

# REPRESENTATIONS OF QUIVERS VIA REFLECTION FUNCTORS

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These are the notes for a course on representations of quivers for second year students in Paderborn in summer 2007. My aim was to provide a basic introduction without using any advanced methods. It turns out that a good knowledge of linear algebra is sufficient for proving Gabriel's theorem. Thus we classify the quivers of finite representation type and study their representations using reflection functors. The course was complemented by problem sessions run by Andrew Hubery; see his homepage for interesting exercises and further material. I wish to thank him for many useful discussions on the subject and the students of this course for their enthusiasm.

Further material has been added after giving this course in Bielefeld in summer 2010. This includes a discussion of regular representations and wild phenomena. In particular, two classical examples are covered: representations of the Kronecker quiver and representations of the Klein four group.

## 1. REPRESENTATIONS OF QUIVERS

In this section we introduce our basic concepts: quivers and their representations. Throughout we fix a field  $k$ .

**1.1. Quivers.** A *quiver* is a directed graph, which is assumed to be finite. More precisely, a quiver is a quadruple  $Q = (Q_0, Q_1, s, t)$  consisting of a finite set  $Q_0$  of *vertices*, a finite set  $Q_1$  of *arrows*, and two maps  $s, t: Q_1 \rightarrow Q_0$ . An arrow  $\alpha \in Q_1$  *starts* at  $s(\alpha)$  and *terminates* at  $t(\alpha)$ . We sometimes write  $\alpha: s(\alpha) \rightarrow t(\alpha)$ .

A *non-trivial path* of length  $r \geq 1$  in  $Q$  is a sequence  $\xi = \xi_r \dots \xi_1$  of arrows satisfying  $t(\xi_p) = s(\xi_{p+1})$  for  $1 \leq p < r$ . We write

$$i_1 \xrightarrow{\xi_1} i_2 \xrightarrow{\xi_2} \dots \xrightarrow{\xi_r} i_{r+1}.$$

The path  $\xi$  starts at  $s(\xi) = s(\xi_1)$  and terminates at  $t(\xi) = t(\xi_r)$ . For each vertex  $i$ , we have in addition the *trivial path*  $\varepsilon_i$  of length zero with  $s(\varepsilon_i) = i = t(\varepsilon_i)$ .

For a pair  $i, j$  of vertices, let  $Q(i, j)$  denote the set of paths  $\xi$  with  $s(\xi) = i$  and  $t(\xi) = j$ . The obvious composition of paths induces maps

$$Q(i, \xi): Q(i, s(\xi)) \longrightarrow Q(i, t(\xi)) \quad \text{and} \quad Q(\xi, j): Q(t(\xi), j) \longrightarrow Q(s(\xi), j)$$

with  $Q(i, \xi)(\mu) = \xi\mu$  and  $Q(\xi, j)(\nu) = \nu\xi$ .

**1.2. Representations.** Let  $Q$  be a quiver. A *representation* of  $Q$  is a collection

$$X = (X_i, X_\alpha)_{i \in Q_0, \alpha \in Q_1}$$

consisting of a vector space  $X_i$  for each vertex  $i$  and a linear map  $X_\alpha: X_{s(\alpha)} \rightarrow X_{t(\alpha)}$  for each arrow  $\alpha$ . A *morphism* of representations  $\phi: X \rightarrow Y$  is a collection  $\phi = (\phi_i)_{i \in Q_0}$  of linear maps  $\phi_i: X_i \rightarrow Y_i$  for each vertex  $i$  such that  $Y_\alpha \phi_{s(\alpha)} = \phi_{t(\alpha)} X_\alpha$  for each arrow  $\alpha$ . In other words, for each arrow  $\alpha$  we have a commutative diagram

$$\begin{array}{ccc} X_{s(\alpha)} & \xrightarrow{\phi_{s(\alpha)}} & Y_{s(\alpha)} \\ \downarrow X_\alpha & & \downarrow Y_\alpha \\ X_{t(\alpha)} & \xrightarrow{\phi_{t(\alpha)}} & Y_{t(\alpha)} \end{array}$$

The *composition* of  $\phi$  with  $\psi: Y \rightarrow Z$  is given by  $(\psi\phi)_i = \psi_i\phi_i$  for each vertex  $i$ . For each representation  $X$ , we have the *identity morphism*  $\text{id}_X: X \rightarrow X$  with  $(\text{id}_X)_i = \text{id}_{X_i}$  for all  $i$ . The set of morphisms  $X \rightarrow Y$  we denote by  $\text{Hom}(X, Y)$  and we write  $\text{End}(X)$  for the set of endomorphisms  $X \rightarrow X$ .

This defines a category  $\text{Rep}(Q, k)$ . We denote by  $\text{rep}(Q, k)$  the full subcategory with objects the finite dimensional representations. Here, a representation  $X$  is *finite dimensional* if each vector space  $X_i$  is finite dimensional.

**1.3. The category of representations.** Various concepts that are defined for vector spaces carry over to representations of  $Q$  by applying the vector space definition *point-wise*, that is, for each vertex  $i$ .

Fix a pair  $X, Y$  of representations of  $Q$ . We call  $X$  a *subrepresentation* of  $Y$  and write  $X \subseteq Y$ , if  $X_i$  is a subspace of  $Y_i$  for each vertex  $i$  and  $X_\alpha(x) = Y_\alpha(x)$  for each arrow  $\alpha$  and  $x \in X_{s(\alpha)}$ .

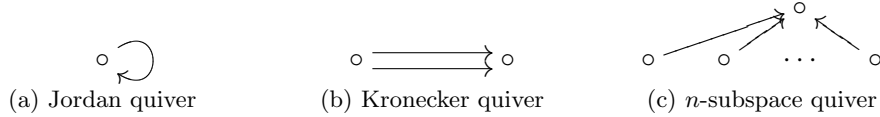
Given a morphism  $\phi: X \rightarrow Y$ , its *kernel*  $\text{Ker } \phi$  is by definition the subrepresentation of  $X$  with  $(\text{Ker } \phi)_i = \text{Ker } \phi_i$  for each vertex  $i$ . The *cokernel*  $\text{Coker } \phi$  and the *image*  $\text{Im } \phi$  are defined analogously. Note that  $\phi$  is a *monomorphism* if and only if  $\text{Ker } \phi = 0$ , while  $\phi$  is an *epimorphism* if and only if  $\text{Coker } \phi = 0$ . The morphism  $\phi$  is an *isomorphism* if

each  $\phi_i$  is an isomorphism. One defines addition and scalar multiplication for morphisms  $X \rightarrow Y$  point-wise and that makes  $\text{Hom}(X, Y)$  into a vector space.

The *dimension vector* of a finite dimensional representation  $X$  is the vector  $\dim X$  in  $\mathbb{Z}^{Q_0}$  with

$$(\dim X)_i = \dim X_i \quad (i \in Q_0).$$

**1.4. Some special quivers.** We consider three quivers in more detail. The repre-



sentations of the *Jordan quiver* are endomorphisms of a single vector space and their classification can be formulated in terms of Jordan blocks. The representations of the *Kronecker quiver* are pairs of linear maps up to simultaneous conjugation. The  *$n$ -subspace quiver* has  $n + 1$  vertices and its representations are basically configurations of  $n$  subspaces of a fixed vector space.

**1.5. Duality.** Let  $Q^{\text{op}}$  denote the *opposite quiver* which is obtained from  $Q$  by reversing all arrows. The vector space duality  $D = \text{Hom}_k(-, k)$  induces a *duality*

$$D: \text{Rep}(Q, k) \longrightarrow \text{Rep}(Q^{\text{op}}, k).$$

Given a representation  $X$  of  $Q$ , let  $(DX)_i = D(X_i)$  and  $(DX)_\alpha = D(X_\alpha)$  for  $i \in Q_0$  and  $\alpha \in Q_1$ . For a morphism  $\phi: X \rightarrow Y$ , let  $(D\phi)_i = D(\phi_i)$ .

Let  $V, W$  be a pair of vector spaces. Recall that there is a canonical monomorphism  $\varepsilon_V: V \rightarrow D^2V$  defined by  $\varepsilon_V(x)(\phi) = \phi(x)$  for  $x \in V$  and  $\phi \in DV$ . This induces an isomorphism

$$\text{Hom}(W, DV) \xrightarrow{\sim} \text{Hom}(V, DW)$$

by sending  $\phi: W \rightarrow DV$  to  $(D\phi)\varepsilon_V$ .

**Lemma 1.5.1.** *For  $X \in \text{Rep}(Q, k)$  and  $Y \in \text{Rep}(Q^{\text{op}}, k)$ , there is a canonical monomorphism  $\varepsilon_X: X \rightarrow D^2X$  which induces a natural isomorphism*

$$\text{Hom}(Y, DX) \xrightarrow{\sim} \text{Hom}(X, DY)$$

by sending  $\phi: Y \rightarrow DX$  to  $(D\phi)\varepsilon_X$ .

*Proof.* Use the linear maps  $\varepsilon_{X_i}$  and  $\text{Hom}(Y_i, DX_i) \xrightarrow{\sim} \text{Hom}(X_i, DY_i)$ . □

**1.6. Simple representations.** A *simple* representation is defined to be a non-zero representation with no proper subrepresentations.

Given a vertex  $i$ , let  $S(i)$  be the representation with

$$S(i)_j = \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \quad \text{and} \quad S(i)_\alpha = 0$$

for  $j \in Q_0$  and  $\alpha \in Q_1$ . This representation is simple.

**Lemma 1.6.1.** *Let  $X$  be a representation and suppose that  $i$  is a vertex with  $X_i \neq 0$  and  $X_\alpha = 0$  for each arrow  $\alpha$  starting at  $i$ . Then  $S(i)$  is a subrepresentation of  $X$ .*

*Proof.* The assumption on  $X$  implies  $\text{Hom}(S(i), X) \cong X_i$ . □

Suppose that  $Q$  has no *oriented cycles*, that is, non-trivial paths from a vertex to itself. Then for any simple representation  $S$  of  $Q$ , there exists a unique vertex  $i$  such that  $S \cong S(i)$ . This follows from Lemma 1.6.1. On the other hand, there are additional simple representations if  $Q$  has oriented cycles.

**Example 1.6.2.** The finite dimensional simple representations of the Jordan quiver are parametrised by the monic irreducible polynomials over  $k$ . More precisely, the representation corresponding to such a polynomial  $\sum_{i=0}^d \lambda_i t^i$  of degree  $d$  is the pair  $(X, \phi)$  consisting of the vector space  $X = k^d$  and the endomorphism  $\phi: X \rightarrow X$  with  $\phi(e_i) = e_{i+1}$  for  $1 \leq i < d$  and  $\phi(e_d) = \sum_{i=1}^d -\lambda_{i-1} e_i$ .

**1.7. Projective and injective representations.** For each vertex  $i$ , we define a *projective representation*  $P(i)$  and an *injective representation*  $I(i)$ .

Given any set  $X$ , we denote by  $k[X]$  the vector space with basis  $X$ , that is,  $k[X]$  is the set of linear combinations  $\sum_p \lambda_p x_p$  with  $x_p \in X$ ,  $\lambda_p \in k$ , and almost all  $\lambda_p = 0$ . For a map  $\phi: X \rightarrow Y$ , let  $k[\phi]: k[X] \rightarrow k[Y]$  be the linear map sending  $\sum_p \lambda_p x_p$  to  $\sum_p \lambda_p \phi(x_p)$ .

Now define

$$P(i)_j = k[Q(i, j)] \quad \text{and} \quad P(i)_\alpha = k[Q(i, \alpha)]$$

for  $j \in Q_0$  and  $\alpha \in Q_1$ . Dually, we define

$$I(i)_j = Dk[Q(j, i)] \quad \text{and} \quad I(i)_\alpha = Dk[Q(\alpha, i)].$$

Note that  $I(i) = D\bar{P}(i)$  where  $\bar{P}(i)$  refers to the projective representation of  $Q^{\text{op}}$ .

**Lemma 1.7.1.** *Let  $X$  be a representation of  $Q$ . Then there are natural isomorphisms*

$$\text{Hom}(P(i), X) \cong X_i \quad \text{and} \quad \text{Hom}(X, I(i)) \cong DX_i.$$

*Proof.* The isomorphism  $\text{Hom}(P(i), X) \rightarrow X_i$  sends  $\phi$  to  $\phi_i(\varepsilon_i)$ . Its inverse map sends  $x \in X_i$  to the morphism  $\phi: P(i) \rightarrow X$  with

$$\phi_j(\xi) = X_{\xi_r} \dots X_{\xi_1}(x)$$

for a basis element  $\xi = \xi_r \dots \xi_1$  of  $P(i)_j$ .

The isomorphism  $\text{Hom}(X, I(i)) \cong DX_i$  follows from the first using Lemma 1.5.1 since

$$\text{Hom}(X, I(i)) = \text{Hom}(X, D\bar{P}(i)) \cong \text{Hom}(\bar{P}(i), DX) \cong DX_i. \quad \square$$

**Lemma 1.7.2.** (1) *The representations  $P(i)$  ( $i \in Q_0$ ) are pairwise non-isomorphic.*  
 (2) *The representations  $I(i)$  ( $i \in Q_0$ ) are pairwise non-isomorphic.*

*Proof.* Fix two vertices  $i, j$  and suppose  $P(j) \cong P(i)$ . Then

$$k = S(i)_i \cong \text{Hom}(P(i), S(i)) \cong \text{Hom}(P(j), S(i)) \cong S(i)_j = \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

by Lemma 1.7.1. Thus  $j = i$ . The proof for the  $I(i)$  is analogous.  $\square$

**Lemma 1.7.3.** *Suppose  $Q$  has no oriented cycles. Then  $P(i)$  and  $I(i)$  are finite dimensional with  $\text{End}(P(i)) \cong k \cong \text{End}(I(i))$ .*

*Proof.* If  $Q$  has no oriented cycles then  $Q(i, j)$  is finite for all pairs of vertices  $i, j$  because the quiver is finite. We have

$$\text{End}(P(i)) \cong P(i)_i = k[Q(i, i)] = k$$

by Lemma 1.7.1, and a similar argument works for  $I(i)$ .  $\square$

**Remark 1.7.4.** The representation  $P(i)$  is a *projective object* in the sense that for every epimorphism  $X \rightarrow Y$  the induced map  $\text{Hom}(P(i), X) \rightarrow \text{Hom}(P(i), Y)$  is surjective. Dually,  $I(i)$  is an *injective object* in the sense that for every monomorphism  $X \rightarrow Y$  the induced map  $\text{Hom}(Y, I(i)) \rightarrow \text{Hom}(X, I(i))$  is surjective. This follows from Lemma 1.7.1.

## 2. DIRECT SUM DECOMPOSITIONS

In this section we consider finite dimensional representations. We show that each representation decomposes essentially uniquely into indecomposable representations.

**2.1. Direct sums.** Let  $X_1, \dots, X_r$  be a finite number of representations. A *direct sum*

$$X = X_1 \oplus \dots \oplus X_r$$

is a representation  $X$  together with morphisms  $\iota_i: X_i \rightarrow X$  and  $\pi_i: X \rightarrow X_i$  for  $1 \leq i \leq r$  such that  $\sum_{i=1}^r \iota_i \pi_i = \text{id}_X$  and  $\pi_i \iota_i = \text{id}_{X_i}$  for all  $i$ . Note that we can identify each  $X_i$  via  $\iota_i$  with a subrepresentation of  $X$ . Then we obtain

$$(2.1.1) \quad X = \sum_{i=1}^r X_i \quad \text{and} \quad X_i \cap \sum_{i' \neq i} X_{i'} = 0 \quad \text{for} \quad 1 \leq i \leq r.$$

Here  $\sum_{i \in I} X_i$  refers to the smallest subrepresentation of  $X$  containing  $X_i$  for all  $i \in I$ . Conversely, if  $X_1, \dots, X_r$  is a family of subrepresentations of a representation  $X$  satisfying (2.1.1) then  $X = X_1 \oplus \dots \oplus X_r$ . In that case we take for  $\iota_i: X_i \rightarrow X$  the inclusion morphism and let  $\pi_i = (\rho_i \iota_i)^{-1} \rho_i$ , where  $\rho_i: X \rightarrow X / \sum_{i' \neq i} X_{i'}$  denotes the canonical morphism.

A family of subrepresentations  $X_1, \dots, X_r$  of  $X$  satisfying (2.1.1) is called a *direct sum decomposition* of  $X$ . Note that one can check (2.1.1) point-wise for each vertex. Thus we have a decomposition  $X = X_1 \oplus \dots \oplus X_r$  if and only if we have a vector space decomposition  $X_j = (X_1)_j \oplus \dots \oplus (X_r)_j$  for each vertex  $j$ .

**Lemma 2.1.1.** *Let  $X = X_1 \oplus \dots \oplus X_r$  and  $Y = Y_1 \oplus \dots \oplus Y_s$ . Then we have induced vector space decompositions*

$$\bigoplus_{i=1}^r \text{Hom}(X_i, Y) = \text{Hom}(X, Y) = \bigoplus_{j=1}^s \text{Hom}(X, Y_j).$$

*Proof.* Let  $\iota_i^* = \text{Hom}(\iota_i, Y)$  and  $\pi_i^* = \text{Hom}(\pi_i, Y)$  for  $1 \leq i \leq r$ . Then we have  $\sum_{i=1}^r \pi_i^* \iota_i^* = \text{id}_{\text{Hom}(X, Y)}$  and  $\iota_i^* \pi_i^* = \text{id}_{\text{Hom}(X_i, Y)}$  for all  $i$ . This proves the first equality. The argument for the second equality is analogous.  $\square$

It follows from Lemma 2.1.1 that a direct sum of  $X_1, \dots, X_r$  is unique up to an isomorphism. Thus we may speak of *the* direct sum and the notation  $X_1 \oplus \dots \oplus X_r$  is well-defined. We write  $X^r = X \oplus \dots \oplus X$  for the direct sum of  $r$  copies of a representation  $X$ .

A representation  $X$  is called *indecomposable* if  $X \neq 0$  and  $X = X_1 \oplus X_2$  implies  $X_1 = 0$  or  $X_2 = 0$ .

**2.2. Fitting's lemma.** We fix a representation  $X$  and study the ring of endomorphisms  $\text{End}(X)$ .

**Lemma 2.2.1.** *Let  $X$  be a representation and  $\phi$  an endomorphism.*

- (1) *For large enough  $r$ , we have  $X = \text{Im } \phi^r \oplus \text{Ker } \phi^r$ .*
- (2) *If  $X$  is indecomposable, then  $\phi$  is either an automorphism or nilpotent.*

*Proof.* Because  $X$  is finite dimensional, we may choose  $r$  large enough so that  $\text{Im } \phi^r = \text{Im } \phi^{r+1}$ . Thus  $\phi^r: \text{Im } \phi^r \rightarrow \text{Im } \phi^{2r}$  is an isomorphism and we denote by  $\psi$  its inverse. Furthermore, let  $\iota_1: \text{Im } \phi^r \rightarrow X$  and  $\iota_2: \text{Ker } \phi^r \rightarrow X$  denote the inclusions. We put  $\pi_1 = \psi\phi^r: X \rightarrow \text{Im } \phi^r$  and  $\pi_2 = \text{id}_X - \psi\phi^r: X \rightarrow \text{Ker } \phi^r$ . Then  $\iota_1\pi_1 + \iota_2\pi_2 = \text{id}_X$  and  $\pi_i\iota_i = \text{id}_{X_i}$  for  $i = 1, 2$ . Thus  $X = \text{Im } \phi^r \oplus \text{Ker } \phi^r$ . Part (2) is an immediate consequence of (1).  $\square$

A ring is called *local* if the sum of two non-units is again a non-unit.

**Proposition 2.2.2.** *A representation  $X$  is indecomposable if and only if  $\text{End}(X)$  is local.*

*Proof.* Let  $X$  be indecomposable and  $\phi, \phi' \in \text{End}(X)$ . Suppose  $\phi + \phi'$  is invertible, say  $\rho(\phi + \phi') = \text{id}_X$ . If  $\phi$  is non-invertible then  $\rho\phi$  is non-invertible. Thus  $\rho\phi$  is nilpotent, say  $(\rho\phi)^r = 0$ , by Lemma 2.2.1. We obtain

$$(\text{id}_X - \rho\phi)(\text{id}_X + \rho\phi + \dots + (\rho\phi)^{r-1}) = \text{id}_X.$$

Therefore  $\rho\phi' = \text{id}_X - \rho\phi$  is invertible whence  $\phi'$  is invertible.

If  $X = X_1 \oplus X_2$  with  $X_i \neq 0$  for  $i = 1, 2$ , then we have idempotent endomorphisms  $\varepsilon_i$  of  $X$  with  $\text{Im } \varepsilon_i = X_i$ . Clearly, each  $\varepsilon_i$  is non-invertible but  $\text{id}_X = \varepsilon_1 + \varepsilon_2$ .  $\square$

The assumption on  $X$  to be finite dimensional is necessary.

