The Dixmier map for nilpotent super Lie algebras

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Abstract

In this article we prove that there exists a Dixmier map for nilpotent super Lie algebras. In other words, if we denote $\mathrm{Prim}(\mathcal{U}(\mathfrak{g}))$ the set of (graded) primitive ideals of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ and $\mathcal{A}d_0$ the adjoint group of $\mathfrak{g}_0$, we prove that the usual Dixmier map for nilpotent Lie algebras can be naturally extended to the context of nilpotent super Lie algebras, *i.e.* there exists a bijective map

$$I : \mathfrak{g}_0^*/\mathcal{A}d_0 \to \mathrm{Prim}(\mathcal{U}(\mathfrak{g})),$$

defined by sending the equivalence class $[\lambda]$ of a functional λ to a primitive ideal $I(\lambda)$ of $\mathcal{U}(\mathfrak{g})$, and which coincides with the Dixmier map in the case of nilpotent Lie algebras. We prove in fact a stronger result, which more or less can be interpreted as saying that the mapping I is invariant only over the coadjoint orbits of the super adjoint group of $\mathfrak{g}$. One key fact in the construction is the existence of polarizations for super Lie algebras, generalizing the concept defined for Lie algebras. As a corollary of the previous description, we obtain that $\mathcal{U}(\mathfrak{g})/I(\lambda) \simeq \mathrm{Cliff}_q(k) \otimes A_p(k)$, where $(p, q) = (\dim(\mathfrak{g}_0/\mathfrak{g}_0^\lambda)/2, \dim(\mathfrak{g}_1/\mathfrak{g}_1^\lambda))$, and also an explicit description of the stabilizers of the primitive ideals of $\mathcal{U}(\mathfrak{g})$.

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Introduction

The aim of this article is to extend the Kirillov orbit method *à la Dixmier* for nilpotent Lie algebras to the context of nilpotent super Lie algebras. More precisely, we shall prove the following results. Let $\mathfrak{g}$ be a nilpotent super Lie algebra over an uncountable algebraically closed field of characteristic 0. First, for every linear functional $\lambda \in \mathfrak{g}_0^*$ there exists a so called polarization $\mathfrak{h}$ of $\mathfrak{g}$ at λ (see Subsection 4.4) such that the induced module $\mathrm{ind}(\lambda|_{\mathfrak{h}}, \mathfrak{g})$ is simple and the kernel of its structure morphism is a (graded) primitive ideal of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ (see Theorem 5.4). Moreover, the previously constructed ideal does not depend on the polarization (see Theorem 5.5), and it will be denoted $I(\lambda)$. Conversely, for every (graded) primitive ideal I of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, there exists a linear functional $\lambda \in \mathfrak{g}_0^*$ such that $I = I(\lambda)$ (see Theorem 5.7) and we further have that $\mathcal{U}(\mathfrak{g})/I \simeq \mathrm{Cliff}_q(k) \otimes A_p(k)$, where $(p, q) = (\dim(\mathfrak{g}_0/\mathfrak{g}_0^\lambda)/2, \dim(\mathfrak{g}_1/\mathfrak{g}_1^\lambda))$ and $\mathfrak{g}^\lambda = (\mathfrak{g}_0^\lambda, \mathfrak{g}_1^\lambda)$ is the kernel of the superantisymmetric bilinear form determined by λ on $\mathfrak{g}$ (see Proposition 5.13). A similar version of this last result was proved without any assumption on the field by A. Bell and I. Musson in [BM90], but without any determination of the indices (p, q). Finally, we have that $I(\lambda) = I(\lambda')$ if and only if λ and λ' are in the same coadjoint orbit on $\mathfrak{g}_0^*$ under the action of the adjoint group $\mathcal{A}d_0$ of $\mathfrak{g}_0$ (see Proposition 5.10). We even strengthen this result proving that a derivation $d \in \mathrm{Der}(\mathfrak{g})$ preserves an ideal $I(\lambda)$ if and only if there exists a homogeneous element $x \in \mathfrak{g}$ of the same degree as d such that $d.\lambda = \mathrm{ad}(x).\lambda$ (see Proposition 5.11), which, together with the previous result, roughly means that the map I is "invariant only over the orbits" of the adjoint super group of $\mathfrak{g}$ (*cf.* Remark 5.12). Summarizing, if we denote $\mathrm{Prim}(\mathcal{U}(\mathfrak{g}))$ the set of primitive ideals of $\mathcal{U}(\mathfrak{g})$, these results can be equivalently restated as saying that the map

$$I : \mathfrak{g}_0^*/\mathcal{A}d_0 \to \mathrm{Prim}(\mathcal{U}(\mathfrak{g})),$$

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given by sending the equivalence class $[\lambda]$ of a functional λ to $I(\lambda)$ is well-defined and bijective. If $\mathfrak{g}$ is just a nilpotent Lie algebra, the previous results are exactly the statements of the Dixmier map, which were gradually proved (together with generalizations to the solvable and other cases) by N. Conze, J. Dixmier, M. Duflo and M. Vergne to say a few names (*cf.* [Dix96], Ch. 6 and the references therein, especially the Supplementary remarks at §6.6).

One of our main motivations to consider this extension is to study representations arising from (noncommutative) supersymmetric gauge field theory in physics, that could be found proceeding in an analogous manner to the one used for Yang-Mills theory in the Ph.D. thesis of the author, in which the Kirillov orbit method for nilpotent Lie algebras was used extensively (*cf.* [HS10], where these results were published).

We would like to make a few comments. At first glance, it could seem that what we have proved follows from the work [Kac77] of V. Kac (*cf.* Thm. 7′ on p. 82, where he claims similar results for completely solvable super Lie algebras). Nonetheless, as stated by A. Sergeev in [Ser99], Theorems 7 and 7′ in [Kac77] contain a mistake. In particular the proof of one key point of them, namely item (a) of the first of these theorems, does not hold, as explained by Sergeev in Sec. 5 of the aforementioned article. We would like to remark however that items (b), (c) and (d) of Theorem 7 still hold for nilpotent super Lie algebras, as it can be deduced from the results in [Ser99]. On the other hand, we mention that E. Letzter has proved in [Let92] that there is a bijection between the set of (graded) primitive ideals of the enveloping algebra of a completely solvable super Lie algebra and the set of primitive ideals of the enveloping algebra of the Lie algebra of even elements of the given super Lie algebra which implies the existence of a bijection I as before. We would like to remark that our manner of proceeding is completely different: our proof follows more or less the pattern of the nonsuper case, which we also find more explicit. It has allowed us, for instance, to study the quotients of the enveloping algebra of a nilpotent super Lie algebra by its primitive ideals and other properties related to our motivations (*cf.* Subsection 5.2). Finally, we would also like to say it also generalizes some results proved by S. Mukherjee in [Muk04] (*cf.* Thm. 11.1 of the mentioned paper).

A lot of what we state at the beginning will be the superized version of properties already known for Lie algebras or associative algebras. Since many of the proofs of these result are more or less the same as for the non-super case, we shall only state the superized versions of them and put a standard reference whose proof can also be applied in the super world with at most minor changes, that will be shortly explained. Such changes could include the use of homogeneous elements instead of general ones, and the usually consequent use of super commutators instead of commutators, the appearance of inessential (and obvious) signs (*cf.* Lemma 3.6 for a typical case) and the replacement of the use of previous results for algebras by their superized versions for super algebras. This should not make the reader believe that in the graded world the standard theory of rings follows verbatim, as one can notice by reading [NVO04] or [CM84]. In the particular case of enveloping algebras of super Lie algebras, there were also several obstacles, as the construction of polarizations (because the super version of [Dix96], Prop. 1.12.10, does not yield polarizations of solvable super Lie algebras), the characterization of simple modules (because [Dix96], Lemma 2.6.4 does not hold in the super case), the impossibility to exponentiate odd derivations (*cf.* Remark 5.12), etc. There are thus a collection of proofs that were not so clear at first sight, because they rely on other results inside a lengthy chain of generalizations, and others that were not clear at all, at least to the author. We shall include these.

The article is organized as follows. In the first section we recall some generalities on super (associative) algebras, their ideals, representations and localizations. In order to fix notation, we also provide a short reminder on super vector spaces. In Section 2, we recall general facts on super Lie algebras and their representations. We also remind the usual definition of the universal enveloping algebra, and discuss some of its properties. In the next section we provide some results on ideals and representations of enveloping algebras of super Lie algebras that will be used in the construction of the Dixmier map. As already explained, in these three sections we shall provide a list of results that will be used in the sequel and that will be the corresponding superized version of well-known facts for plain algebras and Lie algebras.

In Section 4 we provide a definition of polarization for super Lie algebras, which resembles more to the definition for Lie algebras than to the one implicit in [Kac77] for super Lie algebras (*i.e.* a maximal subordinate subalgebra). In order to do that, we first recall some definitions on bilinear forms on super vector spaces. Moreover, we remind the basic facts on polarizations of Lie algebras and some results proved by Sergeev in [Ser99]. At the end, we give the main result of the section, namely the fact that solvable super Lie algebras have polarizations and derive some consequences. In the final section we state and prove the main results of this article, described at the beginning, and we derive several consequences.

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1 Generalities on super algebras

From now on, we choose k to be an uncountable algebraically closed field of characteristic 0. All unadorned tensor products $\otimes$ are over k.

In the first subsection we recall the basic definitions of super vector spaces. Since the terminology sometimes varies in the literature, this section is also useful to fix the notation and some simple results we shall use in the sequel. We more or less follow the conventions of [DM99], which we suggest as a reference.

In Subsection 1.2 we will provide the definitions and some basic results on super algebras and their representations, stating several results concerning the two-sided ideals of a super algebra, that shall be crucial tu us. Finally, in the last subsection we will remind some useful basic facts on localization of super algebras.

1.1 Super vector spaces

We recall that a *super vector space* over k is a k-vector space V provided with a $\mathbb{Z}/2\mathbb{Z}$-grading of the form $V = V_0 \oplus V_1$. An element $v \in V$ is called *homogeneous* if $v \in V_i$ for some $i \in \mathbb{Z}/2\mathbb{Z}$, and more precisely, the elements of V_0 are called *even* and the ones which belong to V_1 are called *odd*. For a nonzero homogeneous vector $v \in V_i$ ($i \in \mathbb{Z}/2\mathbb{Z}$), we write $|v| = i$ and call it the *degree* or *parity* of v. When we speak about the parity of an element, we will always assume that it is homogeneous.

Given two super vector spaces V and W, a *morphism (of super vector spaces)* $f : V \to W$ is a k-linear map between the underlying vector spaces that preserves the grading. The vector space of morphisms from V to W is denoted by $\mathrm{Hom}(V, W)$. It is easy to see that the collection of super vector spaces provided with the previous morphisms is a k-linear category, denoted by sVect_k. A subobject of an object of this category will be called a *sub super vector space* or more simply a *subspace*. We may define the *super dimension* (called *dimension* in [DM99]) $\mathrm{sdim}(V)$ of a super vector space V as the pair $(\dim(V_0), \dim(V_1))$.

This category is provided of a functor Π, called *parity*, which satisfies that $\Pi(V)_i = V_{1-i}$, for $i \in \mathbb{Z}/2\mathbb{Z}$, and if $f : V \to W$, then $\Pi(f) = f$. Moreover, it is easy to see that this category is monoidal, when considering the tensor product given by $(V \otimes W)_i = \oplus_{j\in\mathbb{Z}/2\mathbb{Z}} V_j \otimes V_{i-j}$, for $i \in \mathbb{Z}/2\mathbb{Z}$, and the unit given by the super vector space of super dimension $(1, 0)$, which we shall denote k (instead of $\underline{1}$ in [DM99]). We may also consider the *internal hom* in the category which is the super vector space $\mathcal{H}om(V, W)$ (instead of $\underline{\mathrm{Hom}}$ in [DM99]) such that $\mathcal{H}om(V, W)_0 = \mathrm{Hom}(V, W)$ and $\mathcal{H}om(V, W)_1 = \mathrm{Hom}(V, \Pi(W))$. It is clear that there is an adjunction between the tensor product and the internal hom of the form

$$\mathrm{Hom}(V \otimes W, U) \simeq \mathrm{Hom}(V, \mathcal{H}om(W, U)).$$

In fact, the previous isomorphism is just the degree zero part of a natural isomorphism of super vector spaces

$$\mathcal{H}om(V \otimes W, U) \simeq \mathcal{H}om(V, \mathcal{H}om(W, U)).$$

If we consider the flip $V \otimes W \to W \otimes V$ given by $v \otimes w \to (-1)^{|v||w|} w \otimes v$, one sees that sVect_k is in fact a braided monoidal category.

1.2 Super algebras

A *super associative and unitary algebra* A is a vector space A provided with morphisms of super vector spaces $\mu : A \otimes A \to A$, called *product*, and an element $1_A \in A_0$ such that $\mu(\mu(a \otimes b) \otimes c) = \mu(a \otimes \mu(b \otimes c))$ for all $a, b, c \in A$ and $\mu(a \otimes 1_A) = a = \mu(1_A \otimes a)$, for all $a \in A$. As usual, we denote the product by a dot or simply by juxtaposition: $\mu(a \otimes b) = a \cdot b = ab$. By simplicity, super algebra will always denote a super associative and unitary algebra. Every super algebra is provided with an *isomorphism* Σ of order two, defined as $\Sigma(a_0 + a_1) = a_0 - a_1$, where a_i is a homogeneous element of degree i, for $i \in \mathbb{Z}/2\mathbb{Z}$. A *morphism* $\phi : A \to B$ of super algebras A and B is a morphism of the underlying super vector spaces $\phi : A \to B$ such that $\phi(aa') = \phi(a)\phi(a')$, for all $a, a' \in A$, and $\phi(1_A) = 1_B$. The *tensor product* of super algebras is canonically defined following the Koszul's sign rule.

A *left module* of a super algebra A is a super vector space V provided with a morphism of super algebras $\rho : A \to \mathcal{E}nd(V)$. Given two left A-modules V and W, a *morphism* from V to W is a morphism between the underlying super vector spaces $f : V \to W$ such that $f(av) = af(v)$, for all $a \in A$ and $v \in V$. There are similar definitions for right A-modules.

The following examples of super algebras will be of great importance to us.

Example 1.1. *(i) Given $n \in \mathbb{N}_0$, let $A_n(k)$ denote the super algebra over k given by*

$$k\langle q_1, \dots, q_n, p_1, \dots, p_n\rangle / \langle\{[q_i, p_j] - \delta_{ij}1, 1 \le i, j \le n\}\rangle,$$

where q_i and p_i are homogeneous of degree 0, for $i = 1, \dots, n$ (we remark that $A_0(k) = k$). It is thus concentrated in degree zero, so a plain algebra, and it is called the n-th Weyl algebra*.*

(ii) Let V be a vector space over k with a nondegenerate symmetric bilinear form $\langle\,,\rangle$. Define $\mathrm{Cliff}(V, \langle\,,\rangle)$ the super algebra given by $TV/\langle\{v \otimes w + w \otimes v - \langle v, w\rangle 1\}\rangle$, where the elements of V are all of degree 1. Then, it is a super algebra, called the Clifford algebra *of $(V, \langle\,,\rangle)$, and it can be proved that it only depends on the dimension n of V, so it will be also denoted by $\mathrm{Cliff}_n(k)$. Moreover, it is easy to see that $\mathrm{Cliff}_1(k) \simeq k[\epsilon]/(\epsilon^2 - 1)$, where $|\epsilon| = 1$, and that $\mathrm{Cliff}_2(k) \simeq M_2(k)$, where the even and odd parts of the matrix algebra $M_2(k)$ are*

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},$$

respectively. Furthermore, we have the so called Bott periodicity, *i.e.* $\mathrm{Cliff}_{n+2}(k) \simeq \mathrm{Cliff}_n(k) \otimes M_2(k)$ *(see* [Kar78], *Ch. III, Subsec. 3.23). We set $\mathrm{Cliff}_0(k) = M_0(k) = k$.*

The following proposition is known for algebras. The proof for super algebras is more or less similar, but we include it because we do not know a precise reference (*cf.* however [EGH$^+$10], Problem 1.23).

Proposition 1.2. *Let A be a super algebra and let V be a simple module over A, which we assume to have a countable homogeneous basis over k. Then, every A-linear endomorphism of V is given by a multiplication by a scalar in k.*

Proof. By Schur's Lemma in [Rac98], p. 591, we see that $\mathcal{E}nd_A(V)$ is a super field (*i.e.* every nonzero homogeneous element is invertible). So $\mathrm{End}_A(V)$ is a field. Following an idea of Dixmier, let us suppose that there exist an isomorphism $\phi \in \mathrm{End}_A(V)$ such that it is not the same as the multiplication by a scalar in k. This is equivalent to the fact that ϕ is not algebraic over k, for k is algebraically closed. Hence, ϕ is trascendental over k and $\mathrm{End}_A(V)$ contains a copy of the field $k(\phi)$, which has an uncountable basis, because k is not countable. However, since V is simple, any nonzero homogeneous element is a generator of the A-module V, say v, so every endomorphism of V is completely determined by its value at v, which is a linear combination of the countable homogeneous basis of V. So the dimension of $\mathrm{End}_A(V)$ over k is at most countable. This is a contradiction, so ϕ must be given by the multiplication by a scalar in k. The proposition is thus proved. □

Remark 1.3. *We remark the fact that $\mathrm{End}_A(V) \simeq k$ immediately implies that $\mathcal{E}nd_A(V)$ is k or $k[\epsilon]/(\epsilon^2 - 1)$, with $|\epsilon| = 1$ (see* [Var04], *Section 6.2, p. 215). This can be proved as follows. We first show that $\mathcal{E}nd_A(V)_1$ has super dimension less than or equal to 1. Let us suppose that it is not zero. We will prove that is one, i.e. that two homogeneous isomorphisms in $\mathcal{E}nd_A(V)$ degree 1 are linearly dependent. Given $\phi, \psi \in \mathcal{E}nd_A(V)$*

two homogeneous isomorphisms of degree 1, *there exists a nonzero* $c_{\phi,\psi} \in k$ *such that* $\phi \circ \psi = c_{\phi,\psi} 1_V$. *From that we see that* $c_{\phi,\phi} 1_V \circ \psi = \phi \circ \phi \circ \psi = \phi \circ c_{\phi,\psi} 1_V$. *So,* ϕ *is just a scalar multiple in* k *of* ψ, *and hence* $\dim(\mathcal{E}nd_A(V)_1) = 1$. *Moreover, since* $\phi^2 = c_{\phi,\phi} 1_V$, *with* $c_{\phi,\phi} \neq 0$, *we may define* $\epsilon = \phi/\sqrt{c_{\phi,\phi}}$.

A *subalgebra* of a super algebra A is a subspace of the underlying super vector space of A such that it is closed under the product of A. A *left (resp. right, two-sided) ideal* of A is subspace I of the super vector space underlying A such that $ax \in I$ (resp. $xa \in I$, $axa' \in I$, for all $a' \in A$ and) for all $a \in A$ and $x \in I$. A two-sided ideal will be usually called ideal. For clarity, we remark that in this article, the term left (resp. right, two-sided) ideal will always denote a left (resp. right, two-sided) *graded* or *super ideal*, which are sometimes used in the literature, unless we state the opposite.

Given two homogeneous elements $a, b \in A$, the *super commutator* $[a, b]$ of a and b is defined as $ab - (-1)^{|a||b|} ba$. We recall that a homogeneous element $z \in A$ is called *supercentral* if $[z, a] = 0$, for all homogeneous elements $a \in A$. The *super center* of A is the super vector space expanded by the supercentral elements of A. A homogeneous k-linear map d in $\mathcal{E}nd(A)$ is called a *derivation* if it satisfies the super Leibniz identity, *i.e.*

$$d(ab) = d(a)b + (-1)^{|a||d|} a d(b).$$

A super algebra A is called *left (resp. right) noetherian* if any left (resp. right) ideal has a finite set of homogeneous generators. Equivalently, A is left (resp. right) noetherian if it satisfies the ascending chain condition on left (resp. right) ideals. From now on, noetherian will always denote left noetherian, unless we say the contrary. It is obvious to see that if A is noetherian as an algebra, then it is noetherian as a super algebra.

We shall now recall some properties of two-sided ideals of super algebras.

An ideal I of a super algebra A is called *maximal* if $I \neq A$ and it is maximal in the set of all ideals of A different from A with respect to inclusion. It is called *primitive* if it is the annihilator of a simple left A-module. The *(Jacobson) radical* $J(A)$ of A is the intersection of all primitive ideals, or equivalently, the intersection of all maximal ideals of A (*cf.* [CM84], Sec. 4, p. 250). Since $\mathrm{char}(k) \neq 2$, we have that the Jacobson radical of the super algebra A coincides with the Jacobson radical of the underlying algebra of A (*cf.* [CM84], Thm. 4.4, (3)).

