Denote by b the composite $t(\alpha) \to \mathrm{Cok}\, \alpha \overset{b'}{\to} P$ where the first morphism is the natural projection. Consider the following morphism in $\mathrm{Mor}(\mathcal{A})$

$$\left(\begin{pmatrix} a \circ \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right) \colon \alpha \longrightarrow i_2(P) \oplus i_1(P).$$

It is a monomorphism and by a diagram-chasing its cokernel lies in $\mathrm{Mon}(\mathcal{A})$. Then it becomes a conflation in $\mathrm{Mon}(\mathcal{A})$. This shows that the exact category $\mathrm{Mon}(\mathcal{A})$ has enough injective objects. From the argument above, it is direct to conclude that in the exact category $\mathrm{Mon}(\mathcal{A})$ the class of projective objects coincides with the class of injective objects, and projective objects are direct summands of objects of the form $i_1(P) \oplus i_2(P)$ where P is a projective object in $\mathcal{A}$.   □

Remark 2.2. With a slightly modified proof as above, one can show that a similar result holds if the category $\mathcal{A}$ is an exact category. In this case, one replaces $\mathrm{Mon}(\mathcal{A})$ by the category of *inflations* in $\mathcal{A}$; compare [21, section 5] and [20].   □

For a Frobenius abelian category $\mathcal{A}$, we denote by $\underline{\mathrm{Mon}}(\mathcal{A})$ the stable category of $\mathrm{Mon}(\mathcal{A})$ modulo projective objects; it is a triangulated category. We will call it the *stable monomorphism category* of $\mathcal{A}$.

Note that both the functors i_1 and i_2 are fully faithful and send projective objects to projective objects. Then they induce fully faithful triangle functors $i_1 \colon \underline{\mathcal{A}} \to \underline{\mathrm{Mon}}(\mathcal{A})$ and $i_2 \colon \underline{\mathcal{A}} \to \underline{\mathrm{Mon}}(\mathcal{A})$ ([17, p.23, Lemma 2.8]).

3. Tilting objects in Stable Monomorphism Category

In this section, we will show that for a Frobenius abelian category $\mathcal{A}$, a tilting object in the stable category $\underline{\mathcal{A}}$ induces naturally a tilting object in the stable monomorphism category $\underline{\mathrm{Mon}}(\mathcal{A})$.

Following Keller we call that a triangulated category is *algebraic* provided that it is triangle equivalent to the stable category of a Frobenius exact category ([23, subsection 8.7]). One has a well-behaved notion of tilting object in an algebraic triangulated category.

Let $\mathcal{T}$ be an algebraic triangulated category. Denote by [1] the shift functor and by [n] its n-th power for each $n \in \mathbb{Z}$. An object T in $\mathcal{T}$ is a *tilting object* if the following conditions are satisfied:

(T1) $\mathrm{Hom}_{\mathcal{T}}(T, T[n]) = 0$ for $n \neq 0$;
(T2) the smallest *thick* triangulated subcategory of $\mathcal{T}$ containing T is $\mathcal{T}$ itself;
(T3) $\mathrm{End}_{\mathcal{T}}(T)$ is an artin algebra having finite global dimension.

Here we recall that a triangulated subcategory of $\mathcal{T}$ is called *thick* if it is closed under taking direct summands. Note that the notion of tilting object presented here is slightly different from, however closely related to, the ones in [17] and [23].

Recall that an additive category is said to be *idempotent-split* provided that each idempotent $e \colon X \to X$ admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ such that $u \circ v = \mathrm{Id}_Y$ ([17, Chapter I, 3.2]). Note that for an artin algebra A having finite global dimension, the bounded derived category $\mathbf{D}^b(\mathrm{mod}\ A)$ is algebraical and idempotent-split (see the proof of [17, Chapter I, Corollary 4.9]), and it has A_A as its tilting object.

The following remarkable result due to Keller claims that the converse holds true (compare [8, Theorem 1]).

