APPROXIMATIONS AND ADJOINTS IN HOMOTOPY CATEGORIES

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Abstract. We provide a criterion for the existence of right approximations in cocomplete additive categories; it is a straightforward generalisation of a result due to El Bashir. This criterion is used to construct adjoint functors in homotopy categories. Applications include the study of (pure) derived categories. For instance, it is shown that the pure derived category of any module category is compactly generated.

1. Introduction

This note is motivated by recent work of Neeman and Murfet where derived categories of flat modules are studied in various settings [32, 33, 29]. One of the essential ingredients of their work is the construction of approximations and adjoints for categories of complexes. It turns out that El Bashir's proof of the flat cover conjecture [7, 12] leads to a systematic approach yielding such approximations and adjoints. It is our aim in the present work to explain this new approach and some of its consequences.

Let us mention a few applications in this introduction because they are easily stated. Given any additive category $\mathcal{A}$, we denote by $\mathbf{K}(\mathcal{A})$ the category of cochain complexes in $\mathcal{A}$ with morphisms the cochain maps up to homotopy. For a fixed Quillen exact structure on $\mathcal{A}$, we denote by $\mathbf{D}(\mathcal{A})$ the corresponding derived category.

The first result is an analogue of the flat cover conjecture for complexes of quasi-coherent sheaves on a scheme; it has been established in the affine case by Neeman [33] and Enochs et al. [8], and for noetherian schemes by Murfet [29].

Theorem 1.1. *Given any scheme $\mathbb{X}$, the inclusion $\mathbf{K}(\operatorname{Flat}\mathbb{X}) \to \mathbf{K}(\operatorname{Qcoh}\mathbb{X})$ admits a right adjoint.* □

This result is a consequence of the more general Theorem 3.3 which is formulated in the setting of locally presentable categories in the sense of Gabriel and Ulmer [16]. Roughly speaking, any cocomplete category with a sufficiently nice set of generators is locally presentable. In particular, Grothendieck abelian categories and module categories are locally presentable.

Most of the present work is done for locally presentable categories, including the following result which establishes the 'existence' of the derived category of an exact category that is locally presentable.

Theorem 1.2. *Let $\mathcal{A}$ be an exact category that is locally presentable, and suppose that exact sequences are closed under filtered colimits. Then the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ admits a fully faithful right adjoint. In particular, the category $\mathbf{D}(\mathcal{A})$ has small Hom-sets.* □

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Let us consider as an example of particular interest for any ring A the pure exact structure on its module category $\operatorname{Mod} A$; it is the smallest exact structure on $\operatorname{Mod} A$ such that exact sequences are closed under filtered colimits. This yields the pure derived category $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A)$, studied for example by Christensen and Hovey [10]. Note that it contains the usual derived category $\mathbf{D}(\operatorname{Mod} A)$ as a full triangulated subcategory. The triangulated category $\mathbf{D}(\operatorname{Mod} A)$ is well-known to be compactly generated, and the inclusion $\operatorname{proj} A \to \operatorname{Mod} A$ of the category of all finitely generated projective modules induces an equivalence $\mathbf{K}^b(\operatorname{proj} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} A)^c$ onto the full subcategory formed by all compact objects. We have the following analogue for the pure derived category.

Theorem 1.3. *Let A be a ring. The pure derived category $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A)$ is a compactly generated triangulated category and the inclusion $\operatorname{mod} A \to \operatorname{Mod} A$ of the category of all finitely presented modules induces an equivalence $\mathbf{K}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A)^c$ onto the full subcategory formed by all compact objects. Moreover, the canonical functor $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ admits left and right adjoints that are fully faithful.* □

It seems appropriate to comment on the level of generality in this work. Most of our results are stated for locally presentable categories, even though the arguments work as well with little extra effort for the more general class of accessible categories [27, 1]. Also, no attempt has been made to formulate results in terms of Quillen model structures. So we tried to keep the exposition as elementary as possible, concentrating on basic ideas. The interested and educated reader will have no problems to make the appropriate generalisations.

This paper is organised as follows. Section 2 is devoted to studying the existence of right approximations, generalising work of El Bashir. These results are applied in Section 3 where right adjoints of functors between homotopy categories are constructed. In particular, derived categories of exact categories are studied. The special case of a pure derived category is discussed in Section 4. The final Section 5 collects results on left approximations and left adjoints. We end this note by stating a conjecture on fp-injective modules which is an analogue of results of Neeman on flat modules.

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2. Right approximations

Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ a full additive subcategory. In this section we present conditions such that every object Y in $\mathcal{A}$ admits a *right $\mathcal{B}$-approximation*, that is, a morphism $f\colon X \to Y$ with X in $\mathcal{B}$ such that every morphism $X' \to Y$ with X' in $\mathcal{B}$ factors through f. Right approximations in additive categories were introduced by Auslander and Smalø [4], and independently by Enochs, using the term 'precover' [13].

The following theorem is our main result in this section; it is a straightforward generalisation of a result due to El Bashir [12, Theorem 3.2]. In fact, one finds a plethora of criteria for the existence of right approximations in the literature, generalising the existence of flat covers in module categories [7]. To the best of our knowldege, all these criteria can be derived from the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be an additive category that is locally presentable and let $\mathcal{B}$ be a full additive subcategory that is closed under filtered colimits. Suppose there exists a regular cardinal α such that $\mathcal{B}$ is closed under α-pure subobjects or under α-pure quotients. Then each object in $\mathcal{A}$ admits a right $\mathcal{B}$-approximation.*

Recall that an additive category is *locally presentable* if it is cocomplete and admits a generator that is α-presentable for some regular cardinal α [16, 1]. For example, Grothendieck abelian categories and module categories are locally presentable. Further examples include categories of cochain complexes; this will be relevant for our applications.

We need to make the following definition.[1]

Definition 2.2. Let α be a regular cardinal. A morphism $X \to Y$ in an arbitrary category is called

(1) *α-pure monomorphism*, if it is an α-filtered colimit of split monomorphisms,
(2) *α-pure epimorphism*, if it is an α-filtered colimit of split epimorphisms, and
(3) *α-terminal* if for every factorisation $X \to X' \to Y$ the morphism $X \to X'$ is invertible if it is an α-pure epimorphism.

Note that colimits of morphisms in a category $\mathcal{A}$ are taken in the *category of morphisms* $\operatorname{Mor}\mathcal{A}$. The objects of $\operatorname{Mor}\mathcal{A}$ are the morphisms in $\mathcal{A}$ and the morphisms are the obvious commuting squares. The term 'filtered' without prefix is used to mean '$\aleph_0$-filtered'.

We refer to [2] for basic facts about pure morphisms. For instance, suppose that $\mathcal{A}$ is a locally β-presentable category and let $\alpha \geqslant \beta$ be a regular cardinal. Then a morphism $X \to Y$ is an α-pure epimorphism if and only if it induces a surjective map $\operatorname{Hom}_{\mathcal{A}}(C, X) \to \operatorname{Hom}_{\mathcal{A}}(C, Y)$ for every α-presentable object C in $\mathcal{A}$. Thus the usual notion of purity in a module category is obtained by specialising $\alpha = \aleph_0$.