Moreover, I is called *prime* if $I \neq A$ and if whenever $JK \subseteq I$, for J, K ideals of A, then $J \subseteq I$ or $K \subseteq I$. Equivalently, I is prime if $I \neq A$ and for $a, b \in A$ homogeneous elements not in I, we have that $aAb \nsubseteq I$. The super algebra A is *integral* if $A \neq 0$ and the product of two nonzero homogeneous elements is nonzero. An ideal I of the super algebra A is *completely prime* if A/I is integral. The ideal I is called *semiprime* if $I \neq A$ and if in A/I every two-sided nilpotent ideal is null. It is obvious that the intersection of an arbitrary collection of semiprime ideals is semiprime.

It is trivial to see that a completely prime ideal is prime, and that a prime ideal is semiprime. The standard arguments show that a maximal ideal is primitive and that a primitive ideal is prime (*cf.* [Dix96], 3.1.6).

There is a strong relation between the concept of prime or maximal ideal for a super algebra and the same notion for the underlying algebra.

Lemma 1.4 ([CM84], Lemma 5.1 and Thm. 6.3)**.** *Let* A *be a super algebra and* I *an ideal of* A. *The following are equivalent:*

(i) I *is a prime (resp. maximal) ideal of the super algebra* A.

(ii) $I = P \cap \Sigma(P)$, *for some prime (resp. maximal) ideal* P *of the underlying algebra of* A.

Definition/Proposition 1.5. *Let* A *be a noetherian super algebra and let* $I \neq A$ *be an ideal of* A. *Consider* $\mathcal{E}$ *the family of all ideals* J *in* A *satisfying that there exist* $m \in \mathbb{N}$ *such that* $J^m \subseteq I$. *Then* $\mathcal{E}$ *has a largest element, which satisfies that it is the smallest semi-prime ideal of* A *containing* I, *and which will be called the* root *of* I. *On the other hand, if* $\mathcal{P}$ *denotes the set of prime ideals of* A *containing* I, *then* $\mathcal{P}$ *has a finite number of minimal elements. Moreover, their intersection is the root of* I, *and none of the minimal elements of* $\mathcal{P}$ *contains the intersection of the others.*

Proof. The proof for the non-graded case given in [Dix96], Prop. 3.1.8, for the first statement, and Prop. 3.1.10, for the second one, applies word for word. □

Lemma 1.6. *Let A be a super algebra A and let $I \neq A$ be an ideal of A. Given a set of derivations $\mathcal{D}$ of A, define J the super vector space formed by the elements $x \in A$ such that $d_1 \dots d_n x \in I$, for all $d_1, \dots, d_n \in \mathcal{D}$ and $n \in \mathbb{N}_0$. Then, we see that J is the largest ideal of A contained in I which is stable under $\mathcal{D}$. Moreover, if I is prime, J is also. As a consequence, if I is stable under a set of derivations $\mathcal{D}$, then the minimal prime ideals of A containing I and the root of I are stable under $\mathcal{D}$.*

Proof. The fact that J is prime, if I is so, can be deduced as follows. We recall that each derivation $d \in \mathcal{D}$ is an homogeneous morphism. Since I is prime, Lemma 1.4 implies that there exists a prime ideal P of the underlying algebra of A such that $I = P \cap \Sigma(P)$. By [Dix96], Lemma 3.3.2, the ideal Q of the underlying algebra of A formed by the elements $x \in A$ such that $d_1 \dots d_n x \in I$, for all $d_1, \dots, d_n \in \mathcal{D}$ and $n \in \mathbb{N}_0$, is prime. It is easily verified that $J = Q \cap \Sigma(Q)$, so J is a prime ideal of the super algebra A, by Lemma 1.4. The proof for the remaining statement is the same as the one for the non-graded case in [Dix96], Lemma 3.3.3. □

The following is a super version of a well-known theorem of Levitzki.

Lemma 1.7. *Let A be a noetherian super algebra and let I be a left ideal, such that all its homogeneous elements are nilpotent. Then I is nilpotent. Moreover, if $\{0\}$ is a semi-prime ideal of A, then $I = \{0\}$.*

Proof. The proof for the non-graded case given in [AF92], Thm. 15.22, or [Dix96], Lemma 3.1.14, works as well, with the additional assumption that all elements there should be homogeneous (and replacing in the last reference the use of [Dix96], Prop. 3.1.8, by Definition/Proposition 1.5). □

The next proposition is standard. The proof, which we give for clarity, follows the same idea of that given in [Dix96], Prop. 3.1.15, but with some minor changes.

Proposition 1.8. *Let A be a noetherian super algebra with a countable basis and let $I \neq A$ be an ideal. Then, the following are equivalent:*

(i) I is semiprime.

(ii) I is an intersection of primitive ideals.

Proof. The implication $(ii) \Rightarrow (i)$ is trivial. For the converse, let us assume that I is semiprime and set $B = A/I$ and J the radical of B. We shall prove that every homogeneous element of J is nilpotent. Once we have showed this, using the previous lemma, we may conclude that J is nilpotent, and since $\{0\}$ is semiprime in B, we have that $J = 0$. This in turn implies that I is primitive, as needed.

Let us now prove that every homogeneous element a of J is nilpotent. Given a homogeneous element $a \in J$, consider a homogeneous indeterminate x of the same parity as a and the super algebra $C = B \otimes k[x]$. We claim that $C(1 - ax) = C$. Suppose that it is not the case. Then $1 - ax$ should be included in a maximal left ideal of C, so there exist a simple left C-module S and a nonzero element $s \in S$ such that $(1 - ax)s = 0$. Since x is supercentral in C, it defines a nonzero element in $\mathcal{E}nd_C(S)$ (because $ax(s) = s$), which we also denote by x. By the proof of Proposition 1.2 and Remark 1.3, we see that $\mathcal{E}nd_A(S) = k$ or $\mathcal{E}nd_A(S) = k[\epsilon]/(\epsilon^2 - 1)$, with $|\epsilon| = 1$. This implies that, if $y = x^{-1}$, there exists a linear polynomial $p \in k[X]$ such that $p(y) = x$, so $as = ys$, and further $0 = (1 - ap(a))s = (1 - yp(y))s$. Since the radical of the super algebra A coincides with the radical of its underlying algebra, and using [Dix96], Prop. 3.1.12, the latter identity contradicts the fact that $a \in J$. We have thus proved that $C(1 - ax) = C$. As a consequence, there exist $a_0, \dots, a_m \in B$ such that

$$(a_0 + \dots + a_m x^m)(1 - ax) = 1,$$

which implies that $a_0 = 1$, $a_1 = a$, ..., $a_m = a^m$ and $a^{m+1} = 0$, so a is nilpotent, as we wanted to prove. The proposition thus follows. □

We recall that a super algebra A over k is said to be *central simple* if its super center is k and it has no nontrivial two-sided ideals. We remark that we do not require A to be semisimple, as in [Var04], Section 6.2. As examples of central simple super algebras we have $A_n(k)$ (see [FD93], Part III, Exercise 26) and $\mathrm{Cliff}_n(k)$, for $n \in \mathbb{N}_0$ (see [Var04], Section 6.2, p. 215). We refer to [Lam80], Ch. 4, §2, or [Var04], Section 6.2, for a more detailed study on (finite dimensional) central simple super algebras. The following analogous result to the Azumaya-Nakayama's Theorem will be used in the sequel.

Lemma 1.9. *Let A be a central simple super algebra, B a super algebra, $\mathcal{I}$ the set of ideals of B and $\mathcal{I}'$ the set of ideals of $A \otimes B$. Then, the map from $\mathcal{I}$ to $\mathcal{I}'$ given by $I \mapsto A \otimes I$ is a bijection. Also, the super center of the tensor product is given by $\mathcal{Z}(A \otimes B) = \mathcal{Z}(B)$. Moreover, I is a maximal (resp. prime) ideal of B if and only if $A \otimes I$ is a maximal (resp. prime) ideal of $A \otimes B$.*

Proof. The proof of the first two statements is analogous to the one given in [FD93], Thm. 3.5 and Lemma 3.7, but taking into account that all elements must be homogeneous and one should use super commutators instead of commutators (*cf.* [Lam80], proof of Thm. 2.3). This immediately implies the assertion concerning maximal ideals. The proof of the statement for prime ideals is the same as the one given for [Dix96], Lemma 4.5.1. □

As a direct corollary of the previous result we have (*cf.* [Lam80], Thm. 2.3):

Corollary 1.10. *Let A and B be two central simple super algebras. Then the tensor product $A \otimes B$ is also a central simple super algebra.*

1.3 Localization of super algebras

In this subsection we shall recall some facts on localization of super algebras. Even though some of these results may be stated in more general forms, we restrict ourselves to the cases we need. We refer to [NVO04] for a more comprehensive exposition. Furthermore, some of the results we will state here are the obvious generalizations (with the standard proofs) of those that can be found for instance in [Dix96], Ch. 3, §6, for the case of algebras.

If S is a subset of homogeneous elements of A, it is said to *allow of an arithmetic of fractions* if

(i) $1 \in S$, $0 \notin S$ and S is multiplicative closed,

(ii) If $a \in A$ is a homogeneous element and $s \in S$ are such that $sa = 0$ (resp. $as = 0$), then there exists $s' \in S$ such that $as' = 0$ (resp. $s'a = 0$).

(iii) For $s \in S$ and $a \in A$ homogeneous elements, there exist $t \in S$ and $b \in A$ (resp. $t' \in S$ and $b' \in A$) such that $ta = bs$ (resp. $at' = sb'$).

The left (resp. right) versions of conditions (ii) and (iii) are called the *graded right Ore conditions* (resp. the *graded left Ore conditions*) and if only the left (resp. right) versions are satisfied A is said to *allow of an arithmetic of left (resp. right) fractions*. They are equivalent to the usual left (resp. right) Ore conditions (see [NVO04], Lemma 8.1.1). Moreover, the left (resp. right) condition (ii) is always satisfied if A is left (resp. right) noetherian as an algebra. If the algebra A allows of an arithmetic of fractions, the left and right localization rings $S^{-1}A$ and AS^{-1} can be defined in the obvious way, they are super algebras and in fact coincide (see [NVO04], Prop. 8.1.2). We denote any of them by A_S. If z is a homogeneous element and $S_z = \{z^n : n \in \mathbb{N}_0\}$, then we will usually write A_z instead of A_{S_z}. From now on we shall restrict to the case that S does not contain zero divisors and that all of its elements are of degree 0. Then, we have the following result:

Lemma 1.11. *Let A be a super algebra and S a set of elements of degree 0 allowing of an arithmetic of fractions, and let I be a two-sided ideal of A satisfying that, if $sa \in I$, for $s \in S$ and $a \in A$, then $a \in I$. Let us consider IS^{-1} and $S^{-1}I$ the subspaces of the super vector space underlying A_S expanded by the homogeneous elements as^{-1} and $s^{-1}a$ respectively, where $a \in I$ is homogeneous and $s \in S$. Then, $S^{-1}I \subseteq IS^{-1}$ and IS^{-1} is in fact a two-sided ideal of A_S.*

Proof. The standard proof given in [Dix96], Lemma 3.6.14 works in this case as well, taking into account that all elements there should be homogeneous. □

Proposition 1.12. *Let A be a super algebra and S a set of elements of degree 0 allowing of an arithmetic of fractions. Define $\mathcal{I}_S$ the set of two-sided ideals of A_S and $\mathcal{I}$ the set of two-sided ideals of A satisfying the following property: either $as \in I$ or $sa \in I$, for $s \in S$ and $a \in A$ homogeneous, implies that $a \in I$. Then, if $I \in \mathcal{I}$, it holds that $S^{-1}I = IS^{-1}$, which we simply denote I_S, and the maps from $\mathcal{I} \to \mathcal{I}_S$ given by $I \mapsto I_S$ and $\mathcal{I}_S \to \mathcal{I}$ by $I' \mapsto I' \cap A$ are mutually inverse. Furthermore, if $I \in \mathcal{I}$ is prime, so is I_S.*

Proof. The standard proof given in [Dix96], Prop. 3.6.15 also applies in this case, taking into account that all elements there should be homogeneous, and replacing the use of [Dix96], Lemma 3.6.14 by the previous lemma. □

Finally, we have the following simple result.

Proposition 1.13. *Let A be a super algebra and S a set of elements of degree 0 allowing of an arithmetic of fractions. Then any derivation $d : A \to A$ can be extended to a unique derivation $d_S : A_S \to A_S$.*

Proof. The proof in [Dix96], Prop. 3.6.18 also applies in this case, taking into account that all elements there should be homogeneous and the appearance of harmless signs due to the Koszul's sign rule. □

2 Generalities on super Lie algebras

In the first subsection we shall provide the basic definitions and results on super Lie algebras, most of those can be found in [Kac77] or [Sch79], which we suggest as a reference. We will also recall the standard relation between super Lie algebras and super algebras given by the universal enveloping algebra. In the last subsection we shall focus on representations of super Lie algebras, stating some results on induced modules that will be used all throughout the paper.

2.1 Basic facts on super Lie algebras

A *super Lie algebra* over the field k is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ provided with a morphism

$$[\,,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

called *Lie bracket*, such that

- the bracket is *superskewsymmetric*; *i.e.* $[x, y] = -(-1)^{|x||y|}[y, x]$, for all nonzero homogeneous $x, y \in \mathfrak{g}$,
- the bracket satisfies the super Jacobi identity; *i.e.* $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$, for all nonzero homogeneous $x, y, z \in \mathfrak{g}$,

Instead of the most common denomination "Lie superalgebra", which appears in [Kac77], we prefer to use the more systematic terminology in [DM99]. From now on, even though it is not necessary in many definitions, we will suppose that the underlying vector space of the super Lie algebra is finite dimensional.

A *morphism* $\phi : \mathfrak{g} \to \mathfrak{g}'$ between two super Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ is a morphism of the underlying super vector spaces $\mathfrak{g} \to \mathfrak{g}'$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$, for all homogeneous $x, y \in \mathfrak{g}$. We thus have the category of super Lie algebras.

The following proposition is direct.

Proposition 2.1 ([Var04], Section 3.1, p. 89). *A super vector space $\mathfrak{g}$ is a super Lie algebra if and only if $\mathfrak{g}_0$ is a Lie algebra, $\mathfrak{g}_1$ is a module over $\mathfrak{g}_0$, there exists a $\mathfrak{g}_0$-equivariant linear map $S^2\mathfrak{g}_1 \to \mathfrak{g}_0$, and it holds that*

$$[x, [x, x]] = 0, \qquad \forall\, x \in \mathfrak{g}_1.$$

Example 2.2. *(i) Given a super algebra A, we may regard it as a super Lie algebra with the bracket given by the* super commutator $[a, b] = ab - (-1)^{|a||b|}ba$. *We denote this structure as* $\mathrm{sLie}(A)$.

(ii) Given a super vector space V of super dimension (n, m) we may consider the super vector space $\mathcal{E}nd(V)$. It has structure of super algebra for the product given by composition, and thus of a super Lie algebra provided with the Lie bracket given by the super commutator. It is denoted $\mathfrak{gl}(V)$ or $\mathfrak{gl}(n|m)$.

An *algebraic super group* over k is a pair $(G_0, \mathfrak{g}, \rho)$ given by an algebraic group G_0, by a super Lie algebra $\mathfrak{g}$, both over k, and by a representation of G_0 on $\mathfrak{g}$ by automorphisms of super Lie algebras, such that the Lie algebra of G_0 is $\mathfrak{g}_0$, and the differential of ρ coincides with the restriction to $\mathfrak{g}_0$ of the adjoint action $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ of $\mathfrak{g}$ on itself. The super Lie algebra $\mathfrak{g}$ is called the *super Lie algebra associated to the super group* $(G_0, \mathfrak{g}, \rho)$. We will usually write an algebraic super group more simply $(G_0, \mathfrak{g})$ to simplify the notation.

A *morphism of algebraic super groups* from $(G_0, \mathfrak{g}, \rho)$ to $(H_0, \mathfrak{h}, \sigma)$ is a pair (f_0, ϕ) formed of a morphism of algebraic groups $f : G_0 \to H_0$ and of a morphism of super Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ satisfying that the differential of f coincides with the restriction of $\phi|_{\mathfrak{g}_0}$ and that the map ϕ is equivariant, *i.e.* $\sigma(f(g))(\phi(x)) = \phi(\rho(g)(x))$, for all $x \in \mathfrak{g}$ and $g \in G_0$. The definition of algebraic super group (and the corresponding morphisms between them) that we use is the algebraic version of the so called *super Harish-chandra pair* (*cf.* [Kos77] or [Kos83]).

Example 2.3. *Given V a super vector space, $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$ is an algebraic group, denoted by $\mathrm{GL}(V)_0$. It acts on the super Lie algebra $\mathfrak{gl}(V)$ by conjugations in such a way that $(\mathrm{GL}(V)_0, \mathfrak{gl}(V))$ is an algebraic super group.*

Given a super Lie algebra $\mathfrak{g}$, we can consider a super algebra associated to it, called the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$, which is defined as the quotient of the tensor algebra $T\mathfrak{g}$ by the ideal generated by $\{x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]\}$ for all homogeneous $x, y \in \mathfrak{g}$. The $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{U}(\mathfrak{g})$ is induced from the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$. This super algebra satisfies the universal property $\mathrm{Hom}(\mathcal{U}(\mathfrak{g}), A) \simeq \mathrm{Hom}(\mathfrak{g}, \mathrm{sLie}(A))$, where the first morphism space if of super algebras and the second one is of super Lie algebras. It is provided with an increasing filtration of super vector spaces $\{F^{\bullet}\mathcal{U}(\mathfrak{g})\}_{\bullet \in \mathbb{N}_0}$ coming from the filtration of the tensor algebra $T\mathfrak{g}$ given by its usual grading. It is easy to prove that the underlying algebra of the super algebra $\mathcal{U}(\mathfrak{g})$, for $\mathfrak{g}$ a finite dimensional super Lie algebra, is noetherian (see [Beh87], Prop. 3.1, (i)), so *a fortiori* the super algebra $\mathcal{U}(\mathfrak{g})$ is noetherian.

As well as for enveloping algebras over Lie algebras, there is a PBW theorem for super Lie algebras.

Theorem 2.4 ([Ros65], Thm. 2.1)**.** *Let $\{x_1, \dots, x_s\}$ be an ordered basis of $\mathfrak{g}$ consisting of homogeneous elements. Then the set of all products of the form*

$$x_1^{p_1} \dots x_s^{p_s}$$

where $x_i^0 = 1$, $p_i \in \mathbb{N}_0$ and $p_i \leq 1$ whenever x_i is odd, is a basis for $\mathcal{U}(\mathfrak{g})$.

We recall that an *anti-automorphism* of a super algebra A is an isomorphism ϕ of the underlying super vector space satisfying that $\phi(xy) = (-1)^{|x||y|}\phi(y)\phi(x)$, for all homogeneous elements $x, y \in A$, and $\phi(1) = 1$. The enveloping algebra $\mathcal{U}(\mathfrak{g})$ is provided with an anti-automorphism α, called *principal*, such that $\alpha(x) = -x$, for $x \in \mathfrak{g}$. In fact, it is easily proved that

$$\alpha(x_1 \dots x_n) = (-1)^{n + \sum_{i<j} |x_i||x_j|} x_n \dots x_1,$$

where $x_1, \dots, x_n$ are homogeneous elements of the super Lie algebra $\mathfrak{g}$, and $n \in \mathbb{N}$.

A *sub super Lie algebra* of a super Lie algebra $\mathfrak{g}$ is a super vector space $\mathfrak{h} \subseteq \mathfrak{g}$ closed under the bracket operation, *i.e.* if $x, y \in \mathfrak{h}$, then $[x, y] \in \mathfrak{h}$. Analogously, a *super Lie ideal* of a super Lie algebra $\mathfrak{g}$ is a super vector space $\mathfrak{k} \subseteq \mathfrak{g}$ that satisfies that, for all $x \in \mathfrak{g}$ and $y \in \mathfrak{k}$, $[x, y] \in \mathfrak{k}$. Equivalently, we could have given the previous two definitions just in terms of homogeneous elements. Since we shall often work in the "super" context, we will usually use the shorter terms subalgebra and ideal, unless we need to make the distinction.

Given a super vector space $V \subseteq \mathfrak{g}$ (resp. a set of homogeneous elements $S \subseteq \mathfrak{g}$), the *super centralizer* of V (resp. S) is the super vector space $\mathcal{C}(V)$ (resp. $\mathcal{C}(S)$) expanded by the homogeneous elements $x \in \mathfrak{g}$ such that $[x, y] = 0$, for all homogeneous $y \in V$ (resp. $y \in S$). It is easily seen to be a subalgebra of $\mathfrak{g}$. The super centralizer of $\mathfrak{g}$ is called the *super center* of the super Lie algebra $\mathfrak{g}$, which is denoted $\mathcal{Z}(\mathfrak{g})$, and it is an ideal of $\mathfrak{g}$.

On the other hand, given two super vector spaces $V, W \subseteq \mathfrak{g}$ (resp. two sets of homogeneous elements $S, T \subseteq \mathfrak{g}$), the *super commutator* $[V, W]$ (resp. $[S, T]$) of V and W (resp. of S and T) is the super vector space expanded by $[x, y]$, for all the homogeneous elements $x \in V$ and $y \in W$ (resp. $x \in S$ and $y \in T$). If in the previous definition V and W are ideals, the super commutator is also an ideal. In particular, $[\mathfrak{g}, \mathfrak{g}]$ is called the *derived algebra* of $\mathfrak{g}$.