**Example 2.2.3.** Let  $k[t]$  denote the polynomial ring in one variable and consider the following representation of the Kronecker quiver.

$$X : k[t] \begin{array}{c} \xrightarrow{\cdot t} \\ \xrightarrow{\text{id}} \end{array} k[t]$$

The endomorphism ring of  $X$  is isomorphic to  $k[t]$ . Thus  $X$  is indecomposable but  $\text{End}(X)$  is not local.

**2.3. The radical.** Given a pair  $X, Y$  of representations, we define the *radical*

$$\text{Rad}(X, Y) = \left\{ \phi \in \text{Hom}(X, Y) \left| \begin{array}{l} \tau\phi\sigma \text{ is non-invertible for every pair } Z \xrightarrow{\sigma} X \\ \text{and } Y \xrightarrow{\tau} Z \text{ with } Z \text{ indecomposable} \end{array} \right. \right\}.$$

**Lemma 2.3.1.** *Let  $X, Y$  be a pair of representations.*

- (1)  *$\text{Rad}(X, Y)$  is a subspace of  $\text{Hom}(X, Y)$ .*
- (2)  *$\text{Rad}(X, Y_1 \oplus Y_2) = \text{Rad}(X, Y_1) \oplus \text{Rad}(X, Y_2)$ .*
- (3)  *$\text{Rad}(X_1 \oplus X_2, Y) = \text{Rad}(X_1, Y) \oplus \text{Rad}(X_2, Y)$ .*
- (4) *If  $X$  and  $Y$  are indecomposable, then  $\text{Hom}(X, Y) \setminus \text{Rad}(X, Y)$  equals the set of isomorphisms  $X \rightarrow Y$ .*

*Proof.* (1) Let  $\phi_1, \phi_2 \in \text{Rad}(X, Y)$ . Choose  $\sigma \in \text{Hom}(Z, X)$  and  $\tau \in \text{Hom}(Y, Z)$  with  $Z$  indecomposable. Then  $\tau\phi_1\sigma$  and  $\tau\phi_2\sigma$  are non-invertible, and therefore  $\tau(\phi_1 + \phi_2)\sigma = \tau\phi_1\sigma + \tau\phi_2\sigma$  is non-invertible, since  $\text{End}(Z)$  is local by Proposition 2.2.2. Thus  $\phi_1 + \phi_2$  belongs to  $\text{Rad}(X, Y)$ .

(2) Let  $Y = Y_1 \oplus Y_2$  and  $\phi = (\phi_i) \in \text{Hom}(X, Y)$  with  $\phi_i \in \text{Hom}(X, Y_i)$  for  $i = 1, 2$ . Choose  $\sigma \in \text{Hom}(Z, X)$  and  $\tau = (\tau_i) \in \text{Hom}(Y, Z)$  with  $Z$  indecomposable and  $\tau_i \in \text{Hom}(Y_i, Z)$  for  $i = 1, 2$ . Then  $\tau\phi\sigma = \tau_1\phi_1\sigma + \tau_2\phi_2\sigma$ .

If  $\phi_i \in \text{Rad}(X, Y_i)$  for  $i = 1, 2$ , then  $\tau_i\phi_i\sigma$  is non-invertible for  $i = 1, 2$ , and therefore  $\tau\phi\sigma$  is non-invertible, since  $\text{End}(Z)$  is local by Proposition 2.2.2. Thus  $\phi$  belongs to  $\text{Rad}(X, Y)$ . Conversely, let  $\phi \in \text{Rad}(X, Y)$  and fix  $i \in \{1, 2\}$ . Then  $\phi_i \in \text{Rad}(X, Y_i)$  because we can put  $\tau_j = 0$  for  $j \neq i$  and have that  $\tau_i\phi_i\sigma = \tau\phi\sigma$  is non-invertible.

(3) Analogous to part (2).

(4) Let  $\phi \in \text{Hom}(X, Y) \setminus \text{Rad}(X, Y)$ . Choose  $\sigma \in \text{Hom}(Z, X)$  and  $\tau \in \text{Hom}(Y, Z)$  with  $Z$  indecomposable such that  $\tau\phi\sigma$  is invertible. Then  $\sigma$  is invertible because  $X$  is indecomposable, and  $\tau$  is invertible because  $Y$  is indecomposable. Thus  $\phi$  is invertible. It is clear that an isomorphism  $X \rightarrow Y$  does not belong to  $\text{Rad}(X, Y)$ .  $\square$

#### 2.4. The Krull-Remak-Schmidt theorem.

**Theorem 2.4.1.** *Let  $X$  be a finite dimensional representation. Then there exists a decomposition  $X = X_1^{a_1} \oplus \dots \oplus X_r^{a_r}$  with the  $X_i$  pairwise non-isomorphic indecomposable representations and each  $a_i \geq 1$ . If  $X = Y_1^{b_1} \oplus \dots \oplus Y_s^{b_s}$  is another such decomposition, then  $r = s$  and, after reordering,  $X_i \cong Y_i$  and  $a_i = b_i$  for  $i \leq r$ .*

*Proof.* Induction on dimension shows that  $X$  decomposes into a finite direct sum of indecomposable representations. Suppose that  $X = X_1^{a_1} \oplus \dots \oplus X_r^{a_r}$  is such a direct sum decomposition with the  $X_i$  pairwise non-isomorphic indecomposable representations and each  $a_i \geq 1$ . Let  $Y$  be indecomposable and consider

$$\frac{\dim \text{Hom}(X, Y) - \dim \text{Rad}(X, Y)}{\dim \text{Hom}(Y, Y) - \dim \text{Rad}(Y, Y)}.$$

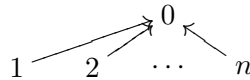
We see by Lemma 2.3.1 that this number equals  $a_i$  if  $Y \cong X_i$  (there is at most one such  $i$ ) and 0 otherwise. In particular, this number is independent of the decomposition.  $\square$

The Krull-Remak-Schmidt theorem says that the classification of finite dimensional representations reduces to the classification of indecomposable representations. There is a similar statement about morphisms between representations. Let  $X = X_1 \oplus \dots \oplus X_r$  and  $Y = Y_1 \oplus \dots \oplus Y_s$  be two representations with their decompositions into indecomposable representations. Then we have

$$\text{Hom}(X, Y) = \bigoplus_{i,j} \text{Hom}(X_i, Y_j)$$

by Lemma 2.1.1. Thus each morphism  $\phi: X \rightarrow Y$  can be written uniquely as a matrix  $\phi = (\phi_{ij})$  where each entry  $\phi_{ij}: X_i \rightarrow Y_j$  is a morphism between indecomposable representations.

**Example 2.4.2.** Each representation  $X$  of the  $n$ -subspace quiver



admits a unique decomposition  $X = X' \oplus X(1) \oplus \dots \oplus X(n)$  such that  $X(i)$  is a direct sum of copies of  $S(i)$  for  $1 \leq i \leq n$  and  $X'$  is a subspace representation (that is, each linear map  $X'_i \rightarrow X'_0$  is injective).

### 3. REFLECTION FUNCTORS

In this section we introduce reflection functors. These functors form our basic tool for classifying representations in terms of their dimension vectors.

**3.1. Orientations.** A vertex  $i$  of  $Q$  is called a *sink* (resp. *source*) if there is no arrow in  $Q$  starting (resp. ending) at  $i$ .

Given any vertex  $i$ , the quiver  $\sigma_i Q$  is obtained from  $Q$  by reversing all arrows which start or end at  $i$ .

An ordering  $i_1, \dots, i_n$  of the vertices of  $Q$  is called *admissible* if for each  $p$  the vertex  $i_p$  is a sink for  $\sigma_{i_{p-1}} \dots \sigma_{i_1} Q$ . In that case we have

$$\sigma_{i_n} \dots \sigma_{i_1} Q = Q.$$

**Lemma 3.1.1.** *There exists an admissible ordering of the vertices of  $Q$  if and only if there are no oriented cycles in  $Q$ .*

*Proof.* We show one implication by induction on the number of vertices. So suppose  $Q$  has no oriented cycles and let  $i_n$  be the starting vertex of a path of maximal length. Then  $i_n$  is a source and we remove it from  $Q$ . There is an admissible ordering  $i_1, \dots, i_{n-1}$  of the remaining vertices and we get an admissible ordering  $i_1, \dots, i_n$  of the vertices of  $Q$ .  $\square$

**3.2. The Euler form.** Let  $n = \text{card } Q_0$ . The *Euler form* is the bilinear form

$$\langle -, - \rangle: \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad \langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

We obtain on  $\mathbb{Z}^n$  a *symmetric bilinear form* by defining

$$(3.2.1) \quad (x, y) = \langle x, y \rangle + \langle y, x \rangle.$$

Suppose that  $Q$  has no *loops* (that is, arrows from a vertex to itself). The *reflection* with respect to a vertex  $i$  is by definition the map

$$\sigma_i: \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \quad \text{with} \quad \sigma_i(x) = x - \frac{2(x, e_i)}{(e_i, e_i)} e_i$$

where  $e_i$  is the  $i$ th coordinate vector. It is easily checked that the  $\sigma_i$  are automorphisms of order two preserving the bilinear form  $(-, -)$ .

For the set  $\mathbb{Z}^n$  we use the *partial order* which is defined as follows:

$$x \leq y \iff x_i \leq y_i \quad \text{for all } i.$$

**3.3. Reflection functors.** Given a pair of quivers  $Q$  and  $Q'$ , a *functor*  $F: \text{Rep}(Q, k) \rightarrow \text{Rep}(Q', k)$  is an assignment such that  $F \text{id}_X = \text{id}_{FX}$  for every representation  $X$  and  $F(\psi\phi) = (F\psi)(F\phi)$  for every pair  $\phi: X \rightarrow Y$  and  $\psi: Y \rightarrow Z$  of morphisms.

Let  $i$  be a vertex of  $Q$ . We define a pair of *reflection functors*

$$S_i^+, S_i^-: \text{Rep}(Q, k) \longrightarrow \text{Rep}(\sigma_i Q, k).$$

To this end fix representations  $X, X'$  of  $Q$  and a morphism  $\phi: X \rightarrow X'$ .

(1) If the vertex  $i$  is a sink of  $Q$ , then we construct  $S_i^+$  as follows. We define  $S_i^+ X = Y$  by letting  $Y_j = X_j$  for a vertex  $j \neq i$ , and letting  $Y_i$  be the kernel of the map  $\xi = (X_\alpha)$  in the following sequence

$$Y_i \xrightarrow{\check{\xi}} \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} X_{s(\alpha)} \xrightarrow{\xi} X_i$$

where  $\check{\xi}$  denotes the inclusion map of the kernel. For an arrow  $\alpha$  in  $Q$ , let  $Y_\alpha = X_\alpha$  if  $t(\alpha) \neq i$ , and  $Y_\alpha: Y_i \rightarrow X_{s(\alpha)} = Y_{s(\alpha)}$  be the map  $\check{\xi}$  followed by the canonical projection onto  $X_{s(\alpha)}$  if  $t(\alpha) = i$ . For the morphism  $S_i^+ \phi = \psi$ , let  $\psi_j = \phi_j$  if  $j \neq i$  and let  $\psi_i: Y_i \rightarrow Y'_i$  be the restriction of the map

$$(\phi_{s(\alpha)}): \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} X_{s(\alpha)} \longrightarrow \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} X'_{s(\alpha)}.$$

(2) If the vertex  $i$  is a source of  $Q$ , then we construct dually  $S_i^-$  as follows. We define  $S_i^- X = Y$  by letting  $Y_j = X_j$  for a vertex  $j \neq i$ , and letting  $Y_i$  be the cokernel of the map  $\xi = (X_\alpha)$  in the following sequence

$$X_i \xrightarrow{\xi} \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X_{t(\alpha)} \xrightarrow{\hat{\xi}} Y_i$$

where  $\hat{\xi}$  denotes the canonical map onto the cokernel. For an arrow  $\alpha$  in  $Q$ , let  $Y_\alpha = X_\alpha$  if  $s(\alpha) \neq i$ , and  $Y_\alpha: Y_{t(\alpha)} = X_{t(\alpha)} \rightarrow Y_i$  be the restriction of  $\hat{\xi}$  to  $X_{t(\alpha)}$  if  $s(\alpha) = i$ . For the morphism  $S_i^- \phi = \psi$ , let  $\psi_j = \phi_j$  if  $j \neq i$  and let  $\psi_i: Y_i \rightarrow Y'_i$  be the map which is induced by

$$(\phi_{t(\alpha)}): \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X_{t(\alpha)} \longrightarrow \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X'_{t(\alpha)}.$$

(3) Let  $i$  be a sink of  $Q$ . Then we define a natural monomorphism

$$(3.3.1) \quad \iota_i X: S_i^- S_i^+ X \longrightarrow X$$

by letting  $(\iota_i X)_j = \text{id}_{X_j}$  for a vertex  $j \neq i$ , and letting  $(\iota_i X)_i$  be the canonical map

$$(S_i^- S_i^+ X)_i = \text{Coker } \check{\xi} \cong \text{Im } \xi \longrightarrow X_i.$$

(4) Let  $i$  be a source of  $Q$ . Then we define a natural epimorphism

$$(3.3.2) \quad \pi_i X: X \longrightarrow S_i^+ S_i^- X$$

by letting  $(\pi_i X)_j = \text{id}_{X_j}$  for a vertex  $j \neq i$ , and letting  $(\pi_i X)_i$  be the canonical map

$$X_i \longrightarrow \text{Im } \xi \cong \text{Ker } \hat{\xi} = (S_i^+ S_i^- X)_i.$$

**Lemma 3.3.1.** *For each vertex  $i$ ,  $S_i^+$  and  $S_i^-$  are functors.*  $\square$

**Lemma 3.3.2.** *Let  $X, X'$  be representations of  $Q$  and  $i$  be a vertex.*

- (1)  $S_i^\pm(X \oplus X') = S_i^\pm X \oplus S_i^\pm X'$ .
- (2)  $X = (S_i^- S_i^+ X) \oplus \text{Coker } \iota_i X$  and  $X = (S_i^+ S_i^- X) \oplus \text{Ker } \pi_i X$ .
- (3) *If  $\text{Coker } \iota_i X = 0$ , then  $\dim S_i^+ X = \sigma_i(\dim X)$ .*
- (4) *If  $\text{Ker } \pi_i X = 0$ , then  $\dim S_i^- X = \sigma_i(\dim X)$ .*

*Proof.* (1) Use that  $S_i^\pm$  is a functor satisfying  $S_i^\pm(\phi + \psi) = S_i^\pm\phi + S_i^\pm\psi$  for any pair of parallel morphisms  $\phi, \psi$ .

(2) The canonical map  $\rho'_i: X_i \rightarrow \text{Coker } \xi$  has a section  $\rho_i: \text{Coker } \xi \rightarrow X_i$ , that is,  $\rho'_i \rho_i = \text{id}_{\text{Coker } \xi}$ . This gives a morphism  $\rho: \text{Coker } \iota_i X \rightarrow X$  if we put  $\rho_j = 0$  for  $j \neq i$ . It is clear that  $\iota_i X: S_i^- S_i^+ X \rightarrow X$  and  $\rho: \text{Coker } \iota_i X \rightarrow X$  give a direct sum decomposition of  $X$ . The proof for  $X = (S_i^+ S_i^- X) \oplus \text{Ker } \pi_i X$  is similar.

(3) If  $\text{Coker } \iota_i X = 0$ , then we have

$$\dim Y_i = \sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} \dim X_{s(\alpha)} - \dim X_i$$

and  $\dim Y_j = \dim X_j$  for  $j \neq i$ . Thus  $\dim Y = \sigma_i(\dim X)$ . The proof of (4) is similar.  $\square$

Note that the representations  $\text{Coker } \iota_i X$  and  $\text{Ker } \pi_i X$  are concentrated at the vertex  $i$ . Thus they are direct sums of copies of the simple representation  $S(i)$ .

**Lemma 3.3.3.** *Let  $i$  be a sink and  $X$  an indecomposable representation of  $Q$ . Then the following are equivalent:*

- (1)  $X \not\cong S(i)$ .
- (2)  $S_i^+ X$  is indecomposable.
- (3)  $S_i^+ X \neq 0$ .
- (4)  $S_i^- S_i^+ X \cong X$ .
- (5) *The map  $(X_\alpha): \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} X_{s(\alpha)} \rightarrow X_i$  is an epimorphism.*
- (6)  $\sigma_i(\dim X) > 0$ .
- (7)  $\dim S_i^+ X = \sigma_i(\dim X)$ .

*Proof.* Apply Lemma 3.3.2.  $\square$

**Remark 3.3.4.** There is an analogue of Lemma 3.3.3 for a source of  $Q$  and the corresponding functor  $S_i^-$ .

The following theorem is a consequence and summarises the basic properties of the reflection functors.

**Theorem 3.3.5.** *The functors  $S_i^+$  and  $S_i^-$  induce mutually inverse bijections between the isomorphism classes of indecomposable representations of  $Q$  and the isomorphism classes of indecomposable representations of  $\sigma_i Q$ , with the exception of the simple representation  $S(i)$ , which is annihilated by these functors. Moreover,  $\dim S_i^\pm X = \sigma_i(\dim X)$  for every indecomposable representation  $X$  not isomorphic to  $S(i)$ .*

**3.4. Coxeter functors.** Let  $Q$  be a quiver without oriented cycles and let  $i_1, \dots, i_n$  be an admissible ordering of the vertices of  $Q$ . The *Coxeter functor* with respect to this ordering is the functor

$$C^+ = S_{i_n}^+ \dots S_{i_1}^+ : \text{Rep}(Q, k) \longrightarrow \text{Rep}(Q, k).$$

We also define

$$C^- = S_{i_1}^- \dots S_{i_n}^- : \text{Rep}(Q, k) \longrightarrow \text{Rep}(Q, k).$$

For  $r \in \mathbb{Z}$ , we write

$$C^r = \begin{cases} (C^+)^r & \text{if } r > 0, \\ \text{Id} & \text{if } r = 0, \\ (C^-)^{-r} & \text{if } r < 0. \end{cases}$$

Note that in general  $C^r C^s \neq C^{r+s}$ .

**Lemma 3.4.1.** *The functors  $C^+$  and  $C^-$  do not depend on the choice of the admissible ordering of the vertices of  $Q$ .*

*Proof.* First observe that  $S_i^+ S_j^+ = S_j^+ S_i^+$  if  $i$  and  $j$  are sinks with respect to some orientation and if both are not joined by an arrow. Now fix two admissible orderings  $i_1, \dots, i_n$  and  $i'_1, \dots, i'_n$  of the vertices of  $Q$ . Let  $i_1 = i'_m$ . Then  $i'_1, \dots, i'_{m-1}$  are not joined to  $i_1$  by an arrow. Therefore

$$S_{i'_m}^+ \dots S_{i'_1}^+ = S_{i'_{m-1}}^+ \dots S_{i'_1}^+ S_{i_1}^+.$$

Applying a similar argument for  $i_2$ , then  $i_3$ , and so on, we obtain

$$S_{i'_n}^+ \dots S_{i'_1}^+ = S_{i_n}^+ \dots S_{i_1}^+. \quad \square$$

For simplicity we assume in the following that  $Q_0 = \{1, \dots, n\}$  with  $1, \dots, n$  an admissible ordering.

**Lemma 3.4.2.** *Let  $i$  be a vertex.*

- (1)  $\dim P(i) = \sigma_1 \dots \sigma_{i-1}(e_i)$  and  $\dim I(i) = \sigma_n \dots \sigma_{i+1}(e_i)$ .
- (2)  $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$  and  $I(i) \cong S_n^+ \dots S_{i+1}^+ S(i)$

*Proof.* We provide the proof for  $P(i)$ ; the proof for  $I(i)$  is dual.

- (1) For  $0 \leq l < i$ , one shows by induction that

$$(3.4.1) \quad \sigma_{i-l} \dots \sigma_{i-1}(e_i) = \sum_{j=0}^l \text{card } Q(i, i-j) e_{i-j}$$

in  $\mathbb{Z}^n$ . For  $l = i - 1$ , we then obtain  $\sigma_1 \dots \sigma_{i-1}(e_i) = \dim P(i)$  because there are no paths from  $i$  to  $j$  in  $Q$  for  $j > i$ .

(2) We use the first part, in particular (3.4.1), and apply Lemma 3.3.3. An induction yields for  $0 \leq l < i$  that

$$\dim S_l^+ \dots S_1^+ P(i) = \sigma_{l+1} \dots \sigma_{i-1}(e_i).$$

Thus  $S_{i-1}^+ \dots S_1^+ P(i) \cong S(i)$  and therefore  $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$ .  $\square$

**Proposition 3.4.3.** *Let  $X$  be an indecomposable representation of  $Q$ .*

- (1)  $C^+ X = 0$  if  $X \cong P(i)$  for some vertex  $i$ , and  $C^- C^+ X \cong X$  otherwise.
- (2)  $C^- X = 0$  if  $X \cong I(i)$  for some vertex  $i$ , and  $C^+ C^- X \cong X$  otherwise.

