

# Hardy inequality and heat semigroup estimates for Riemannian manifolds with singular data

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## Abstract

Upper bounds are obtained for the heat content of an open set  $D$  in a geodesically complete Riemannian manifold  $M$  with Dirichlet boundary condition on  $\partial D$ , and non-negative initial condition. We show that these upper bounds are close to being sharp if (i) the Dirichlet-Laplace-Beltrami operator acting in  $L^2(D)$  satisfies a strong Hardy inequality with weight  $\delta^2$ , (ii) the initial temperature distribution, and the specific heat of  $D$  are given by  $\delta^{-\alpha}$  and  $\delta^{-\beta}$  respectively, where  $\delta$  is the distance to  $\partial D$ , and  $1 < \alpha < 2, 1 < \beta < 2$ .

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Proofs of Theorems 1, 2, 3 and 4</b>	<b>5</b>

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<b>3</b>	<b>Singular initial temperature and singular specific heat</b>	<b>9</b>
<b>4</b>	<b>The special case calculation for a ball in <math>\mathbb{R}^3</math></b>	<b>11</b>

## 1 Introduction

Let  $D$  be a smooth, connected,  $m$ - dimensional Riemannian manifold and let  $\Delta$  be the Laplace-Beltrami operator on  $D$ . It is well known (see [11], [14]) that the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \quad x \in D, \quad t > 0, \quad (1)$$

has a unique minimal positive fundamental solution  $p(x, y; t)$  where  $x \in D, y \in D, t > 0$ . This solution, the Dirichlet heat kernel for  $D$ , is symmetric in  $x, y$ , strictly positive, jointly smooth in  $x, y \in D$  and  $t > 0$ , and it satisfies the semigroup property

$$p(x, y; s + t) = \int_D p(x, z; s)p(z, y; t)dz, \quad (2)$$

for all  $x, y \in D$  and  $t, s > 0$ , where  $dz$  is the Riemannian measure on  $D$ . Equation (1) with the initial condition

$$u(x; 0^+) = \psi(x), \quad x \in D, \quad (3)$$

has a solution

$$u_\psi(x; t) = \int_D p(x, y; t)\psi(y)dy, \quad (4)$$

for any function  $\psi$  on  $D$  from a variety of function spaces like  $C_b(D)$  or  $L^p(D)$ ,  $1 \leq p \leq \infty$ . Note that  $u_\psi \in C_b(D)$  if  $\psi \in C_b(D)$  or that  $u_\psi \in L^p(D)$  if  $\psi \in L^p(D)$ . Initial condition (3) is understood in the sense that  $u_\psi(\cdot; t) \rightarrow \psi(\cdot)$  as  $t \rightarrow 0^+$  where the convergence is appropriate for the function space of initial conditions. For example, if  $\psi \in C_b(D)$  then the convergence is locally uniform, or if  $\psi \in L^p(D)$ ,  $1 \leq p \leq \infty$  then the convergence is in the norm of  $L^p(D)$ . In general, (4) is not the unique solution of (1)-(3). However, it has the following distinguished property: if  $\psi \geq 0$  then  $u_\psi$  is the minimal non-negative solution of that problem (and if  $\psi$  is signed then  $u_\psi = u_{\psi_+} - u_{\psi_-}$ ). If  $D$  is an open subset of another Riemannian manifold  $M$  and if the boundary  $\partial D$  of  $D$  in  $M$  is smooth then the minimality property of  $u_\psi$  implies that, for any  $t > 0$ ,

$$\lim_{x \rightarrow \partial D} u_\psi(x; t) = 0. \quad (5)$$

If  $\partial D$  is non-smooth then (5) can still be understood in a weak sense. Expression (4) makes sense for any non-negative measurable function  $\psi$  on  $D$ , provided the value  $+\infty$  is allowed for  $u_\psi$ . It is known that if  $u_\psi \in L^1_{loc}(D \times \mathbb{R}_+)$  then  $u_\psi$  is a smooth function in  $D \times \mathbb{R}_+$  and it solves (1) (see p. 201 in [14]). For any two non-negative measurable functions  $\psi_1, \psi_2$  on  $D$ , we define for  $t > 0$

$$Q_{\psi_1, \psi_2}(t) = \iint_{D \times D} dx dy p(x, y; t)\psi_1(x)\psi_2(y). \quad (6)$$

Using the properties of the Dirichlet heat kernel we have for  $0 < s < t$

$$Q_{\psi_1, \psi_2}(t) = \int_D u_{\psi_1}(x; s)u_{\psi_2}(x; t - s)dx. \quad (7)$$

Assuming that  $D$  is an open subset of a complete Riemannian manifold  $M$ ,  $Q_{\psi_1, \psi_2}(t)$  has the following physical interpretation: it is the amount of heat in  $D$  at time  $t$  if  $D$  has initial temperature distribution  $\psi_1$ , and a specific heat  $\psi_2$ , while the  $\partial D$  is kept at fixed temperature 0.