Lemma 3.1. (Keller) *Let $\mathcal{T}$ be an idempotent-split algebraic triangulated category with a tilting object T. Then there is a triangle equivalence*

$$\mathcal{T} \simeq \mathbf{D}^b(\mathrm{mod}\ \mathrm{End}_{\mathcal{T}}(T)).$$

Proof. Set $A = \mathrm{End}_{\mathcal{T}}(T)$. By [23, Theorem 8.51 a)] there is a triangle functor $F' \colon \mathcal{T} \to \mathbf{D}(A')$ sending T to A', where A' is a differential graded algebra with the only nonzero cohomology $H^0(A') \simeq A$ and $\mathbf{D}(A')$ is the (unbounded) derived

category of differential graded (right) modules on A'. Note that by [23, subsection 8.4] there is a triangle equivalence $\mathbf{D}(A') \simeq \mathbf{D}(\operatorname{Mod} A)$ identifying A' with A_A, where Mod A is the category of (unnecessarily finitely generated) right A-modules. Consequently, there is a triangle functor $F\colon \mathcal{T} \to \mathbf{D}(\operatorname{Mod} A)$ sending T to A. Using (T1) and (T2) and applying Beilinson Lemma ([17, p.72, Lemma 3.4]), the triangle functor F is fully faithful. Then we may view $\mathcal{T}$ as a triangulated subcategory of $\mathbf{D}(\operatorname{Mod} A)$; moreover, since $\mathcal{T}$ is idempotent-split, it is necessarily a thick subcategory of $\mathbf{D}(\operatorname{Mod} A)$. By (T3) the artin algebra A has finite global dimension, and then the smallest thick triangulated subcategory of $\mathbf{D}(\operatorname{Mod} A)$ containing A_A is $\mathbf{D}^b(\operatorname{mod} A)$. From this we conclude that the essential image of F is $\mathbf{D}^b(\operatorname{mod} A)$. Therefore F induces the required equivalence. $\square$

Our first main observation states that a tilting object in the stable category $\underline{\mathcal{A}}$ induces naturally a tilting object in the stable monomorphism category $\underline{\operatorname{Mon}}(\mathcal{A})$. Recall that for an artin algebra A, $T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is the 2×2 upper triangular matrix algebra ([3, Chapter III, section 2]).

Theorem 3.2. *Let $\mathcal{A}$ be a Frobenius abelian category such that T is a tilting object in its stable category $\underline{\mathcal{A}}$. Then $T' = i_1(T) \oplus i_2(T)$ is a tilting object in $\underline{\operatorname{Mon}}(\mathcal{A})$; moreover, we have an isomorphism $\operatorname{End}_{\underline{\operatorname{Mon}}(\mathcal{A})}(T') \simeq T_2(\operatorname{End}_{\underline{\mathcal{A}}}(T))$ of algebras.*

Proof. Recall that $i_1\colon \underline{\mathcal{A}} \to \underline{\operatorname{Mon}}(\mathcal{A})$ and $i_2\colon \underline{\mathcal{A}} \to \underline{\operatorname{Mon}}(\mathcal{A})$ are fully faithful triangle functors. Note that for objects A and B in $\mathcal{A}$, $\operatorname{Hom}_{\underline{\operatorname{Mon}}(\mathcal{A})}(i_2(A), i_1(B)) = 0$. So to check the condition (T1) for T', it suffices to show that $\operatorname{Hom}_{\underline{\operatorname{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n]) = 0$ for $n \neq 0$. For this end, note that since i_2 is a triangle functor, we have

$$i_2(T)[n] \simeq i_2(T[n]) = T[n] \overset{\operatorname{Id}_{T[n]}}{\longrightarrow} T[n].$$

Thus a morphism in $\operatorname{Hom}_{\underline{\operatorname{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n])$ is of the form $(0, f)$, where $f\colon T \to T[n]$ is a morphism in $\underline{\mathcal{A}}$. By the condition (T1) for T, f factors through a projective object P in $\mathcal{A}$. Therefore the morphism $(0, f)$ factors through $i_1(P)$, which is projective in $\operatorname{Mon}(\mathcal{A})$; see Lemma 2.1. Hence $(0, f) = 0$ in the stable monomorphism category $\underline{\operatorname{Mon}}(\mathcal{A})$.