*Proof.* (1) We have  $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$  by Lemma 3.4.2. Now apply reflection functors and use Lemma 3.3.3 to obtain

$$C^+ P(i) = S_n^+ \dots S_i^+ S(i) = 0.$$

If  $X \not\cong S_1^- \dots S_{i-1}^- S(i)$  for all  $i$ , then

$$C^- C^+ X = S_1^- \dots S_n^- S_n^+ \dots S_1^+ X \cong S_1^- \dots S_{n-1}^- S_{n-1}^+ \dots S_1^+ X \cong \dots \cong X,$$

again by Lemma 3.3.3. The proof of (2) is analogous.  $\square$

**3.5. Preprojective and preinjective representations.** Let  $Q$  be a quiver without oriented cycles. We introduce three classes of representations.

**Definition 3.5.1.** Let  $X$  be an indecomposable representation of  $Q$ .

- (1)  $X$  is *preprojective* if  $X \cong C^r P(i)$  for some vertex  $i$  and some  $r \leq 0$ .
- (2)  $X$  is *preinjective* if  $X \cong C^r I(i)$  for some vertex  $i$  and some  $r \geq 0$ .
- (3)  $X$  is *regular* if  $C^r X \neq 0$  for all  $r \in \mathbb{Z}$ .

Note that  $X$  is preprojective if and only if  $C^r X = 0$  for some  $r > 0$ , and  $X$  is preinjective if and only if  $C^r X = 0$  for some  $r < 0$ . This is an immediate consequence of Proposition 3.4.3.

**Proposition 3.5.2.** *An indecomposable representation is preprojective, preinjective or regular. Given indecomposable representations  $X, Y$  with  $X$  preprojective or preinjective, we have  $X \cong Y$  if and only if  $\dim X = \dim Y$ . Moreover,*

- (1)  $C^r P(i) \cong C^s P(j) \neq 0$  implies  $i = j$  and  $r = s$ ;
- (2)  $C^r I(i) \cong C^s I(j) \neq 0$  implies  $i = j$  and  $r = s$ .

*Proof.* The first assertion is an immediate consequence of Proposition 3.4.3. For the rest we use reflection functors, in particular Lemma 3.3.3. Suppose  $\dim X = \dim Y$  and let  $X$  be preprojective, say  $X \cong C^r P(i)$ . We know  $\dim P(i)$  from Lemma 3.4.2 and have therefore

$$\dim Y = (\sigma_n \dots \sigma_1)^r \sigma_1 \dots \sigma_{i-1}(e_i).$$

Using reflection functors we obtain  $S_{i-1}^+ \dots S_1^+ C^{-r}(Y) \cong S(i)$  and this gives

$$Y \cong C^r S_1^- \dots S_{i-1}^- S(i) \cong C^r P(i) \cong X.$$

The proof for preinjective  $X$  is analogous.

(1) If  $C^r P(i) \cong C^s P(j) \neq 0$  then  $P(i) \cong C^{s-r} P(j)$  and therefore  $s - r \leq 0$  by Proposition 3.4.3. The same argument gives  $r - s \leq 0$  whence  $r = s$ . We obtain that  $P(i) \cong P(j)$  and this implies  $i = j$  by Lemma 1.7.2. The proof of (2) is analogous.  $\square$

#### 4. DYNKIN AND EUCLIDEAN DIAGRAMS

A finite graph arises from a quiver when one forgets the orientation of its arrows. In this section we classify finite graphs using properties of quadratic forms. For graphs of Dynkin or Euclidean type we study the corresponding root systems.

**4.1. Finite graphs.** Let  $\Gamma$  be a finite graph with set of vertices  $\{1, \dots, n\}$ . The finite number of edges joining two vertices  $i$  and  $j$  is denoted by  $d_{ij} = d_{ji}$ . The graph  $\Gamma$  induces a *symmetric bilinear form*

$$(-, -): \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad (e_i, e_j) = \begin{cases} -d_{ij} & \text{if } i \neq j, \\ 2 - 2d_{ii} & \text{if } i = j, \end{cases}$$

where  $e_i$  is the  $i$ th coordinate vector, and a *quadratic form*

$$q: \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad q(x) = \sum_{i=1}^n x_i^2 - \sum_{i < j} d_{ij} x_i x_j.$$

Note that  $\Gamma$ ,  $(-, -)$  and  $q$  determine each other, since  $q(x) = \frac{1}{2}(x, x)$  and  $(x, y) = q(x + y) - q(x) - q(y)$ . The *radical* of the form  $q$  is by definition the set

$$\text{rad } q = \{x \in \mathbb{Z}^n \mid (x, -) = 0\}.$$

A vector  $x \in \mathbb{Z}^n$  is *sincere* if  $x_i \neq 0$  for all  $i$ .

**Remark 4.1.1.** Let  $Q$  be a quiver whose underlying graph is  $\Gamma$ . Then the symmetric bilinear form (3.2.1) for  $Q$  coincides with the one defined for  $\Gamma$ .

**Definition 4.1.2.** Let  $q: \mathbb{Z}^n \rightarrow \mathbb{Z}$  be a quadratic form.

- (1)  $q$  is *positive definite* if  $q(x) > 0$  for all non-zero  $x \in \mathbb{Z}^n$ .
- (2)  $q$  is *positive semi-definite* if  $q(x) \geq 0$  for all  $x \in \mathbb{Z}^n$ .

**Lemma 4.1.3.** *Let  $\Gamma$  be connected and  $y \in \mathbb{Z}^n$  be a positive radical vector. Then  $y$  is sincere and  $q$  is positive semi-definite. For  $x \in \mathbb{Z}^n$ , we have*

$$q(x) = 0 \iff x \in \mathbb{Q}y \iff x \in \text{rad } q.$$

*Proof.* The assumption on  $y$  yields

$$(4.1.1) \quad 0 = (e_i, y) = (2 - 2d_{ii})y_i - \sum_{j \neq i} d_{ij}y_j \quad \text{for } 1 \leq i \leq n.$$

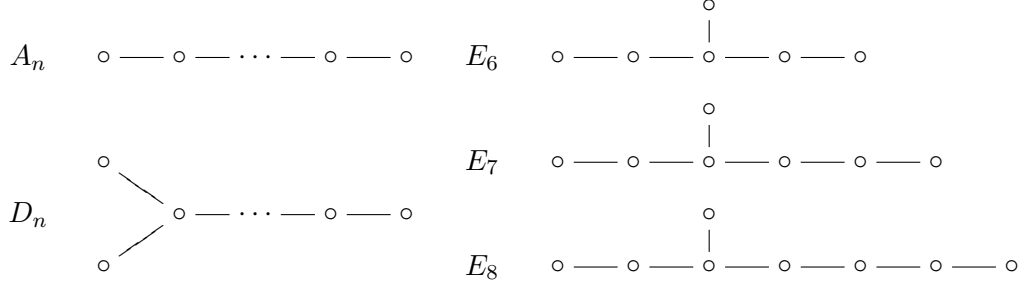
If  $y_i = 0$  then  $\sum_{j \neq i} d_{ij}y_j = 0$ , and since each term is non-negative we have  $y_j = 0$  whenever  $i$  and  $j$  are joined by an edge. It follows that  $y = 0$  since  $\Gamma$  is connected. This is a contradiction and therefore  $y$  is sincere. The next calculation shows that  $q$  is positive semi-definite. For  $x \in \mathbb{Z}^n$ , we have

$$\begin{aligned} q(x) &= \sum_i (2 - 2d_{ii})y_i \frac{1}{2y_i} x_i^2 - \sum_{i < j} d_{ij} x_i x_j \\ &= \sum_{i \neq j} d_{ij} \frac{y_j}{2y_i} x_i^2 - \sum_{i < j} d_{ij} x_i x_j \\ &= \sum_{i < j} d_{ij} \frac{y_j}{2y_i} x_i^2 - \sum_{i < j} d_{ij} x_i x_j + \sum_{i < j} d_{ij} \frac{y_i}{2y_j} x_j^2 \\ &= \sum_{i < j} d_{ij} \frac{y_i y_j}{2} \left( \frac{x_i}{y_i} - \frac{x_j}{y_j} \right)^2 \geq 0, \end{aligned}$$

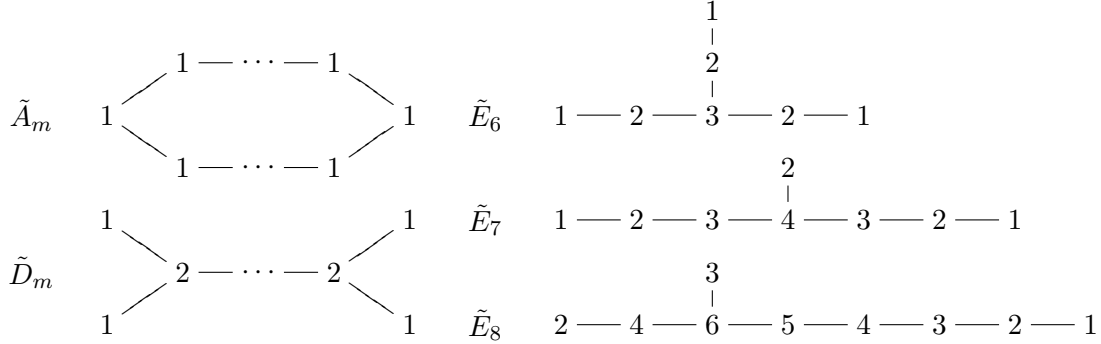
where the second equality follows from (4.1.1). If  $q(x) = 0$  then  $\frac{x_i}{y_i} = \frac{x_j}{y_j}$  whenever there is an edge joining  $i$  and  $j$ . Thus  $x \in \mathbb{Q}y$  since  $\Gamma$  is connected. If  $x \in \mathbb{Q}y$  then  $x \in \text{rad } q$  since  $y \in \text{rad } q$  by assumption. Finally,  $x \in \text{rad } q$  implies  $q(x) = 0$ .  $\square$

4.2. **The classification.** We list the Dynkin and Euclidean diagrams.

*Dynkin diagrams* (with  $n$  vertices).



*Euclidean diagrams* (with  $n = m + 1$  vertices). Each vertex  $i$  is marked with the value  $\delta_i$  of a vector  $\delta \in \mathbb{Z}^n$ . Let  $m \geq 0$  for  $\tilde{A}_m$  and  $m \geq 4$  for  $\tilde{D}_m$ .



**Theorem 4.2.1.** *Let  $\Gamma$  be a connected graph and  $q$  the corresponding quadratic form.*

- (1)  $\Gamma$  is a Dynkin diagram if and only if  $q$  is positive definite.
- (2)  $\Gamma$  is a Euclidean diagram if and only if  $q$  is positive semi-definite but not positive definite. In that case there is a unique positive vector  $\delta \in \mathbb{Z}^n$  with  $\text{rad } q = \mathbb{Z}\delta$ .

*Proof.* We proceed in three steps.

Step 1. *If  $\Gamma$  is Euclidean then  $q$  is positive semi-definite and  $\text{rad } q = \mathbb{Z}\delta$ .* The assertion follows from Lemma 4.1.3 once we have shown that  $\delta$  is a radical vector. This is done by inspection. If  $\Gamma$  has no loops or multiple edges we need to check that

$$0 = (e_i, \delta) = 2\delta_i - \sum_{\substack{j=1 \\ d_{ij} \neq 0}}^n \delta_j \quad \text{for } 1 \leq i \leq n.$$

Finally, since some  $\delta_i = 1$  we have

$$\text{rad } q = \mathbb{Q}\delta \cap \mathbb{Z}^n = \mathbb{Z}\delta.$$

Step 2. *If  $\Gamma$  is Dynkin then  $q$  is positive definite.* This follows from the first part because there exists a Euclidean diagram  $\tilde{\Gamma}$  such that  $\Gamma$  is obtained by deleting some vertex  $e$ . For  $\tilde{\Gamma}$  we have  $q(x) > 0$  for every non-zero vector  $x$  with  $x_e = 0$ .

Step 3. *If  $\Gamma$  is not Dynkin or Euclidean then  $q(x) < 0$  for some  $x \in \mathbb{Z}^n$ .* It is not difficult to find a Euclidean subgraph  $\Gamma'$  with radical vector  $\delta$ . Put  $x = \delta$  if the vertices of  $\Gamma'$  and  $\Gamma$  coincide. Otherwise let  $i$  be a vertex of  $\Gamma \setminus \Gamma'$ , but connected with  $\Gamma'$  by an edge, and take  $x = 2\delta + e_i$ .

□

**4.3. Roots.** Let  $\Gamma$  be a diagram that is Dynkin or Euclidean. We define

$$\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}$$

and a non-zero element of  $\Delta$  is called *root*.

**Proposition 4.3.1.** *Let  $\Gamma$  be Dynkin or Euclidean.*

- (1) *Each  $e_i$  is a root.*
- (2) *If  $x \in \Delta$  and  $y \in \text{rad } q$ , then  $-x, x + y \in \Delta$ .*
- (3) *Every root is positive or negative.*
- (4) *If  $\Gamma$  is Euclidean then  $\Delta/\text{rad } q$  is finite.*
- (5) *If  $\Gamma$  is Dynkin then  $\Delta$  is finite.*

*Proof.* (1) Clear.

(2) We have  $q(y \pm x) = q(y) + q(x) \pm (y, x) = q(x)$ .

(3) Let  $x$  be a root and write  $x = x^+ - x^-$  where  $x^+, x^- \geq 0$  and both have disjoint support. Then  $(x^+, x^-) \leq 0$  and

$$1 \geq q(x) = q(x^+) + q(x^-) - (x^+, x^-) \geq q(x^+) + q(x^-) \geq 0.$$

This implies  $q(x^+) = 0$  or  $q(x^-) = 0$ . Thus one of  $x^+$  and  $x^-$  is sincere if we assume that both vectors are non-zero. This is a contradiction and therefore  $x$  is positive or negative.

(4) Fix a vertex  $e$ . If  $x$  is a root with  $x_e = 0$ , then  $\delta - x$  and  $\delta + x$  are positive at  $e$ . Both vectors are roots by (2) and therefore positive by (3). Thus

$$\{x \in \Delta \mid x_e = 0\} \subseteq \{x \in \mathbb{Z}^n \mid -\delta \leq x \leq \delta\}$$

which is a finite set. If  $x \in \Delta$  then  $x - x_e \delta$  belongs to the finite set  $\{x \in \Delta \mid x_e = 0\}$ .

(5) There exists a Euclidean diagram  $\tilde{\Gamma}$  such that  $\Gamma$  is obtained by deleting some vertex  $e$ . A root  $x$  of  $\Gamma$  can be viewed as a root for  $\tilde{\Gamma}$  with  $x_e = 0$ . Thus the result follows from (4).  $\square$

**Lemma 4.3.2.** *Let  $Q$  be a quiver whose underlying graph is Dynkin or Euclidean. If  $x$  is a positive root and  $\sigma_i(x)$  is not positive, then  $x = e_i$ .*

*Proof.* The root  $\sigma_i(x)$  is not positive by assumption and therefore negative by Proposition 4.3.1. For each vertex  $j \neq i$ , we have  $\sigma_i(x)_j = x_j$  and therefore  $x_j = 0$ . Thus  $x = e_i$ .  $\square$

**4.4. The Coxeter transformation.** Let  $Q$  be a quiver without oriented cycles and fix an admissible ordering  $i_1, \dots, i_n$  of its vertices. The automorphism

$$c: \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \quad \text{with} \quad c(x) = \sigma_{i_n} \dots \sigma_{i_1}(x)$$

is called *Coxeter transformation*. The next lemma shows that  $c$  does not depend on the admissible numbering of the vertices.

**Lemma 4.4.1.** (1)  $c(\dim P(i)) = -\dim I(i)$  for every vertex  $i$ .

(2)  $\{\dim P(i) \mid i \in Q_0\}$  and  $\{\dim I(i) \mid i \in Q_0\}$  form two bases of  $\mathbb{Z}^n$ .

*Proof.* (1) We simplify the labeling of the vertices and let  $i_j = j$  for  $1 \leq j \leq n$ . Then  $\dim P(i) = \sigma_1 \dots \sigma_{i-1}(e_i)$  by Lemma 3.4.2 and we get

$$c(\dim P(i)) = c\sigma_1 \dots \sigma_{i-1}(e_i) = \sigma_n \dots \sigma_i(e_i) = -\sigma_n \dots \sigma_{i+1}(e_i) = -\dim I(i).$$

(2) We have

$$e_i = \dim P(i) - \sum_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} \dim P(t(\alpha)) = \dim I(i) - \sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} \dim I(s(\alpha)). \quad \square$$

**Lemma 4.4.2.** *Let  $x, y \in \mathbb{Z}^n$ .*

- (1)  $\langle \dim P(i), x \rangle = x_i = \langle x, \dim I(i) \rangle$  for each vertex  $i$ .
- (2)  $\langle x, y \rangle = -\langle y, c(x) \rangle = \langle c(x), c(y) \rangle$ .

*Proof.* (1) It is sufficient to check this for each standard basis vector  $x = e_j$ .

(2) It is sufficient to check this for  $x = \dim P(j)$  where  $j$  runs through all vertices; see Lemma 4.4.1. Using (1) and Lemma 4.4.1 we get

$$\langle \dim P(j), y \rangle = \langle y, \dim I(j) \rangle = \langle y, -c(\dim P(j)) \rangle. \quad \square$$

**Lemma 4.4.3.** *Let  $x \in \mathbb{Z}^n$ . Then  $c(x) = x$  if and only if  $x \in \text{rad } q$ .*

*Proof.* We have  $c(x) = x$  iff  $x_i = c(x)_i = \sigma_i(x)_i$  for all  $i$  iff  $(x, e_i) = 0$  for all  $i$  iff  $(x, -) = 0$ .  $\square$

From now on we assume that the underlying graph of  $Q$  is Dynkin or Euclidean. The map  $c$  induces a permutation of the finite set  $\Delta/\text{rad } q$ . Thus  $c^h$  is the identity on  $\Delta/\text{rad } q$  for some  $h > 0$ . In fact,  $c^h$  is the identity on  $\mathbb{Z}^n/\text{rad } q$  since  $e_i \in \Delta$  for all  $i$ .

**Lemma 4.4.4.** *Let  $Q$  be of Dynkin type and  $x \in \mathbb{Z}^n$ . Then there exists  $r \geq 0$  such that  $c^r(x)$  is not positive.*

*Proof.* The vector  $y = \sum_{r=0}^{h-1} c^r(x)$  is fixed by  $c$ , and hence  $y = 0$  by Lemma 4.4.3. Thus  $c^r(x)$  is not positive for some  $r \geq 0$ .  $\square$

**Lemma 4.4.5.** *Let  $Q$  be of Euclidean type and  $x \in \mathbb{Z}^n$ .*

- (1) If  $c^r(x) > 0$  for all  $r \in \mathbb{Z}$  then  $c^h(x) = x$ .
- (2) If  $c^h(x) = x$  then  $\langle \delta, x \rangle = 0$ .

*Proof.* (1) Suppose  $c^h(x) = x + v$  for some non-zero  $v \in \text{rad } q$ . An induction shows that  $c^{lh}(x) = x + lv$  for all  $l \in \mathbb{Z}$ . Thus one finds  $r$  such that  $c^r(x)$  is not positive since the vector  $v$  is sincere and positive or negative.

(2) The vector  $y = \sum_{r=0}^{h-1} c^r(x)$  is fixed by  $c$ , and hence  $y \in \mathbb{Z}\delta$  by Lemma 4.4.3. Now

$$0 = \langle \delta, y \rangle = \sum_{r=0}^{h-1} \langle \delta, c^r(x) \rangle = h \langle \delta, x \rangle$$

since  $c$  preserves the Euler form by Lemma 4.4.2. Thus  $\langle \delta, x \rangle = 0$ .  $\square$

## 5. FINITE REPRESENTATION TYPE

In this section we prove Gabriel's theorem. For quivers of Dynkin or Euclidean type we classify indecomposable representations in terms of their dimension vectors.