This function has been subject of a thorough investigation. Its asymptotic behavior for small  $t$  is well understood if  $D$  has compact closure with  $C^\infty$  boundary, and both  $\psi_1$  and  $\psi_2$  are  $C^\infty$  on the closure  $\overline{D}$  of  $D$ . In that case  $Q_{\psi_1, \psi_2}(t)$  has an asymptotic series in  $t^{1/2}$ , and its coefficients are computable in terms of local geometric invariants [2, 12]. No such series are known if  $D$  is unbounded, or if either the initial data or  $\partial D$  are non-smooth.

In this paper we will obtain upper bounds for the heat content  $Q_{\psi_1, \psi_2}(t)$  under quite general assumptions on  $D$  and on  $\psi_1$  and  $\psi_2$ .

We are particularly interested in the situation where  $D$  is a open subset of another manifold  $M$ , and where  $\psi_1(x)$  and  $\psi_2(x)$  blow up as  $x \rightarrow \partial D$ . In order to guarantee finite heat content for  $t > 0$ , sufficient cooling at  $\partial D$  needs to take place. This will be guaranteed by a condition on  $D$ , that is formulated in terms of a Hardy inequality. Note that in this setting  $Q_{\psi_1, \psi_2}(t)$  may be unbounded as  $t \rightarrow 0^+$ , and one of the interesting points of this study is to obtain the rate of convergence of  $Q_{\psi_1, \psi_2}(t)$  to  $+\infty$  as  $t \rightarrow 0^+$ .

Given a positive measurable function  $h$  on a manifold  $D$ , we say that the Dirichlet Laplacian acting in  $L^2(D)$  satisfies a strong Hardy inequality with weight  $h$  if, for all  $w \in C_c^\infty(D)$ ,

$$\int_D |\nabla w|^2 \geq \int_D \frac{w^2}{h}. \quad (8)$$

Here, and in what follows, we put  $\int_D f = \int_D f(x)dx$  if this does not cause confusion. We also put  $|D| = \int_D 1$ , and  $\|f\|_p = (\int_D |f|^p)^{1/p}$ . A typical example of a Hardy inequality is when  $D$  is an open subset of another manifold  $M$ , and

$$h(x) = c^2 \delta(x)^2, \quad (9)$$

where  $c \geq 2$  is a constant,  $\delta$  is the distance to the boundary,

$$\delta(x) = \min\{d(x, y) : y \in \partial D\},$$

and  $d(x, y)$  is the geodesic distance from  $x$  to  $y$  on  $M$ . Both the validity and applications of Hardy inequalities with weight (9) have been investigated extensively [1], [7], [9], [10], [11], [4]. For example, inequality (8) holds with weight (9) with  $c = 4$  if  $D$  is simply connected with non-empty boundary in  $\mathbb{R}^2$ , with  $c = 2$  if  $D$  is convex in  $\mathbb{R}^m$ , and for some  $c \geq 2$  if  $D$  is bounded with smooth boundary in  $\mathbb{R}^m$ .

In [3] it was shown that if  $D$  has finite measure and satisfies the Hardy inequality with weight  $h$ , and if  $\psi$  is a non-negative measurable function on  $D$ , such that, for some  $q > 1$ ,

$$\|\psi h^{1/q}\|_{q/(q-1)} < \infty, \quad (10)$$

then, for all  $t > 0$ ,

$$Q_{\psi, 1}(t) \leq \left(\frac{q^2}{4(q-1)}\right)^{1/q} \|\psi h^{1/q}\|_{q/(q-1)} (|D| - Q_{1,1}(t))^{1/q} t^{-1/q},$$

where  $Q_{1,1}$  is defined by (6) for  $\psi_1 = \psi_2 = 1$ , that is,

$$Q_{1,1}(t) = \int_D u_1(x; t) dx = \iint_{D \times D} dx dy p(x, y; t).$$

A similar estimate holds for arbitrary open sets  $D \subset \mathbb{R}^m$ , satisfying the Hardy inequality with weight  $h$ . If  $\psi$  is a non-negative measurable function on  $D$  such that, for some  $q > 1$ ,