The crucial input for proving the theorem is the following result due to El Bashir, which he establishes more generally for any Grothendieck abelian category.

Proposition 2.3 ([12, Theorem 2.1])**.** *Let $\mathcal{A}$ be a module category (over a ring with several objects). Given an object Y in $\mathcal{A}$ and a regular cardinal α, the isomorphism classes of α-terminal morphisms $X \to Y$ in $\mathcal{A}$ form a set.*

Proof. Let $\mathcal{C}$ be an additive category and $\mathcal{A} = \operatorname{Mod}\mathcal{C}$ the category of $\mathcal{C}$-*modules*, that is, additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups. Given a $\mathcal{C}$-module X, we define its *cardinality* to be $|X| = \sum_{C \in \mathcal{C}_0} \operatorname{card} X(C)$, where $\mathcal{C}_0$ denotes a representative set of objects in $\mathcal{C}$.

It follows from [7, Theorem 5] that for each cardinal λ, there exists a cardinal κ such that for any morphism $f \colon X \to Y$ in $\mathcal{A}$ with $|X| \geqslant \kappa$ and $|Y| \leqslant \lambda$, there exists an α-pure submodule $0 \neq U \subseteq X$ with $f|_U = 0$. Let $X' = X/U$. Then f admits a factorisation $X \xrightarrow{u} X' \xrightarrow{v} Y$ with u an α-pure epimorphism that is not invertible. Thus any α-terminal morphism $X \to Y$ with $|Y| \leqslant \lambda$ satisfies $|X| < \kappa$. □

Corollary 2.4. *Let $\mathcal{A}$ be an additive category that is locally β-presentable for some regular cardinal β. Given an object Y in $\mathcal{A}$ and a regular cardinal $\alpha \geqslant \beta$, the isomorphism classes of α-terminal morphisms $X \to Y$ in $\mathcal{A}$ form a set.*

Proof. Let $\mathcal{C}$ be the full subcategory formed by all β-presentable objects in $\mathcal{A}$ and denote by $\operatorname{Mod}\mathcal{C}$ the category of $\mathcal{C}$-modules. The functor $F \colon \mathcal{A} \to \operatorname{Mod}\mathcal{C}$ taking

[1] The subsequent definition of α-pure mono/epimorphisms deviates from the standard one in terms of α-presentable objects. The new definition seems to be more practical and coincides with the standard one for $\alpha \gg 0$, provided the category is locally presentable.

an object X to $\operatorname{Hom}_{\mathcal{A}}(-,X)|_{\mathcal{C}}$ is fully faithful and preserves α-filtered colimits for every regular cardinal $\alpha \geqslant \beta$ [16, §7]. Thus F preserves α-pure epimorphisms. Note that the image of F is closed under α-pure quotients [2, Proposition 13]. It follows that F preserves α-terminal morphisms. Now apply Proposition 2.3. □

Proof of Theorem 2.1. We follow El Bashir [12]. Suppose that the category $\mathcal{A}$ is locally β-presentable. We may assume that $\alpha \geqslant \beta$. Fix an object Y in $\mathcal{A}$ and a representative set of α-terminal morphisms $X_i \to Y$ $(i \in I)$ with X_i in $\mathcal{B}$. We claim that the induced morphism $\coprod_{i\in I} X_i \to Y$ is a right $\mathcal{B}$-approximation. To see this, choose a morphism $f\colon X \to Y$ with X in $\mathcal{B}$. Consider the pairs (u,v) of morphisms $X \xrightarrow{u} X' \xrightarrow{v} Y$ with X' in $\mathcal{B}$ such that $f = vu$ and u is an epimorphism. These pairs are partially ordered if one defines $(u_1,v_1) \leqslant (u_2,v_2)$ provided that u_2 factors through u_1. An upper bound of a chain of pairs (u_i,v_i) is obtained by taking its colimit $(\bar{u},\bar{v})$ with $\bar{u} = \operatorname{colim}_i u_i$ and $\bar{v} = \operatorname{colim}_i v_i$. Note that any colimit of epimorphisms is again an epimorphism. In addition, one uses that $\mathcal{B}$ is closed under filtered colimits. In a locally presentable category, the epimorphisms starting in a fixed object form, up to isomorphism, a set [16, Satz 7.14]. Thus we can choose a maximal pair $(\bar{u},\bar{v})$, using Zorn's lemma. The maximality implies that $\bar{v}$ is an α-terminal morphism, since $\mathcal{B}$ is closed under α-pure quotients. Thus f factors through an α-terminal morphism $X_i \to Y$ and therefore through $\coprod_{i\in I} X_i \to Y$. It remains to observe that $\mathcal{B}$ is closed under α-pure quotients if it is closed under α-pure subobjects; this follows from the subsequent Proposition 2.5. □

The following proposition collects some facts which help to apply Theorem 2.1.

Proposition 2.5. *Let $\mathcal{A}$ be a locally presentable category. For a full subcategory $\mathcal{B}$ that is closed under filtered colimits, the following conditions are equivalent:*

(1) *There exists a set $\mathcal{S}$ of objects in $\mathcal{B}$ such that every object of $\mathcal{B}$ is a filtered colimit of objects in $\mathcal{S}$.*
(2) *The category $\mathcal{B}$ is accessible* [27, 1].
(3) *There exists a regular cardinal α such that $\mathcal{B}$ is closed under α-pure subobjects.*

Moreover, these conditions imply that there exists a regular cardinal α such that $\mathcal{B}$ is closed under α-pure quotients.

Proof. (1) ⇒ (2): Fix a regular cardinal α such that $\mathcal{A}$ is locally α-presentable and each object in $\mathcal{S}$ is α-presentable. Denote by $\mathcal{B}_0$ the smallest full subcategory that contains $\mathcal{S}$ and is closed under filtered colimits over diagrams having cardinality less than α. Observe that the objects in $\mathcal{B}_0$ are α-presentable. It is not difficult to check that each filtered colimit of objects in $\mathcal{B}_0$ can be rewritten as an α-filtered colimit of objects in $\mathcal{B}_0$. Thus every object in $\mathcal{B}$ is an α-filtered colimit of objects in $\mathcal{B}_0$. This means that $\mathcal{B}$ is α-accessible.

(2) ⇒ (1): Suppose that $\mathcal{B}$ is α-accessible for some regular cardinal α. Let $\mathcal{S}$ be a representative set of α-presentable objects. It follows that each object in $\mathcal{B}$ is an α-filtered colimit of objects in $\mathcal{S}$. In particular, each object in $\mathcal{B}$ is a filtered colimit of objects in $\mathcal{S}$.

(2) ⇔ (3): See [1, Corollary 2.36].

The last statement follows from [2, Proposition 13]. □

In [14], Enochs and Estrada have shown that quasi-coherent sheaves admit flat covers (by viewing sheaves as representations of appropriate quivers). This is a simple consequence of Theorem 2.1.