### 5.1. The Dynkin case.

**Theorem 5.1.1** (Gabriel). *Let  $Q$  be a quiver whose underlying graph is a Dynkin diagram. Then the assignment  $X \mapsto \dim X$  induces a bijection between the isomorphism classes of indecomposable representations of  $Q$  and the positive roots corresponding to the diagram of  $Q$ . In particular, there are only finitely many isomorphism classes of indecomposable representations.*

*Proof.* Choose an admissible ordering  $i_1, \dots, i_n$  for the vertices of  $Q$ . Suppose  $X$  is an indecomposable representation of  $Q$  with dimension vector  $x = \dim X$ . Let

$$\tau = \sigma_{i_s} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r$$

be the shortest expression such that  $\tau(x)$  is not positive, which exists by Lemma 4.4.4. Now we apply the reflection functors and use Lemma 3.3.3 to obtain

$$S_{i_{s-1}}^+ \dots S_{i_1}^+ (S_{i_n}^+ \dots S_{i_1}^+)^r X \cong S(i_s)$$

and therefore

$$X \cong (S_{i_1}^- \dots S_{i_n}^-)^r S_{i_1}^- \dots S_{i_{s-1}}^- S(i_s).$$

Thus

$$\dim X = (\sigma_{i_1} \dots \sigma_{i_n})^r \sigma_{i_1} \dots \sigma_{i_{s-1}} (e_{i_s})$$

is a positive root. The same argument shows for another indecomposable representation  $X'$  with  $\dim X' = \dim X$  that  $X' \cong X$ .

Now suppose  $x$  is a positive root. Let

$$\tau = \sigma_{i_s} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r$$

be the shortest expression such that  $\tau(x)$  is not positive, which exists by Lemma 4.4.4. We infer from Lemma 4.3.2 that

$$\sigma_{i_{s-1}} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r (x) = e_{i_s}.$$

Let

$$X = (S_{i_1}^- \dots S_{i_n}^-)^r S_{i_1}^- \dots S_{i_{s-1}}^- S(i_s).$$

Another iterated application of reflection functors shows that  $X$  is indecomposable with

$$\dim X = (\sigma_{i_1} \dots \sigma_{i_n})^r \sigma_{i_1} \dots \sigma_{i_{s-1}} (e_{i_s}) = x.$$

There are only finitely many isomorphism classes of indecomposable representations because the number of roots is finite by Proposition 4.3.1.  $\square$

**Remark 5.1.2.** The proof of Theorem 5.1.1 shows that for a quiver of Dynkin type all indecomposable representations are preprojective and preinjective. In fact, the proof simplifies a bit if one uses Coxeter functors.

**Remark 5.1.3.** Given a graph  $\Gamma$  without loops, the corresponding *Weyl group*  $W(\Gamma)$  is the group of automorphisms of  $\mathbb{Z}^n$  which is generated by the reflections  $\sigma_i$ . If  $\Gamma$  is a Dynkin diagram then the roots are precisely the vectors of the form  $\tau(e_i)$  with  $\tau \in W(\Gamma)$  and  $1 \leq i \leq n$ . Clearly, each  $\tau(e_i)$  is a root since the  $\sigma_i$  preserve the quadratic form  $q$ . Conversely, let  $q(x) = 1$ . Then the argument used in the proof of Theorem 5.1.1 shows that  $x$  is of the form  $\tau(e_i)$ .

**5.2. The defect.** Let  $Q$  be a quiver of Euclidean type. The *defect* of a vector  $x \in \mathbb{Z}^n$  is

$$\partial x = \langle \delta, x \rangle = -\langle x, \delta \rangle.$$

The *defect* of a representation  $X$  is  $\partial X = \partial \dim X$ .

**Proposition 5.2.1.** *Let  $X$  be an indecomposable representation.*

- (1)  $X$  is preprojective if and only if  $\partial X < 0$ .
- (2)  $X$  is preinjective if and only if  $\partial X > 0$ .
- (3)  $X$  is regular if and only if  $\partial X = 0$ .

*Proof.* First observe that for every representation  $X$  with  $C^r X \neq 0$  we have

$$\dim C^r X = c^r(\dim X)$$

by Lemma 3.3.3. Now let  $X = C^r P(i)$  be preprojective. Then

$$\partial X = -\langle c^r(\dim P(i)), \delta \rangle = -\langle \dim P(i), \delta \rangle = -\delta_i < 0$$

by Lemmas 4.4.2 and 4.4.3. Similarly preinjectives have positive defect. If  $X$  is regular then  $\partial X = 0$  by Lemma 4.4.5.  $\square$

### 5.3. The Euclidean case.

**Theorem 5.3.1.** *Let  $Q$  be a quiver without oriented cycles and suppose the underlying graph is a Euclidean diagram with  $n$  vertices. Then the assignment  $X \mapsto \dim X$  induces a bijection between the isomorphism classes of indecomposable preprojective or preinjective representations of  $Q$  and the positive roots with non-zero defect corresponding to the diagram of  $Q$ . The preprojective and preinjective indecomposables form  $2n$  countably infinite series  $C^{-r}P(i)$  and  $C^r I(i)$  ( $r \in \mathbb{N}_0, i \in Q_0$ ) of pairwise non-isomorphic representations.*

*Proof.* Choose an admissible ordering  $i_1, \dots, i_n$  for the vertices of  $Q$ . Suppose  $X$  is an indecomposable representation of  $Q$  with dimension vector  $x = \dim X$ . Let  $X \cong C^r P(i_s)$  be preprojective. We have  $\dim P(i_s) = \sigma_{i_1} \dots \sigma_{i_{s-1}}(e_{i_s})$  by Lemma 3.4.2 and therefore  $x = c^r \sigma_{i_1} \dots \sigma_{i_{s-1}}(e_{i_s})$  by Lemma 3.3.3. This is a positive root, and  $\partial x < 0$  by Proposition 5.2.1. A similar argument works if  $X$  is preinjective. The map  $X \mapsto \dim X$  is injective by Proposition 3.5.2.

Now suppose  $x$  is a positive root with  $\partial x \neq 0$ . We know from Lemma 4.4.5 that  $c^t(x)$  is not positive for some  $t \in \mathbb{Z}$ . Suppose first  $t > 0$ . Then there are  $1 \leq s \leq n$  and  $r \geq 0$  such that

$$\tau = \sigma_{i_s} \dots \sigma_{i_1}(\sigma_{i_n} \dots \sigma_{i_1})^r$$

is the shortest expression with  $\tau(x)$  not positive. We infer from Lemma 4.3.2 that

$$\sigma_{i_{s-1}} \dots \sigma_{i_1}(\sigma_{i_n} \dots \sigma_{i_1})^r(x) = e_{i_s}.$$

Let

$$X = (S_{i_1}^- \dots S_{i_n}^-)^r S_{i_1}^- \dots S_{i_{s-1}}^- S(i_s).$$

An iterated application of reflection functors and Lemma 3.3.3 shows that  $X$  is indecomposable with

$$\dim X = (\sigma_{i_1} \dots \sigma_{i_n})^r \sigma_{i_1} \dots \sigma_{i_{s-1}}(e_{i_s}) = x.$$

Moreover,  $X$  is preprojective since  $X \cong C^{-r}P(i_s)$  by Lemma 3.4.2. In case  $t < 0$  a similar argument gives a preinjective representation  $X$  with  $\dim X = x$ .

Finally we show that the preprojectives and preinjectives form  $2n$  countably infinite series of pairwise non-isomorphic representations. This follows essentially from Proposition 3.5.2. It remains to show that  $C^{-r}P(i) \neq 0$  and  $C^rI(i) \neq 0$  for all  $r \geq 0$ . But this follows from Proposition 3.4.3 because  $C^{-r}P(i) = 0$  would imply that  $P(i)$  is preinjective, contradicting the fact that preprojectives and preinjectives have different defect by Proposition 5.2.1. A similar argument works for  $C^rI(i)$ .  $\square$

**Proposition 5.3.2.** *Let  $Q$  be a quiver of Euclidean type  $\tilde{A}_n$  with  $n \geq 0$ . Then there are infinitely many isomorphism classes of indecomposable representations.*

*Proof.* We allow any orientation, in particular an oriented cycle, and fix an arrow  $\alpha_0$ . Now we define for each  $p \geq 1$  a representation  $X = X(p)$  as follows. Let  $X_i = k^p$  for each vertex  $i$ , let  $X_{\alpha_0} = J(p, 0)$  be the Jordan block of size  $p$  with eigenvalue 0 and let  $X_\alpha = \text{id}_{k^p}$  for every arrow  $\alpha \neq \alpha_0$ . Then  $\text{End}(X(p)) \cong k[t]/(t^p)$  and therefore  $X(p)$  is indecomposable.  $\square$

**Corollary 5.3.3** (Gabriel). *Let  $Q$  be a connected quiver. Then there are only finitely many isomorphism classes of indecomposable representations if and only if the underlying graph is a Dynkin diagram.*

*Proof.* If  $Q$  is of Dynkin type then the classification of the indecomposable representations in Theorem 5.1.1 shows that there are only finitely many. If  $Q$  is not of Dynkin type then  $Q$  has a Euclidean subquiver  $Q'$  which has infinitely many indecomposable representations by Theorem 5.3.1 and Proposition 5.3.2. Each representation  $X$  of  $Q'$  can be extended to a representation of  $Q$  by letting  $X_i = 0$  and  $X_\alpha = 0$  for all  $i \in Q_0$  and  $\alpha \in Q_1$  not in  $Q'$ . Thus  $Q$  has infinitely many pairwise non-isomorphic indecomposable representations.  $\square$

## 6. IRREDUCIBLE MORPHISMS

In this section we investigate the morphisms between two representations using the concept of an irreducible morphism. We think of irreducible morphisms as generators, and in some cases, all morphisms are sums of compositions of such irreducible morphisms.

**6.1. The radical.** Let  $X, Y$  be a pair of representations. Recall from Section 2.3 that the radical  $\text{Rad}(X, Y)$  is a subspace of  $\text{Hom}(X, Y)$ . We extend this definition recursively for each  $n \geq 0$  as follows. Let  $\text{Rad}^0(X, Y) = \text{Hom}(X, Y)$  and for  $n > 0$  let  $\text{Rad}^n(X, Y)$  be the set of morphisms  $\phi \in \text{Hom}(X, Y)$  which admit a factorisation  $\phi = \phi''\phi'$  with  $\phi' \in \text{Rad}(X, Z)$  and  $\phi'' \in \text{Rad}^{n-1}(Z, Y)$  for some representation  $Z$ . Note that  $\text{Rad}^1(X, Y) = \text{Rad}(X, Y)$ .

**Lemma 6.1.1.** *Let  $X, Y, Z$  be representations and  $m, n \geq 0$ .*

- (1)  $\text{Rad}^{n+1}(X, Y)$  is a subspace of  $\text{Rad}^n(X, Y)$ .
- (2) For each finite set of representations  $X_i$  and  $Y_j$ , we have

$$\text{Rad}^n\left(\bigoplus_i X_i, \bigoplus_j Y_j\right) = \bigoplus_{i,j} \text{Rad}^n(X_i, Y_j).$$

- (3) If  $\phi \in \text{Rad}^n(X, Y)$  and  $\psi \in \text{Rad}^m(Y, Z)$ , then  $\psi\phi \in \text{Rad}^{n+m}(X, Z)$ .

*Proof.* Use Lemma 2.3.1.  $\square$

**6.2. Irreducible morphisms.** Fix a morphism  $\phi: X \rightarrow Y$  between two representations.

The morphism  $\phi$  is called *split monomorphism* if there exists  $\phi': Y \rightarrow X$  with  $\phi'\phi = \text{id}_X$ , and  $\phi$  is called *split epimorphism* if there exists  $\phi'': Y \rightarrow X$  with  $\phi\phi'' = \text{id}_Y$ .

The morphism  $\phi$  is called *irreducible* if  $\phi$  is neither a split monomorphism nor a split epimorphism and if for any factorisation  $\phi = \phi''\phi'$  the morphism  $\phi'$  is a split monomorphism or the morphism  $\phi''$  is a split epimorphism.

**Lemma 6.2.1.** *An irreducible morphism is a monomorphism or an epimorphism.*

*Proof.* Let  $\phi: X \rightarrow Y$  be irreducible. Consider the canonical factorisation  $X \rightarrow \text{Im } \phi \rightarrow Y$ . If  $X \rightarrow \text{Im } \phi$  is a split monomorphism then  $\phi$  is a monomorphism. If  $\text{Im } \phi \rightarrow Y$  is a split epimorphism then  $\phi$  is an epimorphism.  $\square$

**Lemma 6.2.2.** *Let  $\phi: X \rightarrow Y$  be a morphism between two representations.*

- (1) *If  $X$  is indecomposable, then  $\phi \in \text{Rad}(X, Y)$  if and only if  $\phi$  is not a split mono.*
- (2) *If  $Y$  is indecomposable, then  $\phi \in \text{Rad}(X, Y)$  if and only if  $\phi$  is not a split epi.*
- (3) *If  $X$  and  $Y$  are indecomposable, then  $\phi \in \text{Rad}^1(X, Y) \setminus \text{Rad}^2(X, Y)$  if and only if  $\phi$  is irreducible.*

*Proof.* (1) We apply Lemma 2.3.1. Let  $Y = \bigoplus_{i=1}^r Y_i$  be a decomposition into indecomposable representations. Then  $\text{Rad}(X, Y) = \bigoplus_{i=1}^r \text{Rad}(X, Y_i)$  and we have  $\phi \in \text{Rad}(X, Y)$  if and only if each component  $\phi_i$  belongs to  $\text{Rad}(X, Y_i)$ . Thus  $\phi \notin \text{Rad}(X, Y)$  if and only if  $\phi_i$  is an isomorphism for some  $i$  if and only if  $\phi$  is a split monomorphism.

(2) This is the dual statement of (1).

(3) Combine (1) and (2).  $\square$

We define for indecomposable representations  $X, Y$

$$\text{Irr}(X, Y) = \text{Rad}^1(X, Y) / \text{Rad}^2(X, Y).$$

**Remark 6.2.3.** Let  $\phi: X \rightarrow Y$  be an irreducible morphism between indecomposable representations and let  $Y' = Y \oplus Y$ . Then the morphism  $(\phi, \phi): X \rightarrow Y'$  is not irreducible but belongs to  $\text{Rad}^1(X, Y') \setminus \text{Rad}^2(X, Y')$ .

**Proposition 6.2.4.** *Let  $X, Y$  be indecomposable representations and  $\text{Rad}^n(X, Y) = 0$  for some  $n$ . Then every non-isomorphism  $X \rightarrow Y$  is a sum of compositions of irreducible morphisms between indecomposable representations.*

*Proof.* Suppose  $\phi$  is not irreducible. Because  $\phi$  is not an isomorphism, there exists a factorisation  $\phi = \phi''\phi'$  with  $\phi' \in \text{Rad}(X, Z)$  and  $\phi'' \in \text{Rad}(Z, Y)$  for some representation  $Z$  by Lemma 6.2.2. Let  $Z = \bigoplus_{i=1}^r Z_i$  be a decomposition into indecomposable representations. Then  $\phi = \sum_{i=1}^r \phi_i''\phi_i'$  with  $\phi_i' \in \text{Rad}(X, Z_i)$  and  $\phi_i'' \in \text{Rad}(Z_i, Y)$  for all  $i$ . Next we factorise each non-irreducible  $\phi_i'$  and each non-irreducible  $\phi_i''$  into a composition of two radical morphisms. We continue and obtain in each step a finite sum  $\phi = \sum_i \phi_{in_i} \dots \phi_{i1}$  of compositions of radical morphisms between indecomposable representations. This process stops and all  $\phi_{ij}$  are irreducible because a composition of  $n$  radical morphisms is zero by our assumption on  $X$  and  $Y$ .  $\square$

**6.3. The Harada-Sai lemma.** The *length*  $\ell(X)$  of a representation  $X$  is the maximal number  $n$  such that there exists a chain of subrepresentations

$$0 = X_n \subsetneq \dots \subsetneq X_1 \subsetneq X_0 = X.$$

Note that we have  $\ell(X) = \sum_{i \in Q_0} \dim X_i$  and  $\ell(X) = \ell(U) + \ell(X/U)$  for every subrepresentation  $U \subseteq X$ .

**Lemma 6.3.1** (Harada-Sai). *Let  $\phi_i: X_i \rightarrow X_{i+1}$  with  $1 \leq i \leq 2^n - 1$  be a family of non-isomorphisms between indecomposable representations satisfying  $\ell(X_i) \leq n$  for all  $i$ . Then  $\phi_{2^n-1} \dots \phi_1 = 0$ .*

*Proof.* We show by induction that the length of the image of  $\phi_{2^m-1} \dots \phi_1$  is at most  $n - m$ . This is clear for  $m = 1$  since  $\phi_1$  is not an isomorphism. Let  $\phi' = \phi_{2^{m-1}-1} \dots \phi_1$ ,  $\phi = \phi_{2^m-1}$ , and  $\phi'' = \phi_{2^{m-1}} \dots \phi_{2^{m-1}+1}$ . By the inductive hypothesis, the length of  $\text{Im } \phi'$  and  $\text{Im } \phi''$  is at most  $n - m + 1$ . The assertion follows if either is strictly less. So suppose the images of  $\phi', \phi''$  and  $\phi''\phi\phi'$  each has length  $n - m + 1$ . Then  $\text{Ker}(\phi''\phi) \cap \text{Im } \phi' = 0$  and  $\text{Ker } \phi'' \cap \text{Im}(\phi\phi') = 0$ . On the other hand, we have

$$\begin{aligned} \ell(X_{2^m-1}) &= \ell(\text{Ker}(\phi''\phi)) + \ell(\text{Im}(\phi''\phi)) = \ell(\text{Ker}(\phi''\phi)) + \ell(\text{Im } \phi'), \\ \ell(X_{2^{m-1}+1}) &= \ell(\text{Ker } \phi'') + \ell(\text{Im } \phi'') = \ell(\text{Ker } \phi'') + \ell(\text{Im}(\phi\phi')). \end{aligned}$$

Therefore  $X_{2^m-1} = \text{Ker}(\phi''\phi) \oplus \text{Im } \phi'$  and  $X_{2^{m-1}+1} = \text{Ker } \phi'' \oplus \text{Im}(\phi\phi')$ . Since each is indecomposable,  $\phi''\phi$  is a monomorphism and  $\phi\phi'$  is an epimorphism. Thus  $\phi$  is an isomorphism, contrary to hypothesis.  $\square$

**Proposition 6.3.2.** *Suppose that the length of every indecomposable representation is bounded by  $n$ . Then  $\text{Rad}^{2^n-1}(X, Y) = 0$  for every pair  $X, Y$  of representations.*

*Proof.* We may assume that  $X$  and  $Y$  are indecomposable, by Lemma 6.1.1. An induction shows that any morphism in  $\text{Rad}^r(X, Y)$  can be written as a finite sum  $\sum_i \phi_{ir} \dots \phi_{i1}$  of compositions of radical morphisms between indecomposable representations. Thus  $\text{Rad}^{2^n-1}(X, Y) = 0$  by Lemma 6.3.1.  $\square$

## 7. MORPHISMS BETWEEN PREPROJECTIVE REPRESENTATIONS

We give a combinatorial description of all morphisms between preprojective representations.

Throughout this section  $Q$  denotes a quiver without oriented cycles.

**7.1. Irreducible morphisms between indecomposable projectives.** Given a pair  $i, j$  of vertices, each arrow  $\alpha: i \rightarrow j$  induces a morphism

$$\alpha^*: P(j) \longrightarrow P(i) \quad \text{with} \quad \alpha_i^* = k[Q(\alpha, l)] \quad (l \in Q_0).$$

Note that  $\alpha^*$  is a monomorphism since each map  $Q(\alpha, l)$  is injective. Taking all arrows starting at  $i$  (resp. ending at  $j$ ) we obtain two morphisms

$$\sigma(i): \bigoplus_{\alpha: i \rightarrow i'} P(i') \xrightarrow{(\alpha^*)} P(i) \quad \text{and} \quad \tau(j): P(j) \xrightarrow{(\alpha^*)} \bigoplus_{\alpha: j' \rightarrow j} P(j').$$

**Lemma 7.1.1.** *The morphism  $\sigma(i)$  is a monomorphism and its image is the unique maximal subrepresentation of  $P(i)$ .*

*Proof.* This is a direct consequence of the definition of  $P(i)$ . Note that a subrepresentation  $X \subseteq P(i)$  equals  $P(i)$  if and only if the trivial path  $\varepsilon_i$  belongs to  $X_i$ .  $\square$

**Lemma 7.1.2.** *For a morphism  $\phi: X \rightarrow P(i)$ , the following are equivalent:*

- (1)  $\phi \in \text{Rad}(X, P(i))$ .

- (2)  $\phi$  is not an epimorphism.  
(3)  $\phi$  admits a factorisation  $\phi = \sigma(i)\phi'$ .

*Proof.* (1)  $\Leftrightarrow$  (2): We have  $\phi \in \text{Rad}(X, P(i))$  if and only if  $\phi$  is not a split epimorphism, by Lemma 6.2.2. Thus we need to show that every epimorphism is a split epimorphism. So let  $\phi$  be an epimorphism. Then the map  $\phi_i$  is surjective and we find  $x \in X_i$  with  $\phi_i(x) = \varepsilon_i$ . It follows from Lemma 1.7.1 that there is a morphism  $\phi': P(i) \rightarrow X$  corresponding to  $x$ . Then we have  $\phi\phi' = \text{id}_{P(i)}$  and therefore  $\phi$  is a split epimorphism.