$$\left\| \max\{\psi, 1\} h^{1/q} \right\|_{q/(q-1)} < \infty, \quad (11)$$

then, for all  $t > 0$ ,

$$Q_{\psi,1}(t) \leq a(q) \|\psi h^{1/q}\|_{q/(q-1)} \|h^{1/(q(q-1))}\|_q t^{-1/(q-1)}, \quad (12)$$

where

$$a(q) = 4^{-1/q} \left( \frac{q}{q-1} \right)^{(2q-1)/(q(q-1))}. \quad (13)$$

Below we give a sufficient condition for the finiteness of  $Q_{\psi_1, \psi_2}(t)$  for all  $t > 0$ , and reduce the problem of finding upper bounds for  $Q_{\psi_1, \psi_2}(t)$  to the case  $\psi_1 = \psi_2$ .

**Theorem 1.** *Let  $\psi_1$  and  $\psi_2$  be non-negative and Borel measurable on a manifold  $D$ .*

(i) *If  $Q_{\psi_i, \psi_i}(t) < \infty, i = 1, 2$ , for all  $t > 0$ , then  $Q_{\psi_1, \psi_2}(t) < \infty$  for all  $t > 0$ , and*

$$Q_{\psi_1, \psi_2}(t) \leq (Q_{\psi_1, \psi_1}(t) Q_{\psi_2, \psi_2}(t))^{1/2}, \quad t > 0. \quad (14)$$

(ii) *If  $Q_{\psi_i, 1}(t) < \infty, i = 1, 2$ , for all  $t > 0$ , and if*

$$c_t := \sup_{x \in D} p(x, x; t) < \infty, \quad t > 0,$$

then

$$Q_{\psi_1, \psi_2}(t) \leq c_{t/3} Q_{\psi_1, 1}(t/3) Q_{\psi_2, 1}(t/3) < \infty, \quad t > 0.$$

Our main results are the following three theorems, in which we assume that  $D$  is a Riemannian manifold that satisfies the Hardy inequality with some weight  $h$ , and  $\psi$  is a non-negative measurable function on  $D$ .

**Theorem 2.** *If  $|D| < \infty$ , and if there exists  $1 < q \leq 2$  such that*

$$\|\psi h^{1/q}\|_{q/(q-1)} < \infty, \quad (15)$$

then, for all  $t > 0$ ,

$$Q_{\psi, \psi}(t) \leq \frac{q^{(4-q)/q}}{(2(q-1))^{2/q}} \|\psi h^{1/q}\|_{q/(q-1)}^2 \|1 - u_1(\cdot; t)\|_1^{(2-q)/q} t^{-2/q}. \quad (16)$$

**Theorem 3.** *Suppose there exists  $1 < q \leq 2$  such that (15) holds and that*

$$\|h^{1/q}\|_{q/(q-1)} < \infty.$$

Then, for all  $t > 0$ ,

$$Q_{\psi, \psi}(t) \leq b(q) \|\psi h^{1/q}\|_{q/(q-1)}^2 \|h^{1/q}\|_{q/(q-1)}^{(2-q)/(q-1)} t^{-1/(q-1)}, \quad (17)$$

where

$$b(q) = 2^{(4-3q)/(q(q-1))} \left( \frac{q}{q-1} \right)^{(q^2-4q+2)/(q(1-q))}. \quad (18)$$

**Theorem 4.** *Suppose there exist  $0 \leq r \leq 2$ , and  $1 < q \leq 2$  such that*

$$\|\psi^r\|_q < \infty$$

and

$$\|\psi^{2-r}h^{1/q}\|_{(q-1)/q} < \infty.$$

Then, for all  $t > 0$ ,

$$Q_{\psi,\psi}(t) \leq \left(\frac{q}{4(q-1)}\right)^{1/q} \|\psi^r\|_q \|\psi^{2-r}h^{1/q}\|_{q/(q-1)} t^{-1/q}. \quad (19)$$

In Theorem 5 in Section 3 we use the bounds of Theorems 2 and 4 together with (14) to obtain an upper bound for the heat content of  $D$ , when  $D$  satisfies a Hardy inequality with weight (9), and  $\psi_1(x) = \delta(x)^{-\alpha}$  and  $\psi_2(x) = \delta(x)^{-\beta}$ , where  $\alpha, \beta \in (1, 2)$ . Even though the bounds in e.g. 2 and 4 look very different, both of them are needed to cover the maximal range of  $\alpha$  and  $\beta$  in Theorem 5.