Example 2.6. Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a scheme and denote by $\operatorname{Qcoh}\mathbb{X}$ the category of quasi-coherent $\mathcal{O}_{\mathbb{X}}$-modules. This is a Grothendieck abelian category and therefore locally presentable. Recall that a quasi-coherent $\mathcal{O}_{\mathbb{X}}$-module M is *flat* if the functor $M \otimes_{\mathcal{O}_{\mathbb{X}}} -$ is exact, and let $\operatorname{Flat}\mathbb{X}$ denote the full subcategory consisting of all flat modules in $\operatorname{Qcoh}\mathbb{X}$. It is easily checked that $\operatorname{Flat}\mathbb{X}$ is closed under filtered colimits and $\aleph_0$-pure subobjects. Thus every quasi-coherent $\mathcal{O}_{\mathbb{X}}$-module admits a right $\operatorname{Flat}\mathbb{X}$-approximation. A standard argument [13, §7] shows that one can choose the right approximation to be minimal.

3. Right adjoints

In this section we construct right adjoints of functors between homotopy categories, applying the criterion for the existence of right approximations from the previous section.

Let $\mathcal{A}$ be an additive category. We denote by $\mathbf{C}(\mathcal{A})$ the category of *cochain complexes*, that is, sequences of morphisms $(d^n \colon X^n \to X^{n+1})_{n\in\mathbb{Z}}$ in $\mathcal{A}$ such that $d^n d^{n-1} = 0$ for all $n \in \mathbb{Z}$. The morphisms in this category are the usual cochain maps. The *homotopy category* $\mathbf{K}(\mathcal{A})$ is the category of cochain complexes with morphisms the cochain maps up to homotopy.

Lemma 3.1. *Let $\mathcal{A}$ be a locally presentable additive category. Then the category $\mathbf{C}(\mathcal{A})$ is locally presentable. Moreover, all limits and colimits in $\mathbf{C}(\mathcal{A})$ are computed degreewise.*

Proof. Suppose that $\mathcal{A}$ is locally α-presentable for some regular cardinal α. We denote by $\mathcal{A}^{\mathbb{Z}}$ the category consisting of all sequences of morphisms $(d^n \colon X^n \to X^{n+1})_{n\in\mathbb{Z}}$ in $\mathcal{A}$. Thus $\mathcal{A}^{\mathbb{Z}}$ equals the category of functors $\mathbb{Z} \to \mathcal{A}$, where $\mathbb{Z}$ is viewed as category with exactly one morphism $i \to j$ if and only if $i \leqslant j$. It follows that $\mathcal{A}^{\mathbb{Z}}$ is locally α-presentable [1, Corollary 1.54]. Note that all (co)limits in $\mathcal{A}^{\mathbb{Z}}$ are computed degreewise and that the subcategory $\mathbf{C}(\mathcal{A})$ is closed under (co)limits. Moreover, $\mathbf{C}(\mathcal{A})$ is closed under subobjects. Any α-pure monomorphism in $\mathcal{A}^{\mathbb{Z}}$ is a monomorphism, since $\mathcal{A}^{\mathbb{Z}}$ is locally α-presentable. Thus $\mathbf{C}(\mathcal{A})$ is closed under α-pure subobjects, and it follows from Proposition 2.5 that $\mathbf{C}(\mathcal{A})$ is accessible. In fact, $\mathbf{C}(\mathcal{A})$ is complete and therefore locally presentable [1, Corollary 2.47]. □

Our tool for constructing right adjoints is the following proposition which is due to Neeman.

Proposition 3.2 ([33, Proposition 1.4])**.** *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full triangulated subcategory. Suppose that $\mathcal{S}$ and $\mathcal{T}$ have split idempotents. Then the following are equivalent:*

(1) *The inclusion $\mathcal{S} \to \mathcal{T}$ admits a right adjoint.*
(2) *Every object in $\mathcal{T}$ admits a right $\mathcal{S}$-approximation.* □

Note that any triangulated category has split idempotents provided the category admits countable coproducts [31, Proposition 1.6.8].

The next theorem is the analogue of Theorem 2.1 for homotopy categories.

Theorem 3.3. *Let $\mathcal{A}$ be a locally presentable additive category and $\mathcal{B}$ a full additive subcategory. Suppose that $\mathcal{B}$ is closed under filtered colimits and in addition closed under α-pure subobjects or under α-pure quotients for some regular cardinal α. Then the inclusion $\mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{A})$ admits a right adjoint.*

Proof. We view $\mathbf{C}(\mathcal{B})$ as a full subcategory of $\mathbf{C}(\mathcal{A})$. Colimits in $\mathbf{C}(\mathcal{A})$ are computed degreewise and this implies that $\mathbf{C}(\mathcal{B})$ is closed under filtered colimits and α-pure subobjects or α-pure quotients, respectively. Thus every object in $\mathbf{C}(\mathcal{A})$ admits a right $\mathbf{C}(\mathcal{B})$-approximation by Theorem 2.1, and it follows that every object in $\mathbf{K}(\mathcal{A})$ admits a right $\mathbf{K}(\mathcal{B})$-approximation. Thus we can apply Proposition 3.2 and conclude that the inclusion $\mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{A})$ admits a right adjoint. □

The following result is an application; it has been established in the affine case by Neeman [33] and Enochs et al. [8], and for noetherian schemes by Murfet [29]. Their proofs are different from the one given here.

Corollary 3.4. *Given any scheme $\mathbb{X}$, the inclusion $\mathbf{K}(\operatorname{Flat}\mathbb{X}) \to \mathbf{K}(\operatorname{Qcoh}\mathbb{X})$ admits a right adjoint.* □

Let $\mathcal{A}$ be an *exact category* [35]. Thus $\mathcal{A}$ is an additive category, together with a distinguished class of sequences $X \xrightarrow{u} Y \xrightarrow{v} Z$ of morphisms which are called *exact* and satisfy a number of axioms. Note that the morphisms u and v in each exact sequence as above form a *kernel-cokernel pair*, that is, u is a kernel of v and v is a cokernel of u. A morphism in $\mathcal{A}$ which arises as the kernel in some exact sequence is called *admissible monomorphism*; a morphism arising as a cokernel is called *admissible epimorphism.*

A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is *extension closed* if every exact sequence in $\mathcal{A}$ belongs to $\mathcal{B}$ provided its endterms belongs to $\mathcal{B}$. Any full and extension closed subcategory of $\mathcal{A}$ is exact with respect to the class of sequences which are exact in $\mathcal{A}$.

An object P in $\mathcal{A}$ is *projective* if each admissible epimorphism $Y \to Z$ induces a surjective map $\operatorname{Hom}_{\mathcal{A}}(P, Y) \to \operatorname{Hom}_{\mathcal{A}}(P, Z)$, and the full subcategory of $\mathcal{A}$ formed by these objects is denote by $\operatorname{Proj}\mathcal{A}$. Analogously, the subcategory $\operatorname{Inj}\mathcal{A}$ of injective objects is defined.

