N-TUPLES OF COMPLEX $N \times N$ COMMUTING MATRICES ARE NOT HYPERCYCLIC

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Abstract. We discuss a question raised by N. Feldman on the dynamics of tuples of matrices over $\mathbb{C}$.

1. Introduction

In the last twenty years the subject of hypercyclicity has been studied intensively. Hypercyclicity is a phenomenon occurring only in infinite dimensional topological vector spaces, see for example [5]. That is, if for a continuous and linear operator T acting on a topological vector space X there exists a vector $x \in X$ whose orbit under T is dense in X, i.e.

$$\overline{Orb(T,x)} = \overline{\{x, Tx, T^2x, \dots\}} = X$$

then X is infinite dimensional. However, Feldman in [8], [9] showed that hypercyclic tuples of matrices may exist in finite dimensional vector spaces. Namely, there exist $n \times n$ diagonal matrices $T_1, \dots, T_{n+1}$ with complex entries and vectors $x \in \mathbb{C}^n$ such that the set

$$\{T_1^{k_1} \cdots T_{n+1}^{k_{n+1}} x : k_1, \dots, k_{n+1} \in \mathbb{N} \cup \{0\}\}$$

is dense in $\mathbb{C}^n$. Such vectors $x \in \mathbb{C}^n$ are called hypercyclic for the tuple $(T_1, \dots, T_{n+1})$. In addition, Feldman proved that no n-tuple of diagonal $n \times n$ matrices with complex entries is hypercyclic on $\mathbb{C}^n$. Complementing Feldman's result we showed in [6] that there exist hypercyclic k-tuples of non simultaneously diagonalizable (commuting) $n \times n$ matrices with $k > n$ on $\mathbb{C}^n$. Hence the following question due to Feldman [8] naturally arises.

Question. *Is there a hypercyclic (non simultaneously diagonalizable) n-tuple on $\mathbb{C}^n$?*

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The purpose of the present work is to give a negative answer to the aforementioned question. For simplicity reasons we demonstrate our method, which is completely elementary, for the cases $n = 2, 3, 4$. Recently, Ayadi in [2] has also answered Feldman's question using more sophisticated tools than ours, such as his previous very interesting work on abelian subgroups of $GL(n, \mathbb{C})$, see [3], [4]. We would also like to mention that H. Abels and the third author, among other things, they were able to find the minimal number of matrices either on $\mathbb{C}^n$ or $\mathbb{R}^n$ (the real case is much harder) using algebraic group theory, see [1] and hence they provide yet an alternative approach to Feldman's question. For an account of results on hypercyclicity we refer to the survey articles [10], [11], [12] and the recent books [5], [13]. For results on the dynamics of tuples of matrices see [1], [6], [7], [9], [14], [15], [16].

2. A reduction to Toeplitz matrices

Let us first fix some notation. For $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ the symbol $J_m(\lambda)$ stands for a $m \times m$ matrix in a Jordan form with eigenvalue λ. Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_k$ and multiplicities $n_1, \dots, n_k$ respectively. Then A is similar to a matrix B, where B is the standard Jordan canonical form of A. Up to permutation, B has the following unique structure

$$B = J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_k}(\lambda_k).$$

The next two lemmas are well known with easy proofs, so we just state them.

Lemma 2.1. *Let J, A be $n \times n$ commuting matrices over $\mathbb{C}$ such that $J = J_n(\lambda)$, $\lambda \in \mathbb{C}$. Then*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ 0 & 0 & 0 & a_1 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{pmatrix},$$

where $a_1, \cdots, a_n \in \mathbb{C}$.

Observe that the matrix A in the above lemma is a (triangular) Toeplitz matrix.

Lemma 2.2. *Let A, B be $n \times n$ commuting matrices over $\mathbb{C}$ such that B is in a Jordan canonical form, i.e. $B = J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_k}(\lambda_k)$, for certain $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and $n_1 + \ldots + n_k = n$. Then A has the form*

$A = A_1 \oplus \ldots \oplus A_k$, where A_j is a $n_j \times n_j$ (triangular) Toeplitz matrix as in Lemma 2.1.