Example 2.5. *Given a super Lie algebra* $\mathfrak{g}$*, the super vector space* $\mathrm{Der}(\mathfrak{g})$ *expanded by the homogeneous maps* $d \in \mathcal{H}om(\mathfrak{g}, \mathfrak{g})$ *satifying that*

$$d([x, y]) = [d(x), y] + (-1)^{|x||d|}[x, d(y)]$$

is called the space of derivations *of* $\mathfrak{g}$*. It is a super Lie algebra with the bracket provided by the super commutator. It is clear that the image* $\mathrm{InnDer}(\mathfrak{g})$ *of the morphism of super Lie algebras* $\mathrm{ad} : \mathfrak{g} \to \mathcal{H}om(\mathfrak{g}, \mathfrak{g})$ *is an ideal of* $\mathrm{Der}(\mathfrak{g})$*, called the space of* inner derivations *of* $\mathfrak{g}$*. Note that* ad *induces an obvious identification* $\mathrm{InnDer}(\mathfrak{g})_0 = \mathrm{InnDer}(\mathfrak{g}_0)$.

Example 2.6. *We shall now present another example of algebraic super group that will be used in the sequel. If* $\mathfrak{g}$ *is a finite dimensional super Lie algebra, the set of automorphisms* $\mathrm{Aut}(\mathfrak{g})$ *of* $\mathfrak{g}$ *(we recall that we are considering automorphisms of* $\mathfrak{g}$ *homogeneous of degree* 0*) is a closed algebraic subgroup of the linear algebraic group* $\mathrm{GL}(\mathfrak{g})_0$*, and its associated Lie algebra is* $\mathrm{Der}(\mathfrak{g})_0$*, the degree zero component of the super Lie algebra considered in Example 2.5 (cf.* [TY05]*, Prop. 24.3.7). The algebraic action of* $\mathrm{Aut}(\mathfrak{g})$ *on itself by conjugation induces a morphism of algebraic groups* $\mathrm{Ad} : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{GL}(\mathrm{Der}(\mathfrak{g})_0)$ *(see* [TY05]*, 23.5.2). Moreover, the previous action is just the restriction to* $\mathrm{Der}(\mathfrak{g})_0$ *of the action of* $\mathrm{Aut}(\mathfrak{g})$ *on* $\mathrm{Der}(\mathfrak{g})$ *given by conjugation. Then, the pair* $(\mathrm{Aut}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g}))$ *is an algebraic super group, called the* algebraic super group of automorphisms *of* $\mathfrak{g}$.

As in the non-graded situation, we may consider the *lower central series* of $\mathfrak{g}$ to be the decreasing sequence of ideals defined recursively by $\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}$ and $\mathcal{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})]$ for $i \geq 2$. Furthermore, the *derived series* of $\mathfrak{g}$ is the decreasing sequence of ideals defined recursively by $\mathcal{D}^0(\mathfrak{g}) = \mathfrak{g}$ and $\mathcal{D}^i(\mathfrak{g}) = [\mathcal{D}^{i-1}(\mathfrak{g}), \mathcal{D}^{i-1}(\mathfrak{g})]$ for $i \in \mathbb{N}$. It is easy to see that $\mathcal{D}^i(\mathfrak{g}) \subseteq \mathcal{C}^{i+1}(\mathfrak{g})$, for all $i \in \mathbb{N}_0$. A super Lie algebra $\mathfrak{g}$ is called *solvable* if the there exists $i \in \mathbb{N}_0$ such that $\mathcal{D}^i(\mathfrak{g}) = 0$. Analogously, $\mathfrak{g}$ is said to be *nilpotent* if the there exists $i \in \mathbb{N}_0$ such that $\mathcal{C}^i(\mathfrak{g}) = 0$. It is clear that a nilpotent super Lie algebra is solvable.

The following result indicates that the solvability of a super Lie algebra only relies on its even part.

Proposition 2.7 ([Kac77], Prop. 1.3.3, or [Ser99], Cor. 2.3)**.** *A super Lie algebra* $\mathfrak{g}$ *is solvable if and only if the Lie algebra* $\mathfrak{g}_0$ *is solvable.*

There is also a version of Engel's theorem for super Lie algebras and it is proved in exactly the same way as for (plain) Lie algebras.

Proposition 2.8 ([Sch79], Ch. III, §2, 1., Prop. 1)**.** *Let* $\mathfrak{g}$ *be a subalgebra of* $\mathfrak{gl}(n|m)$ *such that, for every homogeneous* $x \in \mathfrak{g}$*,* $\mathrm{ad}(x)$ *is a nilpotent operator. Then there is a homogeneous vector* v *in the super vector space of super dimension* (n, m) *such that* $x(v) = 0$*, for all* $x \in \mathfrak{g}$*.*

As a corollary of the previous result we have:

Corollary 2.9 ([Sch79], Ch. III, §2, 1., Coro. 1)**.** *A super Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if for every homogeneous* $x \in \mathfrak{g}$*,* $\mathrm{ad}(x)$ *is a nilpotent operator. As a consequence, a super Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* $\mathfrak{g}_0$ *is a nilpotent Lie algebra and the action of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_1$ *is by nilpotent operators.*

2.2 Representations of super Lie algebras

A *left representation* of a super Lie algebra $\mathfrak{g}$ (or a *left* $\mathfrak{g}$*-representation*) is a super vector space V provided with a morphism of super Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Equivalently, a left representation of $\mathfrak{g}$ is a super vector space V provided with a morphism of super vector spaces

$$\rho' : \mathfrak{g} \otimes V \to V$$

such that, for all homogeneous $x, y \in \mathfrak{g}$,

$$\rho'(x \otimes \rho'(y \otimes v)) - (-1)^{|x||y|}\rho'(y \otimes \rho'(x \otimes v)) = \rho'([x, y] \otimes v).$$

It is clear that $\rho'(x \otimes v) = \rho(x)(v)$. We shall usually denote the action by a dot or even by juxtaposition, *i.e.* $\rho(x)(v) = x \cdot v = xv$.

Given a left $\mathfrak{g}$-representation V with structure morphism ρ, the parity changed representation ΠV is defined as follows. The underlying super vector space is just the parity functor Π applied to the underlying super vector space of the $\mathfrak{g}$-representation V. However, the action satisfies the identity

$x.v = (-1)^{|x|}\rho(x)(v)$, for homogeneous $x \in \mathfrak{g}$ and $v \in \Pi V$, and where the left member stands for the action of x on $v \in \Pi V$, but on the right member we are considering the action of x on V.

Given two left representations V and W of $\mathfrak{g}$, a *morphism* $f : V \to W$ is a map of the underlying super vector spaces such that $f(xv) = xf(v)$. We denote the space of such morphisms by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$. We will also consider the super vector space of morphisms $\mathcal{H}om_{\mathfrak{g}}(V, W)$ given by $\mathrm{Hom}_{\mathfrak{g}}(V, W)_0 = \mathrm{Hom}_{\mathfrak{g}}(V, W)$ and $\mathrm{Hom}_{\mathfrak{g}}(V, W)_1 = \mathrm{Hom}_{\mathfrak{g}}(V, \Pi W)$. There are similar definitions for right representations.

A *linear action* (or *representation*) of an algebraic super group $(G_0, \mathfrak{g})$ on a super vector space V is given by a morphism of algebraic super groups $(G_0, \mathfrak{g}) \to (\mathrm{GL}(V)_0, \mathfrak{gl}(V))$. Analogously as before, there are obvious definitions of morphisms of linear representations of algebraic super groups.

It is trivial to see that the category of left representations of $\mathfrak{g}$ is equivalent to the category of left modules of the super algebra $\mathcal{U}(\mathfrak{g})$, since $\mathrm{Hom}(\mathfrak{g}, \mathfrak{gl}(V)) \simeq \mathrm{Hom}(\mathcal{U}(\mathfrak{g}), \mathcal{E}nd(V))$, where the first morphism space if of super Lie algebras and the second one is of super algebras. From now on, we will deal only with left representations and modules, and just call them representations and modules, respectively. Moreover, for a representation V of $\mathfrak{g}$, the corresponding morphisms $\mathfrak{g} \to \mathfrak{gl}(V)$ and $\mathcal{U}(\mathfrak{g}) \to \mathcal{E}nd(V)$ are called the *structure morphisms* of V.

Example 2.10. *The* adjoint representation *of* $\mathfrak{g}$ *in itself given by* $x.y = \mathrm{ad}(x)(y) = [x, y]$ *can be extended by derivations to a representation in* $\mathcal{U}(\mathfrak{g})$*, which is called the* adjoint representation *of* $\mathfrak{g}$ *in* $\mathcal{U}(\mathfrak{g})$*. We shall denote this representation by* $\mathcal{U}(\mathfrak{g})^{\mathrm{ad}}$ *and the structure morphism by* ad*. More generally, if* $\mathfrak{k}$ *is an ideal of* $\mathfrak{g}$*, then it is a subrepresentation of the adjoint representation of* $\mathfrak{g}$ *in itself, and it can also be extended by derivations to a representation of* $\mathfrak{g}$ *in* $\mathcal{U}(\mathfrak{k})$*, which is also called the* adjoint representation *of* $\mathfrak{g}$ *in* $\mathcal{U}(\mathfrak{k})$*.*

The following lemma is the super version of [Dix96], Lemma 2.2.22 and will be needed later. The proof is similar to the non-super case, but we provide it because of the signs, which come from the use of the Koszul's sign rule.

Lemma 2.11 (*cf.* [Dix96], Lemma 2.2.22). *Let* $\mathfrak{k}$ *be an ideal of a super Lie algebra* $\mathfrak{g}$ *and define* $\delta = \mathrm{ad} \circ \alpha$*, where* α *denotes the principal anti-automorphism of* $\mathcal{U}(\mathfrak{g})$ *and* $\mathrm{ad} : \mathcal{U}(\mathfrak{g}) \to \mathcal{E}nd(\mathcal{U}(\mathfrak{k}))$ *is the structure morphism of the adjoint representation of* $\mathfrak{g}$ *in* $\mathcal{U}(\mathfrak{k})$*. For* $p \geq 0$*, consider* $y_1, \dots, y_p \in \mathfrak{g}$ *and* $z \in \mathcal{U}(\mathfrak{k})$ *homogeneous elements and* $n_1, \dots, n_p \in \mathbb{N}$*, then*

$$zy_1^{n_1} \dots y_p^{n_p} = \sum_{0 \leq m_i \leq n_i} (-1)^{\epsilon^{n_1,\dots,n_p}_{m_1,\dots,m_p}} \binom{n_1}{m_1} \dots \binom{n_p}{m_p} y_1^{m_1} \dots y_p^{m_p} \delta(y_1^{n_1-m_1} \dots y_p^{n_p-m_p})(z), \tag{2.1}$$

where

$$\epsilon^{n_1,\dots,n_p}_{m_1,\dots,m_p} = |z| \sum_{i=1}^{p} n_i |y_i| + \sum_{i<j} (n_i - m_i)|y_i| m_j |y_j|.$$

Proof. We proceed by induction on p. The previous identity clearly holds for $p = 0$. Let us assume that it is true for $p - 1$, and we shall prove it for p. This implies that it holds for p but $n_1 = 0$. By induction again, we suppose that (2.1) holds for p and $n_1 - 1$ (and arbitrary $n_2, \dots, n_p$), and we will prove it for n_1. Since $zy_1 = (-1)^{|z||y_1|}(y_1 z + \delta(y_1)(z))$, we obtain that

$$\begin{aligned}
zy_1^{n_1} \dots y_p^{n_p} &= (-1)^{|z||y_1|}(y_1 z + \delta(y_1)(z)) y_1^{n_1-1} \dots y_p^{n_p} \\
&= (-1)^{|y||z|} \sum_{\substack{0 \leq m_1 \leq n_1 - 1 \\ 0 \leq m_i \leq n_i \\ 2 \leq i \leq p}} (-1)^{\epsilon^{n_1-1,\dots,n_p}_{m_1,\dots,m_p}} \binom{n_1 - 1}{m_1} \dots \binom{n_p}{m_p} y_1^{m_1+1} \dots y_p^{m_p} \delta(y_1^{n_1-m_1-1} \dots y_p^{n_p-m_p})(z) \\
&\quad + (-1)^{\delta} \sum_{\substack{0 \leq m_1 \leq n_1 - 1 \\ 0 \leq m_i \leq n_i \\ 2 \leq i \leq p}} (-1)^{\epsilon^{n_1-1,\dots,n_p}_{m_1,\dots,m_p}} \binom{n_1 - 1}{m_1} \dots \binom{n_p}{m_p} y_1^{m_1} \dots y_p^{m_p} \delta(y_1^{n_1-m_1} \dots y_p^{n_p-m_p})(\delta(y_1)(z)) \\
&= \sum_{0 \leq m_i \leq n_i} (-1)^{\epsilon^{n_1,\dots,n_p}_{m_1,\dots,m_p}} \binom{n_1}{m_1} \dots \binom{n_p}{m_p} y_1^{m_1} \dots y_p^{m_p} \delta(y_1^{n_1-m_1} \dots y_p^{n_p-m_p})(z),
\end{aligned}$$

with $\delta = |y||z| + |y_1| \sum_{i=2}^{p} (n_i - m_i)|y_i|$. We remark that we have used the inductive assumption in the third member, and the identities

$$\delta(y_1^{n_1-m_1} \dots y_p^{n_p-m_p}) = (-1)^{|y_1| \sum_{i=2}^{p} (n_i - m_i)|y_i|} \delta(y_1^{n_1-m_1-1} \dots y_p^{n_p-m_p}) \delta(y_1)$$

and

$$\binom{n_1-1}{m_1-1}+\binom{n_1-1}{m_1}=\binom{n_1}{m_1}$$

in the last member. □

A *subrepresentation* of a representation V is a subspace W of the super vector space V such that $\rho(x)(w) \in W$, for all $w \in W$ and all $x \in \mathfrak{g}$. It is clear that any nonzero representation V has at least two different subrepresentations: V and 0, which are called *trivial*. We shall say that a representation V is *irreducible* or *simple* if its only subrepresentations are trivial. We remark that an irreducible representation of the super Lie algebra $\mathfrak{g}$, or equivalently, of the super algebra $\mathcal{U}(\mathfrak{g})$, may have nontrivial subspaces of its underlying vector space which are invariant under the action of $\mathfrak{g}$.

Proposition 2.12 (*cf.* [Dix96], Prop. 2.6.5)**.** *Let V be a $\mathfrak{g}$-representation. Then the following are equivalent*

(i) V is simple.

(ii) V is simple and every $\mathfrak{g}$-linear endomorphism of V is given by the multiplication by a scalar in k.

(iii) $V \neq 0$ and, for any set of homogeneous elements $x_1, \dots, x_n, y_1, \dots, y_n \in V$ satisfying that $x_i \in V_b$ and $y_i \in V_a$ for fixed $a, b \in \mathbb{Z}/2\mathbb{Z}$ and for all $i = 1, \dots, n$, and $x_1, \dots, x_n$ linearly independent over k, there exists a homogeneous $z \in \mathcal{U}(\mathfrak{g})$ such that $zx_i = y_i$, for all $i = 1, \dots, n$.

Proof. The implication $(iii) \Rightarrow (i)$ (and also $(ii) \Rightarrow (i)$) is trivial. On the other hand, the implication $(ii) \Rightarrow (iii)$ follows from the Density Theorem for graded rings (*cf.* [Rac98], Lemma 2, and [ELS04], Thm. 1.3). Finally, the implication $(i) \Rightarrow (ii)$ is a consequence of Proposition 1.2, since every simple $\mathfrak{g}$-representation has a homogeneous countable basis, because the algebra $\mathcal{U}(\mathfrak{g})$ does so. □

Given a subalgebra $\mathfrak{h}$ of a super Lie algebra $\mathfrak{g}$ and a representation W of $\mathfrak{h}$, the *representation of* $\mathfrak{g}$ *induced by* W, denoted by $\operatorname{ind}(W, \mathfrak{g})$, is given by $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(h)} W$ with the left action given by the regular action of $\mathfrak{g}$ on $\mathcal{U}(g)$. It is clear that, given any $\mathfrak{g}$-representation V, there is a canonical isomorphism

$$\operatorname{Hom}_{\mathfrak{h}}(W, V) \simeq \operatorname{Hom}_{\mathfrak{g}}(\operatorname{ind}(W, \mathfrak{g}), V), \tag{2.2}$$

where in the first morphism space we regard V as an $\mathfrak{h}$-representation coming from the inclusion $\mathfrak{h} \subseteq \mathfrak{g}$. From the fact that $\mathcal{U}(\mathfrak{g})$ is free over $\mathcal{U}(\mathfrak{h})$, W is simple if $\operatorname{ind}(W, \mathfrak{g})$ is simple.

We recall that, given a representation V of $\mathfrak{g}$, the *annihilator* of a subspace W of the super vector space underlying V is the left ideal of $\mathcal{U}(\mathfrak{g})$ given by the elements x such that $xw = 0$, for all $w \in W$.

Proposition 2.13. *Let $\mathfrak{h}$ be a subalgebra of a super Lie algebra $\mathfrak{g}$ and let W be a representation of $\mathfrak{h}$ given by $\rho : \mathcal{U}(\mathfrak{h}) \to \mathcal{E}nd(W)$. Set $V = \operatorname{ind}(W, \mathfrak{g})$ the representation of $\mathfrak{g}$ induced by W and $J = \ker(\rho)$. Then,*

(i) The annihilator of W in $\mathcal{U}(\mathfrak{g})$ is the left ideal $\mathcal{U}(\mathfrak{g})J$.

(ii) The kernel of the induced structure morphism $\pi : \mathcal{U}(\mathfrak{g}) \to \mathcal{E}nd(V)$ is the largest two-sided ideal of $\mathcal{U}(\mathfrak{g})$ contained in $\mathcal{U}(\mathfrak{g})J$.

Proof. The proof given in [Dix96], Prop. 5.1.7, works word for word. □

Proposition 2.14. *Let $\mathfrak{h}$ be a subalgebra of a super Lie algebra $\mathfrak{g}$, $\mathfrak{k}$ an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$ and W a representation of $\mathfrak{h}$ given by $\rho : \mathcal{U}(\mathfrak{h}) \to \mathcal{E}nd(W)$ such that $\rho([\mathfrak{g}, \mathfrak{k}]) = 0$. Let $V = \operatorname{ind}(W, \mathfrak{g})$ be the representation of $\mathfrak{g}$ induced by W with structure morphism $\pi : \mathcal{U}(\mathfrak{g}) \to \mathcal{E}nd(V)$. Then the $\mathfrak{k}$-representation on V given by $\pi|\mathfrak{k}$ is a direct sum of copies of the $\mathfrak{k}$-representation on W given by $\rho|_{\mathfrak{k}}$.*

Proof. The proof follows the lines of that in [Dix96], Prop. 5.1.13, but with some changes. One first proves that, given $p \in \mathbb{N}_0$ and homogeneous elements $y \in \mathfrak{k}$ and $x_1, \dots, x_p \in \mathfrak{g}$, we have that

$$yx_1 \dots x_p \in (-1)^{|y|(|x_1|+\dots+|x_p|)} x_1 \dots x_p y + \mathcal{U}(\mathfrak{g})[\mathfrak{g}, \mathfrak{k}].$$

This is done by induction on p. It is obvious for $p = 0$. Let us suppose that it holds for $p-1$, and consider

$$\begin{aligned} yx_1 \dots x_p &= (-1)^{|x_1||y|} x_1 y x_2 \dots x_p + [y, x_1] x_2 \dots x_p \\ &\in (-1)^{|y|(|x_1|+\dots+|x_p|)} x_1 \dots x_p y + x_1 \mathcal{U}(\mathfrak{g})[\mathfrak{g}, \mathfrak{k}] \\ &+ (-1)^{(|y|+|x_1|)(|x_2|+\dots+|x_p|)} x_2 \dots x_p [y, x_1] + \mathcal{U}(\mathfrak{g})[\mathfrak{g}, \mathfrak{k}] \\ &\subseteq (-1)^{|y|(|x_1|+\dots+|x_p|)} x_1 \dots x_p y + \mathcal{U}(\mathfrak{g})[\mathfrak{g}, \mathfrak{k}], \end{aligned}$$

which establishes the claim.