A cochain complex $X = (X^n, d^n)$ in $\mathcal{A}$ is called *acyclic* if for each $n \in \mathbb{Z}$ there is an exact sequence $Z^n \xrightarrow{u^n} X^n \xrightarrow{v^n} Z^{n+1}$ in $\mathcal{A}$ such that $d^n = u^{n+1}v^n$. The full subcategory consisting of all acyclic complexes in $\mathbf{C}(\mathcal{A})$ is denoted by $\mathbf{C}_{\mathrm{ac}}(\mathcal{A})$. The acyclic complexes form a full triangulated subcategory of $\mathbf{K}(\mathcal{A})$ which we denote by $\mathbf{K}_{\mathrm{ac}}(\mathcal{A})$. Following [30, 22], the *derived category* of $\mathcal{A}$ is by definition the Verdier quotient

$$\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})/\mathbf{K}_{\mathrm{ac}}(\mathcal{A}).$$

It is a well-known fact that the derived category of any Grothendieck abelian category has small Hom-sets [3, 5, 15]. On the other hand, there are simple examples of abelian categories where this property fails [9]. The following result establishes the 'existence' of the derived category of an exact category that is locally presentable.

Theorem 3.5. *Let $\mathcal{A}$ be an exact category that is locally presentable, and suppose that exact sequences are closed under filtered colimits. Then the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ admits a fully faithful right adjoint. In particular, the category $\mathbf{D}(\mathcal{A})$ has small Hom-sets.*

The proof of this theorem is based on the following lemma.

Lemma 3.6. *Let $\mathcal{A}$ be an exact category that is locally α-presentable for some regular cardinal α. Suppose that exact sequences are closed under α-filtered colimits. Then the following holds.*

(1) *Let $X' \xrightarrow{f'} Y'$ be an α-pure subobject of $X \xrightarrow{f} Y$ in* $\operatorname{Mor}\mathcal{A}$. *Then* $\operatorname{Ker} f' \to X$ *is an α-pure subobject of* $\operatorname{Ker} f \to X$, *and* $Y \to \operatorname{Coker} f'$ *is an α-pure subobject of* $Y \to \operatorname{Coker} f$.
(2) *The isomorphisms in $\mathcal{A}$ are closed under α-pure subobjects in* $\operatorname{Mor}\mathcal{A}$.

(3) *The admissible monomorphisms in $\mathcal{A}$ are closed under α-pure subobjects in* $\operatorname{Mor}\mathcal{A}$.
(4) $\mathbf{C}_{\mathrm{ac}}(\mathcal{A})$ *is closed under α-pure subobjects in* $\mathbf{C}(\mathcal{A})$.

Proof. (1) This follows from the fact that kernels and cokernels are preserved by taking α-filtered colimits [16, Korollar 7.12].

(2) Let f be an α-pure subobject of an isomorphism. It follows from (1) that f is an epimorphism. On the other hand, there is a morphism g such that the composite gf is an α-pure monomorphism. Thus gf is a regular monomorphism [1, Proposition 2.31] and therefore extremal. It follows that f is an isomorphism.

(3) Consider an admissible monomorphism $X \to Y$ and an α-pure subobject $X' \to Y'$. Thus there is a commuting square

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

such that the vertical morphisms are α-pure monomorphisms. Any split monomorphism is admissible (property a) of an exact category in [35, §2]), and therefore every α-pure monomorphism is admissible, since exact sequences are closed under α-filtered colimits. It follows that the composite $X' \to Y' \to Y$ is an admissible monomorphism (property b) in [35, §2]). Thus $X' \to Y'$ is an admissible monomorphism (property c) in [35, §2]).

(4) First observe that a complex $X = (X^n, d^n)$ is acyclic if and only if for each $n \in \mathbb{Z}$ the morphism $\operatorname{Coker} d^{n-2} \to \operatorname{Ker} d^n$ is invertible and the monomorphism $\operatorname{Ker} d^n \to X^n$ is admissible.

Now fix an α-pure monomorphism $X \to \bar{X}$ in $\mathbf{C}(\mathcal{A})$ such that $\bar{X}$ is acyclic. It follows from (1) that $\operatorname{Coker} d^{n-2} \to \operatorname{Ker} d^n$ is an α-pure subobject of $\operatorname{Coker} \bar{d}^{n-2} \to \operatorname{Ker} \bar{d}^n$ in $\operatorname{Mor}\mathcal{A}$, and that $\operatorname{Ker} d^n \to X^n$ is an α-pure subobject of $\operatorname{Ker} \bar{d}^n \to \bar{X}^n$. Isomorphisms and admissible monomorphisms in $\mathcal{A}$ are closed under α-pure subobjects by (2) and (3). We conclude that X is acyclic. □

Proof of Theorem 3.5. Consider the full subcategory $\mathbf{C}_{\mathrm{ac}}(\mathcal{A})$ of acyclic complexes in $\mathbf{C}(\mathcal{A})$. The assumption on the exact structure implies that $\mathbf{C}_{\mathrm{ac}}(\mathcal{A})$ is closed under filtered colimits, and Lemma 3.6 implies that $\mathbf{C}_{\mathrm{ac}}(\mathcal{A})$ is closed under α-pure subobjects for some regular cardinal α. Thus every object in $\mathbf{C}(\mathcal{A})$ admits a right $\mathbf{C}_{\mathrm{ac}}(\mathcal{A})$-approximation by Theorem 2.1. Applying Proposition 3.2, it follows that the inclusion $\mathbf{K}_{\mathrm{ac}}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ admits a right adjoint. A standard argument [31, Proposition 9.1.18] then shows that the quotient functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ admits a right adjoint. Note that this right adjoint is fully faithful [17, Proposition I.1.3]. Therefore $\mathbf{D}(\mathcal{A})$ has small Hom-sets. □

The following corollary is a straightforward generalisation of Theorem 3.5; its proof requires only minor modifications.

Corollary 3.7. *Let $\mathcal{A}$ be an exact category that is locally presentable, and suppose that exact sequences are closed under filtered colimits. Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$ that is extension closed, closed under filtered colimits, and closed under α-pure subobjects or under α-pure quotients for some regular cardinal α. Then the canonical functor $\mathbf{K}(\mathcal{B}) \to \mathbf{D}(\mathcal{B})$ admits a fully faithful right adjoint.*

Proof. We follow the proof of Theorem 3.5 and assume first that $\mathcal{B}$ is closed under α-pure subobjects.