To check (T2) for T', note that each object α fits into a conflation

$$i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\operatorname{Cok} \alpha)$$

and thus into a triangle

$$i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\operatorname{Cok} \alpha) \longrightarrow i_2(s(\alpha))[1].$$

Here as in section 2, $s(\alpha)$ denotes the source of α. Hence the smallest triangulated subcategory of $\underline{\operatorname{Mon}}(\mathcal{A})$ containing $i_1(\underline{\mathcal{A}})$ and $i_2(\underline{\mathcal{A}})$ is $\underline{\operatorname{Mon}}(\mathcal{A})$ itself. Now applying the condition (T2) of T, we infer that (T2) holds for T'.

Finally to see the condition (T3) for T', it is direct to check that $\operatorname{End}_{\underline{\operatorname{Mon}}(\mathcal{A})}(T') \simeq T_2(\operatorname{End}_{\underline{\mathcal{A}}}(T))$. Note that the algebra $\operatorname{End}_{\underline{\mathcal{A}}}(T)$ has finite global dimension. Then by [3, Chapter III, Proposition 2.6] we infer that $\operatorname{End}_{\underline{\operatorname{Mon}}(\mathcal{A})}(T')$ has finite global dimension. $\square$

We will give an application of Theorem 3.2. Let $A = \oplus_{n \geq 0} A_n$ be a positively graded artin algebra. Denote by c the maximal integer such that $A_c \neq 0$. Consider

the following upper triangular matrix algebra

$$\mathrm{b}(A) = \begin{pmatrix} A_0 & A_1 & \cdots & A_{c-2} & A_{c-1} \\ 0 & A_0 & \cdots & A_{c-3} & A_{c-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}.$$

Here the multiplication of $\mathrm{b}(A)$ is induced from the one of A. This algebra is called the *Beilinson algebra* of A in [11].

Denote by $\mathrm{mod}^{\mathbb{Z}} A$ the category of finitely generated $\mathbb{Z}$-graded A-modules with homomorphisms preserving degrees. We say that A is *graded self-injective* provided that $\mathrm{mod}^{\mathbb{Z}} A$ is a Frobenius category. In fact, this is equivalent to that as a ungraded algebra A is self-injective ([14, 11]). In this case, we denote by $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ the stable category of $\mathrm{mod}^{\mathbb{Z}} A$ modulo projective modules; it is a triangulated category.

We say that a graded algebra A is *right well-graded*, provided that A_c, as a right A_0-module, is sincere in the sense of [3, p.317]. In fact, for a graded self-injective algebra A, it is right well-graded if and only if it is left well-graded; see [11, Lemma 2.2]. In this case we will simply say that the graded algebra A is *well-graded*.

Corollary 3.3. *Let $A = \oplus_{n \geq 0} A_n$ be a positively graded self-injective artin algebra which is well-graded. Suppose that A_0 has finite global dimension. Then there is a triangle equivalence*

$$\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}} A) \simeq \mathbf{D}^b(\mathrm{mod}\, T_2(\mathrm{b}(A))).$$

.

Proof. Note that by [3, Chapter III, Proposition 2.6], the Beilinson algebra $\mathrm{b}(A)$ and then $T_2(\mathrm{b}(A))$ has finite global dimension. By [11, Corollary 1.2] there is a triangle equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathbf{D}^b(\mathrm{mod}\, \mathrm{b}(A))$. In particular, there is a tilting object T in $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ with endomorphism algebra $\mathrm{b}(A)$. We apply Theorem 3.2 to get a tilting object T' in $\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}} A)$ whose endomorphism algebra is isomorphic to $T_2(\mathrm{b}(A))$. Note that the stable monomorphism category $\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}} A)$ is idempotent-split; in fact, it is even a Krull-Schmidt category. Then the result follows immediately from Lemma 3.1. $\square$

4. Stable Monomorphism Category as Singularity Category

In this section we will relate the stable monomorphism category of the (graded) module category of a (graded) self-injective algebra to the (graded) singularity category of the associated (graded) 2×2 upper triangular matrix algebra.

Let A be an artin algebra. Recall that the bounded homotopy category $\mathbf{K}^b(\mathrm{proj}\, A)$ of projective modules is viewed naturally as a triangulated subcategory of $\mathbf{D}^b(\mathrm{mod}\, A)$. Following [31, 32], we call the Verdier quotient triangulated category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(\mathrm{mod}\, A)/\mathbf{K}^b(\mathrm{proj}\, A)$$

the *singularity category* of A; compare [9] and [18].

