THE ALGEBRAS DERIVED EQUIVALENT TO GENTLE CLUSTER TILTED ALGEBRAS

GRZEGORZ BOBIŃSKI AND ASLAK BAKKE BUAN

Abstract. A cluster tilted algebra is known to be gentle if and only if it is cluster tilted of Dynkin type $\mathbb{A}$ or Euclidean type $\tilde{\mathbb{A}}$. We classify all finite dimensional algebras which are derived equivalent to gentle cluster tilted algebras.

We consider finite dimensional algebras over an algebraically closed field k. Dealing with such algebras up to Morita equivalence, we may assume that they are given as path algebras modulo ideals of relations. Gentle algebras form a particularly nice subclass of special biserial algebras. This is a well understood and much studied class of algebras of tame representation type. They occur in various settings related to group algebras of finite groups, and also frequently as test classes when dealing with general problems for finite dimensional algebras.

The special biserial algebras can be combinatorially characterized in terms of their quivers and relations, and so can the subclass of gentle algebras, which we will study here. A prominent class of examples comes from path algebras; a hereditary algebra is easily seen to be gentle if and only if it is the path algebra of a quiver of Dynkin type $\mathbb{A}$ or Euclidean type $\tilde{\mathbb{A}}$.

When dealing with questions of a homological nature, one is frequently inclined to study algebras up to derived equivalence. Two algebras are said to be derived equivalent if their derived categories are equivalent as triangulated categories, and this happens if and only if one can get from one algebra to the other by taking the endomorphism ring of a so called tilting complex. A special case of tilting complexes are tilting modules.

By [20], the class of gentle algebras is closed under derived equivalence. Hence the characterization of hereditary gentle algebras implies also a characterization of gentle algebras derived equivalent to hereditary algebras.

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Recently, a class of algebras with representation theory very similar to that of hereditary algebras has been much studied; these are the cluster tilted algebras. Such an algebra is defined to be the endomorphism ring $\mathrm{End}_{\mathcal{C}_{kQ}}(T)$ of a tilting object T in the cluster category $\mathcal{C}_{kQ}$ of a quiver Q and is then said to be cluster tilted of type Q.

Cluster tilted algebras of type $\mathbb{A}$ were classified in [12], while cluster tilted algebras of type $\tilde{\mathbb{A}}$ were classified in [7]. They are in both cases gentle, and moreover, Assem et. al [1] have shown that no other cluster tilted algebras are gentle. Furthermore, in both cases, also a classification of the derived equivalence classes were given.

The main aim of this paper is to give a complete classification of all finite dimensional algebras which are derived equivalent to gentle cluster tilted algebras. For this we use work of Avella-Alaminos and Geiss [6]. They showed that one can assign to each gentle algebra Λ a function $f_\Lambda \colon \mathbb{N}^2 \to \mathbb{N}$, and that this function is invariant under derived equivalence. The function can be algorithmically computed from the quiver and relations of the algebra. Our classification is described in terms of this function. More precisely, we give necessary and sufficient conditions on f_Λ for Λ to be derived equivalent to a cluster tilted algebra of type $\mathbb{A}$, and similarly we give conditions for type $\tilde{\mathbb{A}}$.

We also point out that in the case of algebras which are derived equivalent to gentle cluster tilted algebras, we have that f_Λ uniquely determines the derived equivalence class of Λ.

Another main tool is a combinatorial description of Brenner–Butler (co)tilting [9] (shortly, BB-(co)tilting). In fact, as a consequence of the proof of our main result, it follows that any derived equivalence between two gentle algebras derived equivalent to cluster tilted algebras can be obtained by repeated BB-tilting or BB-cotilting.

It is known that the cluster algebras are Gorenstein algebras of Gorenstein dimension 1. We show that this property characterizes the gentle cluster algebras among the algebras derived equivalent to gentle cluster algebras.

The paper is organized as follows. In Section 1 we collect facts about gentle algebras, while in Section 2 we present basics about derived equivalence and define Brenner–Butler (co)tilting modules. Next, in Section 3 we define the invariant of Avella–Alaminos and Geiss and in Section 4 we introduce a combinatorial construction used in the proofs. Section 5 is devoted to a presentation of known facts about gentle cluster tilted algebras. We also formulate the main results of the paper there. Finally, the last four sections contain the proofs of the main results.

We refer to [3,5] for general notions, and to [14] for derived categories.

1. Gentle algebras

Throughout the paper k is a fixed algebraically closed field. By $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_+$ we denote the sets of the integers, the non-negative integers and the positive integers, respectively. Finally, if $i, j \in \mathbb{Z}$, then $[i, j] := \{k \in \mathbb{Z} \mid i \leq k \leq j\}$ (in particular, $[i, j] = \varnothing$ if $i > j$).

By a quiver Δ we mean a (non-empty) finite set Δ_0 of vertices and a finite set Δ_1 of arrows together with two maps $s = s_\Delta, t = t_\Delta : \Delta_1 \to \Delta_0$ which assign to $\alpha \in \Delta_1$ the starting vertex $s\alpha$ and the terminating vertex $t\alpha$, respectively. A vertex x of a quiver Δ is said to be adjacent to an arrow α if $x \in \{s\alpha, t\alpha\}$. Similarly, arrows α and β are said to be adjacent if $\{s\alpha, t\alpha\} \cap \{s\beta, t\beta\} \neq \varnothing$. A quiver Δ is called connected if for all $x, y \in \Delta_0$, $x \neq y$, there exists a sequence $(\alpha_1, \ldots, \alpha_n)$ of arrows such that x is adjacent to α_1, α_i is adjacent to α_{i+1} for each $i \in [1, n-1]$, and y is adjacent to α_n. If Δ is a quiver and $\Delta_1' \subset \Delta_1$, then by the subquiver of Δ generated by Δ_1' we mean the quiver $(\{s\alpha, t\alpha \mid \alpha \in \Delta_1'\}, \Delta_1')$.

Fix a quiver Δ. If $n \in \mathbb{N}_+$, then by a path in Δ of length n we mean a sequence $\omega = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \Delta_1$ for each $i \in [1, n]$ and $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, n-1]$. In the above situation we put $s\omega := s\alpha_n$ and $t\omega := t\alpha_1$. Moreover, for each $x \in \Delta_0$ we introduce the trivial path $\mathbf{1}_x$ at x of length 0 such that $s\mathbf{1}_x := x =: t\mathbf{1}_x$. For a path ω we denote by $\ell(\omega)$ its length. If ω' and ω'' are paths in Δ of lengths n' and n'', respectively, such that $s\omega' = t\omega''$, then we define the composition $\omega' \cdot \omega''$ of ω' and ω'', which is a path in Δ of length $n' + n''$, in the obvious way (in particular, $\omega \cdot \mathbf{1}_{s\omega} = \omega = \mathbf{1}_{t\omega} \cdot \omega$ for each path ω). We say that a path ω_0 is a subpath of a path ω if there exist paths ω' and ω'' such that $\omega = \omega' \cdot \omega_0 \cdot \omega''$.

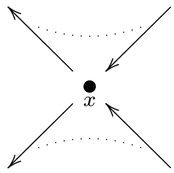
By a quiver with (monomial) relations we mean a pair $\boldsymbol{\Delta} = (\Delta, R)$ consisting of a quiver Δ and a set R of paths in Δ. Given a quiver with relations $\boldsymbol{\Delta}$ we define the algebra $k\boldsymbol{\Delta}$ in the following way. As a vector space $k\boldsymbol{\Delta}$ has a basis formed by the paths in Δ which do not have a subpath from R. If ω' and ω'' are two such paths, then their product is either $\omega' \cdot \omega''$ provided $s\omega' = t\omega''$ and $\omega' \cdot \omega''$ does not have a subpath from R, or 0 elsewhere. If $R = \varnothing$, then one writes $k\Delta$ instead of $k\boldsymbol{\Delta}$ and we call $k\Delta$ the path algebra of Δ. By abuse of terminology we will also call $k\boldsymbol{\Delta}$ the path algebra of $\boldsymbol{\Delta}$.

By a gentle quiver we mean a quiver with relations $\boldsymbol{\Delta}$ such that Δ is connected, R consists of paths of length 2, and the following conditions are satisfied:

(1) for each vertex x there are at most two arrows α such that $s\alpha = x$ and at most two arrows β such that $t\beta = x$,
(2) for each arrow α there is at most one arrow β such that $s\beta = t\alpha$ and $(\beta, \alpha) \notin R$ and at most one arrow γ such that $t\gamma = s\alpha$ and $(\alpha, \gamma) \notin R$,

(3) for each arrow α there is at most one arrow β such that $(\beta, \alpha) \in R$ (in particular, $s\beta = t\alpha$) and at most one arrow γ such that $(\alpha, \gamma) \in R$ (in particular, $t\gamma = s\alpha$),
(4) there exists $n \in \mathbb{N}$ such that every path ω in Δ of length n has a subpath from R (i.e., $\dim_k k\mathbf{\Delta} < \infty$).

In other words, conditions (1)–(3) mean that the most complicated situation which can appear in the neighborhood of a given vertex x is the following



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where the dotted lines denote relations. An algebra Λ is called gentle if and only if there exists a gentle quiver $\mathbf{\Delta}$ such that Λ is isomorphic to $k\mathbf{\Delta}$.

2. Derived equivalences and Brenner–Butler tilting modules

For a finite dimensional algebra Λ denote by $D^b(\Lambda)$ the bounded derived category of the category of finite dimensional right Λ-modules. Then $D^b(\Lambda)$ has a structure of a triangulated category with the suspension functor Σ given by the shift of complexes. We say that finite dimensional algebras Λ and Λ' are derived equivalent if $D^b(\Lambda)$ and $D^b(\Lambda')$ are derived equivalent as triangulated categories. Rickard [19] has showed that this happens if and only if there exists a tilting complex T in $D^b(\Lambda)$ such that Λ' is isomorphic to $\operatorname{End}_{D^b(\Lambda)}(T)$. Recall that if Λ is a finite dimensional algebra, then a complex T in $D^b(\Lambda)$ is called tilting if $\operatorname{Hom}_{D^b(\Lambda)}(T, \Sigma^i T) = 0$ for all $i \in \mathbb{Z}$, $i \neq 0$, and T generates (as a triangulated category) the full subcategory of $D^b(\Lambda)$ formed by the perfect complexes, where a complex is called perfect if it is quasi-isomorphic to a bounded complex of projective modules. A module is called tilting if it is a tilting complex, when viewed as a complex concentrated in degree 0. In the paper, we consider a special class of tilting modules, co called Brenner-Butler (co)tilting modules [9, Chapter 2]. We describe them for gentle algebras, but their definition generalizes to arbitrary finite dimensional algebras.