(2)  $\Leftrightarrow$  (3): Apply Lemma 7.1.1.  $\square$

**Lemma 7.1.3.** *Let  $i$  be a vertex and  $X$  an indecomposable representation.*

- (1) *If  $X \rightarrow P(i)$  is an irreducible morphism then there is an arrow  $i \rightarrow j$  such that  $X \cong P(j)$ .*  
(2) *Suppose  $P(i)$  is simple. If  $P(i) \rightarrow X$  is an irreducible morphism then there is an arrow  $j \rightarrow i$  such that  $X \cong P(j)$ .*

*Proof.* (1) Let  $\phi: X \rightarrow P(i)$  be irreducible. Then  $\phi \in \text{Rad}(X, P(i))$  by Lemma 6.2.2 and we obtain a factorisation  $\phi = \sigma(i)\phi'$  by Lemma 7.1.2. The morphism  $\sigma(i)$  is not a split epimorphism and therefore  $\phi'$  is a split monomorphism. Thus  $X \cong P(j)$  for some arrow  $i \rightarrow j$ .

(2) Let  $\phi: P(i) \rightarrow X$  be irreducible. We claim that  $\phi$  factors through  $\tau(i)$ . First observe that  $X \not\cong P(i) = S(i)$ . This implies

$$\xi: \bigoplus_{\alpha: i' \rightarrow i} X_{i'} \xrightarrow{(X_\alpha)} X_i$$

is an epimorphism, by Lemma 3.3.3. We obtain the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{\alpha: i' \rightarrow i} \text{Hom}(P(i'), X) & \xlongequal{\quad} & \text{Hom}(\bigoplus_{\alpha: i' \rightarrow i} P(i'), X) \xrightarrow{\text{Hom}(\tau(i), X)} \text{Hom}(P(i), X) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{\alpha: i' \rightarrow i} X_{i'} & \xrightarrow{\quad \xi \quad} & X_i \end{array}$$

where the vertical maps are taken from Lemma 1.7.1. Thus  $\text{Hom}(\tau(i), X)$  is surjective and therefore  $\phi$  factors through  $\tau(i)$ . Let  $\phi = \phi'\tau(i)$  be a factorisation. The morphism  $\tau(i)$  is not a split monomorphism and therefore  $\phi'$  is a split epimorphism. Thus  $X \cong P(j)$  for some arrow  $j \rightarrow i$ .  $\square$

We denote by  $Q_1(i, j)$  the set of arrows  $i \rightarrow j$  in  $Q$ .

**Lemma 7.1.4.** *The map sending an arrow  $\alpha: i \rightarrow j$  to  $\alpha^*$  induces an isomorphism*

$$f: k[Q_1(i, j)] \xrightarrow{\sim} \text{Irr}(P(j), P(i)).$$

*Proof.* The map  $f$  is well-defined because for each arrow  $\alpha: i \rightarrow j$  the morphism  $\alpha^*$  belongs to  $\text{Rad}(P(j), P(i))$  and induces therefore an element of  $\text{Irr}(P(j), P(i))$ .

To show that  $f$  is an epimorphism choose a radical morphism  $\phi: P(j) \rightarrow P(i)$ . Then  $\phi$  admits a factorisation  $\phi = \sigma(i)\phi'$  by Lemma 7.1.2. Let  $\phi' = (\phi'_\alpha)$  with  $\phi'_\alpha \in \text{Hom}(P(j), P(t(\alpha)))$ . If  $t(\alpha) = j$  then we have  $\phi'_\alpha = \lambda_\alpha \text{id}_{P(j)}$  for some  $\lambda_\alpha \in k$  because  $\text{End}(P(j)) \cong k$  by Lemma 1.7.3. If  $t(\alpha) \neq j$  then  $\phi'_\alpha \in \text{Rad}(P(j), P(t(\alpha)))$ . It follows that  $f(\sum_{\alpha \in Q_1(i, j)} \lambda_\alpha \alpha)$  and  $\phi$  represent the same element in  $\text{Irr}(P(j), P(i))$  since  $\phi - \sum_{\alpha \in Q_1(i, j)} \lambda_\alpha \alpha^* \in \text{Rad}^2(P(j), P(i))$ .

To show that  $f$  is a monomorphism choose  $x = \sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha$  with  $f(x) = 0$ . Thus  $\theta = \sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha^*$  belongs to  $\text{Rad}^2(P(j), P(i))$  and there is a factorization  $\theta = \phi\psi$  such that  $\phi$  and  $\psi$  are radical morphisms. We obtain a second factorisation  $\phi = \sigma(i)\phi'$  by Lemma 7.1.2 and consider  $\phi'\psi: P(j) \rightarrow \bigoplus_{\alpha: i \rightarrow i'} P(i')$ . Then  $(\phi'\psi)_\alpha = \lambda_\alpha \text{id}_{P(j)}$  if  $t(\alpha) = j$ , and  $(\phi'\psi)_\alpha = 0$  if  $t(\alpha) \neq j$ , since

$$\sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha^* = \sigma(i)\phi'\psi = \sum_{\alpha: i \rightarrow i'} \alpha^*(\phi'\psi)_\alpha$$

and  $\sigma(i)$  is a monomorphism. This implies  $\lambda_\alpha = 0$  for all  $\alpha: i \rightarrow j$  since  $(\phi'\psi)_\alpha$  is a radical morphism. Thus  $f$  is a monomorphism.  $\square$

Lemma 7.1.4 shows that for each arrow  $\alpha$  the morphism  $\alpha^*$  is irreducible.

**7.2. More irreducible morphisms.** We fix an arrow  $\alpha: i \rightarrow j$  in  $Q$  and construct another irreducible morphism

$$\alpha_*: P(i) \longrightarrow C^-P(j)$$

as follows. We may assume that  $1, \dots, n$  is an admissible numbering of the vertices of  $Q$ . Let  $\tilde{\alpha}: j \rightarrow i$  be the arrow in  $\tilde{Q} = \sigma_{i-1} \dots \sigma_1 Q$  corresponding to  $\alpha$ . This induces the morphism  $\tilde{\alpha}^*: \tilde{P}(i) \rightarrow \tilde{P}(j)$  between representations of  $\tilde{Q}$  and we define  $\alpha_* = S_1^- \dots S_{i-1}^- \tilde{\alpha}^*$ . Note that we can identify  $P(i) = S_1^- \dots S_{i-1}^- \tilde{P}(i)$  and  $C^-P(j) = S_1^- \dots S_{i-1}^- \tilde{P}(j)$  by Lemma 3.4.2, since  $\tilde{P}(i) = S(i)$  and  $\tilde{P}(j) = S_i^- \dots S_{j-1}^- S(j)$ .

### 7.3. Reflection functors and morphisms.

**Lemma 7.3.1.** *Let  $i$  be a sink and  $X, Y$  indecomposable representations not isomorphic to  $S(i)$ . Then  $S_i^+$  induces isomorphisms*

$$\text{Rad}^n(X, Y) \xrightarrow{\sim} \text{Rad}^n(S_i^+X, S_i^+Y) \quad \text{for } n \geq 0.$$

*In particular,  $S_i^+$  induces isomorphisms*

$$\text{Hom}(X, Y) \xrightarrow{\sim} \text{Hom}(S_i^+X, S_i^+Y) \quad \text{and} \quad \text{Irr}(X, Y) \xrightarrow{\sim} \text{Irr}(S_i^+X, S_i^+Y).$$

*Proof.* We use the natural morphism  $\iota_i Z: S_i^- S_i^+ Z \rightarrow Z$  (3.3.1) which is defined for any representation  $Z$ ; it is a split monomorphism by Lemma 3.3.2. Thus we can identify  $S_i^- S_i^+ X = X$  and  $S_i^- S_i^+ Y = Y$ . Using this identification the inverse for  $\text{Hom}(X, Y) \rightarrow \text{Hom}(S_i^+X, S_i^+Y)$  sends  $\psi \in \text{Hom}(S_i^+X, S_i^+Y)$  to  $S_i^- \psi$ . Now fix  $\phi \in \text{Hom}(X, Y)$ . Clearly,  $\phi$  is an isomorphism if and only if  $S_i^+ \phi$  is an isomorphism. Thus  $S_i^+$  induces a bijection

$$\text{Rad}^1(X, Y) \xrightarrow{\sim} \text{Rad}^1(S_i^+X, S_i^+Y).$$

Next we suppose  $\phi \in \text{Rad}^n(X, Y)$  and  $n > 1$ . Then  $\phi$  admits a factorisation  $\phi = \phi''\phi'$  with  $\phi' \in \text{Rad}^1(X, Z)$  and  $\phi'' \in \text{Rad}^{n-1}(Z, Y)$  for some representation  $Z$ . We know by induction that  $S_i^+ \phi' \in \text{Rad}^1(S_i^+X, S_i^+Z)$  and  $\phi'' \in \text{Rad}^{n-1}(S_i^+Z, S_i^+Y)$ . Thus  $S_i^+ \phi \in \text{Rad}^n(S_i^+X, S_i^+Y)$ . The same argument shows that  $S_i^-$  maps  $\text{Rad}^n(S_i^+X, S_i^+Y)$  to  $\text{Rad}^n(X, Y)$ . This establishes for all  $n > 1$  the isomorphism

$$\text{Rad}^n(X, Y) \xrightarrow{\sim} \text{Rad}^n(S_i^+X, S_i^+Y). \quad \square$$

There are a number of consequences.

**Proposition 7.3.2.** *Let  $X, Y$  be representations having no indecomposable projective direct summand. Then  $C^+$  induces an isomorphism  $\text{Hom}(X, Y) \xrightarrow{\sim} \text{Hom}(C^+X, C^+Y)$ .*

*Proof.* Combine Lemma 7.3.1 and Proposition 3.4.3.  $\square$

**Proposition 7.3.3.** *If  $X$  is an indecomposable preprojective or preinjective representation then  $\text{End}(X) \cong k$ .*

*Proof.* Combine Lemmas 1.7.3 and 7.3.1.  $\square$

**Proposition 7.3.4.** *Let  $X = C^r P(i)$  and  $Y = C^s P(j)$  be two indecomposable preprojective representations. Then we have*

$$\text{Irr}(X, Y) \cong \begin{cases} k[Q_1(j, i)] & \text{if } r = s, \\ k[Q_1(i, j)] & \text{if } r = s + 1, \\ 0 & \text{otherwise.} \end{cases}$$

*The isomorphism sends an arrow  $\alpha: j \rightarrow i$  to  $C^r \alpha^*$  and  $\beta: i \rightarrow j$  to  $C^r \beta_*$ .*

*Proof.* Suppose there exists an irreducible morphism  $X \rightarrow Y$ .

Let  $r \leq s$ . Then  $C^{r-s} P(i) \neq 0$  and we have

$$\text{Irr}(X, Y) \cong \text{Irr}(C^{r-s} P(i), P(j)) \cong k[Q_1(j, i)],$$

where the first isomorphism follows from Lemma 7.3.1, and the second follows from Lemmas 7.1.3 and 7.1.4. In particular,  $r = s$ .

A similar argument works if  $r > s$ . Then  $C^{s-r} P(j)$  is an indecomposable and non-projective representation. We may assume that  $1, \dots, n$  is an admissible numbering of the vertices of  $Q$  and we have

$$\begin{aligned} \text{Irr}(X, Y) &\cong \text{Irr}(P(i), C^{s-r} P(j)) \\ &\cong \text{Irr}(S_1^- \dots S_{i-1}^- S(i), S_1^- \dots S_{i-1}^- S_i^- \dots S_n^- C^{s-r+1} S_1^- \dots S_{j-1}^- S(j)) \\ &\cong \text{Irr}(\tilde{P}(i), \tilde{P}(j)) \cong k[Q_1(i, j)], \end{aligned}$$

where  $\tilde{P}(i) = S(i)$  and  $\tilde{P}(j) = S_i^- \dots S_{j-1}^- S(j)$  denote the indecomposable projective representations of  $\tilde{Q} = \sigma_{i-1} \dots \sigma_1 Q$  corresponding to  $i$  and  $j$  respectively. In particular,  $r = s + 1$ .

It is clear that this argument can be reversed. Thus we have a necessary and sufficient criterion for the existence of irreducible morphisms  $X \rightarrow Y$ .  $\square$

#### 7.4. Reflection functors and exact sequences.

**Lemma 7.4.1.** *Given a vertex  $i$  of  $Q$ , any exact sequence  $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$  induces exact sequences*

$$\begin{aligned} 0 \rightarrow S_i^+ X' \rightarrow S_i^+ X \rightarrow S_i^+ X'' \rightarrow X^+ \rightarrow 0 & \quad (i \text{ a sink}) \\ 0 \rightarrow X^- \rightarrow S_i^- X' \rightarrow S_i^- X \rightarrow S_i^- X'' \rightarrow 0 & \quad (i \text{ a source}) \end{aligned}$$

*such that  $X^+$  and  $X^-$  are direct sums of copies of  $S(i)$ .*

*Proof.* Apply Lemma A.3.  $\square$

**Lemma 7.4.2.** *Suppose  $Q$  has no oriented cycles and fix a vertex  $i$ . Then we have the following exact sequences:*

$$(7.4.1) \quad 0 \longrightarrow \bigoplus_{\alpha: i \rightarrow j} P(j) \xrightarrow{(\alpha^*)} P(i) \longrightarrow S(i) \longrightarrow 0$$

$$(7.4.2) \quad 0 \longrightarrow P(i) \xrightarrow{(\alpha^*)} \bigoplus_{\alpha: j \rightarrow i} P(j) \xrightarrow{(\alpha_*)} C^-P(i) \longrightarrow 0 \quad (i \text{ a sink})$$

$$(7.4.3) \quad 0 \longrightarrow P(i) \xrightarrow{\begin{bmatrix} (\alpha^*) \\ (\beta_*) \end{bmatrix}} \left( \bigoplus_{\alpha: j \rightarrow i} P(j) \right) \oplus \left( \bigoplus_{\beta: i \rightarrow j} C^-P(j) \right) \xrightarrow{[(\alpha_*) \ (C^- \beta^*)]} C^-P(i) \longrightarrow 0$$

*Proof.* For the first sequence, see Lemma 7.1.1.

The second sequence is obtained from the first as follows. Let  $i$  be a sink of  $Q$ . Then  $i$  is a source of  $\tilde{Q} = \sigma_i Q$ . Now use Lemma 7.4.1 and apply  $S_i^-$  to the sequence (7.4.1) for  $\tilde{Q}$  to obtain the sequence (7.4.2) for  $Q$ .

The third sequence is obtained from the second as follows. Assume for simplicity that  $1, \dots, n$  is an admissible labeling of the vertices of  $Q$ . Then  $i$  is a sink for  $\tilde{Q} = \sigma_{i-1} \dots \sigma_1 Q$  and we recall that  $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$  by Lemma 3.4.2. Now use again Lemma 7.4.1 and apply  $S_1^- \dots S_{i-1}^-$  to the sequence (7.4.2) for  $\tilde{Q}$  to obtain the sequence (7.4.3) for  $Q$ .  $\square$

**Lemma 7.4.3.** *Suppose  $Q$  has no oriented cycles. Given a vertex  $i$  and  $r \in \mathbb{Z}$ , we have*

$$(7.4.4) \quad \sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} C^r \alpha_* C^r \alpha^* + \sum_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} C^{r-1} \alpha^* C^r \alpha_* = 0.$$

*Proof.* The exactness of the sequence (7.4.3) implies  $[(\alpha_*) \ (C^- \beta^*)] \begin{bmatrix} (\alpha^*) \\ (\beta_*) \end{bmatrix} = 0$  and therefore

$$[(C^r \alpha_*) \ (C^{r-1} \beta^*)] \begin{bmatrix} (C^r \alpha^*) \\ (C^r \beta_*) \end{bmatrix} = C^r \left( [(\alpha_*) \ (C^- \beta^*)] \begin{bmatrix} (\alpha^*) \\ (\beta_*) \end{bmatrix} \right) = 0,$$

where  $\alpha$  and  $\beta$  run through all arrows with  $t(\alpha) = i = s(\beta)$ . The latter identity is precisely (7.4.4).  $\square$

**7.5. Morphisms between preprojective representations.** We define a new quiver  $\Gamma = \mathbb{Z}Q$  as follows. Let

$$\Gamma_0 = \{i[r] \mid i \in Q_0, r \in \mathbb{Z}\}.$$

For each arrow  $\alpha: i \rightarrow j$  in  $Q$  and  $r \in \mathbb{Z}$ , we have in  $\Gamma$  a pair of arrows

$$\alpha^*[r]: j[r] \longrightarrow i[r] \quad \text{and} \quad \alpha_*[r]: i[r] \longrightarrow j[r-1].$$

Thus

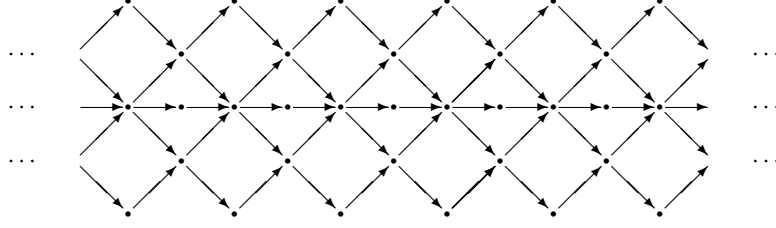
$$\Gamma_1 = \{\alpha^*[r] \mid \alpha \in Q_1, r \in \mathbb{Z}\} \cup \{\alpha_*[r] \mid \alpha \in Q_1, r \in \mathbb{Z}\}.$$

Observe that there exists a chain of irreducible morphisms

$$C^{r_1} P(i_1) \longrightarrow C^{r_2} P(i_2) \longrightarrow \dots \longrightarrow C^{r_m} P(i_m)$$

if and only if there is a path

$$i_1[r_1] \longrightarrow i_2[r_2] \longrightarrow \dots \longrightarrow i_m[r_m]$$

Example:  $\mathbb{Z}Q$  for  $Q$  of Dynkin type  $E_6$ 

in  $\Gamma$ . Here we assume that  $C^r P(i_j) \neq 0$  for  $1 \leq j \leq m$ . This observation follows from Proposition 7.3.4 and motivates the following construction.

We define for each pair of vertices  $i[r]$  and  $j[s]$  in  $\Gamma$  a linear map

$$(7.5.1) \quad \pi: k[\Gamma(i[r], j[s])] \longrightarrow \text{Hom}(C^r P(i), C^s P(j))$$

by induction on the path length. For a path  $\xi: i[r] \rightarrow j[s]$  in  $\Gamma$ , let

$$\pi(\xi) = \begin{cases} \text{id}_{C^r P(i)} & \text{if } \xi = \varepsilon_{i[r]}, \\ C^r \alpha^* & \text{if } \xi = \alpha^*[r], \\ C^r \alpha_* & \text{if } \xi = \alpha_*[r], \\ \pi(\xi_l) \dots \pi(\xi_1) & \text{if } \xi = \xi_l \dots \xi_1, l > 1. \end{cases}$$

**Proposition 7.5.1.** *Let  $C^r P(i)$  and  $C^s P(j)$  be two indecomposable preprojective representations. Then the linear map (7.5.1) is an epimorphism and its kernel is spanned by all elements of the form*

$$(7.5.2) \quad \sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=l}} \tau \alpha_*[t] \alpha^*[t] \sigma + \sum_{\substack{\alpha \in Q_1 \\ s(\alpha)=l}} \tau \alpha^*[t-1] \alpha_*[t] \sigma,$$

where  $l[t]$  runs through all vertices of  $\Gamma$  and  $\sigma, \tau$  run through all paths  $\sigma: i[r] \rightarrow l[t]$  and  $\tau: l[t-1] \rightarrow j[s]$  in  $\Gamma$ .

*Proof.* Lemma 7.4.3 implies that the kernel contains all elements of the form (7.5.2). For the complete proof, see the discussion of preprojective components in [15, 2.3.2].  $\square$

**7.6. The Dynkin case.** The following theorem summarises the structure of the morphisms between representations of quivers of Dynkin type. Note that in this case each indecomposable representation is preprojective and therefore of the form  $C^r P(i)$  for some vertex  $i$  and some  $r \leq 0$ .

**Theorem 7.6.1.** *Let  $Q$  be a quiver whose underlying graph is a Dynkin diagram. Suppose  $X \cong C^r P(i)$  and  $Y \cong C^s P(j)$  are two indecomposable representations.*

- (1) *We have  $\text{End}(X) \cong k$  and every non-isomorphism  $X \rightarrow Y$  is a sum of compositions of irreducible morphisms between indecomposable representations.*
- (2) *There exists an irreducible morphism  $X \rightarrow Y$  if and only if*
  - (a)  *$r = s$  and there exists an arrow  $j \rightarrow i$ , or*
  - (b)  *$r = s + 1$  and there exists an arrow  $i \rightarrow j$ .*

Given two irreducible morphisms  $\phi, \phi': X \rightarrow Y$ , there exists  $\lambda \in k$  with  $\phi' = \lambda\phi$ .