As before, one shows that $\mathbf{C}_{\mathrm{ac}}(\mathcal{B})$ is closed under filtered colimits and α-pure subobjects, viewed as a subcategory of $\mathbf{C}(\mathcal{A})$. Some extra care is needed for the fact that admissible monomorphisms in $\mathcal{B}$ are closed under α-pure subobjects in $\operatorname{Mor}\mathcal{A}$. Note that a morphism in $\mathcal{B}$ is an admissible monomorphism if and only if it is an admissible monomorphism in $\mathcal{A}$ and its cokernel belongs to $\mathcal{B}$. Thus we consider an admissible monomorphism $X \xrightarrow{f} Y$ in $\mathcal{B}$ and an α-pure subobject $X' \xrightarrow{f'} Y'$. Then f' is an admissible monomorphism in $\mathcal{A}$ by Lemma 3.6, and it belongs to $\mathcal{B}$ since $\mathcal{B}$ is closed under α-pure subobjects. Moreover, $\operatorname{Coker} f'$ is an α-pure subobject of $\operatorname{Coker} f$ and belongs therefore to $\mathcal{B}$. It follows that f' is an admissible monomorphism in $\mathcal{B}$.

The rest of the proof goes as before. Thus every object in $\mathbf{C}(\mathcal{B})$ admits a right $\mathbf{C}_{\mathrm{ac}}(\mathcal{B})$-approximation, and it follows that the inclusion $\mathbf{K}_{\mathrm{ac}}(\mathcal{B}) \to \mathbf{K}(\mathcal{B})$ admits a right adjoint.

The case that $\mathcal{B}$ is closed under α-pure quotients is similar and therefore left to the reader. □

Remark 3.8. It seems to be an interesting project to establish in the context of Theorem 3.5 a Quillen model structure on the category $\mathbf{C}(\mathcal{A})$ such that cofibrations are the degreewise admissible monomorphisms and weak equivalences are the quasi-isomorphisms. This would extend the work of Beke in [5]. A strategy for this programme has been pointed out by Maltsiniotis [28].

The following example has been studied by Neeman [32, 33] and Murfet [29].

Example 3.9. Let $\mathbb{X} = (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a scheme. The flat $\mathcal{O}_{\mathbb{X}}$-modules form an extension closed subcategory of $\operatorname{Qcoh}\mathbb{X}$. Thus Corollary 3.7 can be applied. It follows that the canonical functor $\mathbf{K}(\operatorname{Flat}\mathbb{X}) \to \mathbf{D}(\operatorname{Flat}\mathbb{X})$ admits a right adjoint.

Let $\operatorname{Inj}\mathbb{X}$ denote the full subcategory consisting of all injective modules in $\operatorname{Qcoh}\mathbb{X}$. If $\mathbb{X}$ is noetherian and separated, then tensoring with a dualising complex induces an equivalence $\mathbf{D}(\operatorname{Flat}\mathbb{X}) \xrightarrow{\sim} \mathbf{D}(\operatorname{Inj}\mathbb{X})$; this is an 'infinite completion' of Grothendieck duality.

4. The pure derived category

In this section we investigate the derived category of an additive category with respect to its pure exact structure. It seems natural to focus on this exact structure because it is the smallest one such that exact sequences are closed under filtered colimits. An example of particular interest is the pure derived category of a module category.

Locally finitely presented categories in the sense of Crawley-Boevey form a convenient setting for studying purity [11]. Thus we fix an additive category $\mathcal{A}$ that is *locally finitely presented*. This means $\mathcal{A}$ admits filtered colimits and every object in $\mathcal{A}$ can be written as a filtered colimit of some fixed set of finitely presented objects. Recall that an object X is *finitely presented* if the functor $\operatorname{Hom}_{\mathcal{A}}(X, -)$ preserves filtered colimits. Denote by $\operatorname{fp}\mathcal{A}$ the full subcategory formed by all finitely presented objects. We consider the pure exact structure, that is, a sequence $X \to Y \to Z$ of morphisms in $\mathcal{A}$ is *pure exact* if the induced sequence $0 \to \operatorname{Hom}_{\mathcal{A}}(C, X) \to \operatorname{Hom}_{\mathcal{A}}(C, Y) \to \operatorname{Hom}_{\mathcal{A}}(C, Z) \to 0$ is exact for each finitely presented object C. The projective objects with respect to this exact structure are called *pure projective*; they are precisely the direct summands of coproducts of finitely presented objects. The derived category with respect to this exact structure is by definition the *pure derived category*.

The next result combines Corollary 3.7 with recent work of Neeman [32].

Theorem 4.1. *Let $\mathcal{A}$ be a locally finitely presented additive category, endowed with the pure exact structure. Then there exists the following recollement.*

$$\mathbf{K}_{\mathrm{ac}}(\mathcal{A}) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\text{inc}}}}\; \mathbf{K}(\mathcal{A}) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\text{can}}}}\; \mathbf{D}(\mathcal{A})$$

Moreover, the composite $\mathbf{K}(\operatorname{Proj}\mathcal{A}) \xrightarrow{\text{inc}} \mathbf{K}(\mathcal{A}) \xrightarrow{\text{can}} \mathbf{D}(\mathcal{A})$ *is an equivalence.*

Proof. We need to show that the functors $\mathbf{K}_{\mathrm{ac}}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ admit left and right adjoints.

Set $\mathcal{C} = \operatorname{fp}\mathcal{A}$ and consider the category $\operatorname{Mod}\mathcal{C}$ of $\mathcal{C}$-modules, that is, additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. The functor $\mathcal{A} \to \operatorname{Mod}\mathcal{C}$ taking an object X to $\operatorname{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{C}}$ is fully faithful and preserves filtered colimits; it identifies $\mathcal{A}$ with the full subcategory $\operatorname{Flat}\mathcal{C}$ of flat $\mathcal{C}$-modules [11, §1.4]. It follows from Corollary 3.7 that the functors $\mathbf{K}_{\mathrm{ac}}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ admit right adjoints.

The other half of the recollement has been established in [32]. In fact, we can identify the category $\operatorname{Proj}\mathcal{C}$ of projective $\mathcal{C}$-modules with $\operatorname{Proj}\mathcal{A}$. It follows from [32, Proposition 8.1] that the inclusion $\mathbf{K}(\operatorname{Proj}\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ admits a right adjoint. Moreover, $\mathbf{K}(\operatorname{Proj}\mathcal{A})^{\perp} = \mathbf{K}_{\mathrm{ac}}(\mathcal{A})$ by [32, Theorem 8.6], that is, an object Y in $\mathbf{K}(\mathcal{A})$ is acyclic if and only if $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y) = 0$ for all X in $\mathbf{K}(\operatorname{Proj}\mathcal{A})$. Using standard arguments [31, §9], it follows that the inclusion $\mathbf{K}_{\mathrm{ac}}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ admits a left adjoint and that the composite $\mathbf{K}(\operatorname{Proj}\mathcal{A}) \xrightarrow{\text{inc}} \mathbf{K}(\mathcal{A}) \xrightarrow{\text{can}} \mathbf{D}(\mathcal{A})$ is an equivalence. □

Let us reformulate this result in the form which is due to Neeman.