Our second main observation is as follows. Recall that for an artin algebra A, $T_2(A)$ is the 2×2 upper triangular matrix algebra of A.

Theorem 4.1. *Let A be a self-injective algebra. Then we have a triangle equivalence*

$$\underline{\mathrm{Mon}}(\mathrm{mod}\, A) \simeq \mathbf{D}_{\mathrm{sg}}(T_2(A)).$$

Before giving the proof, we recall several notions. Let A be an artin algebra. Following [4, p.400] an acyclic complex $P^\bullet$ of projective A-modules is called *totally acyclic* if the Hom complex $\mathrm{Hom}_A(P^\bullet, A)$ is still acyclic (also see [24, section 7]). An A-module M is said to be *Gorenstein-projective* if there is a totally acyclic complex $P^\bullet$ such that its zeroth cocycle $Z^0(P^\bullet)$ is isomorphic to M ([15, Chapter 10]). Note that a module M is Gorenstein-projective if and only if $\mathrm{Ext}^i_A(M, A) = 0$, $\mathrm{Ext}^i_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A) = 0$ for $i \geq 1$ and the natural map $M \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A)$ is an isomorphism (compare [13, Definition (1.1.2)]). Denote by Gproj A the full subcategory of mod A consisting of Gorenstein-projective modules. Note that projective modules are Gorenstein-projective and thus proj $A \subseteq$ Gproj A. Moreover, by [2, Proposition 5.1] the subcategory Gproj A is closed under extensions and taking direct summands (also see [15]); and then it is direct to see that Gproj A is a Frobenius exact category such that its projective objects are equal to projective A-modules ([5, Proposition 3.8(i)] and [12, Proposition 3.1(1)]). Denote by $\underline{\mathrm{Gproj}}$ A its stable category modulo projective A-modules; it is a triangulated category.

Recall that an artin algebra A is said to be *Gorenstein* if the regular modules $_AA$ and A_A have finite injective dimensions ([18]). In this case the two dimensions are equal and the common value is denoted by G.dim A. We say that the Gorenstein algebra A is *1-Gorenstein* provided that G.dim $A \leq 1$.

For an artin algebra A, denote by sub A the full subcategory of mod A consisting of submodules of projective modules; these modules are called *torsionless modules*.

We note the following well-known result.

Lemma 4.2. *Let A be a 1-Gorenstein algebra. Then we have* Gproj $A =$ sub A.

Proof. The inclusion Gproj $A \subseteq$ sub A is easy. On the other hand, assume that M is a torsionless module. Consider a short exact sequence $0 \to M \to P \to M' \to 0$ with P projective. Since the regular module A_A has injective dimension 1, using dimension shift, we infer that $\mathrm{Ext}^i(M, A) = 0$ for $i \geq 1$. Then by [15, Corollary 11.5.3] (see also [12, Lemma 3.7] and [24, Proposition 7.13]), M is Gorenstein-projective. $\square$

The next observation is essentially due to Li and Zhang ([29, Theorem 1.1]; also see [6, Example 4.14] and [20, Proposition 3.6]). Recall that for an artin algebra A, a (right) module over $T_2(A)$ is identified with a morphism of (right) A-modules; in fact, this yields an equivalence mod $T_2(A) \simeq \mathrm{Mor}(\mathrm{mod}\ A)$ of categories; see [3, Chapter III, Proposition 2.2].

Lemma 4.3. *Let A be a self-injective algebra. Then we have an equivalence of exact categories*

$$\mathrm{Mon}(\mathrm{mod}\ A) \simeq \mathrm{Gproj}\ T_2(A).$$

Proof. We claim first that under the equivalence mod $T_2(A) \simeq \mathrm{Mor}(\mathrm{mod}\ A)$, sub $T_2(A)$ corresponds to Mon(mod A). Note that the regular module $T_2(A)_{T_2(A)}$ corresponds to the monomorphism $\binom{0}{\mathrm{Id}_A} : A \to A \oplus A$. From this one infers that torsionless $T_2(A)$-modules correspond to monomorphisms in mod A. On the other hand, the third paragraph of the proof of Lemma 2.1 already shows that for a monomorphism α, there is a short exact sequence $0 \to \alpha \to \binom{0}{\mathrm{Id}_P} \to \alpha' \to 0$ in Mor(mod A) such that P is a projective A-module. Note that $\binom{0}{\mathrm{Id}_P}$ corresponds to a projective $T_2(A)$-module. Therefore the monomorphism α corresponds to a torsionless $T_2(A)$-module. This completes the proof of the claim.