(3) There exists an integer  $d = d(X, Y) \geq 0$  such that

$$\text{Hom}(X, Y) = \text{Rad}^0(X, Y) = \dots = \text{Rad}^d(X, Y) \supseteq \text{Rad}^{d+1}(X, Y) = 0$$

*Proof.* (1) We have  $\text{End}(X) \cong k$  by Proposition 7.3.3. There are only finitely many isomorphism classes of indecomposable representations by Theorem 5.1.1. Thus the length of the indecomposable representations is bounded by some  $n$  and we have therefore  $\text{Rad}^{2^n-1}(X, Y) = 0$  by Proposition 6.3.2. It follows from Proposition 6.2.4 that every non-isomorphism  $X \rightarrow Y$  is a sum of compositions of irreducible morphisms between indecomposable representations.

(2) See Proposition 7.3.4. The last assertion follows from (3), because  $\text{Irr}(X, Y) \neq 0$  implies  $\text{Rad}^2(X, Y) = 0$ .

(3) Let  $d = d(i[r], j[s])$  be the length of a path  $\xi: i[r] \rightarrow j[s]$  in  $\Gamma$  and put  $d = 0$  if there is no such path. Observe that  $d$  does not depend on the choice of  $\xi$  because any two parallel paths in  $\Gamma$  have the same length. Here one uses that the underlying graph of  $Q$  is a tree. Now we apply (1). The assertion is clear if  $\text{Rad}(X, Y) = 0$ . A non-zero morphism  $\phi \in \text{Rad}(X, Y)$  can be written as a finite sum  $\phi = \sum_l \phi_{ld} \dots \phi_{l2} \phi_{l1}$  of compositions of irreducible morphisms between indecomposable representations, and each chain has length  $d$  since it corresponds to a path  $i[r] \rightarrow j[s]$  in  $\Gamma$ . Thus we have  $\phi \in \text{Rad}^d(X, Y)$  but  $\phi \notin \text{Rad}^{d+1}(X, Y)$ .  $\square$

## 8. THE INFINITE RADICAL

In this section we characterise the quivers of finite representation type in terms of morphisms between their representations. We use some global properties, in particular infinite chains of radical morphisms.

Throughout this section we fix a quiver  $Q$ .

### 8.1. Infinite chains of morphisms.

**Proposition 8.1.1.** *Let  $Q$  be a quiver of Euclidean type. Then there exists an infinite family of non-isomorphisms  $\phi_p: X_p \rightarrow X_{p+1}$ ,  $p \geq 1$ , between indecomposable representations such that  $\phi_n \dots \phi_1 \neq 0$  for all  $n \geq 1$ .*

*Proof.* Suppose first  $Q$  is of type  $\tilde{A}_n$ . We allow any orientation, in particular an oriented cycle. We have an infinite family of indecomposable representations  $X(p)$ ,  $p \geq 1$ , by Proposition 5.3.2. The construction of  $X(p)$  shows that the canonical inclusion  $k^p \rightarrow k^{p+1}$  induces a monomorphism  $\phi_p: X(p) \rightarrow X(p+1)$  for each  $p \geq 1$ .

Now suppose  $Q$  has no oriented cycles. We use for all  $i \in Q_0$  the monomorphism

$$\mu(i): P(i) \xrightarrow{\begin{bmatrix} (\alpha^*) \\ (\beta_*) \end{bmatrix}} \left( \bigoplus_{\alpha: j \rightarrow i} P(j) \right) \oplus \left( \bigoplus_{\beta: i \rightarrow j} C^-P(j) \right) = E(i)$$

in (7.4.3) and observe that  $C^r \mu(i)$  is a monomorphism for all  $r \leq 0$ . This follows from Lemma 7.4.1 and the fact that  $C^r P(i) \neq 0$  for all  $r \leq 0$ ; see Theorem 5.3.1. Now choose a vertex  $i_1$  of  $Q$ . Let  $X_1 = P(i_1)$  and denote by  $\chi_1: X_1 \rightarrow I$  a non-zero morphism to the indecomposable injective representation  $I = I(i_1)$ . Then  $\chi_1$  factors through  $\mu(i_1)$  by Remark 1.7.4, because  $\mu(i_1)$  is a monomorphism. Thus we can choose an indecomposable direct summand  $X_2 = C^{r_2} P(i_2)$  of  $E(i_1)$  corresponding to an arrow

$\alpha_1$  and a morphism  $\chi_2: X_2 \rightarrow I$  such that  $\chi_2\phi_1 \neq 0$  where  $\phi_1 = \mu(i_1)_{\alpha_1}$ . The morphism  $\chi_2$  factors through  $C^{r_2}\mu(i_2)$  because  $C^{r_2}\mu(i_2)$  is a monomorphism. Thus we can choose an indecomposable direct summand  $X_3 = C^{r_3}P(i_3)$  of  $C^{r_2}E(i_2)$  corresponding to an arrow  $\alpha_2$  and a morphism  $\chi_3: X_3 \rightarrow I$  such that  $\chi_3\phi_2\phi_1 \neq 0$  where  $\phi_2 = C^{r_2}\mu(i_2)_{\alpha_2}$ . We continue and obtain an infinite family of morphisms  $\phi_p: X_p \rightarrow X_{p+1}$  such that  $\phi_n \dots \phi_1 \neq 0$  for all  $n \geq 1$ .  $\square$

## 8.2. A characterisation of finite representation type.

**Theorem 8.2.1.** *For a quiver  $Q$ , the following are equivalent:*

- (1) *The number of isomorphism classes of indecomposable representations is finite.*
- (2) *There is a global bound for the length of every indecomposable representation.*
- (3) *We have  $\bigcap_{n \geq 0} \text{Rad}^n(X, Y) = 0$  for every pair  $X, Y$  of representations.*
- (4) *Given an infinite family of non-isomorphism  $\phi_i: X_i \rightarrow X_{i+1}$ ,  $i \geq 1$ , between indecomposable representations, there exists  $n \geq 1$  such that  $\phi_n \dots \phi_1 = 0$ .*

*Proof.* (1)  $\Rightarrow$  (2): Clear.

(2)  $\Rightarrow$  (3): Use Proposition 6.3.2.

(3)  $\Rightarrow$  (4): Suppose  $\psi_n = \phi_n \dots \phi_1 \neq 0$  for all  $n \geq 1$ . Then there exists  $r \geq 1$  such that  $\bigcup_{n \geq 1} \text{Ker } \psi_n = \text{Ker } \psi_r \neq X_1$  because  $X_1$  is finite dimensional. We let  $X = X_1/\text{Ker } \psi_r$  and denote by  $\phi: X \rightarrow X_{r+1}$  the canonical monomorphism. Then the composition  $\phi_n \dots \phi_{r+1}\phi$  is a monomorphism for all  $n > r$ . Choose an indecomposable injective representation  $I$  and a non-zero morphism  $\chi: X \rightarrow I$  which exist by Lemma 1.7.1. Then Remark 1.7.4 implies that  $\chi$  factors through  $\phi_n \dots \phi_{r+1}\phi$  for all  $n > r$ . Thus  $\chi$  belongs to  $\bigcap_{n \geq 0} \text{Rad}^n(X, I)$ .

(4)  $\Rightarrow$  (1): Suppose there are infinitely many indecomposable representations. Then  $Q$  contains an Euclidean subquiver  $Q'$  by Corollary 5.3.3. We obtain from Proposition 8.1.1 an infinite chain  $\phi_i: X_i \rightarrow X_{i+1}$  of non-isomorphisms between indecomposable representations of  $Q'$ , which we can extend to a chain of morphisms for  $Q$  as in the proof of Corollary 5.3.3.  $\square$

## 9. REGULAR REPRESENTATIONS

We study regular representations and concentrate on two particular quivers: the Jordan quiver and the Kronecker quiver. In both cases we provide complete classifications of all representations. Note that the Kronecker quiver is the easiest but also most important case among the quivers of Euclidean type without oriented cycles.

**9.1. Direction of morphisms.** Let  $Q$  be a quiver without oriented cycles. We call a representation *regular* if all its indecomposable direct summands are regular.

**Lemma 9.1.1.** *Let  $X, Y$  be indecomposable representations.*

- (1) *If  $Y$  is (pre)projective and  $X$  is not, then  $\text{Hom}(X, Y) = 0$ .*
- (2) *If  $Y$  is (pre)injective and  $X$  is not, then  $\text{Hom}(Y, X) = 0$ .*

*Proof.* (1) A preprojective representation  $Y$  is of the form  $C^r P(i)$  for some  $r \leq 0$  and some  $i \in Q_0$ . We fix a morphism  $\phi: X \rightarrow Y$  and need to show that  $\phi = 0$ .

Consider first the case  $r = 0$ , that is,  $Y$  is projective and  $X$  is not. We use induction on the length  $l = \ell(i)$  of the longest path in  $Q$  starting at  $i$ . Note that  $\phi$  is not an epimorphism, since every epimorphism splits by Remark 1.7.4. If  $l = 0$ , then  $P(i)$  is

simple and therefore  $\phi = 0$ . If  $l > 0$ , then  $\phi$  factors through  $\bigoplus_{\alpha: i \rightarrow j} P(j) \xrightarrow{(\alpha^*)} P(i)$  by Lemma 7.1.1. We have  $\ell(j) < l$  for each arrow  $i \rightarrow j$ , and therefore  $\phi = 0$ .

Now suppose that  $r < 0$  and that  $X$  is not preprojective. Thus we have  $X \cong C^r C^{-r} X$  by Proposition 3.4.3, and therefore

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(C^r C^{-r} X, Y) \cong \mathrm{Hom}(C^{-r} X, P(i)) = 0,$$

where the second isomorphism follows from Proposition 7.3.2 and the last identity follows from the first part of this proof. Thus  $\phi = 0$ .

(2) is dual to (1).  $\square$

**Remark 9.1.2.** Let  $X, Y$  be indecomposable representations such that  $X = C^r P(i)$  is preprojective and  $Y$  is not. Then

$$\mathrm{Hom}(C^r P(i), Y) \cong \mathrm{Hom}(P(i), C^{-r} Y) \cong (C^{-r} Y)_i.$$

**Lemma 9.1.3.** *Let  $Q$  be a quiver of Euclidean type. For any morphism  $\phi: X \rightarrow Y$  between regular representations, the representations  $\mathrm{Ker} \phi$ ,  $\mathrm{Coker} \phi$ , and  $\mathrm{Im} \phi$  are regular.*

*Proof.* We apply Lemma 9.1.1. The image  $\mathrm{Im} \phi$  is a subrepresentation of  $Y$  and admits therefore no preinjective direct summand. On the other hand,  $\mathrm{Im} \phi$  is a quotient of  $X$  and has therefore no preprojective direct summand. Thus  $\mathrm{Im} \phi$  is regular.

In order to show that  $\mathrm{Ker} \phi$  is regular, we compute the defect and apply Proposition 5.2.1. We have  $\partial \mathrm{Ker} \phi = \partial X - \partial \mathrm{Im} \phi = 0$  since  $\mathrm{Im} \phi$  is regular.  $\mathrm{Ker} \phi$  is a subrepresentation of  $X$  and admits therefore no preinjective direct summand. Thus any preprojective direct summand  $P$  of  $\mathrm{Ker} \phi$  would imply  $\partial \mathrm{Ker} \phi \leq \partial P < 0$ . It follows that  $\mathrm{Ker} \phi$  is regular, and the dual argument shows that  $\mathrm{Coker} \phi$  is regular.  $\square$

**9.2. Jordan quiver representations.** We study the representations of the following Jordan quiver



and assume throughout that the field  $k$  is algebraically closed. By definition, such a representation is a pair  $(V, \phi)$  consisting of a vector space  $V$  and an endomorphism  $\phi: V \rightarrow V$ . For each integer  $p \geq 1$  and  $\lambda \in k$ , we have the representation

$$J_{p,\lambda} = (k^p, J(p, \lambda)) \quad \text{where} \quad J(p, \lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

denotes the Jordan block of size  $p$  with eigenvalue  $\lambda$  (sending a standard basis vector  $e_i$  to  $\lambda e_i + e_{i-1}$ ).

**Theorem 9.2.1** (Jordan normal form). *The finite dimensional indecomposable representations of the Jordan quiver are, up to isomorphism, precisely the representations  $J_{p,\lambda}$  with  $p \geq 1$  and  $\lambda \in k$ .  $\square$*

Fix  $\lambda \in k$ . For each pair of integers  $p, q \geq 1$ , we define a *standard morphism*

$$\phi_{p,q}: J_{p,\lambda} \longrightarrow J_{q,\lambda} \quad \text{with} \quad \phi_{p,q}(e_i) = \begin{cases} e_i & \text{if } p \leq q, \\ e_{i-(p-q)} & \text{if } p > q, \end{cases}$$

where  $\{e_1, \dots, e_p\}$  denotes the standard basis of  $k^p$  and  $e_i = 0$  for  $i \leq 0$ . If  $p > q$ , these morphisms induce an exact sequence

$$0 \longrightarrow J_{p-q, \lambda} \xrightarrow{\phi_{p-q, p}} J_{p, \lambda} \xrightarrow{\phi_{p, q}} J_{q, \lambda} \longrightarrow 0.$$

Let us identify for each  $q < p$  the representation  $J_{q, \lambda}$  with the image of  $\phi_{q, p}$ . Thus we obtain a chain of subrepresentations

$$J_{1, \lambda} \subseteq J_{2, \lambda} \subseteq \dots \subseteq J_{p-1, \lambda} \subseteq J_{p, \lambda}.$$

A representation is called *uniserial* if its subrepresentations are linearly ordered, that is, for each pair of subrepresentations  $U, V$  one has  $U \subseteq V$  or  $V \subseteq U$ .

**Lemma 9.2.2.** *Each representation  $J_{p, \lambda}$  is uniserial. More precisely, each proper subrepresentation is of the form  $J_{q, \lambda}$  for some  $1 \leq q < p$ , and  $J_{p, \lambda}/J_{q, \lambda} \cong J_{p-q, \lambda}$ .*

*Proof.* The proof is by induction on  $p$ . The assertion is clear for  $p = 1$ ; so let  $p > 1$ . Each proper  $J(p, \lambda)$ -invariant subspace of  $k^p$  contains  $e_1$ . Thus each proper subrepresentation of  $J_{p, \lambda}$  contains  $J_{1, \lambda}$ , which is the kernel of  $\phi_{p, p-1}$ . The representation  $J_{p-1, \lambda}$  is uniserial and the epimorphism  $\phi_{p, p-1}$  induces an inclusion preserving bijection between the subrepresentations of  $J_{p, \lambda}$  containing  $J_{1, \lambda}$  and all subrepresentations of  $J_{p-1, \lambda}$ . This yields the description of all subrepresentations of  $J_{p, \lambda}$ . The isomorphism  $J_{p, \lambda}/J_{q, \lambda} \xrightarrow{\sim} J_{p-q, \lambda}$  is induced by  $\phi_{p, p-q}$ .  $\square$

**Lemma 9.2.3.** *Fix two representations  $J_{p, \lambda}$  and  $J_{q, \mu}$ .*

- (1) *Let  $\lambda = \mu$ . Then  $\{\phi_{i, q} \phi_{p, i} \mid 1 \leq i \leq \min(p, q)\}$  is a basis for  $\text{Hom}(J_{p, \lambda}, J_{q, \mu})$ .*
- (2) *Let  $\lambda \neq \mu$ . Then  $\text{Hom}(J_{p, \lambda}, J_{q, \mu}) = 0$  and  $\text{Ext}(J_{p, \lambda}, J_{q, \mu}) = 0$ .*

*Proof.* (1) Let  $\lambda = \mu$  and set  $\Phi_{p, q} = \{\phi_{i, q} \phi_{p, i} \mid 1 \leq i \leq \min(p, q)\}$ . The proof is by induction on  $q$ . The assertion is clear for  $q = 1$ ; so let  $q > 1$ . The exact sequence

$$0 \longrightarrow J_{q-1, \mu} \xrightarrow{\phi_{q-1, q}} J_{q, \mu} \xrightarrow{\phi_{q, 1}} J_{1, \mu} \longrightarrow 0$$

induces an exact sequence

$$(9.2.1) \quad 0 \longrightarrow \text{Hom}(J_{p, \lambda}, J_{q-1, \mu}) \xrightarrow{(J_{p, \lambda}, \phi_{q-1, q})} \text{Hom}(J_{p, \lambda}, J_{q, \mu}) \xrightarrow{(J_{p, \lambda}, \phi_{q, 1})} \text{Hom}(J_{p, \lambda}, J_{1, \mu})$$

by Lemma A.1. The map  $(J_{p, \lambda}, \phi_{q-1, q})$  takes the basis  $\Phi_{p, q-1}$  of  $\text{Hom}(J_{p, \lambda}, J_{q-1, \mu})$  to  $\Phi = \{\phi_{i, q} \phi_{p, i} \mid 1 \leq i \leq \min(p, q-1)\}$  which is a subset of  $\Phi_{p, q}$ . Using the exactness of (9.2.1), it follows that  $\Phi_{p, q} = \Phi \cup \{\phi_{p, q}\}$  is a basis of  $\text{Hom}(J_{p, \lambda}, J_{q, \mu})$ .

(2) The proof goes by induction on  $p$  and  $q$ , using for  $\text{Hom}(-, -)$  the sequence (9.2.1) and its contravariant counterpart. For  $\text{Ext}(-, -)$  one uses Lemma A.4. The case  $p = 1 = q$  is clear.  $\square$

**Lemma 9.2.4.** *For a morphism  $\phi: J_{p, \lambda} \rightarrow J_{q, \lambda}$ , the following are equivalent:*

- (1) *The morphism  $\phi$  is irreducible.*
- (2) *The representation  $\text{Ker } \phi \oplus \text{Coker } \phi$  is simple.*
- (3)  *$|p - q| = 1$  and  $\phi$  is a monomorphism or an epimorphism.*

*Proof.* (1)  $\Rightarrow$  (2): An irreducible morphism is either a monomorphism or an epimorphism. It suffices to discuss the case that  $\phi$  is an epimorphism; the other case is dual. If  $\text{Ker } \phi$  is not simple and  $S \subseteq \text{Ker } \phi$  is a simple subrepresentation, then  $\phi$  can be written as composite  $J_{p, \lambda} \rightarrow J_{p, \lambda}/S \rightarrow J_{q, \lambda}$  of two proper epimorphisms. This is a contradiction, and therefore  $\text{Ker } \phi \oplus \text{Coker } \phi$  is simple.

(2)  $\Rightarrow$  (3): Clear.

(3)  $\Rightarrow$  (1): It suffices to consider an epimorphism  $J_{q+1,\lambda} \rightarrow J_{q,\lambda}$ ; the dual argument works for a monomorphism  $J_{p,\lambda} \rightarrow J_{p+1,\lambda}$ . Let  $J_{q+1,\lambda} \xrightarrow{\alpha} X \xrightarrow{\beta} J_{q,\lambda}$  be a factorisation and fix a decomposition  $X = \bigoplus_i X_i$  into indecomposable representations. Then  $\beta_{i_0} \alpha_{i_0}$  is an epimorphism for at least one index  $i_0$ . It follows that  $X_{i_0} = J_{r,\lambda}$  for some  $r \geq q$ . If  $r = q$ , then  $\beta_{i_0}$  is an isomorphism, and therefore  $\beta$  is a split epimorphism. Otherwise, we obtain a factorisation  $J_{q+1,\lambda} \xrightarrow{\alpha_{i_0}} X_{i_0} \xrightarrow{\beta'_{i_0}} J_{q+1,\lambda} \xrightarrow{\phi_{q+1,q}} J_{q,\lambda}$  of the epimorphism  $\beta_{i_0} \alpha_{i_0}$ . It follows that  $\beta'_{i_0} \alpha_{i_0}$  is an epimorphism and hence an isomorphism. Thus  $\alpha$  is a split monomorphism.  $\square$

**9.3. Kronecker quiver representations.** We study the representations of the following Kronecker quiver

$$1 \rightrightarrows 2$$

and assume throughout that the field  $k$  is algebraically closed. By definition, such a representation is a pair of linear maps between two vector spaces. Let us list the indecomposable representations.