The above lemmas point out that in order to understand the dynamics of tuples of commuting matrices it suffices to know how the iterates of (triangular) Toeplitz matrices behave. Let us briefly explain why. Suppose that $B_1, \ldots, B_n$ are complex $n \times n$ commuting matrices. Then there is a similarity transformation matrix P such that $P^{-1}B_1P = J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_k}(\lambda_k)$, for certain $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $n_1 + \ldots + n_k = n$. In other words $P^{-1}B_1P$ is just the Jordan canonical form of B_1. Observe now that the matrices $P^{-1}B_1P, \ldots, P^{-1}B_nP$ are pairwise commuting and by Lemmas 2.1,2.2 we have $P^{-1}B_jP = A_{j,1} \oplus \ldots \oplus A_{j,k}$ for $j = 2, \ldots, n$ and each $A_{j,\rho}$ is a $n_j \times n_j$ (triangular) Toeplitz matrix, $j = 2, \ldots, n$, $\rho = 1, \ldots, k$. It is easy to check that hypercyclicity is preserved under similarity. Therefore, we conclude that the tuple

$$(B_1, \ldots, B_n)$$

is hypercyclic if and only if the tuple

$$(J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_k}(\lambda_k), A_{2,1} \oplus \ldots \oplus A_{2,k}, \ldots, A_{n,1} \oplus \ldots \oplus A_{n,k})$$

is hypercyclic. It is also useful to observe that

$$(A_{j,1} \oplus \ldots \oplus A_{j,k})^m = A_{j,1}^m \oplus \ldots \oplus A_{j,k}^m, \;\; m \in \mathbb{N}.$$

Hence, Toeplitz matrices come naturally into play when dealing with dynamics of commuting matrices.

3. Iterates of Toeplitz matrices

Let $n \in \mathbb{N}$ and consider the $n \times n$ Toeplitz matrices

$$A_i = \begin{pmatrix} a_{i,1} & a_{i,2} & a_{i,3} & a_{i,4} & \cdots & a_{i,n} \\ 0 & a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,n-1} \\ 0 & 0 & a_{i,1} & a_{i,2} & \cdots & a_{i,n-2} \\ 0 & 0 & 0 & a_{i,1} & \cdots & a_{i,n-3} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{i,1} \end{pmatrix}$$

for $i = 1, 2, \ldots, m$. We first begin by illustrating how to determine the elements of A_i^k for some power k without needing to go through the laborious process of raising A_i to the power k. Define the $n \times n$ nilpotent shift matrices U_p for $p = 0, 1, 2, \ldots$ by setting

$$u_{i,j} = \begin{cases} 1, & j = i + p \\ 0, & \text{otherwise.} \end{cases}$$

Observe that U_0 is the $n\times n$ identity matrix and $U_p = U_1^p$ which implies that $U_pU_q = U_qU_p = U_{p+q}$ for any $p, q \in \{0, 1, 2, \ldots\}$. Also, U_p is the $n \times n$ zero matrix for any $p \geq n$. As a result we have that

$$A_i = \sum_{j=1}^{n} a_{i,j} U_{j-1}.$$

It now follows from the multinomial theorem that

$$\begin{aligned} A_i^k &= \left(\sum_{j=1}^{n} a_{i,j} U_{j-1} \right)^k \\ &= \sum_{\substack{k_{i,1},\ldots,k_{i,n} \\ k_{i,1}+\ldots+k_{i,n}=k}} \binom{k}{k_{i,1},\ldots,k_{i,n}} a_{i,1}^{k_{i,1}} \ldots a_{i,n}^{k_{i,n}} \, U_1^{k_{i,2}+2k_{i,3}+\ldots+(n-1)k_{i,n}} \end{aligned}$$

where

$$\binom{k}{k_{i,1}, k_{i,2}, \ldots, k_{i,n}} = \frac{k!}{k_{i,1}!k_{i,2}!\ldots k_{i,n}!}$$

is the multinomial coefficient. This example illustrates the ease with which one can compute the entries of the matrix where now attention is paid on the set of configurations which make the expression $k_{i,2} + 2k_{i,3} + \ldots + (n-1)k_{i,n}$ equal to a given integer which by nature is more of a combinatorial problem.