Now, if we consider homogeneous elements $y \in \mathfrak{k}$, $u \in \mathcal{U}(\mathfrak{g})$ and $w \in W$, the previous result tells us that $yu = (-1)^{|u||y|} uy + \sum_{i \in I} u_i c_i$, for $u_i \in \mathcal{U}(\mathfrak{g})$ and$c_i \in [\mathfrak{g}, \mathfrak{k}]$. This yields that

$$\pi(y)(u \otimes_{\mathcal{U}(\mathfrak{h})} w) = (-1)^{|u||y|} uy \otimes_{\mathcal{U}(\mathfrak{h})} w + \sum_{i \in I} u_i \rho(c_i) \otimes_{\mathcal{U}(\mathfrak{h})} w = (-1)^{|u||y|} u \otimes_{\mathcal{U}(\mathfrak{h})} \rho(y) w,$$

for $\rho([\mathfrak{g}, \mathfrak{k}]) = 0$. If we consider a homogeneous basis $\{z_j\}_{j \in J}$ of $\mathcal{U}(\mathfrak{g})$ over $\mathcal{U}(\mathfrak{h})$, we have that the $\mathfrak{k}$-representation on V given by $\pi|_{\mathfrak{k}}$ is a direct sum of the $\mathfrak{k}$-representations $k.z_j \otimes W$, where $\pi(y)(z_j \otimes w) = (-1)^{|z_j||y|} z_j \otimes \rho(y) w$. It is clear that $k.z_j \otimes W$ is isomorphic to W if z_j is even and to W^{Σ} if z_j is odd, where W^{Σ} denotes the $\mathfrak{k}$-representation on W with structure morphism $\rho \circ \Sigma$. Finally, we note that the map of super vector spaces $W \to W^{\Sigma}$ given by $w_0 + w_1 \mapsto w_0 - w_1$ is an isomorphism of $\mathfrak{k}$-representations. The proposition is thus proved. □

We recall that a Lie algebra is called *algebraic* if it is the Lie algebra of an algebraic group. Also, given $\mathfrak{g}_0$ a finite dimensional Lie algebra, the set of automorphisms $\mathrm{Aut}(\mathfrak{g}_0)$ of the Lie algebra $\mathfrak{g}_0$ is a linear algebraic group with (algebraic) Lie algebra $\mathrm{Der}(\mathfrak{g}_0)$ (see [TY05], Prop. 24.3.7). The algebraic action of $\mathrm{Aut}(\mathfrak{g}_0)$ on itself by conjugation induces a morphism of algebraic groups $\mathrm{Ad} : \mathrm{Aut}(\mathfrak{g}_0) \to \mathrm{GL}(\mathrm{Der}(\mathfrak{g}_0))$ (see [TY05], 23.5.2). Let $\mathfrak{ad}_0$ be the smallest algebraic subalgebra of the Lie algebra $\mathrm{Der}(\mathfrak{g}_0)$ satisfying that $\mathrm{InnDer}(\mathfrak{g}_0) \subseteq \mathfrak{ad}_0$. The *adjoint (algebraic) group* $\mathcal{A}d_0$ of $\mathfrak{g}_0$ is the smallest algebraic subgroup of $\mathrm{GL}(\mathfrak{g}_0)$ (or $\mathrm{Aut}(\mathfrak{g}_0)$) whose Lie algebra contains $\mathrm{InnDer}(\mathfrak{g}_0)$, or equivalently, it is the irreducible algebraic subgroup of $\mathrm{GL}(\mathfrak{g}_0)$ (or $\mathrm{Aut}(\mathfrak{g}_0)$) with Lie algebra $\mathfrak{ad}_0$ (see [TY05], 24.8.1-2). The action of $\mathrm{Aut}(\mathfrak{g}_0)$ on $\mathrm{Der}(\mathfrak{g}_0)$ preserves the ideal $\mathrm{InnDer}(\mathfrak{g}_0)$, and thus preserves also the ideal $\mathfrak{ad}_0$. Hence, the action of $\mathrm{Aut}(\mathfrak{g}_0)$ on itself by conjugations preserves $\mathcal{A}d_0$. Note that $\mathcal{A}d_0$ acts naturally on $\mathfrak{g}_0$. When $\mathrm{InnDer}(\mathfrak{g}_0) = \mathfrak{ad}_0$, we will simply say that $\mathcal{A}d_0$ is the *adjoint group* of $\mathfrak{g}_0$. The previous identity is equivalent to say that $\mathfrak{g}_0$ is algebraic, which is satisfied for every nilpotent Lie algebra.

The preceding paragraph can be extended to the case of algebraic super groups. Let now $\mathfrak{ad}'$ be the smallest algebraic subalgebra of the Lie algebra $\mathrm{Der}(\mathfrak{g})_0$ (or $\mathfrak{gl}(\mathfrak{g})_0$) satisfying that $\mathrm{InnDer}(\mathfrak{g}_0) \subseteq \mathfrak{ad}'$, and let $\mathcal{A}d'$ be the irreducible algebraic subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathfrak{ad}'$. The latter can be equivalently defined as the smallest algebraic subgroup of $\mathrm{GL}(\mathfrak{g}_0) \times \mathrm{GL}(\mathfrak{g}_1)$ such that its Lie algebra contains the Lie algebra $\mathrm{InnDer}(\mathfrak{g}_0)$ (see [TY05], 24.8.1-2). This immediately implies that $\mathfrak{ad}' = \mathfrak{ad}_0$ and $\mathcal{A}d' = \mathcal{A}d_0$ (see [TY05], Prop. 28.4.5). Arguing as before, we see that the action of $\mathrm{Aut}(\mathfrak{g})$ on $\mathrm{Der}(\mathfrak{g})_0$ preserves the ideal $\mathrm{InnDer}(\mathfrak{g}_0)$, and thus preserves also the ideal $\mathfrak{ad}_0$. So, the action of $\mathrm{Aut}(\mathfrak{g})$ on itself by conjugations preserves $\mathcal{A}d_0$ and we also have that $\mathcal{A}d_0$ acts naturally on $\mathfrak{g}$.

Define the super vector space $\mathfrak{ad} = \mathfrak{ad}_0 \oplus \mathfrak{g}_1$, where the indices also denote the parity of its elements. It has a unique structure of super Lie algebra satisfying that the restriction of its bracket to the even part is just the usual bracket of $\mathfrak{ad}_0$, the bracket of an element of $\mathfrak{ad}_0$ and an element of $\mathfrak{g}_1$ is given by the action previously defined, and the bracket of two elements of $\mathfrak{g}_1$ is given by the element of $\mathrm{InnDer}(\mathfrak{g})_0 \subseteq \mathfrak{ad}_0$ defined by the usual bracket in $\mathfrak{g}$. Moreover, it is easily verified that $\mathcal{A}d_0$ acts on $\mathfrak{ad}$ by automorphisms of super Lie algebras, so that $(\mathcal{A}d_0, \mathrm{ad})$ is an algebraic super group, called the *adjoint algebraic super group* of $\mathfrak{g}$, and the canonical inclusion $(\mathcal{A}d_0, \mathrm{ad}) \hookrightarrow (\mathrm{Aut}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g}))$ is a morphism of super Lie groups. We see that $\mathrm{InnDer}(\mathfrak{g}) = \mathfrak{ad}$ if and only if $\mathrm{InnDer}(\mathfrak{g}_0) = \mathfrak{ad}_0$, in which case we will simply also say that $(\mathcal{A}d_0, \mathfrak{ad})$ is the *adjoint super group* of $\mathfrak{g}$. By the previous comments, that is equivalent to say that $\mathfrak{g}_0$ is an algebraic Lie algebra, which is satisfied for instance if $\mathfrak{g}$ is nilpotent.

Let us suppose that we have further an action of an algebraic super group $(H_0, \mathfrak{h})$ on a super Lie algebra $\mathfrak{g}$ by automorphisms, *i.e.* that there is a morphism of super Lie groups $(H_0, \mathfrak{h}) \to (\mathrm{Aut}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g}))$. It is easy to see that this action induces an action of $(H_0, \mathfrak{h})$ on $\mathcal{U}(\mathfrak{g})$, which preserves the filtration of the enveloping algebra. This can be applied, to the standard action of the adjoint algebraic super group $(\mathcal{A}d_0, \mathfrak{ad})$ of $\mathfrak{g}$ on itself given in the previous paragraph to see that it induces an action on $\mathcal{U}(\mathfrak{g})$.

The following result will be used in the sequel.

Proposition 2.15. *Let $\mathfrak{g}$ be a super Lie algebra and $\mathcal{A}d_0$ the adjoint algebraic group of $\mathfrak{g}_0$. For every element $a \in \mathrm{Aut}(\mathfrak{g})$, denote by $a_{\mathcal{U}}$ the induced automorphism of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. For any ideal I of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ and $a \in \mathcal{A}d_0$, we have that $a_{\mathcal{U}}(I) = I$.*

Proof. The proof given in [Dix96], Prop. 2.4.17, applies word for word. □

Remark 2.16. *It is easy to see that if $x \in \mathfrak{ad}$ is an element of the super Lie algebra associated to the adjoint algebraic super group of $\mathfrak{g}$, then x preserves I, i.e. $x(I) \subseteq I$, which can be proved as follows. If $x \in \mathfrak{ad}_0$, it is the differential of an element of the adjoint algebraic group $\mathcal{A}d_0$, which preserves the ideal by the proposition. If $x \in \mathfrak{ad}_1$, it is the inner derivation induced by an element $\tilde{x} \in \mathfrak{g}$. Since I is an ideal of $\mathcal{U}(\mathfrak{g})$ it must hold that $[\tilde{x}, I] \subseteq I$, and the claim follows.*

3 Generalities on enveloping algebras of super Lie algebras

In the first subsection we shall present some basic facts on ideals of enveloping algebras of super Lie algebras. Then, in the next subsection we provide a useful criterion for a representation of an enveloping algebra of a super Lie algebra to be simple, analogous to the well-known one for plain Lie algebras, which will allow us to prove the primitivity of the ideals to be considered in Section 5. Finally, in the last subsection we shall give the needed results on primitive ideals of enveloping algebras that will be used to prove the claimed theorems leading to the Dixmier map for nilpotent super Lie algebras.

3.1 Ideals of enveloping algebras of super Lie algebras

We shall now state two results that we will need in the sequel.

Proposition 3.1 ([Let89], Prop. 3.3)**.** *Let I be an ideal of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a nilpotent super Lie algebra $\mathfrak{g}$, distinct from $\mathcal{U}(\mathfrak{g})$. Then, I is primitive if and only if it is maximal.*

Proposition 3.2. *Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$, and I an ideal of $\mathcal{U}(\mathfrak{g})$. Then, if I is a (semi)prime ideal of $\mathcal{U}(\mathfrak{g})$, $I \cap \mathcal{U}(\mathfrak{k})$ is a (semi)prime ideal of $\mathcal{U}(\mathfrak{k})$*

Proof. The proof for the non-graded case given in [Dix96], Prop. 3.3.4, works as well in this case, replacing the use of [Dix96], Lemma 3.3.3, by Lemma 1.6. □

3.2 On the simplicity of representations of super Lie algebras

Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$ and let V be a representation of $\mathfrak{k}$ with structure morphism $\sigma : \mathcal{U}(\mathfrak{k}) \to \mathcal{E}nd(V)$. Following [Dix96], the *stabilizer of σ in $\mathfrak{g}$* is the subspace of the super vector space underlying $\mathfrak{g}$ expanded by the homogeneous elements $y \in \mathfrak{g}$ satisfying that there exists a homogeneous endomorphism $s \in \mathcal{E}nd(V)$ of the same degree as y such that

$$\sigma([y, x]) = [s, \sigma(x)]$$

for all homogeneous elements $x \in \mathfrak{k}$, and where we remark that $[s, \sigma(x)]$ is the super commutator in $\mathcal{E}nd(V)$. It is denoted $\mathfrak{st}(\sigma, \mathfrak{g})$ or $\mathfrak{st}(V, \mathfrak{g})$. Analogously, given an ideal I of $\mathcal{U}(\mathfrak{k})$ we may define the *stabilizer of I in $\mathfrak{g}$*, denoted by $\mathfrak{st}(I, \mathfrak{g})$, as the super vector space expanded by the homogeneous elements $x \in \mathfrak{g}$ such that $\mathrm{ad}(x)(I) \subseteq I$ (*i.e.* such that $\mathrm{ad}(x)(z) \in I$ for all homogeneous elements $z \in I$). It is clear that both $\mathfrak{st}(\sigma, \mathfrak{g})$ and $\mathfrak{st}(I, \mathfrak{g})$ are subalgebras of $\mathfrak{g}$ containing $\mathfrak{k}$. Moreover, it is easy to see that if $\mathfrak{k}$ is an ideal of a super Lie algebra $\mathfrak{g}$, U a representation of $\mathfrak{k}$ such that its structure morphism $\sigma : \mathcal{U}(\mathfrak{k}) \to \mathcal{E}nd(U)$ has kernel I, then $\mathfrak{st}(\sigma, \mathfrak{g}) \subseteq \mathfrak{st}(I, \mathfrak{g})$ (*cf.* [Dix96], Prop. 5.3.3).

Lemma 3.3. *Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$, U a simple representation of $\mathfrak{k}$ with structure morphism $\sigma : \mathcal{U}(\mathfrak{k}) \to \mathcal{E}nd(U)$, $\mathfrak{h} = \mathfrak{st}(\sigma, \mathfrak{g})$, $\rho : \mathcal{U}(\mathfrak{h}) \to \mathcal{E}nd(W)$ a representation of $\mathfrak{h}$ such that the $\mathfrak{k}$-representation on W given by $\rho|_{\mathfrak{k}}$ is a direct sum of copies of the $\mathfrak{k}$-representation on U given by σ, and let V be the induced representation $\operatorname{ind}(V, \mathfrak{g})$ with structure morphism π. Let V_n be the super vector space expanded by the classes of $x \otimes w$, for homogeneous elements $x \in F^n\mathcal{U}(\mathfrak{g})$ and $w \in W$. It is an exhaustive increasing filtration of the super vector space V. Given $n \in \mathbb{N}$ and $t \in V_n \setminus \{0\}$, there exists $z \in \mathcal{U}(\mathfrak{k})$ such that $zt \in V_{n-1} \setminus \{0\}$.*

Proof. The proof follows the pattern for the non-super case given in [Dix96], Prop. 5.3.5, but since there are several differences we give it.

Let $\{x_1, \dots x_m\}$ be a homogeneous basis of a complement of $\mathfrak{h}$ in $\mathfrak{g}$ and write $t = \sum_{|\bar{n}| \leq p} \bar{x}^{\bar{n}} \otimes_{\mathcal{U}(\mathfrak{h})} w_{\bar{n}}$, where $\bar{x}^{\bar{n}} = x_1^{n_1} \dots x_m^{n_m}$ and $w_{\bar{n}} \in W$. If $w_{\bar{n}} = 0$ for all $\bar{n}$ such that $|\bar{n}| = \sum_{i=1}^m n_i = p$, there is nothing to prove. It suffices thus to prove the lemma for the case that there exists some $\bar{n}_0$ such that $w_{\bar{n}_0}$ does not vanish. We may suppose that the homogeneous element t further satisfies that $t = \sum_{|\bar{n}| = p} \bar{x}^{\bar{n}} \otimes_{\mathcal{U}(\mathfrak{h})} w_{\bar{n}}$, where all nonvanishing $w_{\bar{n}} \in W$ are homogeneous of the same degree. This can be proved as follows. First, since $\mathcal{U}(\mathfrak{k})$ preserves the filtration defined on V, we may ignore the terms indexed by $\bar{n}$ with $|\bar{n}| < p$. Second, if we write $t = t^0 + t^1$, where t^i is the sum of the terms $\bar{x}^{\bar{n}} \otimes_{\mathcal{U}(\mathfrak{h})} w_{\bar{n}}$ such that $|w_{\bar{n}}| = i$, for $i \in \mathbb{Z}/2\mathbb{Z}$, then the fact that $\mathcal{U}(\mathfrak{k})$ preserves the filtration defined on V implies that we may proceed stepwise, as we wanted to prove.

Since W can be written as a direct sum $\oplus_{\lambda \in \Lambda} W_\lambda$, for W_λ a $\mathfrak{k}$-representation isomorphic to U, set $\zeta_\lambda : W \to U$ the unique epimorphism of $\mathfrak{k}$-modules with kernel $\oplus_{\lambda' \neq \lambda} W_{\lambda'}$. Let $u \in U$ be an homogeneous element of the same parity as $w_{\bar{n}_0}$. By Proposition 2.12, there exists en even element $z \in \mathcal{U}(\mathfrak{k})$ and elements $\xi_{\lambda, w_{\bar{n}}} \in k$ such that $z\zeta_\lambda(w_{\bar{n}}) = \xi_{\lambda, w_{\bar{n}}} u$, for all λ and $\bar{n}$. By Lemma 2.11, we see that $zt \equiv \sum_{|\bar{n}|=p} \bar{x}^{\bar{n}} \otimes_{\mathcal{U}(\mathfrak{h})} zw_{\bar{n}}$ (mod. V_{p-1}), so by changing t by zt, we can further assume that t satisfies that $\zeta_\lambda(w_{\bar{n}}) = \xi_{\lambda,\bar{n}} u$, for some $\xi_{\lambda,\bar{n}} \in k$ and that there exists λ_0 such that $\xi_{\lambda_0, \bar{n}_0} \neq 0$.

Choose $i_0 \in \{1, \dots, m\}$ such that $\bar{n}_0 = (n_{0,1}, \dots, n_{0,m})$ satisfies that $n_{0,i_0} \neq 0$, and define $\bar{n}'_0 = \bar{n}_0 - e_{i_0}$, where e_{i_0} is the vector of $\mathbb{N}_0^m$ which has 1 in the i_0-th place and zero elsewhere. Given a homogeneous element $z \in \mathcal{U}(\mathfrak{k})$, Lemma 2.11 tells us that

$$zt \equiv \sum_{|\bar{n}|=p} (-1)^{|z||\bar{x}^{\bar{n}}|} \bar{x}^{\bar{n}} \otimes_{\mathcal{U}(\mathfrak{h})} zw_{\bar{n}} - \sum_{j=1}^{m} \sum_{|\bar{n}|=p} (-1)^{|z||\bar{x}^{\bar{n}}| + |x_j| \sum_{i>j} n_i |x_i|} n_j \bar{x}^{\bar{n}-e_j} \otimes_{\mathcal{U}(\mathfrak{h})} [x_j, z] w_{\bar{n}} (\text{mod. } V_{p-2}). \tag{3.1}$$

Suppose now that the statement of the lemma does not hold, *i.e.* that $\mathcal{U}(\mathfrak{k})t \cap V_{p-1} = 0$. If $zu = 0$, we get that $zt \in V_{p-1}$ and by the assumption it must vanish. Hence, we conclude that the component coming from $\bar{x}^{\bar{n}'_0}$ in (3.1) is given by

$$\sum_{j=1}^{m} (-1)^{|x_j|(|z| + \sum_{i>j} n'_{0,i}|x_i|)} (n'_{0,j} + 1)[x_j, z] w_{\bar{n}'_0 + e_j} = 0.$$

By applying ζ_{λ_0} to this equality we obtain

$$\sum_{j=1}^{m} (-1)^{|x_j|(|z| + \sum_{i>j} n'_{0,i}|x_i|)} (n'_{0,j} + 1)[x_j, z] \xi_{\lambda_0, \bar{n}'_0 + e_j} u = 0,$$

which can be rewritten as $[y, z]u = 0$, for $y = \sum_{j=1}^m (-1)^{|x_j|(|z| + \sum_{i>j} n'_{0,i}|x_i|)} (n'_{0,j} + 1) \xi_{\lambda_0, \bar{n}'_0 + e_j} x_j \in \mathfrak{g}$. It is direct to see that, if $y = y_0 + y_1$ is the decomposition of y in homogeneous elements, the previous vanishing identity is equivalent to $[y_i, z]u = 0$, for $i \in \mathbb{Z}/2\mathbb{Z}$. Define $y'_0 = y_0$, $y'_1 = (-1)^{|z|} y_1$ and $y' = y'_0 + y'_1$. Moreover, the homogeneous component y'' of y' of degree $|x_{i_0}|$ does not belong to $\mathfrak{h}$ since $\xi_{\lambda_0, \bar{n}_0} \neq 0$ and it does not depend on z.