Theorem 2 has a curious consequence. We claim that if a manifold  $D$  has finite measure  $|D|$ , and is stochastically complete then no Hardy inequality holds on  $D$  (which confirms the philosophy that the Hardy inequality corresponds to cooling that comes from the boundary). Indeed, stochastic completeness means that  $u_1 \equiv 1$ . In this case,  $\|1 - u_1(\cdot; t)\|_1 = 0$  so that we obtain from (16) that  $Q_{\psi,\psi}(t) = 0$  whenever function  $\psi$  satisfies the condition (15) for some  $q \in (1, 2)$ . However, if  $h$  is finite then it is easy to construct a non-trivial function  $\psi$  that satisfies (15): choose any measurable set  $S$  with finite positive measure such that  $h$  is bounded on  $S$ , and let  $\psi = 1_S$ . Then (15) holds with any  $q > 1$  while  $Q_{\psi,\psi}(t) > 0$  so that we obtain contradiction. Of course, without the finiteness of  $|D|$ , the Hardy inequality may hold on stochastically complete manifolds like  $\mathbb{R}^m \setminus \{0\}$ .

This paper is organized as follows. In Section 2 we will prove Theorems 1, 2, 3 and 4. In Section 3 we will state and prove Theorem 5. Finally in Section 4 we obtain very refined asymptotics in the special case of the ball in  $\mathbb{R}^3$  with  $\psi_1(x) = \delta(x)^{-\alpha}$ ,  $\alpha < 2$ ,  $\psi_2(x) = \delta(x)^{-\beta}$ ,  $\beta < 2$ , and  $\alpha + \beta > 3$  (Theorem 7). This special case shows that the bound obtained in Theorem 5 is close to being sharp. Moreover it suggests formulae for the first few terms in the asymptotic series of a compact Riemannian manifold  $D$  with the singular data above.

## 2 Proofs of Theorems 1, 2, 3 and 4

*Proof of Theorem 1.* In both parts, it suffices to prove the claims for non-negative functions  $\psi_1, \psi_2$  from  $L^2(D)$ . Arbitrary non-negative measurable functions  $\psi_1, \psi_2$  can be approximated by monotone increasing sequences of non-negative functions from  $L^2(D)$ , whence the both claims follow by the monotone convergence theorem.

To prove part (i) we use symmetry and the semigroup property, and obtain by (7) for  $s = t/2$  that

$$\begin{aligned} Q_{\psi_1,\psi_2}(t) &= \int_D u_{\psi_1}(x; t/2) u_{\psi_2}(x; t/2) dx \\ &\leq \left( \int_D u_{\psi_1}^2(x; t/2) dx \right)^{1/2} \left( \int_D u_{\psi_2}^2(x; t/2) dx \right)^{1/2} \\ &= (Q_{\psi_1,\psi_1}(t) Q_{\psi_2,\psi_2}(t))^{1/2}. \end{aligned}$$

It follows from (2) that

$$p(x, y; t) \leq (p(x, x; t)p(y, y; t))^{1/2} \leq c_t. \quad (20)$$

To prove part (ii) we have by (20) that

$$\begin{aligned} p(x, y; t) &= \iint_{D \times D} dz_1 dz_2 p(x, z_1; t/3) p(z_1, z_2; t/3) p(z_2, y; t/3) \\ &\leq c_{t/3} u_1(x; t/3) u_1(y; t/3). \end{aligned} \quad (21)$$

This together with definition (6) completes the proof.  $\square$

For the proofs of Theorems 2, 3, 4, choose a sequence  $\{D_n\}$  that consists of precompact open subsets of  $D$  with smooth boundaries such that  $\overline{D}_n \subset D_{n+1}$  and  $\bigcup_n D_n = D$ . Obviously, Hardy inequality (8) remains true in any  $D_n$  with the same weight  $h$ , because  $C_c^\infty(D_n) \subset C_c^\infty(D)$ . Moreover, we claim that (8) holds for any function  $w \in C(\overline{D}_n) \cap C^1(D_n)$  that satisfies the boundary condition  $w|_{\partial D_n} = 0$ . Indeed, if  $\int_{D_n} |\nabla w|^2 = \infty$  then (8) is trivially satisfied. If  $\int_{D_n} |\nabla w|^2 < \infty$  then  $w$  belongs to the Sobolev space  $W^{1,2}(D_n)$ . Extend function  $w$  to  $D_{n+1}$  by setting  $w = 0$  in  $D_{n+1} \setminus \overline{D}_n$ . Due to the boundary condition  $w|_{\partial D_n} = 0$ , we obtain that  $w_n \in W^{1,2}(D_{n+1})$ . Since  $w$  is compactly supported in  $D_{n+1}$ , it follows that  $w \in W_0^{1,2}(D_{n+1})$  where  $W_0^{1,2}(\Omega)$  is the closure  $C_c^\infty(\Omega)$  in  $W^{1,2}(\Omega)$ . Since the Hardy inequality (8) holds for functions from  $C_c^\infty(D_{n+1})$ , passing to the limit in  $W^{1,2}(D_{n+1})$  and using Fatou's lemma, we obtain that  $w$  also satisfies (8).