Let $\mathbf{\Delta}$ be a gentle quiver without loops (i.e., there are no arrows α in Δ such that $s\alpha = t\alpha$) and $\Lambda := k\mathbf{\Delta}$. Let x be a vertex in Δ such that for each $\alpha \in \Delta_1$ with $s\alpha = x$ there exists (necessarily unique) $\beta_\alpha \in \Delta_1$ with $t\beta_\alpha = x$ and $(\alpha, \beta_\alpha) \notin R$. Observe that this condition is satisfied if there are no arrows starting at x or there are two arrows terminating at x. We define a quiver with relations $\mathbf{\Delta}' = (\Delta', R')$, which we call the quiver with relations obtained from $\mathbf{\Delta}$ by applying the reflection at x, in the following way: $\Delta'_0 = \Delta_0$, $\Delta'_1 = \Delta_1$,

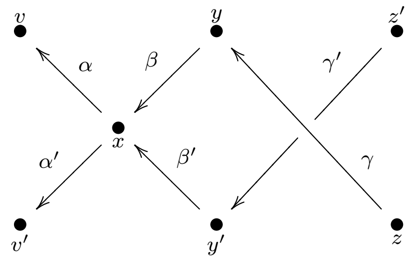
$$s_{\Delta'}\alpha := \begin{cases} x & t_\Delta\alpha = x, \\ s_\Delta\beta_\alpha & s_\Delta\alpha = x, \\ s_\Delta\alpha & \text{otherwise}, \end{cases}$$

$$t_{\Delta'}\alpha := \begin{cases} s_\Delta\alpha & t_\Delta\alpha = x, \\ x & \exists\, \beta \in \Delta_1 : t_\Delta\beta = x \wedge s_\Delta\beta = t_\Delta\alpha \wedge (\beta,\alpha) \in R, \\ t_\Delta\alpha & \text{otherwise}, \end{cases}$$

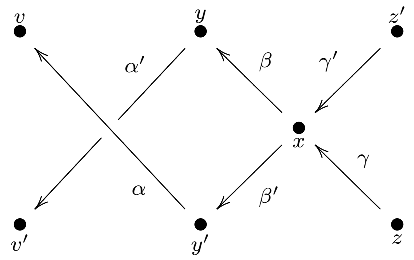
and

$$\begin{aligned} R' := \{(\alpha,\beta) \in R \mid t_\Delta\alpha \neq x \wedge s_\Delta\alpha \neq x\} \cup \{(\alpha,\beta_\alpha) \mid s_\Delta\alpha = x\} \cup \\ \{(\alpha,\beta) \mid t_\Delta\alpha = x \wedge \exists\, \gamma \in \Delta_1 : \\ \gamma \neq \alpha \wedge t_\Delta\gamma = x \wedge s_\Delta\gamma = t_\Delta\beta \wedge (\gamma,\beta) \in R\}. \end{aligned}$$

For example, if Δ is the following quiver



and $R = \{(\alpha,\beta),(\beta,\gamma),(\alpha',\beta'),(\beta',\gamma')\}$, then Δ' is the following quiver



and $R' = \{(\alpha,\beta'),(\beta',\gamma),(\alpha',\beta),(\beta,\gamma')\}$ (in fact, the above example indicates all possible changes which can appear). Then $k\mathbf{\Delta'} \simeq \operatorname{End}_\Lambda(T)$, where

$$T := \tau^{-1}S_x \oplus \bigoplus_{\substack{y\in\Delta_0 \\ y\neq x}} \Lambda \cdot \mathbf{1}_y,$$

S_x is the quotient of $\Lambda \cdot \mathbf{1}_x$ modulo its unique maximal submodule, and τ^{-1} is the quasi-inverse of the Auslander–Reiten translation. Moreover, T is a tilting module, which we call the Brenner–Butler tilting (shortly, BB-tilting) module at x.

We define coreflections and BB-cotilting modules (which are tilting modules in the sense of our definition) dually. If Λ and Λ' are gentle algebras, then we say that Λ and Λ' are BB-equivalent if and only if there exists a sequence $(\Lambda_0, \ldots, \Lambda_n)$, $n \in \mathbb{N}$, of algebras such that $\Lambda_0 = \Lambda$, $\Lambda_n = \Lambda'$, and for each $i \in [1, n]$ there exists a BB-(co)tilting Λ_{i-1}-module T with $\mathrm{End}_{\Lambda_{i-1}}(T) \simeq \Lambda_i$. Obviously, if Λ and Λ' are BB-equivalent, then Λ and Λ' are derived equivalent. We will show that for the class of algebras we consider these two notions coincide.

3. The invariant of Avella-Alaminos and Geiss

If $\mathbf{\Delta}$ is a gentle quiver, then there exist functions $\sigma, \tau : \Delta_1 \to \{\pm 1\}$ such that the following conditions are satisfied:

(1) if $\alpha, \beta \in \Delta_1$, $s\alpha = s\beta$ and $\alpha \neq \beta$, then $\sigma\alpha = -\sigma\beta$,
(2) if $\alpha, \beta \in \Delta_1$, $t\alpha = t\beta$ and $\alpha \neq \beta$, then $\tau\alpha = -\tau\beta$,
(3) if $\alpha, \beta \in \Delta_1$ and $s\alpha = t\beta$, then $(\alpha, \beta) \in R$ if and only if $\sigma\alpha = \tau\beta$.

The functions σ and τ are not uniquely determined by $\mathbf{\Delta}$. From now on we always assume that given a gentle quiver we are also given functions σ and τ as above. If $\mathbf{\Delta}$ is a gentle quiver and $\omega = (\alpha_1, \ldots, \alpha_n)$ is a path in Δ of positive length, then we put $\sigma\omega := \sigma\alpha_n$ and $\tau\omega := \tau\alpha_1$.

Now we fix a gentle quiver $\mathbf{\Delta}$. Following Avella-Alaminos and Geiss [6] we will define a function $f_{\mathbf{\Delta}} : \mathbb{N}^2 \to \mathbb{N}$, which will be also denoted f_Λ provided Λ is (isomorphic to) the path algebra of $\mathbf{\Delta}$.

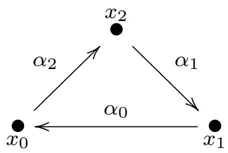
By a path in $\mathbf{\Delta}$ of positive length we mean a path $(\alpha_1, \ldots, \alpha_n)$ in Δ of positive length such that $(\alpha_i, \alpha_{i+1}) \notin R$ (equivalently, $\sigma\alpha_i = -\tau\alpha_{i+1}$) for each $i \in [1, n-1]$. Moreover, for each $x \in \Delta_0$ we introduce two paths $\mathbf{1}_{x,1}$ and $\mathbf{1}_{x,-1}$ of length 0 such that $s\mathbf{1}_{x,\varepsilon} := x =: t\mathbf{1}_{x,\varepsilon}$, $\sigma\mathbf{1}_{x,\varepsilon} := \varepsilon$ and $\tau\mathbf{1}_{x,\varepsilon} := -\varepsilon$. A path ω in $\mathbf{\Delta}$ is called maximal if there is no $\alpha \in \Delta_1$ such that $s\alpha = t\omega$ and $\sigma\alpha = -\tau\omega$, and there is no $\beta \in \Delta_1$ such that $t\beta = s\omega$ and $\tau\beta = -\sigma\omega$ (such objects were called permitted threads in [6]). By $\mathcal{M} = \mathcal{M}_{\mathbf{\Delta}}$ we denote the set of all maximal paths in $\mathbf{\Delta}$.

By an antipath in $\mathbf{\Delta}$ of positive length we mean a path $(\alpha_1, \ldots, \alpha_n)$ of positive length in Δ such that $(\alpha_i, \alpha_{i+1}) \in R$ (equivalently, $\sigma\alpha_i = \tau\alpha_{i+1}$) for each $i \in [1, n-1]$. Moreover, for each $x \in \Delta_0$ we introduce two antipaths $\mathbf{1}'_{x,1}$ and $\mathbf{1}'_{x,-1}$ of length 0 such that $s\mathbf{1}'_{x,\varepsilon} := x =: t\mathbf{1}'_{x,\varepsilon}$ and $\sigma\mathbf{1}'_{x,\varepsilon} := \varepsilon =: \tau\mathbf{1}'_{x,\varepsilon}$. An antipath ω is called maximal if there is no $\alpha \in \Delta_1$ such that $s\alpha = t\omega$ and $\sigma\alpha = \tau\omega$ and there is no $\beta \in \Delta_1$ such that $t\beta = s\omega$ and $\tau\beta = \sigma\omega$ (these objects correspond to forbidden threads in the terminology of [6]). By $\mathcal{N} = \mathcal{N}_{\mathbf{\Delta}}$ we denote the set of all maximal antipaths in $\mathbf{\Delta}$.

If $\omega \in \mathcal{M}$, then there exists unique $\omega' \in \mathcal{N}$ such that $t\omega' = t\omega$ and $\tau\omega' = -\tau\omega$. Moreover, the function $\phi_{\mathbf{\Delta}} : \mathcal{M} \to \mathcal{N}$ obtained in this way is a bijection. Similarly, we obtain a bijection $\psi_{\mathbf{\Delta}} : \mathcal{N} \to \mathcal{M}$ by associating with $\omega \in \mathcal{N}$ the unique $\omega' \in \mathcal{M}$ such that $s\omega' = s\omega$ and $\sigma\omega' = -\sigma\omega$. Finally, we define a bijection $\Phi_{\mathbf{\Delta}} : \mathcal{N} \to \mathcal{N}$ by $\Phi_{\mathbf{\Delta}} := \phi_{\mathbf{\Delta}} \circ \psi_{\mathbf{\Delta}}$. This bijection induces an action of $\mathbb{Z}$ on $\mathcal{N}$ and we denote by $\mathcal{N}/\mathbb{Z}$ the set of the orbits with respect to this action. If $\mathcal{O} \in \mathcal{N}/\mathbb{Z}$, then we put $p(\mathcal{O}) := |\mathcal{O}|$ and $q(\mathcal{O}) := \sum_{\omega \in \mathcal{O}} \ell(\omega)$.

Let $\mathcal{C} = \mathcal{C}_{\mathbf{\Delta}}$ be the set of arrows $\alpha \in \Delta_1$ such that (α) is not a subpath of a maximal antipath in $\mathbf{\Delta}$. For each $\alpha \in \mathcal{C}$ there exists unique $\beta \in \mathcal{C}$ such that $t\beta = s\alpha$ and $\tau\beta = \sigma\alpha$. In this way we obtain a bijection $\Psi_{\mathbf{\Delta}} : \mathcal{C} \to \mathcal{C}$, which induces an action of $\mathbb{Z}$ on $\mathcal{C}$. If $\mathcal{O} \in \mathcal{C}/\mathbb{Z}$, then we put $p(\mathcal{O}) := 0$ and $q(\mathcal{O}) := |\mathcal{O}|$. In other words, if $\mathcal{O} = \mathbb{Z} \cdot \alpha$, then $q(\mathcal{O})$ is the minimal $q \in \mathbb{N}_+$ such that there exists an antipath $(\alpha_0, \ldots, \alpha_q)$ with $\alpha_0 = \alpha = \alpha_q$ (note that $\alpha_i \neq \alpha_j$ for all $i, j \in [0, q-1]$, $i \neq j$, but it may happen that $s\alpha_i = s\alpha_j$ for some $i, j \in [0, q-1]$, $i \neq j$).

For $p, q \in \mathbb{N}$ we denote by $f_{\mathbf{\Delta}}(p, q)$ the number of $\mathcal{O} \in \mathcal{N}/\mathbb{Z} \cup \mathcal{C}/\mathbb{Z}$ such that $p(\mathcal{O}) = p$ and $q(\mathcal{O}) = q$. Observe that $f_{\mathbf{\Delta}}(0, 3)$ counts the number of the cycles



such that $\alpha_0 \neq \alpha_1 \neq \alpha_2 \neq \alpha_0$ and $(\alpha_0, \alpha_1), (\alpha_1, \alpha_2), (\alpha_2, \alpha_0) \in R$. We will call such configurations (more precisely, the orbits in $\mathcal{C}/\mathbb{Z}$ consisting of 3 arrows) triangles. Observe that $x_0 \neq x_1 \neq x_2 \neq x_0$ in the above situation, since otherwise we would have paths of arbitrary length in $\mathbf{\Delta}$.