Let us define $s \in \operatorname{End}(U)$ given by $s(zu) = [y'', z]u$, for $z \in \mathcal{U}(\mathfrak{k})$. We recall that the degree of s coincides with the degree of y''. It is well-defined by the previous considerations. Given homogeneous elements $z \in \mathcal{U}(\mathfrak{k})$ and $x \in \mathfrak{k}$,

$$[s, x](zu) = s(xzu) - (-1)^{|s||x|} xs(zu) = [y'', xz]u - (-1)^{|y''||x|} x[y'', z]u = [y'', x]zu,$$

which means that $y'' \in \mathfrak{h}$, which is contradiction. The lemma is thus proved. □

Theorem 3.4. *Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$, W a simple representation of $\mathfrak{k}$ with structure morphism $\sigma : \mathcal{U}(\mathfrak{k}) \to \mathcal{E}nd(W)$, $\mathfrak{h} = \mathfrak{st}(\sigma, \mathfrak{g})$, and $\rho : \mathcal{U}(\mathfrak{h}) \to \mathcal{E}nd(V)$ a representation of $\mathfrak{h}$ such that the $\mathfrak{k}$-representation on V given by $\rho|_{\mathfrak{k}}$ is a direct sum of copies of the $\mathfrak{k}$-representation on W given by σ. Then the induced representation $\mathrm{ind}(V, \mathfrak{g})$ is simple.*

Proof. The proof for the non-super case in [Dix96], Thm. 5.3.6, works word for word in this case, replacing the use of [Dix96], Lemma 5.3.5 by Lemma 3.3. □

3.3 Primitive ideals of enveloping algebras of super Lie algebras

Proposition 3.5. *Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$ and V a simple representation of $\mathfrak{g}$ with structure morphism π. Suppose that the $\mathfrak{k}$-representation on V given by $\pi|_{\mathfrak{k}}$ has a simple subrepresentation U with structure morphism σ. Let $\mathfrak{h} = \mathfrak{st}(U, \mathfrak{g})$. Then there exists a simple $\mathfrak{h}$-representation W with structure morphism ρ such that the $\mathfrak{k}$-representation on W given by $\rho|_{\mathfrak{k}}$ is a direct sum of copies of the representation U, and $\mathrm{ind}(W, \mathfrak{g})$ is isomorphic to V.*

Proof. The proof for Lie algebras given in [Dix96], Prop. 5.4.1, also applies for this case, under the assumption that all elements are homogeneous, using super commutators instead of commutators, and replacing the use of [Dix96], Propositions 5.1.3 and 5.1.10, and Lemma 5.3.5 by Identity (2.2) and the posterior explanation, and Lemma 3.3, respectively. □

Lemma 3.6. *Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$, $x_1, \dots, x_p$ a homogeneous basis of a complement of $\mathfrak{k}$ in $\mathfrak{g}$, I an ideal of $\mathcal{U}(\mathfrak{g})$, $K = I \cap \mathcal{U}(\mathfrak{k})$, $A = \mathcal{U}(\mathfrak{g})/I$ and $B = \mathcal{U}(\mathfrak{k})/K$. For all $\bar{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$, we shall denote by $\bar{x}^{\bar{n}}$ the class of $x_1^{n_1} \dots x_p^{n_p}$ modulo I. We regard $\mathbb{N}^p$ as a poset with the order given in paragraph 2.6.1 of* [Dix96]*, and write*

$$A_{\bar{n}} = \sum_{\bar{n}' \leq \bar{n}} \bar{x}^{\bar{n}'} B, \qquad A^-_{\bar{n}} = \sum_{\bar{n}' < \bar{n}} \bar{x}^{\bar{n}'} B.$$

Then,

(i) $A_{\bar{n}}$ and $A^-_{\bar{n}}$ are left and right B-submodules of A. The union of all the modules $A_{\bar{n}}$ is A.

(ii) The annihilators in B of the left and right B-module structures on $A_{\bar{n}}/A^-_{\bar{n}}$ coincide. We denote them by $\mathrm{ann}_{\bar{n}}$. Moreover, $\mathrm{ann}_{\bar{n}}$ is a two-sided ideal of B, for all $\bar{n} \in \mathbb{N}^p$.

(iii) Let $\bar{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$ and $\bar{n}' = (n'_1, \dots, n'_p) \in \mathbb{N}^p$ satisfy that $n_i \leq n'_i$, for $i = 1, \dots, p$. Then $\mathrm{ann}_{\bar{n}} \subseteq \mathrm{ann}_{\bar{n}'}$.

(v) The set of nontrivial $\mathrm{ann}_{\bar{n}}$ has a finite number of minimal elements. If ann denotes their intersection, then $\mathrm{ann} \subseteq \mathrm{ann}_{\bar{n}}$, for all $\bar{n}$ such that $\mathrm{ann}_{\bar{n}} \neq \{0\}$.

(vi) If L is a left ideal of B such that $AL = A$, then L contains a power of ann.

Proof. The proof for Lie algebras given in [Dix96], Lemma 5.4.2, also applies for this case, replacing the use of [Dix96], Lemma 2.2.22 by Lemma 2.11 (which introduce inoffensive extra signs). □

Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$, and I and J ideals of $\mathcal{U}(\mathfrak{k})$. We recall that J is *generic* for I (relative to $\mathfrak{g}$), whenever I is the set of elements $z \in \mathcal{U}(\mathfrak{k})$ such that $[x_1, [\dots, [x_n, z] \dots]] \in J$, for all $n \in \mathbb{N}_0$ and $x_1, \dots, x_n \in \mathfrak{g}$. Equivalently, I is generic for J if it is the largest ideal of $\mathcal{U}(\mathfrak{k})$ contained in J such that $[\mathfrak{g}, I] \subseteq I$ (*cf.* [Dix96], Prop. 3.3.10 and 3.3.11).

Proposition 3.7. *Let $\mathfrak{k}$ be an ideal of a super Lie algebra $\mathfrak{g}$, I a maximal ideal of $\mathcal{U}(\mathfrak{g})$, and $K = I \cap \mathcal{U}(\mathfrak{k})$ Then there exists a primitive ideal J of $\mathcal{U}(\mathfrak{k})$ which is generic for K and such that for every simple representation U of $\mathfrak{k}$ with kernel J and structure morphism σ, there exists a simple representation W of $\mathfrak{st}(U, \mathfrak{g})$ with structure morphism ρ such that the $\mathfrak{k}$-representation on W given by $\rho|_{\mathfrak{k}}$ is a direct sum of copies of the $\mathfrak{k}$-representation U and $\mathrm{ind}(W, \mathfrak{g})$ is simple with kernel I.*

Proof. The proof for Lie algebras given in [Dix96], Prop. 5.4.3, also applies for this case, with the additional assumption that the element considered there should be homogeneous and replacing the use of [Dix96], Propositions 3.1.15, 3.3.4 and 5.4.1, and Lemma 5.4.2 by Propositions 1.8, 3.2 and 3.5, and Lemma 3.6, respectively. □

4 Polarizations

The aim of this section is to prove that polarizations exist for solvable super Lie algebras. In order to do so, we first provide some easy results on bilinear forms on super vector spaces. Then, we recall the basic facts on polarizations of Lie algebras. Finally, we will recall some of the ideas of A. Sergeev used to study irreducible finite dimensional representations of solvable super Lie algebras. As a consequence, we shall derive that all solvable super Lie algebras have polarizations. We would like to remark that M. Duflo has proved this result using a different idea (*cf.* [BBB07])

4.1 Bilinear forms on super vector spaces

Let V be a super vector space provided with an *even bilinear form* $\langle\,,\rangle$, *i.e.* a morphism of super vector spaces $\langle\,,\rangle : V \otimes V \to k$, and let W be a subspace of V. We remark that the homogeneity of the map $\langle\,,\rangle$ is equivalent to the fact that $\langle v, w\rangle = 0$, for all $v, w \in V$ of different parity. We suppose moreover that $\langle\,,\rangle$ is either *superantisymmetric* or *supersymmetric*, *i.e.* $\langle v, w\rangle = -(-1)^{|v||w|}\langle w, v\rangle$, or $\langle v, w\rangle = (-1)^{|v||w|}\langle w, v\rangle$, for all $v, w \in V$ homogeneous, respectively. Furthermore, we see that an even superantisymmetric (resp. supersymmetric) bilinear form on V is equivalent to give an antisymmetric (resp. symmetric) bilinear form on V_0 and a symmetric (resp. antisymmetric) bilinear form on V_1. From now on, all bilinear forms will be even unless otherwise stated.

It is easy to prove that

$$\mathrm{sdim}(W) + \mathrm{sdim}(W^{\perp_{\langle,\rangle}}) = \mathrm{sdim}(V) + \mathrm{sdim}(W \cap V^{\perp_{\langle,\rangle}}),$$

where $W^{\perp_{\langle,\rangle}}$ denotes the subspace of V perpendicular to W with respect to the form $\langle\,,\rangle$:

$$W^{\perp_{\langle,\rangle}} = \{v \in V : \langle v, w\rangle = 0\}.$$

We remark that $W^{\perp_{\langle,\rangle}} = W_0^{\perp_{\langle,\rangle|_{V_0}}} \oplus W_1^{\perp_{\langle,\rangle|_{V_1}}}$. From now on, unless it is necessary to specify the super vector space and its form $\langle\,,\rangle$, we denote the perpendicular space to a subspace W only by $W^\perp$.

We recall that $W \subseteq V$ is called *totally isotropic* if $W \subseteq W^\perp$. Moreover, a totally isotropic subspace $W \subseteq V$ is *maximal totally isotropic* if it is maximal in the set of totally isotropic subspaces of the super vector space V with respect to the inclusion. We note that this gives that $W \subseteq W^\perp$, but it does not necessarily imply that $W = W^\perp$. All the previous definitions specialize to the usual ones for a vector space V with subspace W if we consider V as a super vector space with $V_1 = 0$.

It is easy to see that a subspace W of a super vector space V provided with a superantisymmetric or supersymmetric even bilinear form $\langle\,,\rangle$ is (maximal) totally isotropic if and only if each W_i is a (maximal) totally isotropic subspace of the vector space V_i provided with $\langle\,,\rangle|_{V_i}$, for $i \in \mathbb{Z}/2\mathbb{Z}$. This implies that we may thus restrict to the study of symmetric and antisymmetric forms on vector spaces.

If V is a vector space provided with an antisymmetric bilinear form and W is a subspace then the following conditions are equivalent (see [Dix96], 1.12.1):

- W is maximal in the set of totally isotropic subspaces with respect to the inclusion,
- $\dim(W) = (\dim(V) + \dim(V^\perp))/2$,
- $W \supset W^\perp$,
- $W = W^\perp$.

On the other hand, let us assume that V is a vector space with a symmetric bilinear form. Taking the quotient by $V^\perp$, we may then restrict to the situation where the form is non-degenerate. In this case, a totally isotropic subspace W of V is maximal totally isotropic if and only if $W = W^\perp$, for $\dim(V)$ even, and $\dim(W) = \dim(W^\perp) - 1$, for $\dim(V)$ odd. Hence, for a vector space V with an antisymmetric or symmetric bilinear form, the dimensions of all maximal totally isotropic subspaces coincide. In consequence, the super dimensions of all maximal totally isotropic subspaces of a super vector space coincide.

4.2 Polarizations of Lie algebras

Let us first state the standard results about polarizations of plain Lie algebras.

A subalgebra $\mathfrak{h}_0$ of a Lie algebra $\mathfrak{g}_0$ is said to be *subordinate* to a functional $\lambda_0 \in \mathfrak{g}_0^*$ if $\lambda_0([\mathfrak{h}_0, \mathfrak{h}_0]) = 0$ (*cf.* [Dix96], 1.12.7). Equivalently, $\mathfrak{h}_0$ is a totally isotropic subspace of $\mathfrak{g}_0$ provided with the alternating bilinear form A_{λ_0} given by $v \otimes w \mapsto \lambda_0([v, w])$. Moreover, we say that $\mathfrak{h}_0$ is a *polarization* of $\mathfrak{g}_0$ at λ_0 if it is a subalgebra of $\mathfrak{g}_0$ and it is a maximal totally isotropic subspace of the vector space $\mathfrak{g}_0$ provided with A_{λ_0} (*cf.* [Dix96], 1.12.8). By the previous subsection, if $\mathfrak{g}_0^{\lambda_0}$ denotes the kernel of A_{λ_0}, to be a maximally totally isotropic subspace is the same as to be totally isotropic and of dimension $(\dim(\mathfrak{g}_0) + \dim(\mathfrak{g}_0^{\lambda_0}))/2$.

Proposition 1.12.10 in [Dix96] implies that, given any linear functional λ_0 on any solvable Lie algebra $\mathfrak{g}_0$, a polarization of $\mathfrak{g}_0$ at λ_0 always exists (we remark that one requires the assumption that k is algebraically closed).

Remark 4.1. *We point out the easy fact that if $\mathfrak{k}_0$ is an ideal of a Lie algebra $\mathfrak{g}_0$ subordinate to a functional $\lambda_0 \in \mathfrak{g}_0^*$, then $\mathfrak{k}_0$ should be included in every polarization of $\mathfrak{g}_0$ at λ_0. Indeed, if $\mathfrak{h}_0$ is a polarization at λ_0, then $\mathfrak{h}_0 + \mathfrak{k}_0$ is also a subordinate subalgebra of λ_0. The maximality of $\mathfrak{h}_0$ implies that $\mathfrak{k}_0 \subseteq \mathfrak{h}_0$.*

4.3 Polarizations of super Lie algebras in the sense of Sergeev

We shall now recall some definitions and easy facts from the work of Sergeev.

Let $\mathfrak{g}$ be a solvable super Lie algebra. Define $L_{\mathfrak{g}}$ to be the vector space of functionals given by the elements $\lambda \in \mathrm{Hom}(\mathfrak{g}, k)$ such that $\lambda([\mathfrak{g}_0, \mathfrak{g}_0]) = 0$. We remark that the condition $\lambda \in \mathrm{Hom}(\mathfrak{g}, k)$ is equivalent to say that $\lambda : \mathfrak{g} \to k$ is a k-linear map between the underlying vector spaces such that $\lambda(\mathfrak{g}_1) = 0$. Analogously to the case of Lie algebras, a functional as before determines a symmetric bilinear form $B_\lambda : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to k$ given by $B_\lambda(x, y) = \lambda([x, y])$. A *polarization in the sense of Sergeev* of $\mathfrak{g}$ at $\lambda \in L_{\mathfrak{g}}$ is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}_0 = \mathfrak{g}_0$ and $\mathfrak{h}_1$ is a maximal totally isotropic subspace for the symmetric bilinear form B_λ.

Remark 4.2. *Note that if we see a Lie algebra as a super Lie algebra with $\mathfrak{g}_1 = 0$, $L_{\mathfrak{g}}$ is the vector space of functionals having (classical) polarization $\mathfrak{g}_0$. This coincides with the notion of polarization in the sense of Sergeev in that case.*

Remark 4.3. *Note that, if $\mathfrak{h}$ is a subspace of the super vector space underlying the super Lie algebra $\mathfrak{g}$ such that $\mathfrak{h}_0 = \mathfrak{g}_0$, then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{h}_1$ is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$.*

Lemma 4.4 ([Ser99], Lemma 2.4)**.** *Let $\mathfrak{g}_0$ be a solvable Lie algebra and V be a finite dimensional $\mathfrak{g}_0$-module provided with a $\mathfrak{g}_0$-invariant symmetric bilinear form. Given W a $\mathfrak{g}_0$-submodule of V, which is totally isotropic with respect to the bilinear form, then there exists a $\mathfrak{g}_0$-submodule of V which is a maximal totally isotropic subspace containing W.*

The previous result applied to the case $V = \mathfrak{g}_1$ and $W = 0$ (and taking into account Remark 4.3) implies:

Lemma 4.5 ([Ser99], Lemma 1.1)**.** *Let $\mathfrak{g}$ be a solvable super Lie algebra. Given any $\lambda \in L_{\mathfrak{g}}$, there exists a polarization in the sense of Sergeev of $\mathfrak{g}$ at λ.*

4.4 Polarizations of super Lie algebras

Let us define $\mathcal{L}_{\mathfrak{g}}$ to be the vector space of functionals given by the elements $\lambda \in \mathrm{Hom}(\mathfrak{g}, k)$. Given $\lambda \in \mathcal{L}_{\mathfrak{g}}$, it defines a superantisymmetric bilinear form $\langle\,,\rangle_\lambda$ on $\mathfrak{g}$ by the formula $\langle x, y\rangle = \lambda([x, y])$, for $x, y \in \mathfrak{g}$, so *a fortiori* an antisymmetric bilinear form A_λ on $\mathfrak{g}_0$ and a symmetric bilinear form B_λ on $\mathfrak{g}_1$. Denote by $\mathfrak{g}^\lambda$ the kernel $\{x \in \mathfrak{g} : \lambda([x, y]) = 0, \forall y \in \mathfrak{g}\}$ of $\langle\,,\rangle_\lambda$, which is a sub super Lie algebra.

V. Kac in [Kac77], p. 83, has defined a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be *subordinate* to λ if $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$ and $\mathfrak{h} \supset \mathfrak{g}^\lambda$. We define a *polarization* of $\mathfrak{g}$ at $\lambda \in \mathcal{L}_{\mathfrak{g}}$ to be a subordinate subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that it is a maximal totally isotropic subspace of the super vector space $\mathfrak{g}$ with respect to the even bilinear form $\langle\,,\rangle_\lambda$. By the considerations given in Subsection 4.1, we see that $\mathfrak{h}_0$ should be a polarization of the Lie algebra $\mathfrak{g}_0$ at $\lambda|_{\mathfrak{g}_0}$ and the super dimension of all polarizations at λ coincide.

Notice that $L_{\mathfrak{g}} \subseteq \mathcal{L}_{\mathfrak{g}}$ and that λ defines two one-dimensional $\mathfrak{h}$-representations $F_{\lambda,\mathfrak{h},i} = k.v_{\lambda,\mathfrak{h},i}$ with $|v_{\lambda,\mathfrak{h},i}| = i$, for $i \in \mathbb{Z}/2\mathbb{Z}$, by $x.v_{\lambda,\mathfrak{h},i} = \lambda(x)v_{\lambda,\mathfrak{h},i}$, for all $x \in \mathfrak{h}$. The structure morphism of this representation is given by $\lambda|_{\mathfrak{h}}$.

Remark 4.6. *If the super Lie algebra $\mathfrak{g}$ is just a Lie algebra, this definition obviously coincides with the classical one given on Subsection 4.2. On the other hand, if $\lambda \in L_{\mathfrak{g}}$, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a polarization at λ if and only if it is a polarization in the sense of Sergeev at λ. This tells us that the new definition of polarization is an extension of the previous ones.*

Proposition 4.7. *Let $\mathfrak{g}$ be a solvable super Lie algebra. Every functional $\lambda \in \mathcal{L}_{\mathfrak{g}}$ has a polarization at λ.*

Proof. Let $V \subseteq \mathfrak{g}_1$ be a $\mathfrak{g}_0$-submodule such that it is maximal totally isotropic with respect to the symmetric bilinear form B_λ. Such a submodule exists due to Lemma 4.4. Now, consider the super commutator $[V, V] \subseteq \mathfrak{g}_0$. It is easy verified that $[V, V]$ is a Lie ideal of the Lie algebra $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is a solvable Lie algebra, by Proposition 2.7, there should exist a polarization $\mathfrak{p}$ of $\mathfrak{g}_0$ at $\lambda|_{\mathfrak{g}_0}$. By definition of V, we must have that $\lambda([V, V]) = 0$, and so, by Remark 4.1 the Lie ideal $[V, V]$ of $\mathfrak{g}_0$ should be included in any polarization of $\mathfrak{g}_0$ at $\lambda|_{\mathfrak{g}_0}$, so $[V, V] \subseteq \mathfrak{p}$. Since V is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$, it is *a fortiori* also an $\mathfrak{p}$-submodule. This implies that the subspace $\mathfrak{h}$ of $\mathfrak{g}$ defined as $\mathfrak{h}_0 = \mathfrak{p}$ and $\mathfrak{h}_1 = V$ is a subalgebra of $\mathfrak{g}$. By construction, $\mathfrak{h}_i$ is a maximal totally isotropic subspace of $\mathfrak{g}_i$ provided with $\langle\, , \rangle|_{\mathfrak{g}_i}$, for $i \in \mathbb{Z}/2\mathbb{Z}$. Hence, $\mathfrak{h}$ is a polarization of $\mathfrak{g}$ at λ. □

Remark 4.8. *We remark that Remark 4.1 also extends to this situation: if $\mathfrak{k}$ is an ideal of a super Lie algebra $\mathfrak{g}$ subordinate to a functional $\lambda \in \mathcal{L}_{\mathfrak{g}}$, then $\mathfrak{k}$ is included in every polarization of $\mathfrak{g}$ at λ. The proof given there extends to this case word for word.*

The following is a result of M. Duflo, and it can be seen as the analogous superized result to Lemma 4.4. We reproduce his proof because it does not seem to appear elsewhere.

Lemma 4.9 (*cf.* [BBB07], Cor. 5.2)**.** *Let $\mathfrak{g}$ be a solvable super Lie algebra and V be a finite dimensional $\mathfrak{g}$-module provided with a $\mathfrak{g}$-invariant even superantisymmetric or supersymmetric bilinear form. Given W a $\mathfrak{g}$-submodule of V, which is totally isotropic with respect to the bilinear form, then there exists a $\mathfrak{g}$-submodule of V which is a maximal totally isotropic subspace containing W.*

Proof. Without loss of generality, let us suppose that the bilinear form B on V is superantisymmetric. If B is supersymmetric then, we may consider ΠV instead of V.

Define the super vector space $\mathfrak{h} = V_0 \oplus V_1 \oplus k.z$, where we regard the homogeneous elements of V with the same degree as in V and z in even degree. It is easy to see that $\mathfrak{h}$ is a super Lie algebra if we define $[v, v'] = B(z, z')z$, for all $v, v' \in V$, and we declare z to be supercentral. Moreover, $\mathfrak{g}$ acts on $\mathfrak{h}$ by derivations, so we may consider super Lie algebra given by the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$, and since z is even, the functional $\lambda = z^*$ belongs to $\mathcal{L}_{\mathfrak{g}\ltimes\mathfrak{h}}$. Since the semidirect product of solvable super Lie algebras is also solvable, we see that $\mathfrak{g} \ltimes \mathfrak{h}$ is solvable. Finally, it is clear that the polarizations of $\mathfrak{g} \ltimes \mathfrak{h}$ at λ are in bijection (taking the intersection with V) with the $\mathfrak{g}$-modules W of the statement. □

Remark 4.10. *We may apply the lemma to the following situations:*

(i) Given an ideal $\mathfrak{k}$ of a solvable super Lie algebra $\mathfrak{g}$, and a functional $\lambda \in \mathcal{L}_{\mathfrak{k}}$, then the previous result (for the $\mathfrak{g}$-invariant even superantisymmetric bilinear form $\langle\, , \rangle_\lambda$ on $\mathfrak{k}$) implies that there exist a polarization of $\mathfrak{k}$ at λ that is invariant under $\mathfrak{g}$.