Assume for a moment that the statements of the theorems have been proved in each domain  $D_n$ . Then one can take the limit in (16), (17), (19) as  $n \rightarrow \infty$ , and obtain the statements for  $D$ . Indeed, the left hand side of these inequalities is  $Q_{\psi,\psi}^{D_n}(t) = \iint_{D_n \times D_n} dx dy p_{D_n}(x, y; t) \psi(x) \psi(y)$ , where  $p_{D_n}$  is the Dirichlet heat kernel for  $D_n$ . This converges to  $Q_{\psi,\psi}^D(t)$  as  $n \rightarrow \infty$ . The right hand sides of (16), (17), (19) contain various  $L^p(D_n)$ -norms that can be estimated from above by the  $L^p(D)$ -norms. The only exception is the term  $\left\| 1 - \int_{D_n} dy p_{D_n}(\cdot, y; t) \right\|_1$  in (16) that is decreasing as  $n \rightarrow \infty$ . If  $|D| < \infty$  then  $1 \in L^1(D)$  so that the passage to the limit is justified by the dominated convergence theorem.

Hence, it suffices to prove each of the statements for  $D_n$  instead of  $D$ . Renaming  $D_n$  back to  $D$ , we assume in all three proofs that  $D$  is a precompact open domain with smooth boundary in another manifold.

Another observation is that all inequalities (16), (17), (19) survive the increasing monotone limits in  $\psi$ . So it suffices to prove them when  $\psi$  is bounded and has a compact support in  $D$ , which will be assumed below. Furthermore, since all the statements of Theorems 2, 3, 4 are homogeneous with respect to  $\psi$ , we can assume that  $0 \leq \psi \leq 1$ . If  $\psi \equiv 0$  then there is nothing to prove; hence, we assume that  $\psi$  is non-trivial. Then  $u_\psi(x; t)$  is smooth and bounded in  $\overline{D} \times (0, +\infty)$  and positive in  $D \times (0, +\infty)$ .

*Proof of Theorem 2.* Let  $\nu$  be the outwards normal vector field on  $\partial D$ . Using the Green's

formula, we obtain

$$\begin{aligned}
-\frac{d}{dt} \int_D u_\psi^q &= -q \int_D u_\psi^{q-1} \frac{\partial u_\psi}{\partial t} \\
&= -q \int_D u_\psi^{q-1} \Delta u_\psi \\
&= -q \int_{\partial D} u_\psi^{q-1} \frac{\partial u_\psi}{\partial \nu} + q \int_D (\nabla u_\psi^{q-1}, \nabla u_\psi) \\
&= q(q-1) \int_D u_\psi^{q-2} |\nabla u_\psi|^2,
\end{aligned} \tag{22}$$

where we have used that  $q > 1$  and, hence  $u_\psi^{q-1} = 0$  on  $\partial D$ . Observing that  $u_\psi^{q/2} \in C(\bar{D}) \cap C^1(D)$ ,

$$|\nabla u_\psi^{q/2}|^2 = \frac{q^2}{4} u_\psi^{q-2} |\nabla u_\psi|^2,$$

and applying the Hardy inequality (8) to  $u_\psi^{q/2}$ , we obtain that

$$-\frac{d}{dt} \int_D u_\psi^q = \frac{4(q-1)}{q} \int_D |\nabla(u_\psi^{q/2})|^2 \geq \frac{4(q-1)}{q} \int_D \frac{u_\psi^q}{h}. \tag{23}$$

By Hölder's inequality we have that

$$\begin{aligned}
Q_{\psi,\psi}(t) &= \int_D u_\psi \psi \\
&\leq \left( \int_D \left( \frac{u_\psi}{h^{1/q}} \right)^q \right)^{1/q} \left( \int (\psi h^{1/q})^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \\
&= \left( \int_D \frac{u_\psi^q}{h} \right)^{1/q} \|\psi h^{1/q}\|_{q/(q-1)}.
\end{aligned} \tag{24}$$