The following is the main result of [6].

Theorem 3.1. *Let Λ and Λ' be gentle algebras. If Λ and Λ' are derived equivalent, then $f_\Lambda = f_{\Lambda'}$.*

It is worth to observe the following.

Lemma 3.2. *Let $\mathbf{\Delta}$ be a gentle quiver. Then*

$$\sum_{\mathcal{O} \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} \cup \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}} p(\mathcal{O}) = 2|\Delta_0| - |\Delta_1| \quad \text{and} \quad \sum_{\mathcal{O} \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} \cup \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}} q(\mathcal{O}) = |\Delta_1|.$$

Proof. The latter observation is obvious, for the proof of the former one first observe that

$$\sum_{\mathcal{O} \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} \cup \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}} p(\mathcal{O}) = \sum_{\mathcal{O} \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z}} p(\mathcal{O}) = |\mathcal{N}_{\mathbf{\Delta}}| = |\mathcal{M}_{\mathbf{\Delta}}|.$$

Next, if $(x, \varepsilon) \in \Delta_0 \times \{\pm 1\}$, then either there exists $\omega \in \mathcal{M}_{\mathbf{\Delta}}$ such that $s\omega = x$ and $\sigma\omega = \varepsilon$ or there exists $\alpha \in \Delta_1$ such that $t\alpha = x$ and $\tau\alpha = -\varepsilon$. One easily observes that in this way we may define a bijection between $\Delta_0 \times \{\pm 1\}$ and $\mathcal{M}_{\mathbf{\Delta}} \cup \Delta_1$, which finishes the proof. □

Now we characterize, in terms of the above invariant, classes of gentle quivers, which will play an important role in our considerations. For

$p, q \in \mathbb{N}$ we denote by $[p, q]$ the characteristic function of the subset $\{(p, q)\}$ of $\mathbb{N}^2$.

A gentle quiver $\mathbf{\Delta}$ is said to be of tree type if $|\Delta_0| = |\Delta_1| + 1$. Recall [2] that the gentle quivers of tree type are precisely the gentle quivers whose path algebras are derived equivalent to the path algebras of Dynkin quivers of type $\mathbb{A}$.

Lemma 3.3. *If $\mathbf{\Delta}$ is a gentle quiver of tree type, then*

$$f_{\mathbf{\Delta}} = [|\Delta_0| + 1, |\Delta_0| - 1|].$$

Proof. See [6, Section 7]. □

Consequently, we have the following characterization of the gentle quivers of tree type.

Proposition 3.4. *A gentle quiver $\mathbf{\Delta}$ is of tree type if and only if $f_{\mathbf{\Delta}} = [p + 2, p]$ for some $p \in \mathbb{N}$.*

Proof. If $f_{\mathbf{\Delta}} = [p + 2, p]$, then $|\Delta_1| = p$ and $|\Delta_0| = \frac{1}{2}(|\Delta_0| + (p + 2)) = p+1$, hence $\mathbf{\Delta}$ is of tree type. The inverse implication follows from the previous lemma. □

Let Δ be a connected quiver. An arrow $\alpha \in \Delta_1$ is called a branch arrow if the quiver $\Delta \setminus \{\alpha\}$ is not connected, otherwise we call α a cycle arrow. Obviously Δ contains a cycle arrow if and only if $|\Delta_0| \leq |\Delta_1|$. We say that Δ is a 1-cycle quiver if $|\Delta_0| = |\Delta_1|$. If Δ is a 1-cycle quiver and there are no branch arrows in Δ, then we call Δ a cycle. Observe that if Δ is a 1-cycle quiver, then the subquiver of Δ generated by the cycle arrows is a cycle, which we call the cycle of Δ. We will always assume that given a cycle Δ we are also given its orientation, i.e., we may speak about clockwise and anticlockwise oriented arrows and relations.

A gentle quiver $\mathbf{\Delta}$ is called a 1-cycle gentle quiver if Δ is a 1-cycle quiver. We have the following characterization of the 1-cycle gentle quivers.

Proposition 3.5. *A gentle quiver $\mathbf{\Delta}$ is a 1-cycle gentle quiver if and only if $f_{\mathbf{\Delta}} = [p + r, p] + [q - r, q]$ for some $p, q, r \in \mathbb{N}$.*

Proof. Analogous to the proof of Proposition 3.4 (in particular, we have to use results of [6, Section 7]). □

We present a combinatorial interpretation of the numbers in the above proposition for a 1-cycle gentle quiver $\mathbf{\Delta}$ without branch arrows (i.e., Δ is a cycle). Let Δ_1' and Δ_1'' denote the sets of the clockwise and the anticlockwise oriented arrows, respectively. We divide the clockwise oriented arrows into two classes $\Delta_1'^{(1)}$ and $\Delta_1'^{(2)}$ in the following way:

$$\Delta_1'^{(1)} := \{\alpha \in \Delta_1' \mid \text{there exists } \beta \in \Delta_1 \text{ such that } (\beta, \alpha) \in R\}$$

(i.e., $\Delta_1'^{(1)}$ consists of the clockwise oriented arrows α such that $t\alpha$ is the middle vertex of a zero relation) and $\Delta_1'^{(2)} := \Delta_1' \setminus \Delta_1'^{(1)}$. Next for each $\alpha \in \Delta_1'$ we define $\omega_\alpha \in \mathcal{N}_{\mathbf{\Delta}}$: we put $\omega_\alpha := \mathbf{1}'_{t\alpha,-\tau\alpha}$ if $\alpha \in \Delta_1'^{(1)}$ (note that ω_α is the maximal antipath ω in $\mathbf{\Delta}$ such that $t\omega = t\alpha$ and $\tau\omega = -\tau\alpha$ in this case), and ω_α is the maximal antipath ω in $\mathbf{\Delta}$ such that $t\omega = t\alpha$ and $\tau\omega = \tau\alpha$ if $\alpha \in \Delta_1'^{(2)}$. We define the sets $\Delta_1''^{(1)}$ and $\Delta_1''^{(2)}$, and the paths ω_α for $\alpha \in \Delta_1''$, similarly. Finally, we put

$$\mathcal{O}' := \{\omega_\alpha \mid \alpha \in \Delta_1'^{(1)} \cup \Delta_1''^{(2)}\} \quad \text{and} \quad \mathcal{O}'' := \{\omega_\alpha \mid \alpha \in \Delta_1'^{(2)} \cup \Delta_1''^{(1)}\}.$$

Proposition 3.6. *Let $\mathbf{\Delta}$ be a 1-cycle gentle quiver without branch arrows. Using the above notation we have the following.*

(1) *If $\mathcal{O}' \neq \varnothing \neq \mathcal{O}''$, then*

$$\mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} = \{\mathcal{O}', \mathcal{O}''\} \qquad \textit{and} \qquad \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z} = \varnothing.$$

(2) *If either $\mathcal{O}' = \varnothing$ or $\mathcal{O}'' = \varnothing$, then*

$$\mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} = \{\mathcal{O}' \cup \mathcal{O}''\} \qquad \textit{and} \qquad \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z} = \{\Delta_1\}.$$

In particular,

$$f_{\mathbf{\Delta}} = [|\Delta_1'| - r, |\Delta_1'|] + [|\Delta_1''| + r, |\Delta_1''|],$$

where $r := |\Delta_1'^{(1)}| - |\Delta_1''^{(1)}|$ is the difference between the numbers of the clockwise and the anticlockwise oriented relations.

Proof. Observe that if $x \in \Delta_0$ and $\varepsilon \in \{\pm 1\}$, then there exists $\omega \in \mathcal{N}_{\mathbf{\Delta}}$ such that $t\omega = x$ and $\tau\omega = \varepsilon$ if and only if there is no $\alpha \in \Delta_1$ such that $s\alpha = x$ and $\sigma\alpha = \varepsilon$. This implies that $\mathcal{N}_{\mathbf{\Delta}} = \mathcal{O}' \cup \mathcal{O}''$. Indeed, if $\omega \in \mathcal{N}_{\mathbf{\Delta}}$, then there exists $\alpha \in \Delta_1$ such that $t\alpha = t\omega$ (otherwise, there exist $\beta', \beta'' \in \Delta_1$ such that $\beta' \neq \beta''$ and $s\beta' = t\omega = s\beta''$, thus either $\sigma\beta' = \tau\omega$ or $\sigma\beta'' = \tau\omega$). Now, if there is $\alpha \in \Delta_1$ such that $t\alpha = t\omega$ and $\tau\alpha = \tau\omega$, then $\alpha \in \Delta_1'^{(2)} \cup \Delta_1''^{(2)}$ and $\omega = \omega_\alpha$. In the other case, $\tau\alpha = -\tau\omega$ for unique $\alpha \in \Delta_1$ such that $t\alpha = t\omega$. Since Δ is a cycle, there is unique $\beta \in \Delta_1$ such that $s\beta = t\omega$. The maximality of ω implies that $\sigma\beta = -\tau\omega = \tau\alpha$, thus $\alpha \in \Delta_1'^{(1)} \cup \Delta_1''^{(1)}$ and $\omega = \mathbf{1}_{t\alpha,-\tau\alpha} = \omega_\alpha$.

Observe that $\mathcal{C}_{\mathbf{\Delta}} \neq \varnothing$ if and only if the arrows of Δ form an oriented cycle such that $(\alpha, \beta) \in R$ for all $\alpha, \beta \in \Delta_1$ with $s\alpha = t\beta$. This means that $\mathcal{C}_{\mathbf{\Delta}} \neq \varnothing$ if and only if $\mathcal{C}_{\mathbf{\Delta}} = \Delta_1$. Moreover, if this is the case, then $\mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z} = \{\Delta_1\}$ and either $\Delta_1 = \Delta_1'^{(1)}$ or $\Delta_1 = \Delta_1''^{(1)}$.

Finally, the formula for $\mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z}$ follows by an analysis of the action of $\mathbb{Z}$ on $\mathcal{N}_{\mathbf{\Delta}}$. □

A special role among the 1-cycle gentle quivers is played by the gentle quivers of type $\tilde{\mathbb{A}}$, i.e., the gentle quivers whose path algebras are derived equivalent to the path algebras of Euclidean quivers of type $\tilde{\mathbb{A}}$. We will need the following facts about the gentle quivers of type $\tilde{\mathbb{A}}$.

Proposition 3.7. *Let $\mathbf{\Delta}$ be 1-cycle gentle quiver.*

(1) *The quiver* $\mathbf{\Delta}$ *is of type* $\tilde{\mathbb{A}}$ *if and only if* $f_{\mathbf{\Delta}} = [p, p] + [q, q]$ *for some* $p, q \in \mathbb{N}$.
(2) *Let* Δ' *be the cycle of* Δ *and* $\mathbf{\Delta}' := (\Delta', \{(\alpha, \beta) \mid \alpha, \beta \in \Delta'_1\})$. *Then* $\mathbf{\Delta}$ *is of type* $\tilde{\mathbb{A}}$ *if and only if the number of the clockwise oriented relations in* $\mathbf{\Delta}'$ *equals the number of the anticlockwise oriented relations in* $\mathbf{\Delta}'$.