(ii) More generally, given solvable super Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$ such that $\mathfrak{h}$ acts by derivations on $\mathfrak{k}$, and a functional $\lambda \in \mathcal{L}_{\mathfrak{k}}$, consider the super Lie algebra given by the semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{k}$, which is obviously solvable. Since $\mathfrak{k}$ seen inside of $\mathfrak{g}$ is an ideal, the previous item implies that there exists a polarization of $\mathfrak{k}$ at λ invariant under the action of $\mathfrak{h}$ (cf. [Dix96], *Prop. 1.12.10, (iii)).*

5 The Dixmier map for nilpotent super Lie algebras

We are now in position to prove the main results stated at the beginning of the introduction.

5.1 The main theorems

We first recall the following result.

Lemma 5.1 ([BM90], Lemmas 1.10 and 2.2)**.** *Let $\mathfrak{g}$ be a nilpotent super Lie algebra with super center $\mathfrak{z} = kz \neq \mathfrak{g}$, where z is even. Then, there exist elements $x, y \in \mathfrak{g}$ homogeneous of the same parity and an ideal $\mathfrak{k}$ of codimension one in $\mathfrak{g}$ such that*

(i) $[y, x] = z$,

(ii) $\mathfrak{k}$ *is the super centralizer of* y *in* $\mathfrak{g}$ *and* $y \in \mathcal{Z}(\mathfrak{g}/\mathfrak{z})$,

(iii) $\mathfrak{g} = \mathfrak{k} \oplus kx$.

Moreover, if the super center of $\mathfrak{g}/\mathfrak{z}$ consists only of odd elements (so any x and y as before should be odd), then either of the following holds

(1) $\mathfrak{g} = kz \oplus ky$, *with* $[y, y] = z$ *(i.e.* $x = y$*),*

(2) there exists y such that (i), (ii) and (iii) hold and $[y, y] = 0$.

Lemma 5.2. *A nilpotent super Lie algebra $\mathfrak{g}$ of super dimension $(1, 1)$ is isomorphic to one of the following:*

(i) $\mathfrak{g}$ *is supercommutative,*

(ii) $\mathfrak{g} = kz \oplus kc$, *with* $|z| = 0$, $|c| = 1$, $z \in \mathcal{Z}(\mathfrak{g})$ *and* $[c, c] = z$.

Proof. Let us suppose that $\mathfrak{g} = kz \oplus kc$, with $|z| = 0$, $|c| = 1$. By Corollary 2.9, z must be supercentral. Then, the possibilities (i) and (ii) are equivalent to $[c, c] = 0$ or $[c, c] \neq 0$, and the lemma follows. □

The next result is a "super" version of a lemma appearing in [Dix96], whose proof applies to this case as well.

Lemma 5.3. *Let $\mathfrak{a}$ be a supercommutative ideal of $\mathfrak{g}$ with super centralizer $\mathcal{C}(\mathfrak{a})$, and $\mathfrak{h}$ a subalgebra of the super Lie algebra $\mathfrak{g}$ subordinate to λ. Set $\bar{\mathfrak{h}} = (\mathfrak{h} \cap \mathfrak{a}^{\lambda}) + \mathfrak{a}$. Then $\bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}$ subordinate to λ and $\mathfrak{h} \cap \mathcal{C}(\mathfrak{a}) \cap \ker(\lambda)$ is an ideal of the super Lie algebra $\mathfrak{h} + \mathfrak{a}$.*

Proof. The proof given in [Dix96], Lemma 6.1.3, applies word for word. □

Using the lemma we can prove the result:

Theorem 5.4. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra, and $\lambda \in \mathcal{L}_{\mathfrak{g}}$ be a functional. Then there exists a polarization $\mathfrak{h}$ of $\mathfrak{g}$ at λ such that the induced module $\operatorname{ind}(F_{\lambda,\mathfrak{h},i}, \mathfrak{g})$ is simple.*

Proof. The proof is a variation of that given in [Dix96], Thm. 6.1.1, but we avoid the use of the so-called standard polarizations.

We first note that if the super Lie algebra is concentrated in one degree, *i.e.* $\mathfrak{g} = \mathfrak{g}_0$ or $\mathfrak{g} = \mathfrak{g}_1$, then the theorem is immediate: the first case is just the classical result for Lie algebras (see [Dix96], Thm. 6.1.1), and in the second case there is only one polarization $\mathfrak{h} = \mathfrak{g}$, so $\operatorname{ind}(F_{\lambda,\mathfrak{h},i}, \mathfrak{g})$ is one-dimensional and the statement also holds.

We shall now proceed to prove the theorem by induction on the dimension of the underlying vector space of $\mathfrak{g}$. If $\dim(\mathfrak{g}) = 1$ the result follows from the previous paragraph. If $\dim(\mathfrak{g}) = 2$, the only case that does not follow from the previous paragraph is when $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_1) = 1$. Let us suppose that $\mathfrak{g}_0 = k.z$ and $\mathfrak{g}_1 = k.c$. By Lemma 5.2, we see that, up to isomorphism, we have two possibilities: either $[c, c] = 0$ or $[c, c] = z$. Either if we consider the first case for arbitrary λ or the second case for $\lambda = 0$, there is a unique polarization $\mathfrak{h} = \mathfrak{g}$, so the theorem holds, for $\operatorname{ind}(F_{\lambda,\mathfrak{h},i}, \mathfrak{g})$ is one-dimensional. If we regard the second case with $\lambda \neq 0$, we see that $\lambda \in L_{\mathfrak{g}}$ and there is a unique polarization $\mathfrak{h} = \mathfrak{g}_0$. The theorem also holds in this case, because it is a particular case of [Ser99], Cor. 3.2.

Let us suppose that $\dim(\mathfrak{g}) = d > 2$ and that the proposition holds for dimensions (strictly) less than d. By Remark 4.8, we assume there are no ideals of $\mathfrak{g}$ such that λ vanishes on them. In particular, we see that $\mathcal{Z}(\mathfrak{g}) \cap \operatorname{Ker}(\lambda)$ should be trivial. This implies that $\mathcal{Z}(\mathfrak{g})$ should be one-dimensional, because $\mathfrak{g}$ is nilpotent, and included in $\mathfrak{g}_0$. Let $z \in \mathcal{Z}(\mathfrak{g})$ be a nonzero element such that $\lambda(z) = 1$. By the previous lemma, there exists $x, y \in \mathfrak{g}$ homogeneous of the same degree satisfying that $[y, y] = 0$ and $\mathfrak{k} = \mathcal{C}(\{y\})$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus kx$. Consider $\mathfrak{a} = kz \oplus ky$. It is clearly a supercommutative ideal of $\mathfrak{g}$. We see that $\mathfrak{k} = \mathfrak{st}(\lambda|_{\mathfrak{a}}, \mathfrak{g})$.

We will now show the intermediate result that there is a polarization of $\mathfrak{g}$ at λ included in $\mathfrak{k}$ (and including $\mathfrak{a}$). Let $\mathfrak{h}'$ be any polarization of $\mathfrak{g}$ at λ. If $\mathfrak{h}' \subseteq \mathfrak{k}$ we are done. Let us suppose that $\mathfrak{h}' \nsubseteq \mathfrak{k}$. In this case, we may further assume that $x \in \mathfrak{h}'$ (by the proof of [BM90], Lemma 1.10). Define $\mathfrak{h} = (\mathfrak{h}' \cap \mathfrak{a}^{\lambda}) + \mathfrak{a}$. Then, $\mathfrak{a} \subseteq \mathfrak{h} \subseteq \mathfrak{k}$, and, by Lemma 5.3, $\mathfrak{h}$ is subordinate to λ, so a polarization at λ, since $\mathrm{sdim}(\mathfrak{h}) = \mathrm{sdim}(\mathfrak{h}')$.

We have thus a polarization of $\mathfrak{g}$ at λ included in $\mathfrak{k} = \mathfrak{st}(\lambda|_{\mathfrak{a}}, \mathfrak{g})$ (and including $\mathfrak{a}$). By the inductive hypothesis $W = \mathrm{ind}(\lambda|_{\mathfrak{h}}, \mathfrak{k})$ is simple, with structure morphism denoted by ρ. We remark that the representation $\mathrm{ind}(\lambda|_{\mathfrak{h}}, \mathfrak{g})$ is obviously isomorphic to $\mathrm{ind}(W, \mathfrak{g})$. Using Proposition 2.14 we see that the $\mathfrak{a}$-representation on W given by $\rho|_{\mathfrak{a}}$ is a direct sum of copies of the $\mathfrak{a}$-representation given by $\lambda|_{\mathfrak{a}}$. Applying Theorem 3.4, our theorem follows. □

We also have the following theorem, whose proof is an adaptation of that in [Dix96], Thm. 6.1.4.

Theorem 5.5. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra and $\lambda \in \mathrm{Hom}(\mathfrak{g}, k)$. Given two polarizations $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$ at λ, let $\rho_{\mathfrak{h},i}$ and $\rho_{\mathfrak{h}',j}$ be the structure morphisms of the $\mathcal{U}(\mathfrak{g})$-modules $\mathrm{ind}(F_{\lambda,\mathfrak{h},i}, \mathfrak{g})$ and $\mathrm{ind}(F_{\lambda,\mathfrak{h}',j}, \mathfrak{g})$ determined by the polarizations $\mathfrak{h}$ and $\mathfrak{h}'$ and by some $i, j \in \mathbb{Z}/2\mathbb{Z}$, resp. Then, $\ker(\rho_{\mathfrak{h},i}) = \ker(\rho_{\mathfrak{h}',j})$.*

Proof. Is is easy to see that $\ker(\rho_{\mathfrak{h},i}) = \ker(\rho_{\mathfrak{h},j})$, for all $i, j \in \mathbb{Z}/2\mathbb{Z}$, so from now on we will omit the indices i and j.

We first note that if $\mathfrak{g} = \mathfrak{g}_0$ or $\mathfrak{g} = \mathfrak{g}_1$, then the theorem is immediate: the first case is just the classical result for Lie algebras (see [Dix96], Thm. 6.1.4), and in the second case there is only one polarization $\mathfrak{h} = \mathfrak{g}$, so the statement of the theorem holds in both cases.

We shall now proceed by induction on the dimension of $\mathfrak{g}$. For $\dim(\mathfrak{g}) = 1$ the result is a consequence of the previous paragraph. If $\dim(\mathfrak{g}) = 2$, the only case that does not follow from the previous paragraph is when $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_1) = 1$. Let us suppose that $\mathfrak{g}_0 = k.z$ and $\mathfrak{g}_1 = k.c$. By Lemma 5.2, we see that, up to isomorphism, we have two possibilities: either $[c, c] = 0$ or $[c, c] = z$. Either in the first case for arbitrary λ or in the second case for $\lambda = 0$, there is a unique polarization $\mathfrak{h} = \mathfrak{g}$, for which the theorem holds. In the second case with $\lambda \neq 0$, there is also a unique polarization $\mathfrak{h} = \mathfrak{g}_0$, and the statement also follows in this case.

Let us assume that $\dim(\mathfrak{g}) = d > 2$ and that the statement holds for dimensions strictly less than d. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two polarizations of $\mathfrak{g}$ at λ.

If there exists a nonzero ideal $\mathfrak{k}$ subordinate to λ, then $\mathfrak{h}$ and $\mathfrak{h}'$ include $\mathfrak{k}$ by Remark 4.8. Passing to the quotient $\mathfrak{g}/\mathfrak{k}$, we see that $\mathfrak{h}/\mathfrak{k}$ and $\mathfrak{h}'/\mathfrak{k}$ are polarizations of $\mathfrak{g}/\mathfrak{k}$ at the functional $\bar{\lambda}$ induced by λ. Indeed, $\mathfrak{h}/\mathfrak{k}$ and $\mathfrak{h}'/\mathfrak{k}$ are obviously subordinate to $\bar{\lambda}$ and maximal. The theorem follows in this case by inductive hypothesis.

We thus suppose that there is no nonzero ideal subordinate to λ. Since $\mathfrak{g}$ is nilpotent, $\mathcal{Z}(\mathfrak{g})$ is a nonzero ideal, so $\dim(\mathcal{Z}(\mathfrak{g})) = 1$, $\mathcal{Z}(\mathfrak{g}) \subseteq \mathfrak{g}_0$ and $\lambda(\mathcal{Z}(\mathfrak{g})) \neq 0$. Set $\mathcal{Z}(\mathfrak{g}) = k.z$. By Lemma 5.1, there exists $x, y \in \mathfrak{g}$ homogeneous of the same degree satisfying that $[y, y] = 0$ and $\mathfrak{k} = \mathcal{C}(\{y\})$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus kx$. Consider $\mathfrak{a} = kz \oplus ky$. It is clearly a supercommutative ideal of $\mathfrak{g}$. Notice that $\mathfrak{k} = \mathfrak{st}(\lambda|_{\mathfrak{a}}, \mathfrak{g})$.

Put $\bar{\mathfrak{h}} = (\mathfrak{h} \cap \mathfrak{a}^{\lambda}) + \mathfrak{a}$ and $\bar{\mathfrak{h}}' = (\mathfrak{h}' \cap \mathfrak{a}^{\lambda}) + \mathfrak{a}$. Lemma 5.3 tells us that $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}'$ are subordinate to λ. Both of them satisfy that $\bar{\mathfrak{h}}, \bar{\mathfrak{h}}' \subseteq \mathfrak{k}$ and also $\mathrm{sdim}(\mathfrak{h}) = \mathrm{sdim}(\bar{\mathfrak{h}})$ and $\mathrm{sdim}(\mathfrak{h}') = \mathrm{sdim}(\bar{\mathfrak{h}}')$. If we restrict to $\mathfrak{k}$, $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}'$ are subalgebras subordinated to $\lambda|_{\mathfrak{k}}$. Furthermore, they are polarizations of $\mathfrak{k}$) at $\lambda|_{\mathfrak{k}}$, because they have the same super dimension as $\mathfrak{h}$ and $\mathfrak{h}'$, resp. By inductive hypothesis, they satisfy that $\ker(\mathrm{ind}(\lambda|_{\bar{\mathfrak{h}}}, \mathfrak{k})) = \ker(\mathrm{ind}(\lambda|_{\bar{\mathfrak{h}}'}, \mathfrak{k}))$, so by Proposition 2.13, we have that $\ker(\mathrm{ind}(\lambda|_{\bar{\mathfrak{h}}}, \mathfrak{g})) = \ker(\mathrm{ind}(\lambda|_{\bar{\mathfrak{h}}'}, \mathfrak{g}))$.

We must then show that $\ker(\rho_{\bar{\mathfrak{h}}}) = \ker(\rho_{\mathfrak{h}})$ in order to conclude the proof. Since, if $\mathfrak{h} \subseteq \mathfrak{k}$, then $\mathfrak{h} = \bar{\mathfrak{h}}$, we shall assume that $\mathfrak{h} \nsubseteq \mathfrak{k}$. In this case, we may further suppose that $x \in \mathfrak{h}$ (by the proof of [BM90], Lemma 1.10). Set $\mathfrak{n} = \mathfrak{h} + \mathfrak{a}$. Since $\lambda([x, y]) = \lambda(z) \neq 0$, we see that $y \notin \mathfrak{h}$. Also note that $z \in \mathfrak{h}$ (by Remark 4.8). We see that $\mathfrak{h}, \bar{\mathfrak{h}} \subseteq \mathfrak{n}$ are polarizations of $\mathfrak{n}$ at $\lambda|_{\mathfrak{n}}$, because they are subordinated to $\lambda|_{\mathfrak{n}}$ and of the appropiate super dimension. By Proposition 2.13 we see that it suffices to prove that $\ker(\mathrm{ind}(\lambda|_{\mathfrak{h}}, \mathfrak{n})) = \ker(\mathrm{ind}(\lambda|_{\bar{\mathfrak{h}}}, \mathfrak{n}))$. Since, by Lemma 5.3, $\mathfrak{h} \cap \mathfrak{k} \cap \ker(\lambda)$ is an ideal in $\mathfrak{n}$, by inductive hypothesis we will suppose that the former is trivial. Then, $\dim(\mathfrak{h} \cap \mathfrak{k}) \leq 1$, so $\mathfrak{h} \cap \mathfrak{k} = \mathcal{Z}(\mathfrak{g})$, and analogously for $\bar{\mathfrak{h}}$. This implies that

$$\mathfrak{n} = kz \oplus ky \oplus kx,$$
$$\mathfrak{h} = kz \oplus kx,$$
$$\bar{\mathfrak{h}} = kz \oplus ky.$$

If $|x| = |y| = 0$, the statement follows from [Dix96], Lemma 6.1.2, (iii). If $|x| = |y| = 1$, the statement follows from [Ser99], Lemma 1.2, (2). The theorem is thus proved. □

From Theorem 5.4 we see that given $\lambda \in \mathcal{L}_{\mathfrak{g}}$, there exist a primitive ideal $I(\lambda)$ of $\mathcal{U}(\mathfrak{g})$ given as the kernel of the structure morphism of the representation $\mathrm{ind}(\lambda|_{\mathfrak{h}}, \mathfrak{g})$, for some polarization $\mathfrak{h}$ of $\mathfrak{g}$ at λ. In fact, by Theorem 5.5, the ideal does not depend on the polarization.

The following lemma will be useful when dealing with polarizations in an inductive process.

Lemma 5.6. *Let $\mathfrak{g}$ be a super Lie algebra with super center $\mathfrak{z} = kz \neq \mathfrak{g}$, where z is even, and let $x, y \in \mathfrak{g}$ be homogeneous of the same parity and $\mathfrak{k}$ be an ideal of codimension one in $\mathfrak{g}$ satisfying the properties (i), (ii), (iii) and (2) stated in Lemma 5.1. Given $\lambda \in \mathcal{L}_{\mathfrak{g}}$ satisfying that $\lambda(z) = 1$, define $\lambda' = \lambda|_{\mathfrak{k}} \in \mathcal{L}_{\mathfrak{k}}$. We can further suppose that $\lambda(y) = 0$, for $\lambda(y) = 0$ if $|y| = 1$, and we may change y by $y - \lambda(y)z$ when y is even. Then, if $\mathfrak{h}$ is a polarization of $\mathfrak{k}$ at λ', it is also a polarization of $\mathfrak{g}$ at λ.*

Proof. It is obvious that $\mathfrak{h}$ is subordinate to λ. The (unique) case $\dim(\mathfrak{g}) = 2$ is also clear, so we will suppose that $\dim(\mathfrak{g}) > 2$, and prove that $\mathfrak{h}$ is maximal totally isotropic, *i.e.* that if $v \in \mathfrak{g}$ is a homogeneous element satisfying that $\lambda([v, \mathfrak{h}]) = 0$, then $v \in \mathfrak{h}$. By Remark 4.8 and the fact that ky is an ideal of the super Lie algebra $\mathfrak{k}$, we see that $y \in \mathfrak{h}$. Thus, the assumption that $\lambda([v, \mathfrak{h}]) = 0$ implies that $\lambda([v, y]) = 0$. Using that $y \in \mathcal{Z}(\mathfrak{g}/\mathcal{Z}(\mathfrak{g}))$ (so $[\mathfrak{g}, y] = kz$) and $\lambda(z) = 1$, we conclude that the previous vanishing identity is equivalent to $[v, y] = 0$, *i.e.* $v \in \mathfrak{k}$, which in turn implies that $v \in \mathfrak{h}$. □

The following result tells us that every primitive ideal is of the form $I(\lambda)$, for some functional $\lambda \in \mathcal{L}_{\mathfrak{g}}$. The proof is a variation of the one given in [Dix96], Thm. 6.1.7.

Theorem 5.7. *Let I be a primitive ideal of the enveloping algebra of a nilpotent super Lie algebra $\mathcal{U}(\mathfrak{g})$. Then, there exists $\lambda \in \mathcal{L}_{\mathfrak{g}}$ such that $I = I(\lambda)$.*

Proof. We first remark that if $\mathfrak{g} = \mathfrak{g}_0$ or $\mathfrak{g} = \mathfrak{g}_1$, then the theorem is immediate, because the first case is just the classical result for Lie algebras (see [Dix96], Thm. 6.1.7), and in the second case, since $\mathcal{U}(\mathfrak{g})$ is finite dimensional, all irreducible representations are finite dimensional, so the statement follows from [Ser99], Thm. 1.3, 1).