Let us now turn to the product $A_1^{k_1} \ldots A_m^{k_m}$ which is

$$\sum_{\substack{k_{1,1},\ldots,k_{1,n} \\ k_{1,1}+\ldots+k_{1,n}=k_1}} \cdots \sum_{\substack{k_{m,1}\ldots k_{m,n} \\ k_{m,1}+\ldots+k_{m,n}=k_m}} \prod_{j=1}^{m} \binom{k_j}{k_{j,1},\ldots,k_{j,n}} a_{j,1}^{k_{j,1}} \ldots a_{j,n}^{k_{j,n}} \times$$
$$\times U_1^{k_{1,2}+\ldots+k_{m,2}+2(k_{1,3}+\ldots+k_{m,3})+\ldots+(n-1)(k_{1,n}+\ldots+k_{m,n})}.$$

The only difficulty in such a computation is on determining the set of configurations which make the expression

$$k_{1,2} + \ldots + k_{m,2} + 2(k_{1,3} + \ldots + k_{m,3}) + \ldots + (n-1)(k_{1,n} + \ldots + k_{m,n})$$

equal to a given integer.

4. 2×2 MATRICES

For this section let B_1, B_2 be 2×2 non-simultaneously diagonalizable commuting matrices . Hence B_1 has only one eigenvalue and the same is true for B_2 as well. Then from the analysis in section 2, there exists a

2×2 similarity transformation matrix P such that both $A_1 = P^{-1}B_1P$, $A_2 = P^{-1}B_2P$ are triangular Toeplitz matrices, i.e.

$$A_1 = \begin{pmatrix} a_{1,1} & a_{1,2} \\ 0 & a_{1,1} \end{pmatrix}, A_2 = \begin{pmatrix} a_{2,1} & a_{2,2} \\ 0 & a_{2,1} \end{pmatrix}.$$

Then for a multi-index $k = (k_1, k_2)$, $k_1, k_2 \in \mathbb{N}$ define

$$A(k) = \begin{pmatrix} a_1(k) & a_2(k) \\ 0 & a_1(k) \end{pmatrix} := A_1^{k_1} A_2^{k_2}.$$

A direct computation shows the following, provided that $a_{1,1} \neq 0$, $a_{2,1} \neq 0$.

Lemma 4.1.

$$\begin{aligned} a_1(k) &= a_{1,1}^{k_1} a_{2,1}^{k_2} \\ a_2(k) &= a_1(k) \sum_{j=1}^{2} k_j \frac{a_{j,2}}{a_{j,1}} \end{aligned}$$

Lemma 4.2. *The tuple (A_1, A_2) is hypercyclic if and only if the set*

$$\left\{ \begin{pmatrix} \sum_{j=1}^{2} k_j \frac{a_{j,2}}{a_{j,1}} \\ a_{1,1}^{k_1} a_{2,1}^{k_2} \end{pmatrix} : k_1, k_2 \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$.

Proof. Assume that (A_1, A_2) is hypercyclic with hypercyclic vector $(z_1, z_2) \in \mathbb{C}^2$. Necessarily we have $a_{1,1} \neq 0$ and $a_{2,1} \neq 0$. Then the set

$$\{(a_1(k)z_1 + a_2(k)z_2, a_1(k)z_2 : k = (k_1, k_2) \in \mathbb{N} \times \mathbb{N}\}$$

is dense in $\mathbb{C}^2$. From the last fact, it follows that $z_2 \neq 0$. Take $(w_1, w_2) \in \mathbb{C}^2$. From our hypothesis and Lemma 4.1 there exist sequences $\{k_1(n)\}$, $\{k_2(n)\}$ of positive integers such that

$$a_1(k(n))z_2 \to w_2 z_2$$

$$a_1(k(n))z_1 + a_2(k(n))z_2 \to w_2 z_1 + w_1 w_2 z_2,$$

where $k(n) = (k_1(n), k_2(n))$. Since $z_2 \neq 0$, from the above we get

$$a_1(k(n)) \to w_2 \quad \text{and} \quad \sum_{j=1}^{2} k_j(n) \frac{a_{j,2}}{a_{j,1}} \to w_1.$$

For the converse implication one works in a similar manner. □

Once we have at our disposal the above lemma, one can follow the argument from [9] in order to conclude that the tuple (A_1, A_2) is not hypercyclic.