Corollary 4.2 ([33, Remark 3.2])**.** *Let A be a ring with several objects. Then there exists the following recollement.*

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Flat} A) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\text{inc}}}}\; \mathbf{K}(\operatorname{Flat} A) \;\overset{\overset{\text{inc}}{\longleftarrow}}{\underset{\longleftarrow}{\longrightarrow}}\; \mathbf{K}(\operatorname{Proj} A)$$

Moreover, the composite $\mathbf{K}(\operatorname{Proj} A) \xrightarrow{\text{inc}} \mathbf{K}(\operatorname{Flat} A) \xrightarrow{\text{can}} \mathbf{D}(\operatorname{Flat} A)$ *is an equivalence.* □

It is a remarkable fact that we have an equivalence $\mathbf{K}(\operatorname{Proj} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Flat} A)$, even though it may happen that flat A-modules have infinite projective dimension. Thus it would be interesting to have necessary and sufficient conditions for an exact category $\mathcal{A}$ having enough projective objects, such that the composite $\mathbf{K}(\operatorname{Proj}\mathcal{A}) \xrightarrow{\text{inc}} \mathbf{K}(\mathcal{A}) \xrightarrow{\text{can}} \mathbf{D}(\mathcal{A})$ is an equivalence.

Let A be a ring. We denote by $\operatorname{mod} A$ the category of finitely presented A-modules and let $\operatorname{proj} A = \operatorname{Proj} A \cap \operatorname{mod} A$. Suppose that A^{op} is *coherent*, that is, the category $\operatorname{mod} A^{\mathrm{op}}$ is abelian. Then we have the following description of the compact objects of $\mathbf{K}(\operatorname{Proj} A)$ which is due to Jørgensen and Neeman [20, 32].

Given any triangulated category $\mathcal{T}$, we denote by $\mathcal{T}^c$ the full subcategory formed by all compact objects.

Proposition 4.3 ([32, Proposition 7.14])**.** *Let A be a ring with several objects and suppose that A^{op} is coherent. Then the triangulated category $\mathbf{K}(\operatorname{Proj} A)$ is compactly generated and there is an equivalence*

$$\mathbf{D}^b(\operatorname{mod} A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{K}^{-,b}(\operatorname{proj} A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\operatorname{Hom}_{A^{\mathrm{op}}}(-,A)} \mathbf{K}(\operatorname{Proj} A)^c. \qquad \square$$

Let $\mathcal{A}$ be a locally finitely presented additive category. In [11], Crawley-Boevey showed that $\mathcal{A}$ admits set-indexed products iff $\operatorname{fp}\mathcal{A}$ admits pseudo-cokernels iff the category $\widecheck{\operatorname{fp}\mathcal{A}}$ is abelian, where $\check{\mathcal{C}} = (\operatorname{mod}\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ for any additive category $\mathcal{C}$.

Theorem 4.4. *Let $\mathcal{A}$ be a locally finitely presented additive category, endowed with the pure exact structure. Suppose that $\mathcal{A}$ admits set-indexed products. Then the derived category $\mathbf{D}(\mathcal{A})$ is a compactly generated triangulated category and the inclusion* $\operatorname{fp}\mathcal{A} \to \mathcal{A}$ *induces an equivalence*

$$\mathbf{D}^b(\widetilde{\operatorname{fp}\mathcal{A}}) \xrightarrow{\sim} \mathbf{D}(\mathcal{A})^c.$$

Proof. View $\mathcal{C} = \operatorname{fp}\mathcal{A}$ as a ring with several objects and identify the category $\operatorname{Proj}\mathcal{C}$ of projective $\mathcal{C}$-modules with $\operatorname{Proj}\mathcal{A}$; see the proof of Theorem 4.1. Now combine the equivalence $\mathbf{K}(\operatorname{Proj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(\mathcal{A})$ from Theorem 4.1 with the description of the compact objects of $\mathbf{K}(\operatorname{Proj}\mathcal{C})$ given in Proposition 4.3. □

Remark 4.5. Let $\mathcal{A}$ be a locally finitely presented additive category that is cocomplete. Then the category $\mathcal{C} = \operatorname{fp}\mathcal{A}$ admits cokernels, and it follows that each object in $\check{\mathcal{C}}$ has injective dimension at most two. Thus the inclusion $\mathcal{C} \to \check{\mathcal{C}}$ induces an equivalence $\mathbf{K}^b(\mathcal{C}) \xrightarrow{\sim} \mathbf{D}^b(\check{\mathcal{C}})$, and therefore the inclusion $\operatorname{fp}\mathcal{A} \to \mathcal{A}$ induces an equivalence $\mathbf{K}^b(\operatorname{fp}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(\mathcal{A})^c$.

Let us exhibit the pure derived category of a module category more closely; see also [10, 38]. Fix a ring A. The category $\mathcal{A} = \operatorname{Mod} A$ of A-modules is locally finitely presented and $\operatorname{fp}\mathcal{A} = \operatorname{mod} A$. Note that $\widetilde{\operatorname{mod} A}$ equals the *free abelian category* $\operatorname{Ab}(A)$ over A, where the ring A is viewed as a category with a single object [18]. To be precise, the functor $A \to \operatorname{Ab}(A)$ taking A to $\operatorname{Hom}_A(A, -)$ has the property that any additive functor $A \to \mathcal{C}$ to an abelian category extends uniquely, up to a unique isomorphism, to an exact functor $\operatorname{Ab}(A) \to \mathcal{C}$.

We denote by $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A)$ the *pure derived category* of $\operatorname{Mod} A$, that is, the derived category with respect to the pure exact structure on $\operatorname{Mod} A$. The usual derived category with respect to all exact sequences in $\operatorname{Mod} A$ is denoted by $\mathbf{D}(\operatorname{Mod} A)$. Recall that $\mathbf{D}(\operatorname{Mod} A)$ is a compactly generated triangulated category with an equivalence $\mathbf{K}^b(\operatorname{proj} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} A)^c$ [21]. We have the following analogue for the pure derived category.

Corollary 4.6. *Let A be a ring. The pure derived category $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A)$ is a compactly generated triangulated category and the inclusions* $\operatorname{mod} A \to \operatorname{Mod} A$ *and* $\operatorname{mod} A \to \operatorname{Ab}(A)$ *induce equivalences*

$$\mathbf{D}^b(\operatorname{Ab}(A)) \xleftarrow{\sim} \mathbf{K}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A)^c.$$

The canonical functor $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ admits left and right adjoints that are fully faithful. The left adjoint preserves compactness and its restriction to compact objects identifies with the inclusion $\mathbf{K}^b(\operatorname{proj} A) \to \mathbf{K}^b(\operatorname{mod} A)$.