We shall now proceed by induction on the dimension of $\mathfrak{g}$. If $\dim(\mathfrak{g}) = 1$, the result is a consequence of the previous considerations. In case $\dim(\mathfrak{g}) = 2$, the only case that does not follow from the previous paragraph is when $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_1) = 1$. Let us suppose that $\mathfrak{g}_0 = k.z$ and $\mathfrak{g}_1 = k.c$. By Lemma 5.2, we have two possibilities up to isomorphism: either $[c, c] = 0$ or $[c, c] = z$. It is not difficult to prove, using Proposition 1.2 and the algebras involved, that all the irreducible representation of $\mathcal{U}(\mathfrak{g})$ are finite dimensional, so our theorem follows from [Ser99], Thm. 1.3, 1).

Let us assume that $\dim(\mathfrak{g}) = d > 2$ and that the statement holds for dimensions strictly less than d. If $I \cap \mathfrak{g} \neq 0$, the theorem follows if we apply induction to the super Lie algebra $\mathfrak{g}/(I \cap \mathfrak{g})$. Thus, we suppose that $I \cap \mathfrak{g} = 0$. Remark 1.3 tells us that the super center of $\mathcal{U}(\mathfrak{g})/I$ is k or $k[\epsilon]/(\epsilon^2 - 1)$, with $|\epsilon| = 1$. This further implies that the super center of $\mathfrak{g}$ can be of super dimension $(1, 0)$ or $(1, 1)$ (and $\mathcal{Z}(\mathfrak{g})$ is nontrivial because $\mathfrak{g}$ is nilpotent). However, the second possibility cannot occur, as already noticed in [BM90], 1.1, p. 403: using that any odd supercentral element y of $\mathfrak{g}$ must satisfy that $y\mathcal{U}(\mathfrak{g})y = y^2\mathcal{U}(\mathfrak{g}) = 0$, y must be in any prime ideal of $\mathcal{U}(\mathfrak{g})$, so *a fortiori* in any maximal ideal of $\mathcal{U}(\mathfrak{g})$. Hence, the super center of $\mathfrak{g}$ has super dimension $(1, 0)$: let z be a nonzero element of it. By Lemma 5.1, there exists $x, y \in \mathfrak{g}$ homogeneous of the same degree and an ideal $\mathfrak{k}$ of $\mathfrak{g}$ such that $[y, y] = 0$, $y \in \mathcal{Z}(\mathfrak{g}/\mathcal{Z}(\mathfrak{g}))$, $[x, y] = z$, $\mathfrak{k}$ is the super centralizer of y and $\mathfrak{g} = \mathfrak{k} \oplus kx$. Define $\mathfrak{a}$ the supercommutative ideal of $\mathfrak{g}$ given by $kz \oplus ky$.

By Proposition 3.7, there exist a primitive ideal J of $\mathcal{U}(\mathfrak{a})$, a simple (one-dimensional) representation U of $\mathfrak{a}$ with structure morphism σ, and a simple representation of W of $\mathfrak{st}(U, \mathfrak{g})$ with structure morphism ρ, such that the $\mathfrak{a}$-representation on W given by $\rho|_{\mathfrak{a}}$ is a direct sum of copies of the $\mathfrak{a}$-representation U, and $\mathrm{ind}(W, \mathfrak{g})$ is simple with kernel I. Since U is one dimensional, we see that $\mathfrak{st}(U, \mathfrak{g})$ is expanded by the homogeneous elements $v \in \mathfrak{g}$ satisfying that $\sigma([v, y]) = 0$. Using that $\sigma(z) \neq 0$ (because if not $z \in I$, which is a contradiction), we see that $\mathfrak{k} = \mathfrak{st}(U, \mathfrak{g})$. Substituting z by $z/\sigma(z)$, if necessary, we may suppose that $\sigma(z) = 1$, so $z - 1 \in I$.

By the inductive assumption, there exists $\lambda' \in \mathcal{L}_{\mathfrak{k}}$ and a polarization $\mathfrak{h}$ of $\mathfrak{k}$ at λ' such that, if ρ' denotes the structure morphism of the $\mathfrak{k}$-representation $\operatorname{ind}(\lambda'|_{\mathfrak{h}}, \mathfrak{k})$, then $\ker(\rho) = \ker(\rho')$. Proposition 2.13 tells us that I is the kernel of $\operatorname{ind}(\lambda'|_{\mathfrak{h}}, \mathfrak{g})$. We now consider a functional $\lambda \in \mathcal{L}_{\mathfrak{g}}$ such that $\lambda|_{\mathfrak{k}} = \lambda'$. It suffices to prove that $\mathfrak{h}$ is also a polarization of $\mathfrak{g}$ at λ, which follows from Lemma 5.6. The theorem is thus proved. □

Lemma 5.8. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra with super center $\mathfrak{z} = kz \neq \mathfrak{g}$, with z even, $x, y \in \mathfrak{g}$ homogeneous of the same parity and $\mathfrak{k}$ an ideal of codimension one in $\mathfrak{g}$ satisfying the properties (i), (ii), (iii) and (2) stated in Lemma 5.1. Define $\bar{\mathfrak{k}} = \mathfrak{k}/ky$, δ the locally nilpotent derivation of $\mathcal{U}(\mathfrak{k})$ induced by $\operatorname{ad}(x)$, and $\bar{u}$ the corresponding image element of $u \in \mathcal{U}(\mathfrak{k})$ under the canonical projection from $\mathcal{U}(\mathfrak{k})$ to $\mathcal{U}(\bar{\mathfrak{k}})$. Then,*

(i) if x is even, there exists a morphism of super algebras $\phi_0 : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\bar{\mathfrak{k}}) \otimes A_1(k)$ given by

$$\phi_0(x) = 1 \otimes p,$$
$$\phi_0(u) = \sum_{n \in \mathbb{N}_0} \overline{\delta^n(u)} \otimes q^n,$$

where $u \in \mathcal{U}(\mathfrak{k})$, and $A_1(k)$ is the super algebra described in Example 1.1, (i). Moreover the morphism induces an isomorphism ψ_0 from $\mathcal{U}(\mathfrak{g})_z$ to $\mathcal{U}(\bar{\mathfrak{k}})_z \otimes A_1(k)$.

(ii) if x is odd, there exists a morphism of super algebras $\phi_1 : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\bar{\mathfrak{k}}) \otimes M_2(k)$ given by

$$\phi_1(x) = \begin{pmatrix} 0 & \overline{\frac{[x,x]}{2}} \\ 1 & 0 \end{pmatrix},$$
$$\phi_1(u) = \begin{pmatrix} \bar{u} & \overline{\delta(u)} \\ 0 & \overline{\Sigma(u)} \end{pmatrix},$$

where $u \in \mathcal{U}(\mathfrak{k})$, and $M_2(k)$ is the super algebra described in Example 1.1, (ii). Moreover, the morphism induces an isomorphism ψ_1 from $\mathcal{U}(\mathfrak{g})_z$ to $\mathcal{U}(\bar{\mathfrak{k}})_z \otimes M_2(k)$.

Furthermore, given $I \neq \mathcal{U}(\mathfrak{g})$ an ideal of $\mathcal{U}(\mathfrak{g})$ such that $z - 1 \in I$, then there exists one and only one ideal J of $\mathcal{U}(\bar{\mathfrak{k}})$ satisfying that $\bar{z} - 1 \in J$ and $\phi_0(I_z) = J_{\bar{z}} \otimes A_1(k)$, if x is even, or $\phi_1(I_z) = J_{\bar{z}} \otimes M_2(k)$, if x is odd. Finally, there is a chain of isomorphisms of super algebras

$$\mathcal{U}(\mathfrak{g})/I \to \mathcal{U}(\mathfrak{g})_z/I_z \to (\mathcal{U}(\bar{\mathfrak{k}})_{\bar{z}}/J_{\bar{z}}) \otimes A_1(k) \to (\mathcal{U}(\bar{\mathfrak{k}})/J) \otimes A_1(k),$$

if x is even, and

$$\mathcal{U}(\mathfrak{g})/I \to \mathcal{U}(\mathfrak{g})_z/I_z \to (\mathcal{U}(\bar{\mathfrak{k}})_{\bar{z}}/J_{\bar{z}}) \otimes M_2(k) \to (\mathcal{U}(\bar{\mathfrak{k}})/J) \otimes M_2(k),$$

if x is odd.

Proof. The morphism ϕ_0 and ϕ_1 are easily obtained from [BM90], Lemmas 1.4, 1.5 and 1.7. The proof that the induced mappings given by localizing at z are isomorphisms is exactly the same as the given in [Dix96], Lemma 4.7.8, (i) replacing the use of [Dix96], Propositions 3.6.15 and 3.6.18, by Propositions 1.12 and 1.13, and the use of [Dix96], Lemma 4.7.6 by [BM90], Lemma 1.4 for x even and Lemma 1.5 for x odd. Finally, the proof of the two chain of isomorphisms is the same as the one given in [Dix96], Lemma 4.7.8, (ii), replacing the use of [Dix96], Proposition 3.6.15 and Lemma 4.5.1 by Proposition 1.12 and Lemma 1.9, respectively. □

Lemma 5.9. *Let $\mathfrak{g}$ be a super Lie algebra with super center $\mathfrak{z} = kz \neq \mathfrak{g}$, where z is even, $x, y \in \mathfrak{g}$ be homogeneous of the same parity and $\mathfrak{k}$ an ideal of codimension one in $\mathfrak{g}$ satisfying the properties (i), (ii), (iii) and (2) stated in Lemma 5.1. Set $\bar{\mathfrak{k}} = \mathfrak{k}/k.y$. Given $\lambda \in \mathcal{L}_{\mathfrak{g}}$ satisfying that $\lambda(z) = 1$ and $\lambda(y) = 0$, define $\lambda' = \lambda|_{\mathfrak{k}} \in \mathcal{L}_{\mathfrak{k}}$ and $\bar{\lambda}'$ the functional induced by λ' on $\bar{\mathfrak{k}}$. Then, $\psi_0(I(\lambda)_z) = I(\bar{\lambda}')_{\bar{z}} \otimes A_1(k)$ if x is even, and $\psi_1(I(\lambda)_z) = I(\bar{\lambda}')_{\bar{z}} \otimes M_2(k)$ if x is odd.*

Proof. The proof for x even is the same as the one appearing in [Dix96], Lemma 6.2.1, with the additional assumption that all elements must be homogeneous. The proof for x odd is exactly the same as in the even case, replacing the use of [Dix96], Lemma 4.7.8 and Prop. 5.1.7, by Lemma 5.8 and Proposition 2.13, respectively, and the appearances of the Weyl algebra $A_1(k)$ by $M_2(k)$ and of $1 \otimes q$ by $1 \otimes e_{12}$, and using that $M_2(k)$ is also a simple super algebra. □

The final results of this section are the following, whose proof follows the lines of [Dix96], Prop. 6.2.3.

Proposition 5.10. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra and let $\mathcal{A}d_0$ be the adjoint group of the Lie algebra $\mathfrak{g}_0$, acting on $\mathfrak{g}_0^*$. Given $\lambda, \lambda' \in \mathcal{L}_{\mathfrak{g}}$, then $I(\lambda) = I(\lambda')$ if and only if λ' and λ lie in the same orbit of $\mathfrak{g}_0^*$ under the coadjoint action of $\mathcal{A}d_0$.*

Proof. For $a \in \mathcal{A}d_0$, we denote $a_{\mathcal{U}}$ the automorphism of $\mathcal{U}(\mathfrak{g})$ induced by a. Suppose that $a(\lambda) = \lambda'$. Then, by transport of structures $a_{\mathcal{U}}(I(\lambda)) = I(\lambda')$, and using Proposition 2.15 we conclude that $I(\lambda) = I(\lambda')$.

Conversely, let us assume that $I(\lambda) = I(\lambda')$, which we denote by I. We shall prove that λ' and λ are in the same orbit of $\mathfrak{g}_0^*$ under the coadjoint action of $\mathcal{A}d_0$. We proceed by induction on the dimension of $\mathfrak{g}$. The result is obvious for $\dim(\mathfrak{g}) = 1$, so let us consider that $\dim(\mathfrak{g}) > 1$. Set $\mathfrak{h}$ and $\mathfrak{h}'$ polarizations at λ and λ', respectively, such that $\mathrm{ind}(\lambda|_{\mathfrak{h}}, \mathfrak{g})$ and $\mathrm{ind}(\lambda'|_{\mathfrak{h}'}, \mathfrak{g})$ are simple.

If $\mathfrak{z}' = I \cap \mathfrak{z} \neq 0$, then, $\lambda(\mathfrak{z}') = \lambda'(\mathfrak{z}') = 0$ and $\mathfrak{z}' \subseteq \mathfrak{h}$ and $\mathfrak{z}' \subseteq \mathfrak{h}'$, so we may consider $\bar{\lambda}, \bar{\lambda}' \in \mathcal{L}_{\mathfrak{g}/\mathfrak{z}'}$ induced by λ and λ', respectively. It is easy to see that $\bar{\mathfrak{h}} = \mathfrak{h}/\mathfrak{z}'$ and $\bar{\mathfrak{h}}' = \mathfrak{h}'/\mathfrak{z}'$ are polarizations at $\bar{\lambda}$ and $\bar{\lambda}'$, respectively, and that $\mathrm{ind}(\bar{\lambda}|_{\bar{\mathfrak{h}}}, \mathfrak{g}/\mathfrak{z}')$ and $\mathrm{ind}(\bar{\lambda}'|_{\bar{\mathfrak{h}}'}, \mathfrak{g}/\mathfrak{z}')$ are simple $\mathcal{U}(\mathfrak{g}/\mathfrak{z}')$-modules. Moreover, the image of I under the projection map $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}/\mathfrak{z}')$ coincides with the kernel $\bar{I} = I(\bar{\lambda}) = I(\bar{\lambda}')$ of the structure morphism of either $\mathrm{ind}(\bar{\lambda}|_{\bar{\mathfrak{h}}}, \mathfrak{g}/\mathfrak{z}')$ or $\mathrm{ind}(\bar{\lambda}'|_{\bar{\mathfrak{h}}'}, \mathfrak{g}/\mathfrak{z}')$, thus $\mathcal{U}(\mathfrak{g})/I \simeq \mathcal{U}(\mathfrak{g}/\mathfrak{z}')/\bar{I}$. Then, by the induction hypothesis, there exists an element $\bar{a}$ in the adjoint group of $(\mathfrak{g}/\mathfrak{z}')_0$ such that $\bar{a}_{\mathcal{U}}(\bar{\lambda}) = \bar{\lambda}'$. Since the adjoint group of $(\mathfrak{g}/\mathfrak{z}')_0$ is the epimorphic image of the adjoint group of $\mathfrak{g}_0$ under the canonical map $\mathrm{GL}_{\mathfrak{z}'}(\mathfrak{g})_0 \to \mathrm{GL}(\mathfrak{g}/\mathfrak{z}')_0$, where $\mathrm{GL}_{\mathfrak{z}'}(\mathfrak{g})_0$ is the algebraic subgroup of $\mathrm{GL}(\mathfrak{g})_0$ preserving the ideal $(\mathfrak{z}')_0$, we see that there exists a in the adjoint group of $\mathfrak{g}_0$ (in fact, in $\mathrm{GL}_{\mathfrak{z}'}(\mathfrak{g})_0$) such that $a(\lambda) = \lambda'$.

Let us thus suppose that $I \cap \mathfrak{z} = 0$, which tells us that $\dim(\mathfrak{z}) = 1$. Assume that $\mathfrak{z} = kz \neq \mathfrak{g}$, where z is even, and consider $x, y \in \mathfrak{g}$ be homogeneous of the same parity and $\mathfrak{k}$ an ideal of codimension one in $\mathfrak{g}$ satisfying the properties stated in Lemma 5.1, such that $\lambda(z) = \lambda'(z) = 1$ and $\lambda(y) = 0$. In the previous statement we have used that $\lambda(z) = \lambda'(z)$, which can be proved as follows. Since $z - \lambda(z) \in I(\lambda)$, $z - \lambda'(z) \in I(\lambda')$ and $I(\lambda) = I(\lambda')$, we conclude that $\lambda(z) - \lambda'(z) \in I(\lambda)$, so it must vanish, for I is maximal.

We will now show that we may further suppose that $\lambda'(y) = 0$. If $|y| = 1$, we have by definition that $\lambda'(y) = 0$. If $|y| = 0$, so $|x| = 0$, we consider the element $\exp(\mathrm{ad}(\lambda'(y)x)) \in \mathcal{A}d_0$:

$$(\exp(\mathrm{ad}(\lambda'(y)x))\lambda')(y) = \lambda'(y - \lambda'(y)[x, y]) = 0,$$

so we may change λ' by other functional in the same coadjoint orbit in order to also have $\lambda'(y) = 0$.

Set $\bar{\mathfrak{k}} = \mathfrak{k}/k.y$ and define $\bar{\lambda}$ and $\bar{\lambda}'$ the functionals induced by λ and λ' on $\bar{\mathfrak{k}}$, respectively. By Lemma 5.9, $I(\bar{\lambda})_{\bar{z}} \otimes A_1(k) = \psi_0(I(\lambda)_z) = I(\bar{\lambda}')_{\bar{z}} \otimes A_1(k)$ if x is even, and $I(\bar{\lambda})_{\bar{z}} \otimes M_2(k) = \psi_1(I(\lambda)_z) = I(\bar{\lambda}')_{\bar{z}} \otimes M_2(k)$ if x is odd. Since both $A_1(k)$ and $M_2(k)$ are central simple super algebras, we get that $I(\bar{\lambda})_{\bar{z}} = I(\bar{\lambda}')_{\bar{z}}$, so $I(\bar{\lambda}) = I(\bar{\lambda}')$. By the inductive hypothesis, there exists $\bar{a}$ in the adjoint group of $\bar{\mathfrak{k}}$ such that $\bar{a}(\bar{\lambda}) = \bar{\lambda}$. Using the result mentioned two paragraphs above about the relationship of the adjoint groups of quotients of super Lie algebras, we conclude that there exists $a^{\sim}$ in the adjoint group of $\mathfrak{k}$ such that $a^{\sim}(\lambda|_{\mathfrak{k}}) = \lambda'|_{\mathfrak{k}}$. By [TY05], Prop. 28.8.5, there exists a in the adjoint group of $\mathfrak{g}$ such that a preserves $\mathfrak{k}$ and $a(\lambda)|_{\mathfrak{k}} = \lambda'|_{\mathfrak{k}}$. If $|x| = 1$, then $a(\lambda) = \lambda'$ and the statement follows. Let $|x| = 0$, and define $c = \lambda'(x) - a(\lambda)(x)$. Then,

$$(\exp(\mathrm{ad}(cy))(a(\lambda))|_{\mathfrak{k}} = (\exp(\mathrm{ad}(cy))(\lambda')|_{\mathfrak{k}},$$

and

$$(\exp(\mathrm{ad}(cy))(a(\lambda))(x) = a(\lambda)(x - c[y, x]) = \lambda'(x).$$

The proposition is thus proved. □

Proposition 5.11. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra and let $d \in \mathrm{Der}(\mathfrak{g})$ be a (homogeneous) derivation of $\mathfrak{g}$, which is also regarded to act on $\mathcal{H}om(g,k)$. Given $\lambda \in \mathcal{L}_{\mathfrak{g}}$, then $I(\lambda)$ is invariant under d, i.e. $d(I(\lambda)) \subseteq I(\lambda)$, if and only if there exists a homogeneous element $x \in \mathfrak{g}$ of the same parity as d such that $d.\lambda = \mathrm{ad}(x).\lambda$, where the dots denote the action of the elements of $\mathrm{Der}(\mathfrak{g})$ on $\mathcal{H}om(\mathfrak{g},k)$ under the coadjoint action of $\mathcal{A}d_0$.*

Proof. Let us suppose that $d.\lambda = \mathrm{ad}(x).\lambda$. We will prove that $I(\lambda)$ is invariant under d. In fact, since $I(\lambda)$ is invariant under $\mathrm{ad}(x)$, d preserves $I(\lambda)$ if and only if $d' = d - \mathrm{ad}(x)$ does it, so it suffices to show that d' preserves $I(\lambda)$. By Remark 4.10, we can choose a polarization $\mathfrak{h}$ of $\mathfrak{g}$ at λ that is also invariant under d'. As usual, we denote the extension of d' to a derivation of $\mathcal{U}(\mathfrak{g})$ by the same letter.