Proposition 4.3. *The tuple (A_1, A_2) is not hypercyclic.*

Proof. Suppose on the contrary that (A_1, A_2) is hypercyclic. By Lemma 4.2 it follows that the set

$$\left\{ \begin{pmatrix} \sum_{j=1}^{2} k_j \frac{a_{j,2}}{a_{j,1}} \\ a_{1,1}^{k_1} a_{2,1}^{k_2} \end{pmatrix} : k_1, k_2 \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$ and hence the set

$$\left\{ \begin{pmatrix} \left(e^{\frac{a_{1,2}}{a_{1,1}}} \right)^{k_1} \left(e^{\frac{a_{2,2}}{a_{2,1}}} \right)^{k_2} \\ a_{1,1}^{k_1} a_{2,1}^{k_2} \end{pmatrix} : k_1, k_2 \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. The last implies that the tuple of diagonal matrices

$$\left(\begin{pmatrix} e^{\frac{a_{1,2}}{a_{1,1}}} & 0 \\ 0 & a_{1,1} \end{pmatrix}, \begin{pmatrix} e^{\frac{a_{2,2}}{a_{2,1}}} & 0 \\ 0 & a_{2,1} \end{pmatrix} \right)$$

is hypercyclic which is a contradiction from the results in [9]. □

From the discussion in section 2 and the proposition above we conclude that the tuple (B_1, B_2) is not hypercyclic.

5. 3×3 MATRICES

In order to simplify things, assume that B_1 is a 3×3 matrix having only one eigenvalue, i.e B_1 is similar to a 3×3 matrix in Jordan form, say A_1 through a similarity matrix P. Let us consider the more general situation where A_1 is a (triangular) Toeplitz matrix. To illustrate how making use of the computation in section 3 can turn a purely algebraic problem (that of determining the elements of A^k) into a more or less combinatorial one, consider raising to the power k the 3×3 matrix

$$A_1 = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,1} & a_{1,2} \\ 0 & 0 & a_{1,1} \end{pmatrix}.$$

From the multinomial theorem

$$A_1^k = \sum_{\substack{k_{1,1}, k_{1,2}, k_{1,3} \\ k_{1,1}+k_{1,2}+k_{1,3}=k}} \binom{k}{k_{1,1}, k_{1,2}, k_{1,3}} a_{1,1}^{k_{1,1}} a_{1,2}^{k_{1,2}} a_{1,3}^{k_{1,3}} \, U_1^{k_{1,2}+2k_{1,3}}$$

so the product matrix is of the form

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ 0 & c_{1,1} & c_{1,2} \\ 0 & 0 & c_{1,1} \end{pmatrix}.$$

To determine $c_{1,1}$ we require $k_{1,2} + 2k_{1,3} = 0$ which implies that $k_{1,2} = k_{1,3} = 0$ and so $k_{1,1} = k$. As a result,

$$c_{1,1} = \binom{k}{k, 0, 0} a_{1,1}^{k} a_{1,2}^{0} a_{1,3}^{0} = a_{1,1}^{k}.$$

To determine $c_{1,2}$ we require $k_{1,2} + 2k_{1,3} = 1$ which implies that $k_{1,3} = 0$ and $k_{1,2} = 1$. As a result,

$$c_{1,2} = \binom{k}{k-1, 1, 0} a_{1,1}^{k-1} a_{1,2}^{1} a_{1,3}^{0} = k a_{1,1}^{k-1} a_{1,2}.$$