Note that by [10, Remark 3.5] (also see [16, 18]) the algebra $T_2(A)$ is 1-Gorenstein and then we can apply Lemma 4.2. By the claim above, we obtain an equivalence

$\mathrm{Mon}(\mathrm{mod}\ A) \simeq \mathrm{Gproj}\ T_2(A)$ of categories. It is direct to check that this equivalence preserves the exact structures, that is, the equivalence and its inverse preserve short exact sequences in $\mathrm{Mon}(\mathrm{mod}\ A)$ and $\mathrm{Gproj}\ T_2(A)$. $\qquad\square$

We will recall the last ingredient in our proof. Let A be an artin algebra. Consider the following composite of natural functors

$$F_A \colon \mathrm{Gproj}\ A \hookrightarrow \mathrm{mod}\ A \hookrightarrow \mathbf{D}^b(\mathrm{mod}\ A) \longrightarrow \mathbf{D}_{\mathrm{sg}}(A)$$

where from the left side, the first functor is the inclusion, the second identifies modules with stalk complexes concentrated in degree zero ([19, p.40, Proposition 4.3]) and the last is the quotient functor. Note that the additive functor F_A vanishes on projective modules and then induces uniquely an additive functor $\underline{\mathrm{Gproj}}\ A \to \mathbf{D}_{\mathrm{sg}}(A)$, which is still denoted by F_A. The following important result is due to Buchweitz ([9, Theorem 4.4.1]) and independently due to Happel ([18, Theorem 4.6]); also see [12, Proposition 3.5 and Theorem 3.8].

Lemma 4.4. (Buchweitz-Happel) *Let A be an artin algebra. Then the functor $F_A \colon \underline{\mathrm{Gproj}}\ A \to \mathbf{D}_{\mathrm{sg}}(A)$ is a fully faithful triangle functor. Moreover, if A is Gorenstein, then the functor F_A is dense and thus a triangle equivalence.*

Proof of Theorem 4.1. By Lemma 4.3 there is an equivalence $\mathrm{Mon}(\mathrm{mod}\ A) \simeq \mathrm{Gproj}\ T_2(A)$ of Frobenius exact categories. Hence we have an induced equivalence of triangulated categories

$$\underline{\mathrm{Mon}}(\mathrm{mod}\ A) \ \simeq \ \underline{\mathrm{Gproj}}\ T_2(A).$$

Note that $T_2(A)$ is Gorenstein by [10, Theorem 3.3] (also see [16, 18]). Thus the result follows directly from Lemma 4.4. $\qquad\square$

We will also need a graded version of Theorem 4.1. Let $A = \oplus_{n\geq 0}A_n$ be a positively graded artin algebra. Denote by $\mathrm{proj}^{\mathbb{Z}}\ A$ the full subcategory of $\mathrm{mod}^{\mathbb{Z}}\ A$ consisting of projective objects. Following [32] one has the *graded singularity category* of A defined by

$$\mathbf{D}_{\mathrm{sg}}^{\mathbb{Z}}(A) = \mathbf{D}^b(\mathrm{mod}^{\mathbb{Z}}\ A)/\mathbf{K}^b(\mathrm{proj}^{\mathbb{Z}}\ A).$$

For a graded module $M = \oplus_{i\in\mathbb{Z}}M_i$ and an integer $d \in \mathbb{Z}$, its *shifted module* $M(d)$ has the same module structure as M while it is graded such that $M(d)_i = M_{d+i}$ for all $i \in \mathbb{Z}$. This defines automorphisms $(d)\colon \mathrm{mod}^{\mathbb{Z}}\ A \to \mathrm{mod}^{\mathbb{Z}}\ A$, which are called *degree-shift functors*. For graded modules M, N, we write $\mathrm{HOM}_A(M, N) = \oplus_{i\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{mod}^{\mathbb{Z}}\ A}(M, N(i))$ and set $\mathrm{EXT}_A^n(-, -)$ to be the n-th right derived functors ([30] and [14]).