Recall the notation of Subsection 4.4, where the one-dimensional representation of $\mathfrak{h}$ determined by λ generated by a vector of parity $i \in \mathbb{Z}/2\mathbb{Z}$ was denoted by $F_{\lambda,\mathfrak{h},i} = k.v_{\lambda,\mathfrak{h},i}$. Since the annihilator of $\mathrm{ind}(F_{\lambda,\mathfrak{h},i},\mathfrak{g})$ is independent of i (by Theorem 5.5), we shall omit this subindex for simplicity. Now, define a map $D' \in \mathcal{E}nd(\mathrm{ind}(\lambda|_{\mathfrak{h}},\mathfrak{g}))$ of the same parity as d' as follows: given a homogeneous element $z \in \mathcal{U}(\mathfrak{g})$, then $D'(z \otimes_{\mathcal{U}(\mathfrak{h})} v_{\lambda,\mathfrak{h}}) = d'(z) \otimes_{\mathcal{U}(\mathfrak{h})} v_{\lambda,\mathfrak{h}}$. It is well-defined because $\lambda \circ d' = 0$. Moreover, the map D' satisfies that $D'(u.v) = d'(u).v + (-1)^{|u||d'|}u.D'(v)$, for all homogeneous elements $u \in \mathcal{U}(\mathfrak{g})$ and $v \in \mathrm{ind}(\lambda|_{\mathfrak{h}},\mathfrak{g})$. Since $I(\lambda)$ is the annihilator of $\mathrm{ind}(\lambda|_{\mathfrak{h}},\mathfrak{g})$, this is turn implies that $I(\lambda)$ is preserved under d': for $u \in I(\lambda)$ homogeneous, we have that $d'(u).v = D'(u.v) - (-1)^{|u||d'|}u.D'(v) = 0$.

Conversely, let us assume that $d(I(\lambda)) \subseteq I(\lambda)$, and to simplify, let us write I instead of $I(\lambda)$. We shall prove that there exists a homogeneous elements $x \in \mathfrak{g}$ of the same parity as d such that $d.\lambda = \mathrm{ad}(x).\lambda$, *i.e.* such that $\lambda \circ d = \lambda \circ \mathrm{ad}(x)$. We proceed by induction on the dimension of $\mathfrak{g}$. The result is obvious for $\dim(\mathfrak{g}) = 1$, so let us consider that $\dim(\mathfrak{g}) > 1$.

If $I \cap \mathfrak{g} \neq 0$, then we see that the ideal $I \cap \mathfrak{g}$ of $\mathfrak{g}$ is preserved by d, *i.e.* $d(I \cap \mathfrak{g}) \subseteq (I \cap \mathfrak{g})$, since both I and $\mathfrak{g}$ are stable under d. So, the derivation d induces a derivation $\bar{d}$ on the super Lie algebra $\mathfrak{g}/(I \cap \mathfrak{g})$, and a *a fortiori* a derivation on the corresponding enveloping algebra $\mathcal{U}(\mathfrak{g}/(I \cap \mathfrak{g}))$, which we also denote by $\bar{d}$. We note that the parity of $\bar{d}$ coincides with that of d. Moreover, the image $\bar{I}$ of I under the projection map $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}/(I \cap \mathfrak{g}))$ is a maximal ideal of $\mathcal{U}(\mathfrak{g}/(I \cap \mathfrak{g}))$ which is stable under $\bar{d}$. By the inductive hypothesis, there exists $\bar{x} \in \mathfrak{g}/(I \cap \mathfrak{g})$ of the same parity as $\bar{d}$ such that the following idenity $\lambda(\bar{d}(-)) = \lambda([\bar{x},-])$ holds on $\mathfrak{g}/(I \cap \mathfrak{g})$. Hence, if $x \in \mathfrak{g}$ is a homogeneous representative of $\bar{x}$, we see that $\lambda(d(-)) = \lambda([x,-])$ holds on $\mathfrak{g}$.

Let us suppose that $I \cap \mathfrak{g} = 0$, which tells us that $\dim(\mathfrak{z}) = 1$. Assume that $\mathcal{Z}(\mathfrak{g}) = kz \neq \mathfrak{g}$, where z is even, and consider $x, y \in \mathfrak{g}$ be homogeneous of the same parity and $\mathfrak{k}$ an ideal of codimension one in $\mathfrak{g}$ satisfying the properties stated in Lemma 5.1, such that $\lambda(z) = 1$, or equivalently, such that $z - 1 \in I$. Since d preserves the super center of $\mathfrak{g}$, z must be an eigenvector of d, say of eigenvalue α, but since $z - 1 \in I$ and d preserves I, then $d(z-1) = \alpha z$ should belong to I. As $I \cap \mathfrak{z} = 0$, we conclude that $\alpha = 0$, $d(z) = 0$.

Define $\mathfrak{a}$ the super Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$ generated by d. It is nilpotent of super dimension $(1,0)$ if d is even and $(1,1)$ if d is odd, and it acts by derivations on $\mathfrak{g}$. We may thus consider the super Lie algebra $\mathfrak{a} \ltimes \mathfrak{g}$ given by the semidirect product. Note that $\mathfrak{g}$ and $\mathfrak{k}$ are ideals of $\mathfrak{a} \ltimes \mathfrak{g}$, since they are invariant under d, which is equivalent to say that they are invariant under $\mathfrak{a}$.

We may further suppose that we can choose y such that it is an eigenvector of d. In order to prove that, we first recall the easy fact that any derivation of a super Lie algebra preserves all terms of the lower central series $\{\mathcal{C}^{\bullet}(\mathfrak{g})\}_{\bullet \in \mathbb{N}}$ of $\mathfrak{g}$. In particular, if $\mathcal{C}^N(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$, then d, or equivalently $\mathfrak{a}$, should preserve the ideal $\mathcal{C}^{N-1}(\mathfrak{g}) \supseteq \mathcal{Z}(\mathfrak{g})$. Since $\mathfrak{a}$ is nilpotent, by Proposition 2.8 the $\mathfrak{a}$-representation $\mathcal{C}^{N-1}(\mathfrak{g})$ has a flag of subrepresentations. This simply tells us that when dealing with the Jordan form of the restriction of d to $\mathcal{C}^{N-1}(\mathfrak{g})$, we may assume that all the elements of the Jordan basis are homogeneous. It may happen that there exists another eigenvector (of zero eigenvalue) appearing in the fixed Jordan basis, outside z. If this is the case, we choose y to be one of these eigenvectors, that satisfies the claim. Suppose otherwise that z is the unique eigenvector in the Jordan basis of $d|_{\mathcal{C}^{N-1}(\mathfrak{g})}$. Hence, there exists an element of the Jordan basis of $d|_{\mathcal{C}^{N-1}(\mathfrak{g})}$ such that its image under d is z. We choose y to be such an element, and then pick x as in Lemma 5.1. Note that y and also x have the same parity as d. Changing d by $d + (-1)^{|x||y|}\mathrm{ad}(x)$, we may thus assume that $d(y) = 0$. In any case, we can thus suppose that y is an eigenvector of d, as we wanted to prove.

We shall first prove that there is a maximal ideal $J \subseteq \mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})$ such that $J \cap \mathcal{U}(\mathfrak{g}) = I$. In order to do that, consider the left ideal $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})I$ of $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})$ generated by I. Since I is invariant under $\mathfrak{a}$, it is in fact a two-sided ideal. By the PBW Theorem, the extension $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g}) \supseteq \mathcal{U}(\mathfrak{g})$ is free and, in consequence, given any left ideal $L \subseteq \mathcal{U}(\mathfrak{g})$, we have that $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})L \cap \mathcal{U}(\mathfrak{g}) = L$. This in turn implies that $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})I \cap \mathcal{U}(\mathfrak{g}) = I$, so $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})I \neq \mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})$. Choose J a maximal ideal of $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})$ containing $\mathcal{U}(\mathfrak{a} \ltimes \mathfrak{g})I$, which satisfies what we stated.

Now, by Proposition 3.7, there exists a primitive ideal K of $\mathcal{U}(\mathfrak{k})$ containing $J \cap \mathcal{U}(\mathfrak{k}) = I \cap \mathcal{U}(\mathfrak{k})$ such that it is invariant under the adjoint action of $\mathfrak{a} \ltimes \mathfrak{g}$, so *a fortiori* invariant under d, and a functional $\tilde{\lambda} \in \mathcal{L}_{\mathfrak{k}}$ provided with a polarization $\mathfrak{h} \subseteq \mathfrak{k}$ also invariant under $\mathfrak{a} \ltimes \mathfrak{g}$ (by Remark 4.10) such that the kernel of the structure morphisms of $\mathrm{ind}(\tilde{\lambda}|_{\mathfrak{h}}, \mathfrak{k})$ is K and of $\mathrm{ind}(\tilde{\lambda}|_{\mathfrak{h}}, \mathfrak{a} \ltimes \mathfrak{g})$ is J, respectively. By the proof of the aforementioned proposition, this also implies that $\mathrm{ind}(\tilde{\lambda}|_{\mathfrak{h}}, \mathfrak{g})$ is I. Set $\lambda' \in \mathcal{L}_{\mathfrak{g}}$ any functional such that $\tilde{\lambda} = \lambda'|_{\mathfrak{k}}$. By Lemma 5.6, $\mathfrak{h}$ is also a polarization of λ'. Furthermore, since the kernel of the structure morphism of $\mathrm{ind}(\lambda'|_{\mathfrak{h}}, \mathfrak{g})$ is $I(\lambda)$, by Proposition 5.10, there exists $g \in \mathcal{A}d_0$, the adjoint group of $\mathfrak{g}_0$, such that $g.\lambda' = \lambda$. In particular, $g.(\lambda'|_{\mathfrak{k}}) = \lambda|_{\mathfrak{k}}$, or, more explicitly, $g.\tilde{\lambda} = \lambda|_{\mathfrak{k}}$. On the other hand, the fact that $K = I(\tilde{\lambda})$ is invariant under $\mathfrak{g}$ yields that $g.K = K$, so $K = g^{-1}.K = g^{-1}.I(\tilde{\lambda}) = I(g.\tilde{\lambda}) = I(\lambda|_{\mathfrak{k}})$.

Since $K = I(\lambda|_{\mathfrak{k}})$ is a primitive ideal of $\mathcal{U}(\mathfrak{k})$ invariant under d, by the inductive hypothesis, there exists a homogeneous element $u \in \mathfrak{k}$ of the same degree as d such that $d.\lambda|_{\mathfrak{k}} = \mathrm{ad}(u).\lambda|_{\mathfrak{k}}$, or equivalently, $\lambda \circ d|_k = \lambda \circ \mathrm{ad}(u)|_{\mathfrak{k}}$. Define $c = \lambda(d(x)) - \lambda([u, x])$. Then,

$$\lambda \circ (d - c.\mathrm{ad}(y))|_{\mathfrak{k}} = \lambda \circ \mathrm{ad}(u)|_{\mathfrak{k}},$$

and

$$\lambda \circ (d - c.\mathrm{ad}(y))(x) = \lambda(d(x) - c[y, x]) = \lambda \circ \mathrm{ad}(u)(x).$$

The proposition is thus proved. □

Remark 5.12. *If the derivation in the previous proposition is even, the statement is then equivalent to Proposition 5.10, and in fact it can be seen as the infinitesimal version of it, but in general they are not equivalent, by the obvious observation that the exponential map of super Lie algebras is only defined on the even part. Moreover, both propositions have the following supergeometrical content, which is just the superized version of* [Dix96], *Prop. 6.2.3. Let $(H_0, \mathfrak{h})$ be an sub algebraic super group of the super group of automorphisms of a nilpotent super Lie algebra $\mathfrak{g}$. It acts then on the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ and on the super variety X defined by $\mathcal{H}om(\mathfrak{g}, k)$. Then, the action of $(H_0, \mathfrak{h})$ on $\mathcal{U}(\mathfrak{g})$ preserves an ideal $I(\lambda)$ if and only if the orbit of a point λ of (the underlying variety of) X under the action of $(H_0, \mathfrak{h})$ is included in the orbit of λ under the action of the adjoint super group of $\mathfrak{g}$.*

5.2 Some consequences

We want to derive some consequences from the main theorems proved before. From Lemma 5.9 we obtain the following proposition, which is analogous to [Dix96], Prop. 6.2.2 (*cf.* [BM90], Thm. A).

Proposition 5.13. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra and $\lambda \in \mathcal{L}_{\mathfrak{g}}$. The primitive ideal $I(\lambda)$ satisfies that $\mathcal{U}(\mathfrak{g})/I(\lambda) \simeq \mathrm{Cliff}_q(k) \otimes A_p(k)$, where $(p, q) = (\dim(\mathfrak{g}_0/\mathfrak{g}_0^\lambda)/2, \dim(\mathfrak{g}_1/\mathfrak{g}_1^\lambda))$ and $\mathfrak{g}^\lambda = (\mathfrak{g}_0^\lambda, \mathfrak{g}_1^\lambda)$ is the kernel of the superantisymmetric bilinear form $\langle\, ,\rangle_\lambda$ determined by λ on $\mathfrak{g}$.*

Proof. We first remark that if $\mathfrak{g} = \mathfrak{g}_0$ or $\mathfrak{g} = \mathfrak{g}_1$, then the proposition is immediate. Indeed, the first case is just the classical result for Lie algebras (see [Dix96], Prop. 6.2.2). In the second case, $\mathfrak{g}^\lambda = \mathfrak{g}$ and the corresponding irreducible $\mathfrak{g}$-representation is one dimensional, so the statement also follows.

We shall now proceed by induction on the dimension of $\mathfrak{g}$. If $\dim(\mathfrak{g}) = 1$, the result is obvious, since $\mathfrak{g}^\lambda = \mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ is a supersymmetric super algebra with a unique maximal ideal I whose quotient is k. In case $\dim(\mathfrak{g}) = 2$, the only case that does not follow from the previous paragraph is when $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_1) = 1$. Let us suppose that $\mathfrak{g}_0 = k.z$ and $\mathfrak{g}_1 = k.c$. By Lemma 5.2, we have two possibilities up to isomorphism: either $[c, c] = 0$ or $[c, c] = z$. It is not difficult to prove that either if we consider the first case for arbitrary λ or the second case for $\lambda = 0$, $\mathfrak{g}^\lambda = \mathfrak{g}$, so the theorem holds, for $\mathrm{ind}(F_{\lambda,\mathfrak{h},i}, \mathfrak{g})$ is one-dimensional. If we regard the second case with $\lambda \neq 0$, we see that $\mathfrak{g}^\lambda = \mathfrak{g}_0$. It can be easily checked that the annihilator is the ideal generated by $z - 1$, whose quotient is $\mathrm{Cliff}_1(k)$, so the statement also holds in this case (*cf.* [BM90], 0.2, (b)).

Let us suppose that $\dim(\mathfrak{g}) > 2$ and let $\mathfrak{z}$ denote the super center of $\mathfrak{g}$. We denote $I(\lambda)$ simply by I and consider $\mathfrak{h}$ a polarization of $\mathfrak{g}$ at λ such that $\mathrm{ind}(\lambda|_{\mathfrak{h}}, \mathfrak{g})$ is simple.

If $I \cap \mathfrak{z} \neq 0$, then $\lambda(I \cap \mathfrak{z}) = 0$ and $(I \cap \mathfrak{z}) \subseteq \mathfrak{h}$, so we may consider $\bar{\lambda} \in \mathcal{L}_{\mathfrak{g}/(I \cap \mathfrak{z})}$ induced by λ. It is easy to see that $\bar{\mathfrak{h}} = \mathfrak{h}/(I \cap \mathfrak{z})$ is a polarization at $\bar{\lambda}$ and that $\mathrm{ind}(\bar{\lambda}|_{\bar{\mathfrak{h}}}, \mathfrak{g}/(I \cap \mathfrak{z}))$ is a simple $\mathcal{U}(\mathfrak{g}/(I \cap \mathfrak{z}))$-module.

Moreover, the image of I under the projection map $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}/(I \cap \mathfrak{z}))$ coincides with the kernel $\bar{I}$ of the structure morphism of $\operatorname{ind}(\bar{\lambda}|_{\bar{\mathfrak{h}}}, \mathfrak{g}/(I \cap \mathfrak{z}))$, thus $\mathcal{U}(\mathfrak{g})/I \simeq \mathcal{U}(\mathfrak{g}/(I \cap \mathfrak{z}))/\bar{I}$. It is also clear that $\mathfrak{g}^{\lambda} \supseteq I \cap \mathfrak{z}$, so $(\mathfrak{g}/(I \cap \mathfrak{z}))^{\bar{\lambda}} = \mathfrak{g}^{\lambda}/(I \cap \mathfrak{z})$ and $(\mathfrak{g}/(I \cap \mathfrak{z}))/(\mathfrak{g}/(I \cap \mathfrak{z}))^{\lambda}) = \mathfrak{g}/\mathfrak{g}^{\lambda}$. Then, the statement follows from the inductive hypothesis.

Let us now assume that $I \cap \mathfrak{z} = 0$, which tells us that $\dim(\mathfrak{z}) = 1$. Suppose that $\mathfrak{z} = kz \neq \mathfrak{g}$, where z is even, and consider $x, y \in \mathfrak{g}$ be homogeneous of the same parity and $\mathfrak{k}$ an ideal of codimension one in $\mathfrak{g}$ satisfying the properties (i), (ii), (iii) and (2) stated in Lemma 5.1, such that $\lambda(z) = 1$ and $\lambda(y) = 0$, so $z - 1 \in I$. Set $\bar{\mathfrak{k}} = \mathfrak{k}/k.y$ and define $\lambda' = \lambda|_{\mathfrak{k}} \in \mathcal{L}_{\mathfrak{k}}$ and $\bar{\lambda}'$ the functional induced by λ' on $\bar{\mathfrak{k}}$. It is direct to check that $\mathfrak{g}^{\lambda} \subseteq \mathfrak{k}$ and moreover $\operatorname{sdim}(\bar{\mathfrak{k}}^{\bar{\lambda}'}) = \operatorname{sdim}(\mathfrak{g}^{\lambda})$. By Lemma 5.9, $\psi_0(I(\lambda)_z) = I(\bar{\lambda}')_{\bar{z}} \otimes A_1(k)$ if x is even, and $\psi_1(I(\lambda)_z) = I(\bar{\lambda}')_{\bar{z}} \otimes M_2(k)$ if x is odd. The corollary thus follows from the inductive assumption and Lemma 5.8. □

We want to determine how the stabilizers of the primitive ideals of $\mathcal{U}(\mathfrak{g})$ are. In order to do that, we first consider the following simple result.

Lemma 5.14. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra, $\mathfrak{k}$ an ideal of $\mathfrak{g}$ such that there exists a homogeneous element $x \in \mathfrak{g}$ satisfying that $\mathfrak{g} = \mathfrak{k} \oplus kx$, and $\lambda \in \mathcal{L}_{\mathfrak{g}}$ a functional satisfying that $\lambda(x) = \lambda([x, x]) = \lambda([x, \mathfrak{k}]) = 0$. Denote $\lambda' \in \mathcal{L}_{\mathfrak{k}}$ the restriction of λ to $\mathfrak{k}$. Then $\mathfrak{st}(I(\lambda'), \mathfrak{g}) = \mathfrak{g}$.*

Proof. The proof given in [Dix96], Lemma 6.2.6, (iii), also holds in this case, where the existence of a polarization invariant under $\operatorname{ad}(x)$ follows from Remark 4.10, (i). □

The following result is analogous to [Dix96], Lemma 6.2.7, and the proof is an adaption to the one given there.

Lemma 5.15. *Let $\mathfrak{g}$ be a super Lie algebra, $\mathfrak{k}$ a nilpotent ideal of $\mathfrak{g}$ such that there exists and $\lambda \in \mathcal{L}_{\mathfrak{k}}$. Denote by $\mathfrak{g}'$ the super vector space expanded by the homogeneous elements $x \in \mathfrak{g}$ satisfying that $\lambda([x, \mathfrak{k}]) = 0$, and notice that it is a subalgebra of $\mathfrak{g}$. Then $\mathfrak{st}(I(\lambda), \mathfrak{g}) \subseteq \mathfrak{g}' + \mathfrak{k}$.*

Proof. Consider $x \in \mathfrak{st}(I(\lambda), \mathfrak{g})$. Then $\operatorname{ad}_{\mathfrak{k}}(x) \in \operatorname{Der}(\mathfrak{k})$ is a derivation that preserves the ideal $I(\lambda)$. By Proposition 5.11, there exists $y \in \mathfrak{k}$ such that $\lambda \circ \operatorname{ad}_{\mathfrak{k}}(x) = \lambda \circ \operatorname{ad}_{\mathfrak{k}}(y)$. Hence, $x - y \in \mathfrak{g}'$, which proves the lemma. □

As a direct consequence from the previous lemmas we obtain that (*cf.* [Dix96], Prop. 6.2.8):

Proposition 5.16. *Let $\mathfrak{g}$ be a nilpotent super Lie algebra, $\mathfrak{k}$ an ideal of $\mathfrak{g}$ and $\lambda \in \mathcal{L}_{\mathfrak{g}}$. Then, $\mathfrak{st}(I(\lambda), \mathfrak{g}) = \mathfrak{g}' + \mathfrak{k}$.*

Proof. By the previous lemma, we need only to show that $\mathfrak{st}(I(\lambda), \mathfrak{g}) \supseteq \mathfrak{g}' + \mathfrak{k}$. It is also direct that $\mathfrak{k} \subset \mathfrak{st}(I(\lambda), \mathfrak{g})$, so let us prove that $\mathfrak{g}' \subset \mathfrak{st}(I(\lambda), \mathfrak{g})$. Take $x \in \mathfrak{g}'$ and consider the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by $\mathfrak{k}$ and , which is nilpotent since $\mathfrak{g}$ is. Lemma 5.14 tells us that $x \in \mathfrak{st}(I(\lambda), \mathfrak{g})$ and the proposition is proved. □

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