To determine $c_{1,3}$ we require $k_{1,2} + 2k_{1,3} = 2$ which implies that either $k_{1,3} = 1$ and $k_{1,2} = 0$ or $k_{1,3} = 0$ and $k_{1,2} = 2$. The first choice gives

$$\binom{k}{k-1, 0, 1} a_{1,1}^{k-1} a_{1,2}^{0} a_{1,3}^{1} = k a_{1,1}^{k-1} a_{1,3}$$

and the second choice gives

$$\binom{k}{k-2, 2, 0} a_{1,1}^{k-2} a_{1,2}^{2} a_{1,3}^{0} = \frac{k(k-1)}{2} a_{1,1}^{k-2} a_{1,2}^{2}.$$

Adding the two together gives

$$c_{1,3} = k a_{1,1}^{k-1} a_{1,3} + \frac{k(k-1)}{2} a_{1,1}^{k-2} a_{1,2}^{2}.$$

Take now B_2, B_3 3×3 matrices such that the tuple (B_1, B_2, B_3) is commuting. From the discussion in section 2 and since B_1 has only one eigenvalue, it follows that $A_j = P^{-1} B_j P$ is triangular and Toeplitz for every $j = 1, 2, 3$. The elements of the matrices A_1, A_2, A_3 are defined as in section 2. Then for a multi-index $k = (k_1, k_2, k_3)$, $k_1, k_2, k_3 \in \mathbb{N}$ define

$$B(k) = \begin{pmatrix} b_1(k) & b_2(k) & b_3(k) \\ 0 & b_1(k) & b_2(k) \\ 0 & 0 & b_1(k) \end{pmatrix} = A_1^{k_1} A_2^{k_2} A_3^{k_3}.$$

We are now ready to follow the scheme developed in section 4. The proofs of the following lemmas and proposition are similar to that in section 4 and are left to the interested reader.

Lemma 5.1. *Denoting by $k=(k_1,k_2,k_3)$ a multi-index, $k_1,k_2,k_3\in\mathbb{N}$, we have*

$$\begin{aligned} b_1(k) &= a_{1,1}^{k_1}a_{2,1}^{k_2}a_{3,1}^{k_3} \\ b_2(k) &= b_1(k)\sum_{j=1}^{3}k_j\frac{a_{j,2}}{a_{j,1}} \\ b_3(k) &= b_1(k)\Big(\sum_{j=1}^{3}k_j\big(\frac{a_{j,3}}{a_{j,1}}-\frac{1}{2}\frac{a_{j,2}^2}{a_{j,1}^2}\big)+\frac{1}{2}\big(\sum_{j=1}^{3}k_j\frac{a_{j,2}}{a_{j,1}}\big)^2\Big) \\ &= b_1(k)\Big(\sum_{j=1}^{3}k_j\big(\frac{a_{j,3}}{a_{j,1}}-\frac{1}{2}\frac{a_{j,2}^2}{a_{j,1}^2}\big)+\frac{1}{2}\frac{b_2(k)^2}{b_1(k)^2}\Big) \end{aligned}$$

Lemma 5.2. *The tuple (A_1,A_2,A_3) is hypercyclic if and only if the set*

$$\left\{\begin{pmatrix} \sum_{j=1}^{3}k_j\big(\frac{a_{j,3}}{a_{j,1}}-\frac{1}{2}\frac{a_{j,2}^2}{a_{j,1}^2}\big) \\ \sum_{j=1}^{3}k_j\frac{a_{j,2}}{a_{j,1}} \\ a_{1,1}^{k_1}a_{2,1}^{k_2}a_{3,1}^{k_3}\end{pmatrix} : k_1,k_2,k_3\in\mathbb{N}\right\}$$

is dense in $\mathbb{C}^3$.

Proposition 5.3. *The tuple (A_1,A_2,A_3) is not hypercyclic.*

6. 4×4 MATRICES

Again, as in the previous section, we only deal with 4×4 triangular Toeplitz matrices. So, let A_1,A_2,A_3,A_4 be 4×4 triangular Toeplitz matrices with elements defined as in section 2. For a multi-index $k=(k_1,k_2,k_3,k_4)$, $k_1,k_2,k_3,k_4\in\mathbb{N}$ we compute the entries of the 4×4 matrix