An acyclic complex $P^{\bullet}$ in $\mathrm{proj}^{\mathbb{Z}}\ A$ is *totally acyclic* if the Hom complex $\mathrm{HOM}_A(P^{\bullet}, A)$ in $\mathrm{proj}^{\mathbb{Z}}\ A^{\mathrm{op}}$ is still acyclic. A graded A-module is called *graded Gorenstein-projective* provided that it is the zeroth cocycle of a totally acyclic complex. Thus we have a full subcategory $\mathrm{Gproj}^{\mathbb{Z}}\ A$ of $\mathrm{mod}^{\mathbb{Z}}\ A$ consisting of graded Gorenstein-projective modules and evidently $\mathrm{proj}^{\mathbb{Z}}\ A \subseteq \mathrm{Gproj}^{\mathbb{Z}}\ A$. As in the ungraded case, the category $\mathrm{Gproj}^{\mathbb{Z}}\ A$ is a Frobenius exact category with its projective objects equal to graded projective A-modules.

Recall that a graded artin algebra A is said to be *graded Gorenstein* if the graded regular modules $_AA$ and A_A have finite injective dimensions in $\mathrm{mod}^{\mathbb{Z}}\ A$ and $\mathrm{mod}^{\mathbb{Z}}\ A^{\mathrm{op}}$, respectively. In this case the two dimensions are the same, which will be denoted by $\mathrm{G.dim}^{\mathbb{Z}}\ A$.

We note the following fact, which guarantees in principle that most results in Gorenstein homological algebra hold true in the graded situation.

Lemma 4.5. *Let A be a positively graded artin algebra, and let M be a graded A-module. Then we have*

(1) *the module M is graded Gorenstein-projective if and only if it is Gorenstein-projective as a ungraded module;*

(2) *the algebra A is graded Gorenstein if and only if it is Gorenstein as a ungraded algebra; in this case, we have $\mathrm{G.dim}^{\mathbb{Z}} A = \mathrm{G.dim}\, A$.*

Proof. For (1), it suffices to note that a graded module M is graded Gorenstein-projective if and only if $\mathrm{EXT}^i_A(M, A) = 0$, $\mathrm{EXT}^i_{A^{\mathrm{op}}}(\mathrm{HOM}_A(M, A), A) = 0$ for $i \geq 1$ and the natural map $M \to \mathrm{HOM}_{A^{\mathrm{op}}}(\mathrm{HOM}_A(M, A), A)$ is an isomorphism of graded modules; moreover, for graded modules M and N we have for each i a natural identification $\mathrm{EXT}^i_A(M, N) = \mathrm{Ext}^i_A(M, N)$ ([30, Corollary 2.4.7]). For (2), note that a graded module M has finite injective dimension in $\mathrm{mod}^{\mathbb{Z}} A$ if and only if it has finite injective dimension as a ungraded module; moreover, the two dimensions are the same ([30, Theorem 2.8.7]). $\square$

One can show the graded analogues of Lemmas 4.2, 4.3 and 4.4. Using these, we have the following graded analogue of Theorem 4.1.

Proposition 4.6. *Let $A = \oplus_{n \geq 0} A_n$ be a positively graded self-injective artin algebra. Denote by $T_2(A)$ the 2×2 upper triangular matrix algebra of A which is graded such that $T_2(A)_n = T_2(A_n)$ for $n \geq 0$. Then we have a triangle equivalence*

$$\underline{\mathrm{Mon}}(\mathrm{mod}^{\mathbb{Z}} A) \simeq \mathbf{D}^{\mathbb{Z}}_{\mathrm{sg}}(T_2(A)).$$

5. The Stable Category of Ringel-Schmidmerier

In the last section, we will give the promised two characterizations to the stable category of Ringel-Schmidmeier ([35]).