For each integer  $r \geq 0$ , there are the preprojective and preinjective representations:

$$P_r : k^r \begin{array}{c} \xrightarrow{[\text{id}]} \\ \xrightarrow{[0]} \end{array} k^{r+1} \quad I_r : k^{r+1} \begin{array}{c} \xrightarrow{[\text{id} \ 0]} \\ \xrightarrow{[0 \ \text{id}]} \end{array} k^r$$

where in each block matrix  $\text{id} = \text{id}_{k^r}$ . We have  $C^{-r}P(1) = P_{2r+1}$  and  $C^{-r}P(2) = P_{2r}$ , while  $C^rI(1) = I_{2r}$  and  $C^rI(2) = I_{2r+1}$ ; see Theorem 5.3.1.

The regular representations are indexed by points of the *projective line*  $\mathbb{P}^1(k)$ , which are by definition pairs  $(\lambda_0 : \lambda_1)$  of elements in  $k$  different from  $(0 : 0)$  and subject to the relation  $(\lambda_0 : \lambda_1) = (\alpha\lambda_0 : \alpha\lambda_1)$  for all  $\alpha \in k$ ,  $\alpha \neq 0$ .

For each integer  $p \geq 1$  and  $\lambda \in \mathbb{P}^1(k)$ , we consider the representations

$$k^p \begin{array}{c} \xrightarrow{J(p,\lambda_0)} \\ \xrightarrow{\text{id}} \end{array} k^p \quad \text{for } \lambda = (\lambda_0 : 1), \text{ and } \quad k^p \begin{array}{c} \xrightarrow{\text{id}} \\ \xrightarrow{J(p,\lambda_1)} \end{array} k^p \quad \text{for } \lambda = (1 : \lambda_1),$$

where  $J(p, \mu)$  denotes the Jordan block of size  $p$  with eigenvalue  $\mu$ . Observe that both representations are isomorphic in case  $\lambda_0 \neq 0 \neq \lambda_1$ . To see this, note that the first representation is isomorphic to

$$k^p \begin{array}{c} \xrightarrow{\text{id}} \\ \xrightarrow{J(p,\lambda_0)^{-1}} \end{array} k^p,$$

and that  $J(p, \lambda_0)^{-1} \cong J(p, \lambda_1)$  since both endomorphisms are indecomposable with same eigenvalue. Thus we denote this representation by  $R_{p,\lambda}$ , and it is well-defined up to an isomorphism.

**Theorem 9.3.1** (Kronecker). *The representations  $P_r, I_r$  ( $r \geq 0$ ), and  $R_{p,\lambda}$  ( $p \geq 1$  and  $\lambda \in \mathbb{P}^1(k)$ ) form, up to isomorphism, a complete list of finite dimensional indecomposable representations of the Kronecker quiver.*

*Proof.* The preprojective and preinjective representations have been classified in Theorem 5.3.1, while the regular ones are classified further below in Proposition 9.3.4.  $\square$

Fix a point  $\lambda \in \mathbb{P}^1(k)$  with  $\lambda = (\lambda_0 : 1)$  or  $\lambda = (1 : \lambda_1)$ . Then for each pair  $p, q \geq 1$  of integers, we have an obvious bijection

$$\mathrm{Hom}(J_{p,\lambda_i}, J_{q,\lambda_i}) \xrightarrow{\sim} \mathrm{Hom}(R_{p,\lambda}, R_{q,\lambda}).$$

In particular, a standard morphism  $\phi_{p,q}: J_{p,\lambda_i} \rightarrow J_{q,\lambda_i}$  induces another standard morphism  $R_{p,\lambda} \rightarrow R_{q,\lambda}$  which we denote by  $\psi_{p,q}$ .

**Lemma 9.3.2.** *Fix two representations  $R_{p,\lambda}$  and  $R_{q,\mu}$ .*

- (1) *Let  $\lambda = \mu$ . Then  $\{\psi_{i,q}\psi_{p,i} \mid 1 \leq i \leq \min(p, q)\}$  is a basis for  $\mathrm{Hom}(R_{p,\lambda}, R_{q,\mu})$ .*
- (2) *Let  $\lambda \neq \mu$ . Then  $\mathrm{Hom}(R_{p,\lambda}, R_{q,\mu}) = 0$  and  $\mathrm{Ext}(R_{p,\lambda}, R_{q,\mu}) = 0$ .*

*Proof.* The case  $\lambda = \mu$  is clear from Lemma 9.2.3. For  $\lambda \neq \mu$ , the same induction argument as in Lemma 9.2.3 can be used. It remains to check the case  $p = 1 = q$ . It is easily checked that  $\mathrm{Hom}(R_{1,\lambda}, R_{1,\mu}) = 0$ . Thus we consider an exact sequence  $0 \rightarrow R_{1,\mu} \rightarrow E \rightarrow R_{1,\lambda} \rightarrow 0$  and need to show that it splits. Suppose first that  $\lambda_1 \neq 0 \neq \mu_1$ . We may assume that  $\lambda_1 = 1 = \mu_1$  and obtain the following commutative diagram of linear maps with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \longrightarrow & k^2 & \longrightarrow & k & \longrightarrow & 0 \\ & & \mu_0 \downarrow \downarrow \mathrm{id} & & \phi_0 \downarrow \downarrow \phi_1 & & \lambda_0 \downarrow \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & k & \longrightarrow & k^2 & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

It follows from Lemma A.3 that  $\phi_1$  is an isomorphism, and we may assume that  $\phi_1 = \mathrm{id}$ . This yields an exact sequence  $0 \rightarrow J_{1,\mu_0} \rightarrow E_0 \rightarrow J_{1,\lambda_0} \rightarrow 0$  of Jordan quiver representations which splits by Lemma 9.2.3, since  $\lambda_0 \neq \mu_0$ . Thus  $\eta$  splits. The case  $\lambda_0 \neq 0 \neq \mu_0$  is analogous, and the remaining case  $\{\lambda, \mu\} = \{(1 : 0), (0 : 1)\}$  is easy.  $\square$

**Lemma 9.3.3.** *Any indecomposable regular representation admits a subrepresentation that is isomorphic to  $R_{1,\lambda}$  for some  $\lambda \in \mathbb{P}^1(k)$ .*

*Proof.* Suppose the representation  $X$  is given by a pair of linear maps  $\phi, \psi: V \rightarrow W$ . Observe that  $\dim V = \dim W$  since  $X$  is regular. If  $\phi$  is bijective, then we may assume it is the identity so that  $\psi$  is a Jordan block, say  $\psi = J(p, \lambda_1)$ . This implies  $X \cong R_{p,\lambda}$ , where  $\lambda = (1 : \lambda_1)$ . Thus  $X$  admits  $R_{1,\lambda}$  as a subrepresentation.

Now suppose that  $\phi$  is not bijective and put  $K = \mathrm{Ker} \phi \cap \mathrm{Ker} \psi$ . Then  $K \rightrightarrows 0$  is a subrepresentation of  $X$  and the inclusion is a split monomorphism. Thus  $K = 0$ , and therefore  $X$  admits  $R_{1,\lambda}$  with  $\lambda = (0 : 1)$  as a subrepresentation.  $\square$

**Proposition 9.3.4.** *The finite dimensional indecomposable and regular representations of the Kronecker quiver are, up to isomorphism, precisely the representations  $R_{p,\lambda}$  with  $p \geq 1$  and  $\lambda \in \mathbb{P}^1(k)$ .*

*Proof.* Fix an indecomposable regular representation  $X$  that is given by a pair of linear maps  $\phi, \psi: V \rightarrow W$ . The proof goes by induction on the dimension of  $X$ . Lemma 9.3.3 yields an exact sequence

$$(9.3.1) \quad 0 \longrightarrow R_{1,\lambda} \longrightarrow X \longrightarrow Y \longrightarrow 0$$

for some  $\lambda \in \mathbb{P}^1(k)$ . If  $Y = 0$ , then  $X \cong R_{1,\lambda}$ . Thus we assume  $Y \neq 0$  and fix a decomposition  $Y = \bigoplus_{i=1}^r Y_i$  into indecomposable representations. Note that  $Y$  is

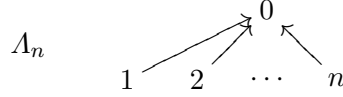
regular by Lemma 9.1.3, so that each  $Y_i$  is of the form  $R_{q_i, \mu_i}$  for some pair  $q_i, \mu_i$ . The sequence (9.3.1) induces for each  $i$  an exact sequence

$$0 \longrightarrow R_{1, \lambda} \longrightarrow X_i \longrightarrow R_{q_i, \mu_i} \longrightarrow 0,$$

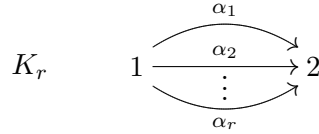
where  $X_i$  denotes the preimage of  $Y_i$  under the epimorphism  $X \rightarrow Y$ . Note that this sequence does not split since  $X$  is indecomposable. Thus  $\mu_i = \lambda$  for all  $i$  by Lemma 9.3.2. We may assume  $\lambda = (\lambda_0 : 1)$  or  $\lambda = (1 : \lambda_1)$ , and therefore one of  $\phi$  and  $\psi$  can be chosen to be the identity, using Lemma A.3 as in the proof of Lemma 9.3.2. It follows that the other map is a Jordan block, say  $J(p, \lambda_i)$ , and therefore  $X \cong R_{p, \lambda}$ .  $\square$

## 10. WILD PHENOMENA

We consider the following  $n$ -subspace quiver:



The representations of  $A_n$  are basically configurations of  $n$  subspaces of a fixed vector space. For  $n \leq 3$ , the underlying diagram is of Dynkin type, while it is of Euclidean type for  $n = 4$ . In this section we demonstrate some wild phenomenon for  $n \geq 5$ . The same phenomenon occurs for the  $r$ -Kronecker quiver



in case  $r \geq 3$ .

Throughout this section we fix a quiver  $Q$ .

**10.1. Total representations.** We assign to each representation  $X$  of  $Q$  its *total representation*, that is, a single vector space together with a distinguished set of endomorphisms. Put  $\bar{X} = \bigoplus_{i \in Q_0} X_i$  and denote for each  $i \in Q_0$  by  $\bar{X}_i$  the canonical projection  $\bar{X} \twoheadrightarrow X_i \hookrightarrow \bar{X}$ . For  $\alpha \in Q_1$  let  $\bar{X}_\alpha$  denote the linear map  $\bar{X} \twoheadrightarrow X_{s(\alpha)} \xrightarrow{X_\alpha} X_{t(\alpha)} \hookrightarrow \bar{X}$ . For a morphism  $\phi: X \rightarrow Y$  of representations, let  $\bar{\phi} = (\phi_i)_{i \in Q_0}$  be the induced linear map  $\bar{X} \rightarrow \bar{Y}$ .

**Lemma 10.1.1.** *Sending a representation to its total representation induces a bijection between all representations of  $Q$  and all families  $(V, \phi_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$  of vector spaces  $V$  with endomorphisms  $\phi_i, \phi_\alpha: V \rightarrow V$  satisfying*

$$\sum_{i \in Q_0} \phi_i = \text{id}_V, \quad \phi_i \phi_j = \delta_{ij} \phi_i \quad (i, j \in Q_0), \quad \text{and} \quad \phi_{t(\alpha)} \phi_\alpha \phi_{s(\alpha)} = \phi_\alpha \quad (\alpha \in Q_1).$$

*Proof.* The inverse map sends a family  $(V, \phi_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$  to the representation  $X$  with  $X_i = \text{Im } \phi_i$  for  $i \in Q_0$  and  $X_\alpha = (X_{s(\alpha)} \hookrightarrow V \xrightarrow{\phi_\alpha} V \twoheadrightarrow X_{t(\alpha)})$  for  $\alpha \in Q_1$ .  $\square$

**Lemma 10.1.2.** *The map  $\text{Hom}(X, Y) \rightarrow \text{Hom}(\bar{X}, \bar{Y})$  sending a morphism  $\phi$  to  $\bar{\phi}$  identifies  $\text{Hom}(X, Y)$  with the subspace of all linear maps  $\psi: \bar{X} \rightarrow \bar{Y}$  satisfying*

$$\psi \bar{X}_i = \bar{Y}_i \psi \quad (i \in Q_0) \quad \text{and} \quad \psi \bar{X}_\alpha = \bar{Y}_\alpha \psi \quad (\alpha \in Q_1).$$

*Proof.* The inverse map sends  $\psi: \bar{X} \rightarrow \bar{Y}$  to the family  $(X_i \mapsto \bar{X} \xrightarrow{\psi} \bar{Y} \rightarrow Y_i)_{i \in Q_0}$ .  $\square$

**10.2. A representation embedding.** A functor  $F: \text{Rep}(Q, k) \rightarrow \text{Rep}(Q', k)$  is called *embedding* if  $F$  induces a bijection  $\text{Hom}(X, Y) \xrightarrow{\sim} \text{Hom}(FX, FY)$  for each pair of representations  $X, Y$  of  $Q$ .

We construct an embedding  $F_Q: \text{Rep}(Q, k) \rightarrow \text{Rep}(K_3, k)$  as composite of two embeddings

$$\text{Rep}(Q, k) \xrightarrow{E} \text{Rep}(\Gamma, k) \xrightarrow{F} \text{Rep}(K_3, k)$$

with intermediate quiver

$$\Gamma \quad \sigma \begin{array}{c} \curvearrowright \\ \circ \\ \curvearrowleft \end{array} \tau.$$

The embedding  $F_Q$  shows that the representations of  $K_3$  are as complicated as the representations of any other quiver; in this sense  $\text{Rep}(K_3, k)$  is *wild*.

Let  $Q_0 = \{1, \dots, n\}$  and  $Q_1 = \{\alpha_1, \dots, \alpha_r\}$ . Fix representations  $X, Y$  of  $Q$  and define the representation  $EX$  of  $\Gamma$  as follows. The underlying vector space is  $\bar{X}^{n+r+2}$ . Any endomorphism of this space can be written as  $(n+r+2) \times (n+r+2)$  block matrix with each entry an element of  $\text{End } \bar{X}$ . So we define

$$(EX)_\sigma = \begin{bmatrix} 0 & \text{id} & 0 & \cdots & 0 \\ 0 & 0 & \text{id} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \text{id} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad (EX)_\tau = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \text{id} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \bar{X}_1 & \text{id} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \bar{X}_2 & \text{id} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{X}_n & \text{id} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{X}_{\alpha_1} & \text{id} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \bar{X}_{\alpha_2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \bar{X}_{\alpha_r} & \text{id} & 0 \end{bmatrix}.$$

For a morphism  $\phi: X \rightarrow Y$ , define a morphism  $EX \rightarrow EY$  by

$$E\phi = \begin{bmatrix} \bar{\phi} & 0 & \cdots & 0 \\ 0 & \bar{\phi} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\phi} \end{bmatrix}.$$

**Lemma 10.2.1.**  $E$  induces an isomorphism  $\text{Hom}(X, Y) \xrightarrow{\sim} \text{Hom}(EX, EY)$ .

*Proof.* The map is certainly injective. In order to show that it is surjective, fix a linear map  $\psi: \bar{X}^{n+r+2} \rightarrow \bar{Y}^{n+r+2}$  and suppose it is a morphism of  $\Gamma$ -representations. The condition  $\psi(EX)_\sigma = (EY)_\sigma \psi$  implies that  $\psi$  is a  $(n+r+2) \times (n+r+2)$  block matrix of the form

$$\begin{bmatrix} \chi & 0 & \cdots & 0 \\ 0 & \chi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \chi \end{bmatrix}$$

for some linear map  $\chi: \bar{X} \rightarrow \bar{Y}$ . The second condition  $\psi(EX)_\tau = (EY)_\tau \psi$  implies that  $\chi \bar{X}_i = \bar{Y}_i \chi$  for all  $i \in Q_0$  and  $\chi \bar{X}_\alpha = \bar{Y}_\alpha \chi$  for all  $\alpha \in Q_1$ . Thus  $\chi = \bar{\phi}$  for some morphism  $\phi: X \rightarrow Y$  by Lemma 10.1.2, and therefore  $\psi = E\phi$ .  $\square$

Let  $X, Y$  be representations of  $\Gamma$ . We identify  $X$  with its underlying vector space and define a representation  $FX$  of the quiver  $K_3$  by

$$X \begin{array}{c} \xrightarrow{X_\sigma} \\ \xrightarrow{X_\tau} \\ \xrightarrow{\text{id}} \end{array} X .$$

Given a morphism  $\phi: X \rightarrow Y$ , define  $F\phi: FX \rightarrow FY$  by  $(F\phi)_1 = \phi = (F\phi)_2$ .

**Lemma 10.2.2.**  $F$  induces an isomorphism  $\text{Hom}(X, Y) \xrightarrow{\sim} \text{Hom}(FX, FY)$ .

*Proof.* The inverse map sends a morphism  $\phi: FX \rightarrow FY$  to  $\phi_1: X \rightarrow Y$ . □

**Proposition 10.2.3.**  $F_Q = FE$  yields an embedding  $\text{Rep}(Q, k) \rightarrow \text{Rep}(K_3, k)$ .

*Proof.* Combine Lemmas 10.2.1 and 10.2.2. □

It is clear that there is a similar embedding  $\text{Rep}(Q, k) \rightarrow \text{Rep}(K_r, k)$  for each  $r > 3$ .

**10.3. From Kronecker to subspace representations.** We construct for  $r \geq 1$  an embedding

$$F_r: \text{Rep}(K_r, k) \longrightarrow \text{Rep}(A_{r+2}, k).$$

Let  $X, Y$  be representations of the  $r$ -Kronecker quiver  $K_r$ . Define  $F_r X$  by specifying an  $r + 2$ -subspace system as follows:

$$\begin{aligned} (F_r X)_0 &= X_1 \times X_2, & (F_r X)_1 &= X_1 \times \{0\}, & (F_r X)_2 &= \{0\} \times X_2, \\ (F_r X)_{i+2} &= \{(v, X_{\alpha_i}(v)) \in X_1 \times X_2 \mid v \in X_1\} & (1 \leq i \leq r). \end{aligned}$$

Given a morphism  $\phi: X \rightarrow Y$ , define  $F_r \phi: F_r X \rightarrow F_r Y$  by  $(F_r \phi)_i(v, w) = (\phi_1(v), \phi_2(w))$  for  $0 \leq i \leq r + 2$ .

**Lemma 10.3.1.**  $F_r$  induces an isomorphism  $\text{Hom}(X, Y) \xrightarrow{\sim} \text{Hom}(F_r X, F_r Y)$ .

*Proof.* The inverse map sends a morphism  $\phi: F_r X \rightarrow F_r Y$  to a morphism  $\psi: X \rightarrow Y$  given by  $\psi_1 = \phi_1$  and  $\psi_2 = \phi_2$ . □

**Proposition 10.3.2.**  $F_3 F_Q$  yields an embedding  $\text{Rep}(Q, k) \rightarrow \text{Rep}(A_5, k)$ . □

## 11. RADICAL SQUARE ZERO REPRESENTATIONS

We fix a quiver  $Q$  and construct functors between radical square zero representations of  $Q$  and representations of the corresponding separated quiver  $Q^s$ .

**11.1. The radical.** Let  $X$  be a representation of  $Q$ . The *radical* of  $X$  is a subrepresentation which we denote by  $\text{Rad } X$ . We define  $(\text{Rad } X)_i = \sum_{\alpha: j \rightarrow i} \text{Im } X_\alpha$  for each vertex  $i$ . The *Jacobson radical*  $\text{rad } X$  of  $X$  is by definition the intersection of all maximal subrepresentations of  $X$ . For  $n \geq 1$ , let  $\text{Rad } X^{n+1} = \text{Rad}(\text{Rad}^n X)$  and  $\text{rad } X^{n+1} = \text{rad}(\text{rad}^n X)$ . Note that each morphism  $X \rightarrow Y$  induces morphisms  $\text{Rad } X \rightarrow \text{Rad } Y$  and  $\text{rad } X \rightarrow \text{rad } Y$ .

**Lemma 11.1.1.**  $\text{rad } X \subseteq \text{Rad } X$  and equality holds if  $\text{Rad}^n X = 0$  for some  $n \geq 1$ .

*Proof.* The representation  $X/\text{Rad } X$  is a direct sum of simple representations, and therefore  $\text{rad}(X/\text{Rad } X) = 0$ . The canonical morphism  $\pi: X \rightarrow X/\text{Rad } X$  sends  $\text{rad } X$  to  $\text{rad}(X/\text{Rad } X)$ . Thus  $\text{rad } X \subseteq \text{Rad } X$ .

Now suppose that  $\text{Rad}^n X = 0$  for some  $n \geq 1$ . Given a maximal subrepresentation  $U \subseteq X$ , it follows that  $\text{Rad}(X/U) = 0$ . Thus  $\text{Rad } X \subseteq U$ , and therefore  $\text{Rad } X \subseteq \text{rad } X$ .  $\square$

Let  $Q$  be a quiver without oriented cycles. If  $n$  equals the length of the longest path in  $Q$ , then  $\text{Rad}^{n+1} X = 0$  for each representation  $X$ . On the other hand, any quiver with an oriented cycle has a simple representation  $X$  such that  $\text{Rad } X = X$ , while  $\text{rad } X = 0$ .