By (23) and (24) we conclude that

$$-\frac{d}{dt} \int_D u_\psi^q \geq \frac{4(q-1)}{q} \|\psi h^{1/q}\|_{q/(q-1)}^{-q} (Q_{\psi,\psi}(t))^q. \tag{25}$$

Note that the function  $t \mapsto Q_{\psi,\psi}(t) = \|u_\psi(\cdot; t/2)\|_2^2$  is decreasing in  $t$ , which, for example, follows from (22) with  $q = 2$ . Integrating differential inequality (25) with respect to  $t$  over the interval  $[t, 2t]$  gives that

$$\int_D u_\psi^q \geq \frac{4(q-1)}{q} \|\psi h^{1/q}\|_{q/(q-1)}^{-q} (Q_{\psi,\psi}(2t))^q t. \tag{26}$$

On the other hand, using  $1 < q \leq 2$  and the Hölder inequality, we obtain

$$\int_D u_\psi^q = \int_D u_\psi^{2-q} u_\psi^{2q-2} \leq \left( \int_D u_\psi \right)^{2-q} \left( \int_D u_\psi^2 \right)^{q-1}$$

that is,

$$\int_D u_\psi^q \leq (Q_{\psi,1}(t))^{2-q} (Q_{\psi,\psi}(2t))^{q-1}. \tag{27}$$

Combining (26) and (27) yields

$$Q_{\psi,\psi}(2t) \leq \frac{q}{4(q-1)} \|\psi h^{1/q}\|_{q/(q-1)}^q (Q_{\psi,1}(t))^{2-q} t^{-1}. \quad (28)$$

Estimating  $Q_{\psi,1}$  by (1), we obtain

$$Q_{\psi,\psi}(2t) \leq \frac{q^{(4-q)/q}}{(4(q-1))^{2/q}} \|\psi h^{1/q}\|_{q/(q-1)}^2 \|1 - u_1(\cdot; t)\|_1^{(2-q)/q} t^{-2/q},$$

which completes the proof.  $\square$

*Proof of Theorem 3.* Since  $\psi \leq 1$ , the condition (11) is satisfied, and we obtain by (12) and (28) that

$$Q_{\psi,\psi}(2t) \leq \frac{q}{4(q-1)} a(q)^{2-q} \|\psi h^{1/q}\|_{q/(q-1)}^2 \|h^{1/q}\|_{q/(q-1)}^{(2-q)/(q-1)} t^{-1/(q-1)}.$$

This completes the proof of Theorem 3 since, by (13) and (18),

$$2^{1/(q-1)} \frac{q}{4(q-1)} a(q)^{2-q} = b(q).$$

$\square$

*Proof of Theorem 4.* By the arithmetic-geometric mean inequality, we have

$$\psi(x)\psi(y) \leq \frac{1}{2} (\psi(x)^r \psi(y)^{2-r} + \psi(x)^{2-r} \psi(y)^r).$$

By non-negativity and symmetry of the Dirichlet heat kernel

$$Q_{\psi,\psi}(t) \leq \int_D u_{\psi^r} \psi^{2-r}. \quad (29)$$

Next, Hölder's inequality yields

$$\int_D u_{\psi^r} \psi^{2-r} \leq \left( \int_D u_{\psi^r}^q \frac{1}{h} \right)^{1/q} \|\psi^{2-r} h^{1/q}\|_{q/(q-1)}. \quad (30)$$

By (23) we have

$$-\frac{d}{dt} \int_D u_{\psi^r}^q \geq \frac{4(q-1)}{q} \int_D u_{\psi^r}^q \frac{1}{h}. \quad (31)$$

Combining (29), (30), (31) we obtain that

$$(Q_{\psi,\psi}(t))^q \leq -\frac{q}{4(q-1)} \frac{d}{dt} \left( \int_D u_{\psi^r}^q \right) \|\psi^{2-r} h^{1/q}\|_{q/(q-1)}^q.$$

Since the function  $t \mapsto Q_{\psi,\psi}(t)$  is decreasing in  $t$ , we obtain by integrating the differential inequality (31) with respect to  $t$  over the interval  $[0, t]$  that

$$t(Q_{\psi,\psi}(t))^q \leq \frac{q}{4(q-1)} \left( \int_D \psi^{r^q} \right) \|\psi^{2-r} h^{1/q}\|_{q/(q-1)}^q,$$

and (19) follows.  $\square$

### 3 Singular initial temperature and singular specific heat

Below we make some further hypothesis on the geometry of  $D$ , and obtain an upper bound for the heat content for a wide class of geometries using Theorems 2 and 4, and (14), if the initial temperature distribution and specific heat are given by  $\delta^{-\alpha}$ ,  $1 < \alpha < 2$ , and  $\delta^{-\beta}$ ,  $1 < \beta < 2$  respectively.