Let k be a field and let $p \geq 2$ be an integer. In particular, the fixed commutative artinian ring R now is the field k. Consider the truncated polynomial algebra $A = k[t]/(t^p)$ with t an indeterminant; it is positively graded such that $\deg t = 1$. Note that A is graded self-injective and moreover it is well-graded. In particular, the category $\mathrm{mod}^{\mathbb{Z}} A$ of finitely generated graded A-modules is Frobenius. Following [35, subsection 0.4], we denote by $\mathcal{S}(\widetilde{p})$ the category of pairs (V, U), where V is a graded module over A and $U \subseteq V$ is a graded submodule, and the morphisms in this category are given by morphisms in the graded module category which respect the inclusion. There is a natural identification $\mathcal{S}(\widetilde{p}) = \mathrm{Mon}(\mathrm{mod}^{\mathbb{Z}} A)$ and then by Lemma 2.1 it is a Frobenius exact category. Hence its stable category $\underline{\mathcal{S}}(\widetilde{p})$ modulo projective objects is triangulated. It will be called the *stable category of Ringel-Schmidmeier.*

We note that the Beilinson algebra $\mathrm{b}(A)$ (in the sense of [11]) of the graded algebra A is isomorphic to the path algebra $k\mathbb{A}_{p-1}$ of the linear quiver $\mathbb{A}_{p-1}$ with $p - 1$ vertices and linear orientation (compare [32, Example 2.9]). Then the 2×2 upper triangular matrix algebra $\Lambda = T_2(\mathrm{b}(A))$ is given by the following quiver with $2p - 2$ vertices subject to the commutativity relation

Let us mention that similar diagrams appeared in [27].

Recall that the 2×2 upper triangular matrix algebra $T_2(A)$ is graded such that $T_2(A)_n = T_2(A_n)$ for $n \geq 0$. Then it is isomorphic, as a graded algebra, to $T_2(k)[t]/(t^p)$, while the latter is graded such that $\deg T_2(k) = 0$ and $\deg t = 1$.

Combining Corollary 3.3 and Proposition 4.6 together, we obtain two characterizations of the stable category of Ringel-Schmidmeier.

Theorem 5.1. *Use the notation above. Then there are triangle equivalences*
$$\mathbf{D}^b(\mathrm{mod}\,\Lambda) \;\simeq\; \underline{\mathcal{S}}(\widetilde{p}) \;\simeq\; \mathbf{D}^{\mathbb{Z}}_{\mathrm{sg}}(T_2(k)[t]/(t^p)).$$

Remark 5.2. Let us remark that taking into account of the results obtained in [25] and [27, Corollary 1.2], one may find a close relation between some results in [26] and Theorem 5.1.

Note that $T = \bigoplus_{i=0}^{p-2}(A/(t^{p-i-1}))(i)$ is a titling object in $\underline{\mathrm{mod}}^{\mathbb{Z}}\,A$, where (i) denote the degree-shift functors. This assertion can be obtained from the proof of [32, Corollary 2.8] or [11, Corollary 1.2]. We apply Theorem 3.2 to deduce that $T' = i_1(T) \oplus i_2(T)$ is a tilting object in $\underline{\mathcal{S}}(\widetilde{p})$. This explicit tilting object is also obtained in [26, Lemma 4.7] via a different method. $\square$

REFERENCES

[1] D.M. Arnold, Abelian Groups and Representations of Finite Partially Ordered Sets, Springer GMS Books in Mathematics, 2000.

[2] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. **86** (1)(1991), 111–152.

[3] M. Auslander, I. Reiten and S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Univ. Press, 1995.

[4] L.L. Avramov and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension,* Proc. London Math. Soc. (3) **85** (2002), no.2, 393–440

[5] A. Beligiannis, *Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras,* J. Algebra **288** (2005), 137-211.

[6] A. Beligiannis, *On algebras of finite Cohen-Macaulay type*, preprint, 2009.

[7] G. Birkhoff, *Subgroups of abelian groups*, Proc. London Math. Soc. II, Ser. **38** (1934), 385–401.

[8] A.I. Bondal and M.M. Kapranov, *Enhanced triangulated categories*, Math. USSR Sbornik **70** (1990), no.1, 93–107.

[9] R.O. Buchweitz, Maximal Cohen-Macaulay Modules and Tate Cohomology over Gorenstein Rings, Unpublished manuscript, 155pp (1987). Available at https://tspace.library.utoronto.ca/handle/1807/16682.

[10] X.W. Chen, *Singularity categories, Schur functors and triangular matrix rings*, Algebra and Representation Theory **12** (2009), 181–191.

[11] X.W. Chen, *Graded self-injective algebras "are" trivial extensions,* J. Algebra **322** (2009), 2601–2606.