$$C(k)=\begin{pmatrix} c_1(k) & c_2(k) & c_3(k) & c_4(k) \\ 0 & c_1(k) & c_2(k) & c_3(k) \\ 0 & 0 & c_1(k) & c_2(k) \\ 0 & 0 & 0 & c_1(k)\end{pmatrix} := A_1^{k_1}A_2^{k_2}A_3^{k_3}A_4^{k_4},$$

since it will be of use to us in the sequel. To determine $c_1(k)$ we require

$$k_{1,2}+\ldots+k_{4,2}+2(k_{1,3}+\ldots+k_{4,3})+3(k_{1,4}+\ldots+k_{4,4})=0$$

which implies that all of the above k's are 0 which implies that $k_{1,1}=k_1$, $k_{2,1}=k_2$, $k_{3,1}=k_3$, $k_{4,1}=k_4$. As a result

$$c_1(k)=a_{1,1}^{k_1}a_{2,1}^{k_2}a_{3,1}^{k_3}a_{4,1}^{k_4}.$$

To determine $c_2(k)$ we require

$$k_{1,2}+\ldots+k_{4,2}+2(k_{1,3}+\ldots+k_{4,3})+3(k_{1,4}+\ldots+k_{4,4})=1$$

which is the case if one of $k_{1,2}$, $k_{2,2}$, $k_{3,2}$, $k_{4,2}$ is 1 and all the rest are 0. Doing the calculation gives

$$c_2(k) = c_1(k) \sum_{j=1}^{4} k_j \frac{a_{j,2}}{a_{j,1}}.$$

To determine $c_3(k)$ we require

$$k_{1,2} + \ldots + k_{4,2} + 2(k_{1,3} + \ldots + k_{4,3}) + 3(k_{1,4} + \ldots + k_{4,4}) = 2$$

which is the case if either one of $k_{1,3}$, $k_{2,3}$, $k_{3,3}$, $k_{4,3}$ is 1 or any one of $k_{1,2}$, $k_{2,2}$, $k_{3,2}$, $k_{4,2}$ is 2 or any two of $k_{1,2}$, $k_{2,2}$, $k_{3,2}$, $k_{4,2}$ are 1 and the rest are 0. Doing the calculation gives

$$c_3(k) = c_1(k) \left(\sum_{j=1}^{4} k_j \frac{a_{j,3}}{a_{j,1}} + \sum_{l,j=1,\ l>j}^{4} k_l k_j \frac{a_{l,2} a_{j,2}}{a_{l,1} a_{j,1}} + \sum_{j=1}^{4} \frac{k_j(k_j-1)}{2} \frac{a_{j,2}^2}{a_{j,1}^2} \right).$$

To determine $c_4(k)$ we require

$$k_{1,2} + \ldots + k_{4,2} + 2(k_{1,3} + \ldots + k_{4,3}) + 3(k_{1,4} + \ldots + k_{4,4}) = 3$$

which is the case if either one of $k_{1,4}$, $k_{2,4}$, $k_{3,4}$, $k_{4,4}$ is 1 or one of $k_{1,3}, k_{2,3}, k_{3,3}, k_{4,3}$ is 1 and one of $k_{1,2}, k_{2,2}, k_{3,2}, k_{4,2}$ is 1 or from the set $k_{1,2}, k_{2,2}, k_{3,2}, k_{4,2}$ either one is 3 or one is 2 and another is 1 or any three are 1. These cover all the configurations which satisfy the above expression. Doing the calculations gives

$$\begin{aligned} c_4(k) &= c_1(k) \Bigg(\sum_{j=1}^{4} k_j \frac{a_{j,4}}{a_{j,1}} + \sum_{j=1}^{4} k_j(k_j-1) \frac{a_{j,2} a_{j,3}}{a_{j,1}^2} \\ &+ \sum_{j,l=1,\ l\neq j}^{4} k_l k_j \frac{a_{j,2}}{a_{j,1}} \frac{a_{l,3}}{a_{l,1}} + \sum_{\substack{j,l,p=1 \\ j\neq l,\, j\neq p,\, l\neq p}}^{4} k_j k_l k_p \frac{a_{j,2} a_{l,2} a_{p,2}}{a_{j,1} a_{l,1} a_{p,1}} \\ &+ \sum_{j=1}^{4} \frac{k_j(k_j-1)(k_j-2)}{6} \frac{a_{j,2}^3}{a_{j,1}^3} + \sum_{j,l=1,\ j\neq l}^{4} \frac{k_j(k_j-1)k_l}{2} \frac{a_{j,2}^2 a_{l,2}}{a_{j,1}^2 a_{l,1}} \Bigg). \end{aligned}$$