Proof. Let us write $\mathcal{A} = \operatorname{Mod} A$. The fact that $\mathbf{D}_{\mathrm{pur}}(\mathcal{A})$ is compactly generated and the description of the compact objects follow from Theorem 4.4 and Remark 4.5. The canonical functor

$$F \colon \mathbf{D}_{\mathrm{pur}}(\mathcal{A}) = \mathbf{K}(\mathcal{A})/\mathbf{K}_{\mathrm{pac}}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})/\mathbf{K}_{\mathrm{ac}}(\mathcal{A}) = \mathbf{D}(\mathcal{A})$$

preserves set-indexed (co)products and admits therefore a left adjoint and a right adjoint, by Brown representability. These adjoints are fully faithful since F is a quotient functor [17, Proposition I.1.3]. The left adjoint preserves compactness since F preserves set-indexed coproducts. Using Theorem 4.1, we may identify in $\mathbf{K}(\mathcal{A})$

$$\mathbf{D}_{\mathrm{pur}}(\mathcal{A}) = {}^{\perp}\mathbf{K}_{\mathrm{pac}}(\mathcal{A}) \quad \text{and} \quad \mathbf{D}(\mathcal{A}) = {}^{\perp}\mathbf{K}_{\mathrm{ac}}(\mathcal{A}).$$

With this identification, the left adjoint of F embeds $\mathbf{K}^b(\operatorname{proj} A)$ into $\mathbf{K}^b(\operatorname{mod} A)$. □

Remark 4.7. The derived categories $\mathbf{D}(\operatorname{Mod} A)$ and $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A)$ are two extremes. More precisely, the exact structures on $\operatorname{Mod} A$ are partially ordered by inclusion. The natural exact structure given by all kernel-cokernel pairs is the unique maximal one, while the pure exact structure is the smallest exact structure such that exact sequences are closed under filtered colimits.

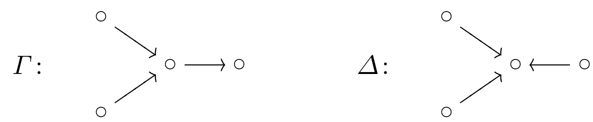
Given any ring A, we view the category $\operatorname{mod} A$ as a ring with several objects and denote it by $\widehat{A}$.

Corollary 4.8. *Let A be a ring. The fully faithful functor $\operatorname{Mod} A \to \operatorname{Mod} \widehat{A}$ sending X to $\operatorname{Hom}_A(-,X)|_{\widehat{A}}$ induces an equivalence $\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} \widehat{A})$.*

Proof. The functor $\operatorname{Mod} A \to \operatorname{Mod} \widehat{A}$ sends pure exact sequences to exact sequences and induces therefore an exact functor $F\colon \mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} \widehat{A})$. The description of the compact objects in Corollary 4.6 implies that F restricts to an equivalence between the subcategories of compact objects; thus F is an equivalence by a standard devisage argument. □

It seems interesting to find out when two rings A and B have equivalent pure derived categories. In view of Corollary 4.8, this reduces to the question when $\widehat{A}$ and $\widehat{B}$ have equivalent derived categories; thus tilting theory applies [37, 22].

Example 4.9. Fix a field k and consider the path algebras $A = k\Gamma$ and $B = k\Delta$ of the following quivers.



Note that both algebras are of finite representation type. Thus $\widehat{A}$ and $\widehat{B}$ are each Morita equivalent to their associated Auslander algebra; see [6] for unexplained terminology.

It follows from work of Ladkani [26] that $\widehat{A}$ and $\widehat{B}$ are both derived equivalent to the incidence algebra kP of the poset $P = D_4 \times A_3$. Thus we have equivalences

$$\mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} kP) \xleftarrow{\sim} \mathbf{D}_{\mathrm{pur}}(\operatorname{Mod} B)$$

even though the categories $\operatorname{Mod} A$ and $\operatorname{Mod} B$ are not equivalent.

5. Left approximations and left adjoints

In this section we discuss briefly the existence of left approximations and left adjoints. In fact, most results are parallel to those previously obtained for right approximations and right adjoints. Given any category $\mathcal{A}$ and a full subcategory $\mathcal{B}$, a morphism $f\colon X \to Y$ in $\mathcal{A}$ is called *left $\mathcal{B}$-approximation* of X if Y belongs to $\mathcal{B}$ and every morphism $X \to Y'$ with Y' in $\mathcal{B}$ factors through f.

The following is the principal existence result for left approximations; it is the analogue of Theorem 2.1. The result is well-known for subcategories of a module category that are closed under $\aleph_0$-pure submodules [23, 36].

Proposition 5.1. *Let $\mathcal{A}$ be a locally presentable category and $\mathcal{B}$ be a full subcategory. Suppose that $\mathcal{B}$ is closed under set-indexed products and α-pure subobjects for some regular cardinal α. Then each object in $\mathcal{A}$ admits a left $\mathcal{B}$-approximation.*

Proof. Fix an object X in $\mathcal{A}$. We may assume that $\mathcal{A}$ is locally α-presentable and that X is β-presentable for some regular cardinal $\beta \geqslant \alpha$. In [1, Theorem 2.33] it is shown that each morphism $X \to Y$ factors through an α-pure monomorphism $Y' \to Y$ such that Y' is β-presentable. Choose a representative set of morphisms $f_i\colon X \to Y_i$ $(i \in I)$ with Y_i in $\mathcal{B}$ and β-presentable. Then it follows from the assumption on $\mathcal{B}$ that the induced morphism $X \to \prod_{i\in I} Y_i$ is a left $\mathcal{B}$-approximation. □

The next lemma will be used in some of the following applications.

Lemma 5.2. *Let $\mathcal{A}$ be a locally presentable abelian category and Z an object in $\mathcal{A}$. Then there exists a regular cardinal α such that the kernel of $\operatorname{Ext}^1_{\mathcal{A}}(Z,-)$ is closed under α-pure subobjects.*

Proof. Choose a regular cardinal β such that $\mathcal{A}$ is locally β-presentable and Z is β-presentable. Then there exists a regular cardinal $\alpha \geqslant \beta$ such that the kernel of each morphism $Y \to Z$ from a β-presentable object Y is α-presentable.

Let $X \to \bar{X}$ be an α-pure monomorphism such that $\operatorname{Ext}^1_{\mathcal{A}}(Z,\bar{X}) = 0$ and fix an exact sequence $\eta\colon 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$. The choice of α implies that the sequence η fits into a commutative diagram of the following form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \Vert & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

such that the upper row is exact and consists of α-presentable objects. The composite $X' \to X \to \bar{X}$ factors through $X' \to Y'$, since $\operatorname{Ext}^1_{\mathcal{A}}(Z,\bar{X}) = 0$. It follows that $X' \to X$ factors through $X' \to Y'$, since $X \to \bar{X}$ is an α-pure monomorphism. Thus the sequence η splits, and we conclude that $\operatorname{Ext}^1_{\mathcal{A}}(Z,X) = 0$. □

The following example is our first application of Proposition 5.1.

Example 5.3. Let $\mathcal{A}$ be a locally presentable abelian category and $\mathcal{C}$ a set of objects. Then the full subcategory

$$\mathcal{C}' = \{X \in \mathcal{A} \mid \operatorname{Ext}^1_{\mathcal{A}}(Z,X) = 0 \text{ for all } Z \in \mathcal{C}\}$$

is closed under set-indexed products and α-pure subobjects for some regular cardinal α, by Lemma 5.2. It follows from Proposition 5.1 that each object in $\mathcal{A}$ admits a left $\mathcal{C}'$-approximation.