[12] X.W. Chen, *Relative singularity categories and Gorenstein-projective modules*, Math. Nach., to appear, arXiv: 0709.1762v1.

[13] L.W. Christensen, Gorenstein Dimension, Lecture Notes in Math. **1747**, Springer-Verlag, Berlin Heidelberg, 2000.

[14] R. Gordon and E.L. Green, *Graded Artin algebras,* J. Algebra **76**(1) (1982), 111–137.

[15] E.E. Enochs and O.M.G. Jenda, Relative Homological Algebra, de Gruyter Expositions in Math. **30**, Walter de Gruyter, Berlin New York, 2000.

[16] R.M. Fossum, P. Griffith and I. Reiten, Trivial Extensions of Abelian Categories, Lecture Notes in Math. **456**, Springer-Verlag, Berlin Heidelberg New York, 1975.

[17] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Mathematical Society Lecture Note Ser. **119**, Cambridge Univ. Press, 1988.

[18] D. HAPPEL, *On Gorenstein algebras,* Progress in Math., vol. **95**, 389–404, Birkhäuser Verlag, Basel, 1991.

[19] R. HARTSHORNE, Residue and Duality, Lecture Notes in Math. **20**, Springer-Verlag, 1966.

[20] O. IYAMA, K. KATO AND J.I. MIYACHI, *Recollement of homotopy categories and Cohen-Macaylay modules*, arXiv:0911.0172v1.

[21] B. KELLER, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), 379–417.

[22] B. KELLER, *Derived categories and their uses*, in: Handbook of Algebra **1**, 671–701, North-Holland, Amsterdam, 1996.

[23] B. KELLER, *Derived categories and tilting*, in: Handbook of Tilting Theory, 49–104, London Math. Soc. Lecture Note Ser. **332**, Cambridge Univ. Press, Cambridge, 2007.

[24] H. KRAUSE, *The stable derived category of a noetherian scheme*, Compositio Math. **141** (2005), 1128–1162.

[25] D. KUSSIN, H. LENZING AND H. MELTZER, *Triangle singularities, ADE-chains and weighted projective lines*, preprint, 2009.

[26] D. KUSSIN, H. LENZING AND H. MELTZER, *Categories of vector bundles and invariant subspaces of nilpotent operators*, arXiv:1002.3797v1.

[27] S. LADKANI, *On derived equivalences of lines, rectangles and triangles,* arXiv:0911.5137v1.

[28] H. LENZING, *Hereditary categories*, in: Handbook of Tilting Theory, 105–146, London Math. Soc. Lecture Note Ser. **332**, Cambridge Univ. Press, Cambridge, 2007.

[29] Z.W. LI AND P. ZHANG, *A construction of Gorenstein-projective modules,* J. Algebra **323** (6) (2010), 1802–1812.

[30] C. NASTASESCU AND F. VAN OYSTAEYEN, Methods of Graded Rings, Lecture Note in Math. **1836**, Springer, 2004.

[31] D. ORLOV, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Proc. Steklov Inst. Math. **246** (3) (2004), 227–248.

[32] D. ORLOV, *Derived categories of coherent sheaves and triangulated categories of singularities,* math.AG/0503632v2.

[33] C.M. RINGEL AND M. SCHMIDMEIER, *Submodule categories of wild representation type*, Jour. Pure and Applied Algebra **205** (2) (2006), 412–422.

[34] C.M. RINGEL AND M. SCHMIDMEIER, *The Auslander-Reiten translation in submodule categories*, Trans. Amer. Math. Soc. **360** (2008), 691–716.

[35] C.M. RINGEL AND M. SCHMIDMEIER, *Invariant subspaces of nilpotent operators I*, Journal Reine Angew. Math. Band **614** (2008), 1–52.

[36] D. SIMSON, *Chain categories of modules and subprojective representations of posets over uniserial algebras*, Rocky Mountain J. Math. **32** (2002), 1627–1650.

Xiao-Wu Chen, Department of Mathematics, University of Science and Technology of China, Hefei 230026, P. R. China

Homepage: http://mail.ustc.edu.cn/~xwchen

Current address: Institut fuer Mathematik, Universitaet Paderborn, 33095, Paderborn, Germany