As in the previous section we do not give proofs of the lemmas and proposition that follow. The proofs are in the spirit of the proofs given in section 4. From the above calculations plus a little work we get the following

Lemma 6.1.

$$\begin{aligned}
c_1(k) &= a_{1,1}^{k_1}a_{2,1}^{k_2}a_{3,1}^{k_3}a_{4,1}^{k_4}\\
c_2(k) &= c_1(k)\sum_{j=1}^{4}k_j\frac{a_{j,2}}{a_{j,1}}\\
c_3(k) &= c_1(k)\Bigg(\sum_{j=1}^{4}k_j(\frac{a_{j,3}}{a_{j,1}}-\frac{1}{2}\frac{a_{j,2}^2}{a_{j,1}^2})+\frac{1}{2}(\sum_{j=1}^{4}k_j\frac{a_{j,2}}{a_{j,1}})^2\Bigg)\\
&= c_1(k)\Bigg(\sum_{j=1}^{4}k_j(\frac{a_{j,3}}{a_{j,1}}-\frac{1}{2}\frac{a_{j,2}^2}{a_{j,1}^2})+\frac{1}{2}\frac{c_2^2}{c_1(k)^2}\Bigg)\\
c_4(k) &= c_1(k)\Bigg(\sum_{j=1}^{4}k_j(\frac{a_{j,4}}{a_{j,1}}-\frac{a_{j,2}a_{j,3}}{a_{j,1}^2}+\frac{1}{3}\frac{a_{j,2}^3}{a_{j,1}^3})\\
&+ (\sum_{j=1}^{4}k_j\frac{a_{j,2}}{a_{j,1}})(\sum_{j=1}^{4}k_j(\frac{a_{j,3}}{a_{j,1}}-\frac{1}{2}\frac{a_{j,2}^2}{a_{j,1}^2}))+\frac{1}{6}(\sum_{j=1}^{4}k_j\frac{a_{j,2}}{a_{j,1}})^3\Bigg)\\
&= c_1(k)\Bigg(\sum_{j=1}^{4}k_j(\frac{a_{j,4}}{a_{j,1}}-\frac{a_{j,2}a_{j,3}}{a_{j,1}^2}+\frac{1}{3}\frac{a_{j,2}^3}{a_{j,1}^3})\\
&+ \frac{c_2(k)}{c_1(k)}(\frac{c_3(k)}{c_1(k)}-\frac{1}{2}\frac{c_2(k)^2}{c_1(k)^2})+\frac{1}{6}\frac{c_2(k)^3}{c_1(k)^3}\Bigg)
\end{aligned}$$

Lemma 6.2. *The tuple (A_1, A_2, A_3, A_4) is hypercyclic if and only if the set*

$$\left\{\begin{pmatrix}\sum_{j=1}^4 k_j(\frac{a_{j,4}}{a_{j,1}}-\frac{a_{j,2}a_{j,3}}{a_{j,1}^2}+\frac{1}{3}\frac{a_{j,2}^3}{a_{j,1}^3})\\ \sum_{j=1}^4 k_j(\frac{a_{j,3}}{a_{j,1}}-\frac{1}{2}\frac{a_{j,2}^2}{a_{j,1}^2})\\ \sum_{j=1}^4 k_j\frac{a_{j,2}}{a_{j,1}}\\ a_{1,1}^{k_1}a_{2,1}^{k_2}a_{3,1}^{k_3}a_{4,1}^{k_4}\end{pmatrix} : k_1,k_2,k_3,k_4\in\mathbb{N}\right\}$$

is dense in $\mathbb{C}^4$.

Proposition 6.3. *The tuple (A_1, A_2, A_3, A_4) is not hypercyclic.*

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