Proof. (1) See [6, Section 7].
(2) This follows directly from [4, Theorem (A)]. □

4. Completion procedure

Let $\mathbf{\Delta}$ be a gentle quiver. We say that a relation $(\alpha, \beta) \in R$ is isolated if $(\alpha, \beta) \in \mathcal{N}_{\mathbf{\Delta}}$. Given a set R_0 of isolated relations, we define a quiver with relations $\mathbf{\Delta}'$, which we call the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0, in the following way: $\Delta'_0 := \Delta_0$, $\Delta'_1 := \Delta_1 \cup \{\gamma_\rho \mid \rho \in R_0\}$ (where γ_ρ, $\rho \in R_0$, are "new" arrows) and

$$R' := R \cup \{(\gamma_\rho, \alpha), (\beta, \gamma_\rho) \mid \rho = (\alpha, \beta) \in R_0\}.$$

The following observation will be crucial.

Lemma 4.1. *Let* $\rho = (\alpha, \beta)$ *be an isolated relation in a gentle quiver* $\mathbf{\Delta}$ *and* $\mathbf{\Delta}'$ *be the quiver obtained from* $\mathbf{\Delta}$ *by completing* ρ. *Then the following hold:*
(1) $\mathbf{\Delta}'$ *is a gentle quiver if and only if* $\mathbb{Z} \cdot \rho \neq \{\rho\}$.
(2) *If* $\mathbb{Z} \cdot \rho \neq \{\rho\}$, *then* $\mathcal{N}_{\mathbf{\Delta}'} = \mathcal{N}_{\mathbf{\Delta}} \setminus \{\rho\}$ *and*
$$\mathcal{N}_{\mathbf{\Delta}'}/\mathbb{Z} = \{\mathcal{O}' \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} \mid \mathcal{O}' \neq \mathbb{Z} \cdot \rho\} \cup \{(\mathbb{Z} \cdot \rho) \setminus \{\rho\}\}.$$
Moreover, $\mathcal{C}_{\mathbf{\Delta}'} = \mathcal{C}_{\mathbf{\Delta}} \cup \{\alpha, \beta, \gamma_\rho\}$ *and*
$$\mathcal{C}_{\mathbf{\Delta}'}/\mathbb{Z} = \mathcal{C}/\mathbb{Z} \cup \{\{\alpha, \beta, \gamma_\rho\}\}.$$

Proof. Let $\omega' := \psi_{\mathbf{\Delta}}\rho$ and $\omega'' := \phi_{\mathbf{\Delta}}^{-1}\rho$.

(1) It is clear that the first three conditions of the definition of a gentle algebra are satisfied by $\mathbf{\Delta}'$. Moreover, $\omega' \cdot (\gamma_\rho) \cdot \omega''$ is a path in Δ, which does not contain a subpath from R, hence the last condition is satisfied if and only if $\omega' \neq \omega''$, that is, if and only if $\mathbb{Z} \cdot \rho \neq \{\rho\}$.

(2) Assume that $\mathbb{Z} \cdot \rho \neq \{\rho\}$. The equalities

$$\mathcal{N}_{\mathbf{\Delta}'} = \mathcal{N}_{\mathbf{\Delta}} \setminus \{\rho\} \qquad \text{and} \qquad \mathcal{C}_{\mathbf{\Delta}'} = \mathcal{C}_{\mathbf{\Delta}} \cup \{\alpha, \beta, \gamma_\rho\}$$

are immediate. Moreover,

$$\Psi_{\mathbf{\Delta}'}\alpha = \beta, \qquad \Psi_{\mathbf{\Delta}'}\beta = \gamma_\rho \qquad \text{and} \qquad \Psi_{\mathbf{\Delta}'}\gamma_\rho = \alpha,$$

while $\Psi_{\mathbf{\Delta}'}\omega = \Psi_{\mathbf{\Delta}}\omega$ for all $\omega \in \mathcal{C}_{\mathbf{\Delta}}$. Observe that our assumption implies that $\omega' \cdot (\gamma_\rho) \cdot \omega'' \in \mathcal{M}_{\mathbf{\Delta}'}$. Consequently,

$$\Phi_{\mathbf{\Delta}'}\omega = \begin{cases} \Phi_{\mathbf{\Delta}}\rho & \omega = \Phi_{\mathbf{\Delta}}^{-1}\rho, \\ \Phi_{\mathbf{\Delta}}\omega & \text{otherwise}, \end{cases}$$

which finishes the proof. □

We list some consequences of the above lemma.

Corollary 4.2. *Let R_0 be a set of isolated relations in a gentle quiver $\mathbf{\Delta}$ and $\mathbf{\Delta}'$ be the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0. Then $\mathbf{\Delta}'$ is a gentle quiver if and only if $\mathcal{O} \not\subseteq R_0$ for each $\mathcal{O} \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z}$.*

Proof. Immediate. □

Corollary 4.3. *Let R_0 be a set of isolated relations in a gentle quiver $\mathbf{\Delta}$ and $\mathbf{\Delta}'$ be the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0. If $\mathbf{\Delta}'$ is a gentle quiver, then*

$$f_{\mathbf{\Delta}'} = |R_0| \cdot [0,3] + \sum_{\mathcal{O} \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} \cup \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}} [p(\mathcal{O}) - m(\mathcal{O}), q(\mathcal{O}) - 2m(\mathcal{O})],$$

where $m(\mathcal{O}) := |R_0 \cap \mathcal{O}|$ for $\mathcal{O} \in \mathcal{N}_{\mathbf{\Delta}}/\mathbb{Z} \cup \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$.

Proof. Follows from the above lemma by induction. □

Now we study the inverse operation.

If Δ is a quiver and $\Delta_1' \subset \Delta_1$, then we put $\Delta \setminus \Delta_1' := (\Delta_0, \Delta_1 \setminus \Delta_1')$. Similarly, if $\mathbf{\Delta}$ is a gentle quiver and $\Delta_1' \subset \Delta_1$, then we denote by $\mathbf{\Delta} \setminus \Delta_1'$ the pair

$$(\Delta \setminus \Delta_1', \{\rho \in R : (\alpha) \text{ is not a subpath of } \rho \text{ for each } \alpha \in \Delta_1'\}).$$

In the above situation we say that $\mathbf{\Delta} \setminus \Delta_1'$ is obtained from $\mathbf{\Delta}$ by removing the arrows from Δ_1'.

Recall that by a triangle in a gentle quiver $\mathbf{\Delta}$ we mean every orbit in $\mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$ consisting of 3 elements. Obviously, if $\mathcal{O}'$ and $\mathcal{O}''$ are different triangles in a gentle quiver $\mathbf{\Delta}$, then $\mathcal{O}' \cap \mathcal{O}'' = \varnothing$.

Corollary 4.4. *Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be pairwise different triangles in a gentle quiver $\mathbf{\Delta}$ and $\alpha_i \in \mathcal{O}_i$ for each $i \in [1,m]$. If $\mathbf{\Delta}' := \mathbf{\Delta} \setminus \{\alpha_i \mid i \in [1,m]\}$ and*

$$f_{\mathbf{\Delta}} = m \cdot [0,3] + \sum_{i \in [1,n]} [p_i, q_i]$$

for some $p_i, q_i \in \mathbb{N}$, $i \in [1,n]$, then $\mathbf{\Delta}'$ is a gentle quiver and

$$f_{\mathbf{\Delta}'} = \sum_{i \in [1,n]} [p_i + m_i, q_i + 2m_i]$$

for some $m_1, \ldots, m_n \in \mathbb{N}$ such that $m_1 + \cdots + m_n = m$.

Proof. One easily checks that $\mathbf{\Delta}'$ is a gentle quiver. Now, for each $i \in [1,n]$ let (β_i, γ_i) be a path in Δ such that $\mathcal{O}_i = \{\alpha_i, \beta_i, \gamma_i\}$. Then $\mathbf{\Delta}$ is (isomorphic to) the quiver obtained from $\mathbf{\Delta}'$ by completing the relations (β_i, γ_i), $i \in [1,n]$, so the claim follows immediately from the previous corollary. □

5. Cluster tilted algebras and the main results

For an acyclic quiver Q, a category called the cluster category was defined in [10]. Let τ_D^{-1} be the quasi-inverse of the Auslander–Reiten translation in the bounded derived category $D^b(kQ)$. Then the cluster category $\mathcal{C} = \mathcal{C}_{kQ}$ is the orbit category $D^b(kQ)/F$, where F is the autoequivalence $\tau_D^{-1}\Sigma$. Cluster categories are canonically triangulated, as shown in [17].

An object T in $\mathcal{C}$ with $\operatorname{Ext}^1_{\mathcal{C}}(T,T) = 0$ and $|Q_0|$ pairwise non-isomorphic indecomposable summands, is called a cluster tilting object. There is a natural embedding of $\operatorname{mod} kQ$ into $\mathcal{C}_{kQ}$. Under this embedding, tilting modules are mapped to tilting objects. The endomorphism ring $\operatorname{End}_{\mathcal{C}}(T)$ of a tilting object, is a cluster tilted algebra of type Q [11]. Here we will consider cluster tilted algebras of types $\mathbb{A}$ and $\tilde{\mathbb{A}}$. They appear in the following theorem [1, Theorem 3.3].

Theorem 5.1. *Let C be a cluster tilted algebra of type Q. Then C is gentle if and only if Q is either of Dynkin type $\mathbb{A}$ or of Eulidean type $\tilde{\mathbb{A}}$.*

We have the following consequence for our class of algebras.

Corollary 5.2. *Let C be a cluster tilted algebra of type Q. If C is derived equivalent to a gentle algebra, then Q is either of Dynkin type $\mathbb{A}$ or of Eulidean type $\tilde{\mathbb{A}}$.*

Proof. [20, Corollary 1.2] implies that C is gentle, hence the claim follows from the previous theorem. □

Now we collect facts about cluster tilted gentle algebras.

The following theorem is a reformulation of [12, Proposition 3.1].

Proposition 5.3. *An algebra Λ is a cluster tilted algebra of type $\mathbb{A}$ if and only if there exists a gentle quiver $\mathbf{\Delta}$ of tree type such that R consists of isolated relations and Λ is isomorphic to the path algebra of the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R.*

As an immediate consequence of the above theorem, Corollary 4.3 and Proposition 3.4 we obtain the following.

Corollary 5.4. *If $\mathbf{\Delta}$ is a gentle quiver such that its path algebra is derived equivalent to a cluster tilted algebra of type $\mathbb{A}$, then*

$$f_{\mathbf{\Delta}} = m \cdot [0,3] + [p+m+2, p]$$

for some $m, p \in \mathbb{N}$.

Moreover, the following result follows from the proof of [12, Theorem].

Theorem 5.5. *Let Λ and Λ' be cluster tilted algebras of type $\mathbb{A}$. Then Λ and Λ' are derived equivalent if and only if Λ and Λ' are BB-equivalent.*

Proof. It is enough to observe that all derived equivalences used in [9] are in fact (co)reflections as defined in Section 2. □

The following theorem was proved in [1].

Proposition 5.6. *An algebra Λ is a cluster tilted algebra of type $\tilde{\mathbb{A}}$ if and only if there exists a gentle quiver $\mathbf{\Delta}$ of type $\tilde{\mathbb{A}}$ such that R consists of isolated relations and Λ is isomorphic to the path algebra of the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R.*

Again, we have the following immediate consequence of the above theorem, Proposition 3.7 (1), and Corollary 4.3.