**11.2. The separated quiver.** We define the *separated quiver*  $Q^s$  as follows. Let  $Q_0 = \{1, 2, \dots, n\}$  be the vertices of  $Q$  and set  $Q_0^s = \{1, \dots, n, 1', \dots, n'\}$ . For each arrow  $\alpha: i \rightarrow j$  of  $Q$ , there is an arrow  $\bar{\alpha}: i \rightarrow j'$  in  $Q^s$ . Thus  $Q_1^s = \{\bar{\alpha} \mid \alpha \in Q_1\}$ .

There are two functors

$$S: \text{Rep}(Q, k) \longrightarrow \text{Rep}(Q^s, k) \quad \text{and} \quad T: \text{Rep}(Q^s, k) \longrightarrow \text{Rep}(Q, k)$$

which are defined as follows. Given a representation  $X$  of  $Q$ , let  $(SX)_i = (X/\text{Rad } X)_i$  and  $(SX)_{i'} = (\text{Rad } X/\text{Rad}^2 X)_i$  for each vertex  $i \in Q_0$ . For each arrow  $\alpha: i \rightarrow j$  of  $Q$ , let  $(SX)_{\bar{\alpha}}: (SX)_i \rightarrow (SX)_{j'}$  be the map which is induced by  $X_\alpha$ . Given a representation  $Y$  of  $Q^s$ , let  $(TY)_i = Y_i \oplus Y_{i'}$  for each vertex  $i \in Q_0$ . For each arrow  $\alpha \in Q_1$ , let  $(TY)_\alpha = \begin{bmatrix} Y_\alpha & 0 \\ 0 & 0 \end{bmatrix}$ .

We call a representation  $X$  *separated* if  $(\text{Rad } X)_i = X_i$  for every sink  $i$ .

**Lemma 11.2.1.** *Each representation  $X$  admits a unique decomposition*

$$X = X' \oplus \left( \bigoplus_{i \text{ a sink}} X(i) \right)$$

such that  $X'$  is separated and each  $X(i)$  is a direct sum of copies of  $S(i)$ .  $\square$

**Proposition 11.2.2.** *The functors  $S$  and  $T$  induce mutually inverse bijections between the isomorphism classes of radical square zero representations of  $Q$  and the isomorphism classes of separated representations of  $Q^s$ .*

*Proof.* It is easily checked that  $TSX \cong X$  for each representation  $X$  of  $Q$  satisfying  $\text{Rad}^2 X = 0$ , and that  $STY \cong Y$  for each separated representation  $Y$  of  $Q^s$ .  $\square$

## 12. REPRESENTATIONS OF THE KLEIN FOUR GROUP

In this section we introduce representations of groups and sketch the parallel with representations of quivers. In particular, we deduce the classification of the representations for the Klein four group from that for the Kronecker quiver.

**12.1. Representations of groups.** Let  $G$  be a group and  $k$  be a field. A *k-linear representation* of  $G$  is a pair consisting of a vector space  $X$  over  $k$  and a group homomorphism  $G \rightarrow \text{Aut}(X)$  into the group of  $k$ -linear automorphisms of  $X$ , sending  $g \in G$  to  $X_g$ . One often writes  $gx = X_g(x)$  for  $x \in X$ . A *morphism* between two representations  $X, Y$  of  $G$  is a  $k$ -linear map  $\phi: X \rightarrow Y$  such that  $\phi X_g = Y_g \phi$  for all  $g \in G$ . We write  $\text{Hom}(X, Y)$  for the set of all morphisms  $X \rightarrow Y$ . The  $k$ -linear representations of  $G$  form a category which we denote by  $\text{Rep}(G, k)$ .

Denote by  $\Gamma(G)$  the multiple loop quiver with  $\Gamma(G)_0 = \{*\}$  and  $\Gamma(G)_1 = G$ . Then the representations of  $G$  can be identified with representations  $X$  of  $\Gamma(G)$  such that

$$X_1 = \text{id}_X \quad \text{and} \quad X_g X_h = X_{gh} \quad \text{for all } g, h \in G.$$

Thus all concepts developed for representations of quivers carry over to representations of groups.

**Example 12.1.1.** (1) The *trivial representation*  $k$  consists of the vector space  $k$  with  $gx = x$  for all  $g \in G$  and  $x \in k$ .

(2) The *regular representation*  $k[G]$  consists of the vector space  $k[G]$  with basis  $G$  and  $g(\sum_h \alpha_h h) = \sum_h \alpha_h gh$  for all  $g \in G$  and  $\sum_h \alpha_h h \in k[G]$

**Lemma 12.1.2.** *Let  $X$  be a representation of  $G$ . The map  $\text{Hom}(k[G], X) \rightarrow X$  sending  $\phi$  to  $\phi(1)$  is bijective.*

*Proof.* The inverse map sends  $x \in X$  to  $\phi_x$  with  $\phi_x(\sum_g \alpha_g g) = \sum_g \alpha_g gx$  □

The vector space duality  $D = \text{Hom}(-, k)$  induces a *duality*

$$\text{Rep}(G, k) \longrightarrow \text{Rep}(G, k),$$

where  $(DX)_g = D(X_{g^{-1}})$  for each  $g \in G$ ; this yields an isomorphism

$$\text{Hom}(X, DY) \cong \text{Hom}(Y, DX)$$

for each pair of representations  $X, Y$ .

**Lemma 12.1.3.** *Let  $G$  be a finite group. Then the map  $k[G] \rightarrow Dk[G]$  sending  $\sum_g \alpha_g g$  to  $\sum_g \alpha_g g^*$  (with  $g^*(h) = \delta_{gh}$  for  $g, h \in G$ ) is an isomorphism.*

*Proof.* The map is bijective since  $G$  is finite, and it is easily checked that the map is a morphism, that is,  $\sum_g \alpha_g (hg)^* = h(\sum_g \alpha_g g^*)$  for all  $h \in G$ . □

**Remark 12.1.4.** It follows from Lemmas 12.1.2 and 12.1.3 that the regular representation  $k[G]$  is a projective and injective object in the sense of Remark 1.7.4.

**12.2. Maschke's theorem.** The representation theory of a group depends heavily on the characteristic of the field. A first instance of this phenomenon is the following.

**Theorem 12.2.1** (Maschke). *Let  $X$  be a representation of a finite group  $G$  such that the characteristic of  $k$  does not divide the order of  $G$ . Then every subrepresentation  $U \subseteq X$  admits a complement, that is, a subrepresentation  $V \subseteq X$  such that  $X = U \oplus V$ . Therefore  $X$  decomposes into a direct sum of simple representations.*

*Proof.* Let  $r = \text{card } G$ . Choose a  $k$ -linear projection  $\pi: X \rightarrow U$  onto  $U$  and let  $\pi' = r^{-1} \sum_{g \in G} X_g \pi X_{g^{-1}}$ . Then  $\pi'$  is a morphism  $X \rightarrow U$  such that  $\pi'(x) = x$  for all  $x \in U$ . Thus  $X = U \oplus \text{Ker } \pi'$ .

The second assertion follows immediately by induction on the dimension of  $X$ . □

**12.3. Elementary abelian  $p$ -groups.** Let  $p > 0$  be prime and denote by  $C_p$  the cyclic group of order  $p$ . For  $r \geq 1$ , we consider the *elementary abelian group*

$$C_p^r = \langle g_1, \dots, g_r \rangle$$

and fix a field  $k$  of characteristic  $p$ . The representations of  $C_p^r$  can be identified with representations  $X$  of the  $r$ -loop quiver  $\Gamma$  (that is,  $\Gamma_0 = \{*\}$  and  $\Gamma_1 = \{\gamma_1, \dots, \gamma_r\}$ ) such that

$$(12.3.1) \quad X_{\gamma_i} X_{\gamma_j} = X_{\gamma_j} X_{\gamma_i} \quad \text{and} \quad X_{\gamma_i}^p = 0 \quad \text{for all} \quad 1 \leq i, j \leq r;$$

the underlying vector space remains unchanged and  $X_{\gamma_i} = X_{g_i} - \text{id}_X$  for each  $i$ . This identification will be used without further mentioning.

Recall that the Jacobson radical  $\text{rad } X$  of a representation  $X$  is the intersection of all its maximal subrepresentations.

**Lemma 12.3.1.** *Let  $X$  be a representation of  $C_p^r$ . Then  $\text{rad } X = \sum_{i=1}^r \text{Im } X_{\gamma_i}$ . Therefore  $\text{rad}^2 X = 0$  if and only if  $X_{\gamma_i} X_{\gamma_j} = 0$  for all  $1 \leq i, j \leq r$ .*

*Proof.* The relations in (12.3.1) imply  $\text{Rad}^{pr} X = 0$ . Then it follows from Lemma 11.1.1 that  $\text{rad } X = \text{Rad } X = \sum_{i=1}^r \text{Im } X_{\gamma_i}$ .  $\square$

We define a functor  $T: \text{Rep}(K_r, k) \rightarrow \text{Rep}(C_p^r, k) \hookrightarrow \text{Rep}(\Gamma, k)$  from representations of the  $r$ -Kronecker quiver

$$K_r \quad 1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \\ \vdots \\ \xrightarrow{\alpha_r} \end{array} 2$$

to representations of  $C_p^r$ . Given a Kronecker representation  $X$ , let  $TX = X_1 \oplus X_2$  and  $(TX)_{\gamma_i} = \begin{bmatrix} X_{\alpha_i} & 0 \\ X_{\alpha_i} & 0 \end{bmatrix}$  for  $1 \leq i \leq r$ . For a morphism  $\phi: X \rightarrow Y$ , let  $T\phi = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}$ .

**Proposition 12.3.2.** *The functor  $T$  induces a bijection between the isomorphism classes of representations of the  $r$ -Kronecker quiver that have no direct summand isomorphic to the simple representation  $S(2)$ , and the isomorphism classes of representations  $X$  of  $C_p^r$  such that  $\text{rad}^2 X = 0$ .*

*Proof.* The assertion is a special case of Proposition 11.2.2 since the  $r$ -Kronecker quiver is the separated quiver of  $\Gamma$ . Note that a Kronecker representation is separated if and only if there is no direct summand isomorphic to  $S(2)$ . On the other hand, a  $\Gamma$ -representation  $X$  satisfies  $\text{Rad}^2 X = 0$  if and only if it is a representation of  $C_p^r$  satisfying  $\text{rad}^2 X = 0$ , by Lemma 12.3.1.

Recall that the inverse of  $T$  is constructed as follows. Let  $X$  be a representation of  $C_p^r$  such that  $X_{\gamma_i} X_{\gamma_j} = 0$  for all  $1 \leq i, j \leq r$ . Then the  $X_{\gamma_i}$  induce a family of linear maps

$$X / \sum_i \text{Im } X_{\gamma_i} \begin{array}{c} \xrightarrow{\bar{X}_{\gamma_1}} \\ \vdots \\ \xrightarrow{\bar{X}_{\gamma_r}} \end{array} \sum_i \text{Im } X_{\gamma_i} .$$

We denote this  $r$ -Kronecker quiver representation by  $SX$ . It is easily checked that  $TSX \cong X$ , and that  $STY \cong Y$  for each separated Kronecker representation  $Y$ .  $\square$

In view of Proposition 10.2.3, the category  $\text{Rep}(C_p^r, k)$  shows some wild behaviour for  $r \geq 3$ . More precisely, for any quiver  $Q$ , there exists a functor

$$F: \text{Rep}(Q, k) \longrightarrow \text{Rep}(C_p^r, k)$$

such that for every pair of representations  $X, Y$  of  $Q$ ,

- (1)  $FX \cong FY$  implies  $X \cong Y$ , and
- (2)  $F$  induces a monomorphism  $\text{Hom}(X, Y) \rightarrow \text{Hom}(FX, FY)$ .

Note that one cannot expect a surjective map  $\text{Hom}(X, Y) \rightarrow \text{Hom}(FX, FY)$ , since  $\text{End}(Z) \cong k$  implies  $Z \cong k$  for each representation  $Z$  of  $C_p^r$ .

**12.4. The Klein four group.** Let  $G = C_2 \times C_2$  be the *Klein four group* and fix a field  $k$  of characteristic 2.

**Lemma 12.4.1.** *Let  $X$  be a representation of  $G$ . Then  $X$  has no direct summand isomorphic to  $k[G]$  if and only if  $X_{\gamma_1}X_{\gamma_2} = 0$ .*

*Proof.* Observe first that  $1, \bar{g}_1 = (g_1 - 1), \bar{g}_2 = (g_2 - 1)$ , and  $\bar{g}_1\bar{g}_2 = \bar{g}_2\bar{g}_1$  form a basis over  $k$  of  $k[G]$ . It follows that each non-zero subrepresentation of  $k[G]$  contains  $\bar{g}_1\bar{g}_2$ .

Suppose there is an element  $x \in X$  such that  $X_{\gamma_1}X_{\gamma_2}(x) \neq 0$ . Using Lemma 12.1.2, there exists a morphism  $\phi: k[G] \rightarrow X$  such that  $\phi(1) = x$ . Then  $\phi(\bar{g}_1\bar{g}_2) = X_{\gamma_1}X_{\gamma_2}(x) \neq 0$ , and therefore  $\text{Ker } \phi = 0$ . Thus  $\phi$  is a split monomorphism, since  $k[G]$  is an injective object; see Remark 12.1.4. The other implication is clear.  $\square$

It follows from Proposition 12.3.2 that the representations of  $G$  are closely related to representations of the Kronecker quiver via the functor  $T: \text{Rep}(K, k) \rightarrow \text{Rep}(G, k) \hookrightarrow \text{Rep}(\Gamma, k)$ , where

$$K \quad 1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \end{array} 2 \quad \text{and} \quad \Gamma \quad \gamma_1 \begin{array}{c} \curvearrowright \circ \curvearrowleft \\ \curvearrowright \circ \curvearrowleft \end{array} \gamma_2 .$$

Given Kronecker representations  $X, Y$ , let  $TX = X_1 \oplus X_2$  and  $(TX)_{\gamma_i} = \begin{bmatrix} X_{\alpha_i}^0 & 0 \\ X_{\alpha_i} & 0 \end{bmatrix}$  for  $i = 1, 2$ . For a morphism  $\phi: X \rightarrow Y$ , let  $T\phi = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}$ .

**Proposition 12.4.2.** *The functor  $T$  induces a bijection between the isomorphism classes of representations of the Kronecker quiver that have no direct summand isomorphic to the simple representation  $S(2)$ , and the isomorphism classes of representations of the Klein four group that have no direct summand isomorphic to the regular representation  $k[G]$ .*

*Proof.* The assertion is a special case of Proposition 12.3.2. Note that a representation  $X$  of  $G$  satisfies  $\text{rad}^2 X = 0$  if and only if there is no direct summand isomorphic to  $k[G]$ , by Lemmas 12.3.1 and 12.4.1.  $\square$

In view of Theorem 9.3.1, we have the following classification, assuming that the field  $k$  is algebraically closed.

**Corollary 12.4.3.** *The representations  $k[G]$ ,  $TP_r$  ( $r \geq 1$ ),  $TI_r$  ( $r \geq 0$ ), and  $TR_{p,\lambda}$  ( $p \geq 1$  and  $\lambda \in \mathbb{P}^1(k)$ ) form, up to isomorphism, a complete list of finite dimensional indecomposable representations of the Klein four group.  $\square$*

### 13. NOTES

In [9] Gabriel proved that a connected quiver admits only finitely many indecomposable representations if and only if it is of Dynkin type. This result and the corresponding classification of the indecomposable representations in terms of their dimension vectors is known as Gabriel's theorem; see also [10]. The proof using reflection functors is due to Bernšteĭn, Gel'fand, and Ponomarev [4].

For the quivers of Euclidean type, the complete classification of their indecomposable representations was established by Donovan and Freislich [8], and independently by Nazarova [14]; see also [7]. For the special case of the Kronecker quiver, the classification is due to Kronecker [13].

The wild behaviour of representations of the  $n$ -subspace quiver for  $n \geq 5$  was first noticed by Brenner [5].

The classification of the representations of the Klein four group was obtained by Bašev [3], and independently by Heller and Reiner [12]. Gabriel's survey [10] discusses radical square zero representations and further connections between representations of finite groups and quivers.

Irreducible morphisms were introduced by Auslander and Reiten in [2] and their structure for algebras of finite representation type is discussed in [11]. Infinite chains of morphisms were used by Auslander when he characterised algebras of finite representation type [1].

The exposition given here is based on various sources. Besides the somewhat classical references given above, Crawley-Boevey's excellent notes on representations of quivers [6] should be mentioned.

#### APPENDIX A. EXACT SEQUENCES

A sequence of morphisms between representations

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_r} X_{r+1}$$

is called *exact* if  $\text{Im } \phi_i = \text{Ker } \phi_{i+1}$  for all  $1 \leq i < r$ .

**Lemma A.1.** *The following are equivalent for a sequence  $Y' \xrightarrow{\phi} Y \xrightarrow{\psi} Y''$ .*

- (1) *The sequence  $0 \rightarrow Y' \xrightarrow{\phi} Y \xrightarrow{\psi} Y''$  is exact.*
- (2) *The induced sequence  $0 \rightarrow \text{Hom}(X, Y') \xrightarrow{(X, \phi)} \text{Hom}(X, Y) \xrightarrow{(X, \psi)} \text{Hom}(X, Y'')$  is exact for each  $X$ .*
- (3) *The morphism  $\phi$  is a kernel of  $\psi$ .* □

**Lemma A.2.** *The following are equivalent for a sequence  $Y' \xrightarrow{\phi} Y \xrightarrow{\psi} Y''$ .*

- (1) *The sequence  $Y' \xrightarrow{\phi} Y \xrightarrow{\psi} Y'' \rightarrow 0$  is exact.*
- (2) *The induced sequence  $0 \rightarrow \text{Hom}(Y'', Z) \xrightarrow{(\psi, Z)} \text{Hom}(Y, Z) \xrightarrow{(\phi, Z)} \text{Hom}(Y', Z)$  is exact for each  $Z$ .*
- (3) *The morphism  $\psi$  is a cokernel of  $\phi$ .* □

**Lemma A.3** (Snake lemma). *Any commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \phi' \downarrow & & \phi \downarrow & & \phi'' \downarrow & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \end{array}$$

*with exact rows induces an exact sequence*

$$0 \rightarrow \text{Ker } \phi' \rightarrow \text{Ker } \phi \rightarrow \text{Ker } \phi'' \rightarrow \text{Coker } \phi' \rightarrow \text{Coker } \phi \rightarrow \text{Coker } \phi'' \rightarrow 0.$$

*Proof.* The proof is given by Jill Clayburgh at the very beginning of the 1980 film *It's My Turn* directed by Claudia Weill.  $\square$

An exact sequence

$$0 \longrightarrow X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

is called *split exact* provided that the following two equivalent conditions are satisfied:

- (1)  $\phi$  is a split monomorphism, that is,  $\text{id}_X = \phi' \phi$  for some  $\phi': Y \rightarrow X$ ;
- (2)  $\psi$  is a split epimorphism, that is,  $\text{id}_Z = \psi \psi'$  for some  $\psi': Z \rightarrow Y$ .

We write  $\text{Ext}(Z, X) = 0$  if any exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  splits.

**Lemma A.4.** *Let  $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$  be an exact sequence and  $X$  a representation.*

- (1) *If  $\text{Ext}(Y', X) = 0 = \text{Ext}(Y'', X)$ , then  $\text{Ext}(Y, X) = 0$ .*
- (2) *If  $\text{Ext}(X, Y') = 0 = \text{Ext}(X, Y'')$ , then  $\text{Ext}(X, Y) = 0$ .*

*Proof.* We prove (1) and the argument for (2) is dual. Let  $0 \rightarrow X \xrightarrow{\phi} E \xrightarrow{\psi} Y \rightarrow 0$  be an exact sequence. Viewing the morphism  $Y' \rightarrow Y$  as an inclusion gives rise to an exact sequence  $0 \rightarrow X \rightarrow E' \rightarrow Y' \rightarrow 0$  by taking  $E' = \psi^{-1}(Y')$ . This sequence splits and therefore  $Y' \rightarrow Y$  factors through  $\psi$ . This yields the following commutative diagram with exact rows

$$\begin{array}{ccccccc} & & & & Y' & \xlongequal{\quad} & Y' \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \xrightarrow{\phi} & E & \xrightarrow{\psi} & Y \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & E'' & \longrightarrow & Y'' \longrightarrow 0 \end{array}$$

by taking for  $E''$  the cokernel of  $Y' \rightarrow E$ . The lower sequence splits. Thus  $X \rightarrow E''$  is a split monomorphism, and therefore  $\phi$  is a split monomorphism. It follows that  $\text{Ext}(Y, X) = 0$ .  $\square$

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