**Theorem 5.** *Let  $D$  be an open set in a smooth complete  $m$ -dimensional Riemannian manifold  $M$ , and suppose that*

- i. *The Ricci curvature on  $M$  is non-negative.*
- ii. *For  $x \in D$ ,*

$$\psi_\alpha(x) = \delta(x)^{-\alpha}.$$

- iii. *There exist constants  $\kappa_D < \infty$  and  $d \in [m-1, m)$  such that*

$$\int_{\{x \in D: \delta(x) < \epsilon\}} 1 \leq \kappa_D \epsilon^{m-d}, \quad 0 < \epsilon \leq \rho_D, \quad (32)$$

where  $\rho_D = \sup\{\delta(x) : x \in D\}$  is the inradius of  $D$ .

- iv. *The strong Hardy inequality (8) holds with (9) for some  $c \geq 2$ .*

If  $1 < \alpha < 2$ ,  $1 < \beta < 2$ , and if  $\epsilon > 0$  is sufficiently small then

$$Q_{\psi_\alpha, \psi_\beta}(t) = O(t^{-4\epsilon + (m-d-\alpha-\beta)/2}), \quad t \rightarrow 0. \quad (33)$$

*Proof.* By (14) it suffices to prove (33) in the special case  $\alpha = \beta$  with  $1 < \alpha < 2$ . In order to estimate  $\|1 - u_1(\cdot; t)\|_1$  in Theorem 2 we rely on the following lower bound for  $u_1$  (Lemma 5 in [5]).

**Lemma 6.** *Let  $M$  be a smooth, geodesically complete Riemannian manifold with non-negative Ricci curvature, and let  $D$  be an open subset of  $M$  with boundary  $\partial D$ . Then for  $x \in D$ ,  $t > 0$*

$$u_1(x; t) \geq 1 - 2^{(2+m)/2} e^{-\delta(x)^2/(8t)}.$$

To prove (33) we first consider the case

$$(2 + m - d)/2 < \alpha < 2. \quad (34)$$

This set of  $\alpha$ 's is non-empty since  $d \in [m-1, m)$ . By (9) we have that

$$\|\psi_\alpha h^{1/q}\|_{q/(q-1)} = c^{2/q} \left( \int_D \delta^{(2-q\alpha)/(q-1)} \right)^{(q-1)/q}. \quad (35)$$

Denote the left hand side of (32) by  $\omega_D(\epsilon)$ . Then we can write the right hand side of (35) as

$$c^{2/q} \left( \int_{\mathbb{R}^+} \omega_D(d\epsilon) \epsilon^{(2-q\alpha)/(q-1)} \right)^{(q-1)/q}. \quad (36)$$

An integration by parts, using (34) shows that (36) is finite for

$$q < \frac{2 - m + d}{\alpha - m + d}. \quad (37)$$

Because of (34) the right hand side of (37) is in  $(1, 2]$ . We now choose  $\epsilon > 0$  such that

$$q = \frac{2 - m + d}{\alpha - m + d} - \epsilon \in (1, 2]. \quad (38)$$

By Lemma 6 and (32) we have that for  $t \rightarrow 0$

$$\|1 - u_1(\cdot; t)\|_1 = O(t^{(m-d)/2}). \quad (39)$$

By Theorem 2 and (35)-(39) we find that for all  $\alpha$  satisfying (34) and all  $\epsilon > 0$  satisfying (38)

$$Q_{\psi_\alpha, \psi_\alpha}(t) = O(t^{-4\epsilon + (m-d-2\alpha)/2}), \quad t \rightarrow 0. \quad (40)$$