Corollary 5.7. *If $\mathbf{\Delta}$ is a gentle quiver such that its path algebra is derived equivalent to a cluster tilted algebra of type $\tilde{\mathbb{A}}$, then*

$$f_{\mathbf{\Delta}} = (m_1 + m_2) \cdot [0, 3] + [p + m_1, p] + [q + m_2, q]$$

for some $m_1, m_2, p, q \in \mathbb{N}$ such that $p + m_1 > 0$ and $q + m_2 > 0$.

Finally, the following theorem is a consequence of [7, the proof of Theorem 5.5].

Theorem 5.8. *Let Λ and Λ' be cluster tilted algebras of type $\tilde{\mathbb{A}}$. Then the following conditions are equivalent:*

(1) *Λ and Λ' are derived equivalent.*
(2) *Λ and Λ' are BB-equivalent.*
(3) *$f_\Lambda = f_{\Lambda'}$.*

Proof. For the proof of implication (1) $\Longrightarrow$ (2) one has to check again that all derived equivalences used in [7] are (co)reflections. Next, implication (2) $\Longrightarrow$ (3) is obvious. Finally, in the proof of [7, Theorem 5.5] the author shows that $f_\Lambda \neq f_{\Lambda'}$ for cluster tilted algebras Λ and Λ' of type $\tilde{\mathbb{A}}$ which are not in the same derived equivalence class, hence the implication (3) $\Longrightarrow$ (1) follows. □

Now we formulate the main results of the paper.

Theorem A. *Let Λ be a gentle algebras. Then Λ is derived equivalent to a cluster tilted algebra of type $\mathbb{A}$ if and only if*

$$f_{\mathbf{\Delta}} = m \cdot [0, 3] + [p + m + 2, p]$$

for some $m, p \in \mathbb{N}$.

Theorem B. *Let Λ be a gentle algebras. Then Λ is derived equivalent to a cluster tilted algebra of type $\tilde{\mathbb{A}}$ if and only if*

$$f_{\mathbf{\Delta}} = (m_1 + m_2) \cdot [0, 3] + [p + m_1, p] + [q + m_2, q]$$

for some $m_1, m_2, p, q \in \mathbb{N}$ such that $p + m_1 > 0$ and $q + m_2 > 0$.

Taking into account Corollary 5.2 we immediately get the following.

Corollary C. *Let Λ be a gentle algebras. Then Λ is derived equivalent to a cluster tilted algebra if and only if either*

$$f_{\mathbf{\Delta}} = m \cdot [0,3] + [p+m+2, p]$$

for some $m, p \in \mathbb{N}$, or

$$f_{\mathbf{\Delta}} = (m_1 + m_2) \cdot [0,3] + [p+m_1, p] + [q+m_2, q]$$

for some $m_1, m_2, p, q \in \mathbb{N}$ such that $p + m_1 > 0$ and $q + m_2 > 0$.

Moreover, we show the following.

Theorem D. *Let Λ and Λ' be gentle algebras derived equivalent to cluster tilted algebras. Then the following conditions are equivalent:*

(1) *Λ and Λ' are derived equivalent.*
(2) *Λ and Λ' are BB-equivalent.*
(3) *$f_\Lambda = f_{\Lambda'}$.*

Recall that an algebra Λ is called Gorenstein if the injective dimensions of Λ both as a left and as a right Λ-module are finite. If this is the case, then these dimensions coincide [15, Lemma 1.2], and this common value is called the Gorenstein dimension $\operatorname{Gdim} \Lambda$ of Λ. Both the gentle and the cluster tilted algebras are Gorenstein [13,18]. Moreover, in the case of cluster tilted algebras we have the following [18].

Theorem 5.9. *Let Λ be a cluster tilted algebra. Then* $\operatorname{Gdim} \Lambda \leq 1$.

The above property characterizes the gentle cluster tilted algebras.

Theorem E. *Let Λ be a gentle algebra derived equivalent to a cluster tilted algebra. Then Λ is cluster titled if and only if* $\operatorname{Gdim} \Lambda \leq 1$.

6. Proof of Theorem A

Let $\mathcal{A}$ denote the class of the gentle quivers $\mathbf{\Delta}$ such that

$$f_{\mathbf{\Delta}} = m \cdot [0,3] + [p+m+2, p]$$

for some $m, p \in \mathbb{N}$. Taking into account Corollary 5.4, the following proposition will imply Theorem A.

Proposition 6.1. *If $\mathbf{\Delta} \in \mathcal{A}$, then $k\mathbf{\Delta}$ is BB-equivalent to a cluster tilted algebra of type $\mathbb{A}$.*

The following observation will be used many times in our considerations without mentioning it explicitly: if $\mathbf{\Delta} \in \mathcal{A}$, then every orbit in $\mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$ is a triangle.

We start with the following lemma.

Lemma 6.2. *Let $\mathbf{\Delta} \in \mathcal{A}$ and $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the pairwise different triangles in $\mathbf{\Delta}$. If $\alpha_i \in \mathcal{O}_i$ for each $i \in [1,m]$ and $\mathbf{\Delta}' := \mathbf{\Delta} \setminus \{\alpha_i \mid i \in [1,m]\}$, then $\mathbf{\Delta}'$ is a gentle quiver of tree type.*

Proof. Follows immediately from Propositions 4.4 and 3.4. □

The quiver $\mathbf{\Delta}'$ described in the above lemma will be called a model of $\mathbf{\Delta}$. Obviously, a model of $\mathbf{\Delta}$ is not uniquely determined by $\mathbf{\Delta}$.

If $\mathbf{\Delta}$ is a gentle quiver, then we call $(\alpha, \beta) \in R$ a branch relation if α or β is a branch arrow. Observe that if $\mathbf{\Delta} \in \mathcal{A}$, then $(\alpha, \beta) \in R$ is a branch relation if and only if both α and β are branch arrows.

We have the following description of the branch and the cycle arrows for the gentle quivers from $\mathcal{A}$.

Lemma 6.3. *Let $\mathbf{\Delta} \in \mathcal{A}$ and $\alpha \in \Delta_1$. Then α is a branch (cycle) arrow if and only if $\alpha \notin \mathcal{C}_{\mathbf{\Delta}}$ ($\alpha \in \mathcal{C}_{\mathbf{\Delta}}$, respectively).*

Proof. Obviously, α is a cycle arrow if $\alpha \in \mathcal{C}_{\mathbf{\Delta}}$. On the other hand, if $\alpha \notin \mathcal{C}_{\mathbf{\Delta}}$ and $\mathbf{\Delta}'$ is a model of $\mathbf{\Delta}$, then $\alpha \in \Delta_1'$. Since $|\Delta_0'| = |\Delta_1'| + 1$ (according to the previous lemma), $\Delta' \setminus \{\alpha\}$ is not connected. This immediately implies that $\Delta \setminus \{\alpha\}$ is not connected and finishes the proof. □

As a consequence we obtain the following characterization of the cluster tilted algebras of type $\mathbb{A}$.

Corollary 6.4. *Let $\mathbf{\Delta} \in \mathcal{A}$. Then the path algebra of $\mathbf{\Delta}$ is a cluster tilted algebra of type $\mathbb{A}$ if and only if there are no branch relations in $\mathbf{\Delta}$.*

Proof. Obviously, if the path algebra of $\mathbf{\Delta}$ is a cluster tilted algebra of type $\mathbb{A}$, then there are no branch relations in $\mathbf{\Delta}$ by Proposition 5.3.

On the other hand, assume that there are no branch relations in $\mathbf{\Delta}$. If $\mathbf{\Delta}'$ is a model of $\mathbf{\Delta}$, then the previous lemma implies that R' consists of isolated relations and $\mathbf{\Delta}$ is obtained from $\mathbf{\Delta}'$ by completing the relations from R'. This finishes the proof according to Proposition 5.3. □

Consequently, in order to prove Proposition 6.1 we only need to show the following.

Proposition 6.5. *Let $\mathbf{\Delta} \in \mathcal{A}$. Then there exists a gentle quiver $\mathbf{\Delta}'$ without branch relations such that $k\mathbf{\Delta}$ and $k\mathbf{\Delta}'$ are BB-equivalent.*

Proof. For a branch relation $(\alpha, \beta) \in R$ let $n_{(\alpha,\beta)}^{\mathbf{\Delta}}$ be the number of the vertices in the connected component of $\Delta \setminus \{\alpha\}$ containing $t\alpha$. Let $r_{\mathbf{\Delta}}$ denote the number of the branch relations in $\mathbf{\Delta}$ and

$$n_{\mathbf{\Delta}} := \min\{n_\rho^{\mathbf{\Delta}} \mid \rho \in R \text{ is a branch relation}\}$$

(by convention $\min \varnothing = \infty$).

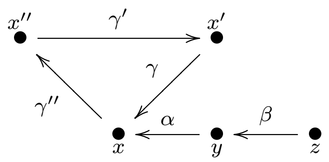
If $r_{\mathbf{\Delta}} > 0$, then we fix a branch relation (α, β) such that $n_{(\alpha,\beta)}^{\mathbf{\Delta}} = n_{\mathbf{\Delta}}$. Observe that we can apply the reflection at $t\alpha$. Indeed, if $\delta \in \Delta_1$ and $s\delta = t\alpha$, then $(\delta, \alpha) \notin R$, since otherwise $n_{(\delta,\alpha)}^{\mathbf{\Delta}} < n_{(\alpha,\beta)}^{\mathbf{\Delta}} = n_{\mathbf{\Delta}}$, which contradicts the minimality of $n_{(\alpha,\beta)}^{\mathbf{\Delta}}$. If $\mathbf{\Delta}'$ is the quiver obtained from $\mathbf{\Delta}$ by applying the reflection at $t\alpha$, then we will show that either

$r_{\mathbf{\Delta}'} < r_{\mathbf{\Delta}}$ or $r_{\mathbf{\Delta}'} = r_{\mathbf{\Delta}}$ and $n_{\mathbf{\Delta}'} < n_{\mathbf{\Delta}}$. Consequently, the claim follows by induction.

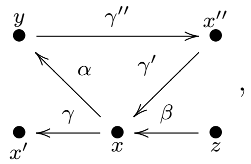
We have to consider three cases:

(1) there exists a cycle arrow γ such that $t\gamma = t\alpha$,
(2) there exists a branch arrow γ such that $\gamma \neq \alpha$ and $t\gamma = t\alpha$,
(3) there is no arrow γ such that $\gamma \neq \alpha$ and $t\gamma = t\alpha$.

(1) First assume that there exists a cycle arrow γ such that $t\gamma = t\alpha$. Then $\gamma \in \mathcal{C}_{\mathbf{\Delta}}$ according to Lemma 6.3. Let $\gamma' := \Psi_{\mathbf{\Delta}}\gamma$ and $\gamma'' := \Psi_{\mathbf{\Delta}}\gamma'$. Observe that $s\gamma'' = t\alpha$. Moreover, if $s\delta = t\alpha$ for some $\delta \in \Delta_1$, then $\delta = \gamma''$. Indeed, if $\delta \neq \gamma''$, then $(\delta, \alpha) \in R$, hence δ is a branch arrow, but this contradicts the minimality of $n^{\mathbf{\Delta}}_{(\alpha,\beta)}$. Consequently, Δ' is obtained from Δ by replacing the subquiver



by the quiver

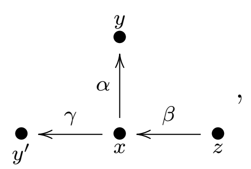


and

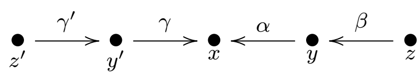
$$R' = (R \setminus \{(\alpha,\beta),(\gamma,\gamma'),(\gamma'',\gamma)\}) \cup \{(\alpha,\gamma'),(\gamma,\beta),(\gamma'',\alpha)\}.$$

Thus $r_{\mathbf{\Delta}'} = r_{\mathbf{\Delta}}$ in this case. Moreover, (γ, β) is a branch relation in $\mathbf{\Delta}'$ and $n^{\mathbf{\Delta}'}_{(\gamma,\beta)} < n^{\mathbf{\Delta}}_{(\alpha,\beta)}$, hence $n_{\mathbf{\Delta}'} < n_{\mathbf{\Delta}}$.