As before, we extend the existence of approximations to homotopy categories.

Corollary 5.4. *Let $\mathcal{A}$ be a locally presentable additive category and $\mathcal{B}$ a full additive subcategory. Suppose that $\mathcal{B}$ is closed under set-indexed products and α-pure subobjects for some regular cardinal α. Then the inclusion $\mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{A})$ admits a left adjoint.*

Proof. We view $\mathbf{C}(\mathcal{B})$ as a full subcategory of $\mathbf{C}(\mathcal{A})$. (Co)limits in $\mathbf{C}(\mathcal{A})$ are computed degreewise and this implies that $\mathbf{C}(\mathcal{B})$ is closed under set-indexed products and α-pure subobjects. Thus every object in $\mathbf{C}(\mathcal{A})$ admits a left $\mathbf{C}(\mathcal{B})$-approximation by Proposition 5.1. Applying the dual statement of Proposition 3.2, it follows that the inclusion $\mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{A})$ admits a left adjoint. □

Example 5.5. Let $\mathcal{A}$ be a Grothendieck abelian category. Then the full subcategory $\operatorname{Inj}\mathcal{A}$ consisting of all injective objects is closed under α-pure subobject for some regular cardinal α. This follows from Lemma 5.2 and a variant of Baer's criterion, because there is a set of objects $\mathcal{C}$ such that an object X in $\mathcal{A}$ is injective if and only if $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)=0$ for all Z in $\mathcal{C}$. It follows from Corollary 5.4 that the inclusion $\mathbf{K}(\operatorname{Inj}\mathcal{A})\to\mathbf{K}(\mathcal{A})$ admits a left adjoint.

The following result about derived categories is the analogue of Corollary 3.7; the proof is almost the same and therefore left to the reader.

Corollary 5.6. *Let $\mathcal{A}$ be an exact category that is locally presentable and let α be a regular cardinal. Suppose that exact sequences are closed under set-indexed products and α-filtered colimits. Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$ that is closed under extensions, set-indexed products, and α-pure subobjects. Then the canonical functor $\mathbf{K}(\mathcal{B})\to\mathbf{D}(\mathcal{B})$ admits a fully faithful left adjoint. In particular, the category $\mathbf{D}(\mathcal{B})$ has small Hom-sets.* □

Example 5.7. Let $\mathcal{A}$ be a locally α-presentable additive category. Consider the α-pure exact structure, that is, a sequence $X\to Y\to Z$ of morphisms in $\mathcal{A}$ is *α-pure exact* if the induced sequence $0\to\operatorname{Hom}_{\mathcal{A}}(C,X)\to\operatorname{Hom}_{\mathcal{A}}(C,Y)\to\operatorname{Hom}_{\mathcal{A}}(C,Z)\to 0$ is exact for each α-presentable object C. The α-pure exact sequences are closed under set-indexed products and α-filtered colimits. In fact, a sequence is α-pure exact if and only if it is an α-filtered colimit of split exact sequences. It follows from Corollary 5.6 that the canonical functor $\mathbf{K}(\mathcal{A})\to\mathbf{D}(\mathcal{A})$ admits a left adjoint.

Example 5.8. Let A be a ring. Consider the category $\operatorname{Fpinj}A$ of all *fp-injective* A-modules, that is, A-modules X such that $\operatorname{Ext}^1_A(-,X)$ vanishes on all finitely presented A-modules. Note that fp-injective modules are precisely the pure submodules of injective modules, whereas flat modules are the pure quotients of projective modules. The fp-injective A-modules form an extension closed subcategory of $\operatorname{Mod}A$ that is closed under set-indexed products and pure submodules. It follows from Corollary 5.6 that the canonical functor $\mathbf{K}(\operatorname{Fpinj}A)\to\mathbf{D}(\operatorname{Fpinj}A)$ admits a left adjoint.

Using fp-injective modules, a result from [19] takes the following form. It seems appropriate to mention this because it stimulated our interest in adjoint functors between homotopy categories.

Given any pair A,B of noetherian rings that admit a dualising complex D, there are equivalences

$$\mathbf{K}(\operatorname{Proj}A)\xrightarrow{\sim}\mathbf{D}(\operatorname{Flat}A)\underset{\operatorname{Hom}_B(D,-)}{\overset{-\otimes_A D}{\rightleftarrows}}\mathbf{D}(\operatorname{Fpinj}B)\xleftarrow{\sim}\mathbf{K}(\operatorname{Inj}B)\ .$$

Let us conclude this note with the following conjecture; it is in analogue of results on flat modules in [32].

Conjecture 5.9. *Given any ring A, the composite $\mathbf{K}(\operatorname{Inj}A)\xrightarrow{\text{inc}}\mathbf{K}(\operatorname{Fpinj}A)\xrightarrow{\text{can}}\mathbf{D}(\operatorname{Fpinj}A)$ is an equivalence. If A is coherent, then $\mathbf{D}(\operatorname{Fpinj}A)$ is compactly generated and the composite $\mathbf{D}(\operatorname{Fpinj}A)\to\mathbf{D}_{\text{pur}}(\operatorname{Mod}A)\to\mathbf{D}(\operatorname{Mod}A)$ induces an equivalence $\mathbf{D}(\operatorname{Fpinj}A)^c\xrightarrow{\sim}\mathbf{D}^b(\operatorname{mod}A)$.*

This conjecture should be formulated more generally as follows. Let $\mathcal{C}$ be a skeletally small abelian category and let $\mathcal{A}=\operatorname{Ex}(\mathcal{C}^{\text{op}},\text{Ab})$ denote the category of exact functors $\mathcal{C}^{\text{op}}\to\text{Ab}$. Categories of this form are ubiquitous [24, 34]. Note that $\mathcal{A}$ is the category of fp-injective objects of the locally coherent Grothendieck category $\operatorname{Lex}(\mathcal{C}^{\text{op}},\text{Ab})$ of left exact functors $\mathcal{C}^{\text{op}}\to\text{Ab}$. The category $\mathcal{A}$ admits set-indexed products and filtered colimits. Moreover, $\mathcal{A}$ admits a canonical exact structure with enough injective objects. To be precise, a sequence $X\to Y\to Z$ of morphisms in $\mathcal{A}$ is *exact* if the induced sequence $0 \to ZC \to YC \to XC \to 0$ of abelian groups is exact for all C in $\mathcal{C}$.

The general form of the above conjecture then says that the canonical functor $\mathbf{K}(\operatorname{Inj}\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is an equivalence and that $\mathbf{D}(\mathcal{A})$ is compactly generated with an equivalence $\mathbf{D}(\mathcal{A})^c \xrightarrow{\sim} \mathbf{D}^b(\mathcal{C})$. This conjecture specialises to Conjecture 5.9 by taking for $\mathcal{C}$ the free abelian category $\operatorname{Ab}(A)$ over a ring A; it has been proved in [25] provided that each object in $\mathcal{C}$ is noetherian; see Theorem 4.4 for another special case.

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