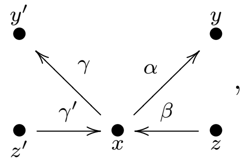
(2) Next, assume that there is a branch arrow γ such that $\gamma \neq \alpha$ and $t\gamma = t\alpha$. By the minimality of $n^{\mathbf{\Delta}}_{(\alpha,\beta)}$ there is no $\delta \in \Delta_1$ such that $s\delta = t\alpha$ (as otherwise either $(\delta, \alpha) \in R$ or $(\delta, \gamma) \in R$). If there is no $\gamma' \in \Delta_1$ such that $(\gamma, \gamma') \in R$, then Δ' is obtained from Δ by replacing the subquiver $\underset{y'}{\bullet} \xrightarrow{\delta} \underset{x}{\bullet} \xleftarrow{\alpha} \underset{y}{\bullet} \xleftarrow{\beta} \underset{z}{\bullet}$ by the quiver



and $R' = (R \setminus \{(\alpha,\beta)\}) \cup \{(\delta,\beta)\}$ (thus $r_{\mathbf{\Delta}'} = r_{\mathbf{\Delta}}$ and $n_{\mathbf{\Delta}'} < n_{\mathbf{\Delta}}$). Otherwise, Δ' is obtained from Δ by replacing the subquiver



by the quiver



and

$$R' = (R \setminus \{(\alpha,\beta),(\gamma,\gamma')\}) \cup \{(\alpha,\gamma'),(\gamma,\beta)\}.$$

Obviously $r_{\mathbf{\Delta}} = r_{\mathbf{\Delta}'}$. Moreover, $n^{\mathbf{\Delta}'}_{(\gamma,\beta)} < n^{\mathbf{\Delta}}_{(\alpha,\beta)}$ (as x and z are not in the connected component of $\Delta' \setminus \{\gamma\}$ containing y'), so $n_{\mathbf{\Delta}'} < n_{\mathbf{\Delta}}$.

(3) Finally assume that there is no arrow γ such that $\gamma \neq \alpha$ and $t\gamma = t\alpha$. If in addition, there is no arrow δ such that $s\delta = t\alpha$, then Δ' is obtained from Δ by replacing the subquiver $\underset{x}{\bullet} \xleftarrow{\alpha} \underset{y}{\bullet} \xleftarrow{\beta} \underset{z}{\bullet}$ by the subquiver $\underset{y}{\bullet} \xleftarrow{\alpha} \underset{x}{\bullet} \xleftarrow{\beta} \underset{z}{\bullet}$ and $R' = R \setminus \{(\alpha,\beta)\}$, hence $r_{\mathbf{\Delta}'} < r_{\mathbf{\Delta}}$. On the other hand, if there is $\delta \in \Delta_1$ such that $s\delta = t\alpha$, then δ is a branch arrow (otherwise, there would be a cycle arrow γ such that $t\gamma = t\alpha$). Moreover, by the minimality of $n^{\mathbf{\Delta}}_{(\alpha,\beta)}$, we have that $(\delta,\alpha) \notin R$ and there is no $\delta' \in \Delta_1$ such that $s\delta' = t\alpha$ and $\delta' \neq \delta$. Thus Δ' is obtained from Δ by replacing the subquiver $\underset{x'}{\bullet} \xleftarrow{\delta} \underset{x}{\bullet} \xleftarrow{\alpha} \underset{y}{\bullet} \xleftarrow{\beta} \underset{z}{\bullet}$ by the subquiver $\underset{x'}{\bullet} \xleftarrow{\gamma} \underset{y}{\bullet} \xleftarrow{\alpha} \underset{x}{\bullet} \xleftarrow{\beta} \underset{z}{\bullet}$ and $R' = (R \setminus \{(\alpha,\beta)\}) \cup \{(\gamma,\alpha)\}$. Consequently, $r_{\mathbf{\Delta}'} = r_{\mathbf{\Delta}}$ and $n_{\mathbf{\Delta}'} < n_{\mathbf{\Delta}}$. This finishes the proof. □

7. Proof of Theorem B

Let $\tilde{\mathcal{A}}$ denote the class of the gentle quivers $\mathbf{\Delta}$ such that

$$f_{\mathbf{\Delta}} = (m_1 + m_2) \cdot [0,3] + [p + m_1, p] + [q + m_2, q]$$

for some $m_1, m_2, p, q \in \mathbb{N}$ such that $p + m_1 > 0$ and $q + m_2 > 0$. Taking into account Corollary 5.7, the following proposition will imply Theorem B.

Proposition 7.1. *If $\mathbf{\Delta} \in \tilde{\mathcal{A}}$, then $k\mathbf{\Delta}$ is BB-equivalent to a cluster tilted algebra of type $\tilde{\mathbb{A}}$.*

Observe that similarly as in the $\mathbb{A}$-case, every orbit in $\mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$ is a triangle provided $\mathbf{\Delta} \in \tilde{\mathcal{A}}$.

We start the proof with the following lemma.

Lemma 7.2. *Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$ and $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the pairwise different triangles in $\mathbf{\Delta}$. If $\alpha_i \in \mathcal{O}_i$ for each $i \in [1,m]$ and $\mathbf{\Delta}' := \mathbf{\Delta} \setminus \{\alpha_i \mid i \in [1,m]\}$, then $\mathbf{\Delta}'$ is a 1-cycle gentle quiver.*

Proof. Follows immediately from Propositions 4.4 and 3.5. □

Again, we call the quiver $\mathbf{\Delta}'$ described in the above lemma a model of $\mathbf{\Delta}$.

Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$ and $\mathcal{O} \in \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$. We say that $\mathcal{O}$ is a branch triangle if for each subset $\mathcal{R}$ of $\mathcal{O}$ such that $|\mathcal{R}| = 2$ we have that $\Delta \setminus \mathcal{R}$ is not connected. Otherwise, we call $\mathcal{O}$ a cycle triangle. A cycle arrow α is called a strongly cycle arrow if either $\alpha \in \Delta_1 \setminus \mathcal{C}$ or α belongs to a cycle triangle. Recall that a vertex x is said to be adjacent to an arrow α if either $x = s\alpha$ or $x = t\alpha$. Similarly, we say that a vertex x is adjacent to a triangle $\mathcal{O}$ if there exists $\alpha \in \mathcal{O}$ such that x is adjacent to α. A vertex x will be called a strongly cycle vertex if it is adjacent to a strongly cycle arrow. Observe that every branch arrow in Δ belongs to $\Delta_1 \setminus \mathcal{C}$.

We have the following characterization of branch/cycle arrows/triangles.

Lemma 7.3. *Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$ and $\mathbf{\Delta}'$ be a model of $\mathbf{\Delta}$. Then the following hold.*

(1) $\Delta_1 \setminus \mathcal{C}_{\mathbf{\Delta}} \subset \Delta_1'$.
(2) *If $\alpha \in \Delta_1 \setminus \mathcal{C}_{\mathbf{\Delta}}$, then α is a branch (cycle) arrow in Δ if and only if α is a branch (cycle, respectively) arrow in Δ'.*
(3) *Let $\mathcal{O} \in \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$. Then $\mathcal{O}$ is a branch triangle if and only if $\mathcal{O} \cap \Delta_1'$ consists of branch arrows in Δ'.*
(4) *Let $\mathcal{O} \in \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$. Then $\mathcal{O}$ is a cycle triangle if and only if $\mathcal{O} \cap \Delta_1'$ contains a cycle arrow in Δ'.*

Proof. The above claims follow directly from the appropriate definitions. □

Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$. One of the consequences of the above fact is that the subquiver of Δ generated by the strongly cycle vertices is connected (this follows since the subquiver of a 1-cycle quiver generated by the cycle arrows is connected). Consequently, given a branch arrow α there exists a component of $\Delta \setminus \{\alpha\}$ which does not contain strongly cycle vertices (and this component is obviously unique). We define $n_\alpha^{\mathbf{\Delta}}$ to be the number of the vertices in this component. We put $n_\alpha^{\mathbf{\Delta}} := \infty$ if α is a cycle arrow. If (α, β) is a branch relation, then we define $n_{(\alpha,\beta)}^{\mathbf{\Delta}} := \min\{n_\alpha, n_\beta\}$. Observe the following: if (α, β) is a branch relation and $n_{(\alpha,\beta)}^{\mathbf{\Delta}} = n_\alpha^{\mathbf{\Delta}}$, then $t\alpha$ is not a strongly cycle vertex (if $t\alpha$ is a strongly cycle vertex, then one easily shows that $n_\beta^{\mathbf{\Delta}} < n_\alpha^{\mathbf{\Delta}}$).

By using arguments analogous to those used in the proof of Proposition 6.5 (using the above modified definition of $n_\rho^{\mathbf{\Delta}}$) we prove the following.

Proposition 7.4. *Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$. Then there exists a gentle quiver $\mathbf{\Delta}'$ without branch relations such that $k\mathbf{\Delta}$ and $k\mathbf{\Delta}'$ are derived equivalent.*

In the next step of our proof we get rid of the branch arrows and the branch triangles.

Proposition 7.5. *Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$. Then there exists a gentle quiver $\mathbf{\Delta}'$ without branch arrows and branch triangles such that $k\mathbf{\Delta}$ and $k\mathbf{\Delta}'$ are derived equivalent.*

Proof. According to the previous lemma we may assume that there are no branch relations in $\mathbf{\Delta}$.

For a strongly cycle vertex x we define the number m_x' in the following way: $m_x' := 0$ if either x is adjacent to a cycle triangle or for each strongly cycle arrow α such that $t\alpha = x$ there is no $\beta \in \Delta_1$ such that $(\alpha, \beta) \in R$. Otherwise, we put $m_x' := m_{s\alpha}' + 1$, where α is the strongly cycle arrow terminating at x (this definition is correct, since $\alpha \notin \mathcal{C}_{\mathbf{\Delta}}$). We define m_x'' dually.

Let $\mathcal{V}$ be the set of the strongly cycle vertices x which are adjacent either to a branch arrow or to a branch triangle. For $x \in \mathcal{V}$ we put $m_x := m_x'$ if either there is a branch triangle adjacent to x or there is a branch arrow terminating at x. Otherwise, we put $m_x := m_x''$. Finally, let $m_{\mathbf{\Delta}} := \min\{m_x \mid x \in \mathcal{V}\}$ and denote by $r_{\mathbf{\Delta}}$ the sum of the numbers of the branch arrows and the branch triangles in $\mathbf{\Delta}$.

Assume that $r_{\mathbf{\Delta}} > 0$ and fix $x \in \mathcal{V}$ with $m_x = m_{\mathbf{\Delta}}$. Observe, that lack of branch relations in $\mathbf{\Delta}$ implies that there may be at most one branch arrow adjacent to x. Consequently, by symmetry we may assume that if there is a branch arrow adjacent to x, then it terminates at x. Let α and β be the strongly cycle arrows adjacent to x. Since there are no branch relations in $\mathbf{\Delta}$, then (up to symmetry) $s\alpha = x = t\beta$ and $(\alpha, \beta) \in R$. Moreover, $\alpha \in \mathcal{C}_{\mathbf{\Delta}}$ if and only if $\beta \in \mathcal{C}_{\mathbf{\Delta}}$. Finally, if $\alpha, \beta \in \mathcal{C}_{\mathbf{\Delta}}$, then they belong to the same triangle. Put $y := s\beta$.

Let $\mathbf{\Delta}'$ be the quiver obtained from $\mathbf{\Delta}$ by applying the reflection at x (we can apply the reflection at x since there are two arrows terminating at x). If $m_x = 0$, then one easily checks that $r_{\mathbf{\Delta}'} < r_{\mathbf{\Delta}}$. On the other hand, if $m_x > 0$ and there is a branch triangle adjacent to x, then $r_{\mathbf{\Delta}'} = r_{\mathbf{\Delta}}$ and $m_{\mathbf{\Delta}'} < m_{\mathbf{\Delta}}$. Finally, if $m_x > 0$ and there is a branch arrow adjacent to x, then $r_{\mathbf{\Delta}''} = r_{\mathbf{\Delta}}$ and $m_{\mathbf{\Delta}''} < m_{\mathbf{\Delta}}$, where $\mathbf{\Delta}''$ is the quiver obtained from $\mathbf{\Delta}'$ by applying the reflection at y (we can apply the reflection at y to $\mathbf{\Delta}'$, since in Δ' there are no arrows starting at y). Consequently, the claim follows by induction. □

Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$ and assume there are neither branch arrows nor branch triangles in $\mathbf{\Delta}$. We investigate its structure more closely.

First observe that for each triangle $\mathcal{O}$ there exists uniquely determined $\gamma_{\mathcal{O}} \in \mathcal{O}$ such that there are no branch arrows in $\Delta \setminus \{\gamma_{\mathcal{O}}\}$. Indeed, one easily observes that $\Delta \setminus \mathcal{O}$ is not connected (this follows since by removing one arrow from each triangle we get a 1-cycle quiver). Now $\gamma_{\mathcal{O}}$ is the arrow in $\mathcal{O}$ such that $\Delta \setminus (\mathcal{O} \setminus \{\gamma_{\mathcal{O}}\})$ is still not connected. It follows from the definition of cycle triangles that $\gamma_{\mathcal{O}}$ is uniquely determined and has the desired property.

Let $\mathbf{\Gamma} := \mathbf{\Delta} \setminus \{\gamma_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}\}$. We call $\mathbf{\Gamma}$ the standard model of $\mathbf{\Delta}$. Note that Γ is a cycle. If $\alpha \in \Delta_1 \setminus \mathcal{C}_{\mathbf{\Delta}}$, then we say that α is clockwise (anticlockwise) oriented if α is clockwise (anticlockwise, respectively) oriented in Γ. Similarly, $\mathcal{O} \in \mathcal{C}_{\mathbf{\Delta}}/\mathbb{Z}$ is said to be clockwise (anticlockwise) oriented if $\mathcal{O} \setminus \{\gamma_{\mathcal{O}}\}$ consists of clockwise (anticlockwise, respectively) oriented arrows in Γ.

By a free relation in $\mathbf{\Delta}$ we mean a relation $(\alpha, \beta) \in R$ such that $\alpha, \beta \notin \mathcal{C}_{\mathbf{\Delta}}$. We claim that if there are free relations in $\mathbf{\Delta}$, then the relations in $\mathbf{\Gamma}$ cannot be equioriented. Indeed, if, for example, the relations in $\mathbf{\Gamma}$ are clockwise oriented, then $f_{\mathbf{\Gamma}} = [p-r, p] + [q+r, q]$ according to Proposition 3.6, where p and q are the numbers of the clockwise and the anticlockwise oriented arrows in $\mathbf{\Gamma}$, respectively, and r is the number of the relations in $\mathbf{\Gamma}$. Consequently, an application of Corollary 4.3 implies that

$$f_{\mathbf{\Delta}} = m \cdot [0, 3] + [p - r - m, p - 2m] + [q + r, q],$$

where m is the number of the completed relations. Now $p - r - m \geq p - 2m$, since $\mathbf{\Delta} \in \tilde{\mathcal{A}}$, hence $r = m$, which means that there are no free relations in $\mathbf{\Delta}$, and this finishes the proof of the claim.

In the next step of our proof we eliminate the free relations.

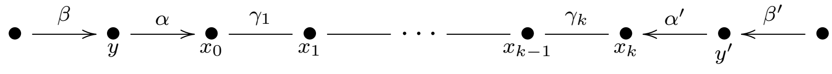
Proposition 7.6. *Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$. Then there exists a gentle quiver $\mathbf{\Delta}'$ without branch arrows, branch triangles and free relations derived equivalent to $\mathbf{\Delta}$.*

Proof. According to Proposition 7.5 we may assume that there are neither branch arrows nor branch triangles in $\mathbf{\Delta}$. We say that a free relation (α, β) in $\mathbf{\Delta}$ is clockwise (anticlockwise) oriented if α and β are clockwise (anticlockwise, respectively) oriented. Let $\mathbf{\Gamma}$ be the standard model of $\mathbf{\Delta}$. For each free relation ρ in $\mathbf{\Delta}$ we denote by k_ρ the minimal distance between ρ and a oppositely oriented relation in $\mathbf{\Gamma}$ (this it well-defined according to the above considerations). We put

$$k_{\mathbf{\Delta}} := \min\{k_\rho \mid \rho \text{ is a free relation in } \mathbf{\Delta}\}$$

and denote by $s_{\mathbf{\Delta}}$ the number of the free relations in $\mathbf{\Delta}$.

Assume that $s_{\mathbf{\Delta}} > 0$ and fix a free relation (α, β) in $\mathbf{\Delta}$ with $k := k_{(\alpha,\beta)} = k_{\mathbf{\Delta}}$. We may assume that ρ is clockwise oriented. Then we have the following subquiver of Γ, where $\mathbf{\Gamma}$ is the standard model of $\mathbf{\Delta}$,



such that $(\alpha', \beta') \in R$. The minimality of k implies that there is no $i \in [1, k-1]$ such that $(\gamma_i, \gamma_{i+1}) \in R$. For the same reason, if $(\gamma_{i+1}, \gamma_i) \in R$ for some $i \in [1, k-1]$, then $\gamma_i, \gamma_{i+1} \in \mathcal{C}_{\mathbf{\Delta}}$ and they belong to the same triangle.

First we show that we may assume that the quiver is ordered in the following sense: there exists $l \in [0,k]$ such that the following conditions are satisfied:

(1) if $i \in [1,l]$, then γ_i is clockwise oriented and $\gamma_i \notin \mathcal{C}_{\mathbf{\Delta}}$,
(2) if $i \in [l+1,k]$ and γ_i is clockwise oriented, then $\gamma_i \in \mathcal{C}_{\mathbf{\Delta}}$.

Indeed, assume not. Then there exists $i \in [1,k-1]$ such that γ_{i+1} is clockwise oriented and either γ_i is anticlockwise oriented or $\gamma_i \in \mathcal{C}_{\mathbf{\Delta}}$. One easily checks that by applying the reflection at x_i we "improve" the configuration (i.e., we decrease the number of the pairs (i,j) such that $i,j \in [1,k]$, $i > j$, γ_i is clockwise oriented, $\gamma_i \notin \mathcal{C}_{\mathbf{\Delta}}$, and either γ_j is anticlockwise oriented or $\gamma_j \in \mathcal{C}_{\mathbf{\Delta}}$), hence the claim follows by induction.

Next, we show that we may assume that $0 = l = k$. Indeed, if $l > 0$, then by applying the reflection at x_0 we obtain the quiver $\mathbf{\Delta}'$ (without branch arrows and branch triangles) with $s_{\mathbf{\Delta}'} = s_{\mathbf{\Delta}}$ and $k_{\mathbf{\Delta}'} < k_{\mathbf{\Delta}}$. Similarly, if $l < k$ and $\mathbf{\Delta}'$ is the quiver obtained from $\mathbf{\Delta}$ by applying either the reflection at x_k (if $\alpha' \notin \mathcal{C}_{\mathbf{\Delta}}$) or the coreflection at x_k (otherwise), then $s_{\mathbf{\Delta}'} = s_{\mathbf{\Delta}}$ and $k_{\mathbf{\Delta}'} < k_{\mathbf{\Delta}}$ (note that if $l < k$ and $\gamma_k \in \mathcal{C}_{\mathbf{\Delta}}$, then the minimality of k implies that $\alpha' \in \mathcal{C}_{\mathbf{\Delta}}$).

Now we have two cases to consider, depending on whether α' belongs or not to $\mathcal{C}_{\mathbf{\Delta}}$. If $\alpha' \in \mathcal{C}_{\mathbf{\Delta}}$, then by applying the reflections at x_0 and y' we obtain the quiver $\mathbf{\Delta}'$ with $s_{\mathbf{\Delta}'} < s_{\mathbf{\Delta}}$. On the other hand, if $\alpha' \notin \mathcal{C}_{\mathbf{\Delta}}$, then we apply the reflections at x_0, y and y', and we obtain the quiver $\mathbf{\Delta}'$ such that $s_{\mathbf{\Delta}'} < s_{\mathbf{\Delta}}$. By induction this finishes the proof. □

In view of the above proposition and Proposition 5.6 the following claim finishes the proof of Proposition 7.1.

Proposition 7.7. *Let $\mathbf{\Delta} \in \tilde{\mathcal{A}}$ be a gentle quiver without branch arrows, branch triangles, and cycle relations. If $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are the pairwise different triangles in $\mathbf{\Delta}$, $\alpha_i \in \mathcal{O}_i$ and $\alpha_i \neq \gamma_{\mathcal{O}_i}$ for each $i \in [1,m]$, then $\mathbf{\Delta}' := \mathbf{\Delta} \setminus \{\alpha_i \mid i \in [1,m]\}$ is a gentle quiver of type $\tilde{\mathbb{A}}$.*

Proof. Observe that there are no relations on the cycle of $\mathbf{\Delta}'$. Consequently, $f_{\mathbf{\Delta}'} = [p,p] + [q,q]$ for some $p,q \in \mathbb{N}_+$ according to Proposition 3.7, hence the claim follows from Corollary 4.3. □

8. Proof of Theorem D

First, we prove the only missing ingredient of the proof. If Δ is a quiver, then by a 3-cycle in Δ we mean every sequence $(\alpha_0, \alpha_1, \alpha_2)$ of arrows such that $\alpha_0 \neq \alpha_1 \neq \alpha_2 \neq \alpha_0$ and $(\alpha_0, \alpha_1, \alpha_2, \alpha_0)$ is a path in Δ. We identify 3-cycles which differ only by a cyclic permutation.

Proposition 8.1. *Let $\mathbf{\Delta} \in \mathcal{A}$ and m be the number of the 3-cycles in Δ. Then*

$$f_{\mathbf{\Delta}} = m \cdot [0,3] + [|\Delta_0| + 1 - m, |\Delta_0| - 1 - 2m].$$

Proof. Let $\mathbf{\Delta}'$ be a model of $\mathbf{\Delta}$. Lemma 3.3 implies that

$$f_{\mathbf{\Delta}'} = [|\Delta_0'| + 1, |\Delta_0'| - 1].$$

Since $|\Delta_0'| = |\Delta_0|$, we obtain using Corollary 4.3 that

$$f_{\mathbf{\Delta}} = m \cdot [0, 3] + [|\Delta_0| + 1 - m, |\Delta_0| - 1 - 2m],$$

where m is the number of the triangles in $\mathbf{\Delta}$. Now, it is easy to observe that there is a bijection between the triangles in $\mathbf{\Delta}$ and the 3-cycles in Δ, which finishes the proof. Indeed, if $\{\alpha, \Psi_{\mathbf{\Delta}}\alpha, \Psi_{\mathbf{\Delta}}^2\alpha\}$ is a triangle in $\mathbf{\Delta}$, then $(\alpha, \Psi_{\mathbf{\Delta}}\alpha, \Psi_{\mathbf{\Delta}}^2\alpha)$ is a 3-cycle. On the other hand, assume that $(\alpha_0, \alpha_1, \alpha_2)$ is a 3-cycle in Δ. If $\{\alpha_0, \alpha_1, \alpha_2\}$ is not a triangle, then there exists a model $\mathbf{\Delta}''$ of $\mathbf{\Delta}$ such that $\alpha_0, \alpha_1, \alpha_2 \in \Delta_1''$. However, $\mathbf{\Delta}''$ is not of tree type, which contradicts Lemma 6.2. □

As an immediate consequence we obtain the following reformulation of [12, Theorem].

Corollary 8.2. *Let Λ and Λ' be cluster tilted algebras of type $\mathbb{A}$. Then Λ and Λ' are derived equivalent if and only if $f_\Lambda = f_{\Lambda'}$.*

Now we can finish the proof of Theorem D. Let Λ and Λ' be gentle algebras derived equivalent to cluster tilted algebras C and C', respectively. Corollary 5.2 implies that C and C' are of types $\mathbb{A}$ or $\tilde{\mathbb{A}}$. Now, using Corollary 5.4 and Proposition 6.1 (in the $\mathbb{A}$-case), and Corollary 5.7 and Proposition 7.1 (in the $\tilde{\mathbb{A}}$-case), we get that we may assume that Λ and Λ' are BB-equivalent to C and C', respectively.

(1) $\Longrightarrow$ (2) Assume that Λ and Λ' are derived equivalent. Then C and C' are derived equivalent, hence C and C' are BB-equivalent according to Theorems 5.5 ($\mathbb{A}$-case) and 5.8 ($\tilde{\mathbb{A}}$-case), thus also Λ and Λ' are BB-equivalent.

(2) $\Longrightarrow$ (3) Follows from Theorem 3.1.

(3) $\Longrightarrow$ (1) Assume that $f_\Lambda = f_{\Lambda'}$. Then $f_C = f_{C'}$, hence C and C' are derived equivalent according to Proposition 8.1 ($\mathbb{A}$-case) and Theorem 5.8 ($\tilde{\mathbb{A}}$-case). Consequently, Λ and Λ' are also derived equivalent, which finishes the proof.

9. Proof of Theorem E

Recall that the gentle algebras are Gorenstein. More precisely, we have the following [13, Theorem 3.4].

Theorem 9.1. *Let Λ be the path algebra of a gentle quiver Δ. Then*

$$\operatorname{Gdim} \Lambda = \max\{\ell(\omega) \mid \omega \in \mathcal{N}_{\mathbf{\Delta}}\}$$

if $\mathcal{N}_{\mathbf{\Delta}} \neq \varnothing$, and $\operatorname{Gdim} \Lambda \leq 1$, otherwise. In particular, $\operatorname{Gdim} \Lambda \leq 1$ if and only if $\ell(\omega) \leq 1$ for each $\omega \in \mathcal{N}_{\mathbf{\Delta}}$.

Now we describe the algebras derived equivalent to cluster tilted algebras of type $\mathbb{A}$ in terms of their quivers.

Proposition 9.2. *An algebra Λ is derived equivalent to a cluster tilted algebra of type $\mathbb{A}$ if and only if there exists a gentle quiver $\mathbf{\Delta}$ of tree type and a subset $R_0 \subset R$ consisting of isolated relations, such that Λ is isomorphic to the path algebra of the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0.*

Proof. If Λ is derived equivalent to a cluster tilted algebra of type $\mathbb{A}$, then Λ is of the form described in the proposition due to Lemma 6.2. On the other hand, if $\mathbf{\Delta}$ is a gentle quiver of tree type, $R_0 \subset R$ consists of isolated relations, and $\mathbf{\Delta}'$ is obtained from $\mathbf{\Delta}$ by completing the relations from R_0, then

$$f_{\mathbf{\Delta}'} = |R_0| \cdot [0, 3] + [|\Delta_0| + 1 - |R_0|, |\Delta_0| - 1 - 2|R_0|]$$

according to Lemma 3.3 and Corollary 4.3. □

We may characterize cluster tilted algebras of type $\mathbb{A}$ among the above class of algebras in the following way.

Corollary 9.3. *Let $\mathbf{\Delta}$ be a gentle quiver of tree type, $R_0 \subset R$ a subset consisting of isolated relations, and Λ the path algebra of the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0. Then the following conditions are equivalent:*

(1) Λ *is cluster tilted.*
(2) $\operatorname{Gdim} \Lambda \leq 1$.
(3) $R_0 = R$.

Proof. (1) $\Longrightarrow$ (2) Follows from Theorem 5.9.

(2) $\Longrightarrow$ (3) Let $\mathbf{\Delta}'$ be the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0. According to Theorem 9.1, $\operatorname{Gdim} \Lambda \leq 1$ implies that $\ell(\omega) \leq 1$ for each $\omega \in \mathcal{N}_{\mathbf{\Delta}'}$, hence $R_0 = R$ according to Lemma 4.1 (note that $\mathcal{C}_{\mathbf{\Delta}} = \varnothing$ since $\mathbf{\Delta}$ is of tree type).

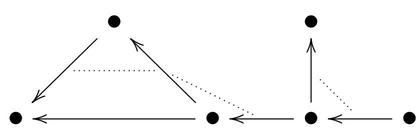
(3) $\Longrightarrow$ (1) Follows from Proposition 5.3. □

Now we study $\tilde{\mathbb{A}}$ case.

Proposition 9.4. *Let Λ be an algebra. If Λ is derived equivalent to a cluster tilted algebra of type $\tilde{\mathbb{A}}$, then there exists a 1-cycle gentle quiver $\mathbf{\Delta}$ and a subset $R_0 \subset R$ consisting of isolated relations, such that Λ is isomorphic to the path algebra of the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0.*

Proof. Follows immediately from Lemma 7.2. □

The converse implication is not true in general. Namely, if $\mathbf{\Delta}$ is the following quiver with relations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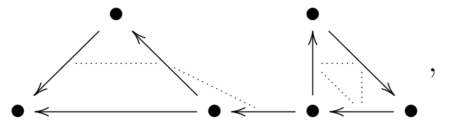
then $\mathbf{\Delta}$ is a 1-cycle gentle quiver (in particular, it is of the above form for $R_0 = \varnothing$), but $f_{\mathbf{\Delta}} = [4,5] + [2,1]$, so $\mathbf{\Delta}$ is not derived equivalent to a cluster tilted algebra. Note however that, if $\mathbf{\Delta}'$ is the following quiver with relations



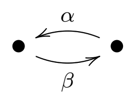
then $f_{\mathbf{\Delta}'} = [0,3] + [3,3] + [2,1]$, hence $\mathbf{\Delta}'$ is derived equivalent to a cluster tilted algebra. In general, in order to obtain a converse of Proposition 9.4, one would need to make assumptions on the relations: The number of the completed clockwise (anticlockwise) relations, must be bigger than the number of the anticlockwise (clockwise, respectively) cycle relations. For this to make sense, one would need an appropriate definition of orientation of branch relations.

We will need the following modification of Proposition 5.6.

Proposition 9.5. *Let Λ be a gentle algebra. Then Λ is a cluster tilted algebra of type $\tilde{\mathbb{A}}$ if and only if there exists a 1-cycle gentle quiver $\mathbf{\Delta}$ such that R consists of isolated relations and Λ is isomorphic to the path algebra of the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R.*

Proof. If Λ is a cluster tilted algebra of type $\tilde{\mathbb{A}}$, then Λ is of the form described in the proposition due to Proposition 5.6. Now assume that $\mathbf{\Delta}$ is a 1-cycle gentle quiver such that R consists of isolated relations, $\mathbf{\Delta}'$ is the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R, and Λ is the path algebra of $\mathbf{\Delta}'$. Extending the construction of the standard model from Section 7 to the considered situation, we show that there exists a model $\mathbf{\Delta}''$ of $\mathbf{\Delta}'$ without cycle relations. Then $\mathbf{\Delta}''$ is of type $\tilde{\mathbb{A}}$ by Proposition 3.7, hence the claim follows from Proposition 5.6. □

We warn the reader that if $\mathbf{\Delta}$ is a 1-cycle gentle quiver such that R consists of isolated relations, then the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R may not be gentle – this happens for example if Δ is the following quiver



and $R = \{(\alpha, \beta)\}$.

We hence obtain the analogue of Corollary 9.3 in the $\tilde{\mathbb{A}}$-case.

Corollary 9.6. *Let $\mathbf{\Delta}$ be a 1-cycle gentle quiver, $R_0 \subset R$ a subset consisting of isolated relations, and Λ the path algebra of the quiver obtained from $\mathbf{\Delta}$ by completing the relations from R_0. If Λ is derived equivalent to a cluster tilted algebra of type $\tilde{\mathbb{A}}$, then the following conditions are equivalent:*

(1) Λ *is cluster tilted.*
(2) $\operatorname{Gdim}\Lambda \leq 1$.
(3) $R_0 = R$.

Proof. (1) $\Longrightarrow$ (2) Follows from Theorem 5.9.

(2) $\Longrightarrow$ (3) First we show that $\mathcal{C}_{\mathbf{\Delta}} = \varnothing$. Indeed, if $\mathcal{C}_{\mathbf{\Delta}} \neq \varnothing$, then $f_{\mathbf{\Delta}} = [0, q] + [p + q, p]$ for some $p \in \mathbb{N}$ and $q \in \mathbb{N}_+$ according to Proposition 3.5. Consequently,

$$f_\Lambda = m \cdot [0, 3] + [0, q] + [p + q - m, p - 2m],$$

where $m := |R_0|$, by Corollary 4.3. Thus Theorem B implies that Λ' is not equivalent to a cluster tilted algebra of type $\tilde{\mathbb{A}}$, which contradicts our assumptions. This finishes the proof of our claim. Now, $\operatorname{Gdim}\Lambda \leq 1$ implies that $\ell(\omega) \leq 1$ for each $\omega \in \mathcal{N}_{\mathbf{\Delta}'}$ according to Theorem 9.1, hence hence $R_0 = R$ according to Lemma 4.1.

(3) $\Longrightarrow$ (1) Follows from the previous proposition. □

Finally, we note that Theorem E follows immediately from Corollaries 9.3 and 9.6.

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Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland
E-mail address: gregbob@mat.uni.torun.pl

Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway
E-mail address: aslakb@math.ntnu.no