Next consider the case

$$1 < \alpha < (2 + m - d)/2. \quad (41)$$

This set of  $\alpha$ 's is again non-empty since  $d \in [m - 1, m)$ . By (32) we have that

$$\|\psi^r\|_q = \left( \int_{\mathbb{R}_+} \omega_D(d\epsilon) \epsilon^{-\alpha r q} \right)^{1/q} < \infty \quad (42)$$

for

$$\alpha r q < m - d, \quad (43)$$

and

$$\|\psi^{2-r} h^{1/q}\|_{q/(q-1)} = \left( \int_{\mathbb{R}_+} \omega_D(d\epsilon) \delta^{(2-\alpha(2-r)q)/(q-1)} \right)^{(q-1)/q} < \infty \quad (44)$$

for

$$\frac{\alpha q(2-r) - 2}{q-1} < m - d. \quad (45)$$

The optimal choice for  $r$  is henceforth given by

$$r = 2(\alpha q - 1)\alpha^{-1}q^{-2}. \quad (46)$$

By (41) we also have that  $\alpha > 1$ . Hence  $r \in (0, 2)$ . The requirements under (43) and (45) become with this choice of  $r$  that

$$q < 2(2\alpha + d - m)^{-1}. \quad (47)$$

Because of (41) the right hand side of (47) is in  $(1, 2)$ . We now choose  $\epsilon > 0$  such that

$$q = 2(2\alpha + d - m)^{-1} - \epsilon > 1. \quad (48)$$

By Theorem 4 and (42)-(47) we find that for all  $\alpha$  satisfying (41) and all  $\epsilon > 0$  satisfying (48)

$$Q_{\psi_\alpha, \psi_\alpha}(t) = O(t^{-2\epsilon + (m-d-2\alpha)/2}), \quad t \rightarrow 0.$$

To prove (33) for the limiting case  $\alpha = \beta = (2 + m - d)/2 := \alpha_c$  we note that  $Q_{\psi, \phi}(t)$  is bilinear and monotone on the positive cone of non-negative and measurable  $\psi$  and  $\phi$ . Moreover for any  $\eta \in (0, 1/2)$  we have that  $\alpha_c + \eta < 2$ , and

$$\psi_{\alpha_c} \leq \psi_{\alpha_c + \eta} + 1.$$

Hence for any  $\eta \in (0, 1/2)$

$$\begin{aligned} Q_{\psi_{\alpha_c}, \psi_{\alpha_c}}(t) &\leq Q_{\psi_{\alpha_c+\eta+1}, \psi_{\alpha_c+\eta+1}}(t) \\ &\leq Q_{\psi_{\alpha_c+\eta}, \psi_{\alpha_c+\eta}}(t) + 2Q_{\psi_{\alpha_c+\eta}, 1}(t) + Q_{1,1}(t). \end{aligned} \quad (49)$$

The last term in the right hand side of (49) is bounded by the measure of  $D$ , and hence  $O(1)$ . Furthermore by (14)

$$Q_{\psi_{\alpha_c+\eta}, 1}(t) \leq (Q_{\psi_{\alpha_c+\eta}, \psi_{\alpha_c+\eta}}(t) Q_{1,1}(t))^{1/2}. \quad (50)$$

For the first term in the right hand side of (49) we use (40) to obtain that for all  $\epsilon \in (0, 1/2)$  satisfying

$$q = 2(2\alpha_c + 2\eta + d - m)^{-1} - \epsilon > 1,$$

and all  $\eta \in (0, 1/2)$

$$Q_{\psi_{\alpha_c+\eta}, \psi_{\alpha_c+\eta}}(t) = O(t^{-4\epsilon + (m-d-2\alpha_c-2\eta)/2}) = O(t^{-4\epsilon - \eta - 1}).$$

By (50),  $Q_{\psi_{\alpha_c+\eta}, 1}(t) = o(Q_{\psi_{\alpha_c+\eta}, \psi_{\alpha_c+\eta}}(t))$  as  $t \rightarrow 0$ . This completes the proof of (40) in the critical case  $\alpha = \alpha_c$ , since both  $\eta$  and  $\epsilon$  were positive, small, and arbitrary.  $\square$

## 4 The special case calculation for a ball in $\mathbb{R}^3$

In this section we show by means of an example that the upper bound obtained in Theorem 5 is close to being sharp for  $\alpha < 2, \beta < 2, \alpha + \beta > 3$ .

**Theorem 7.** *Let  $B_a = \{x \in \mathbb{R}^3 : |x| < a\}$ . If  $\alpha < 2, \beta < 2, \alpha + \beta > 3, J \in \mathbb{N}$  then there exist coefficients  $b_0, b_1, \dots$  depending on  $\alpha$  and  $\beta$  only such that for  $t \rightarrow 0$*

$$\begin{aligned} Q_{\psi_\alpha, \psi_\beta}(t) &= 4\pi c_{\alpha, \beta} a^2 t^{(1-\alpha-\beta)/2} - 4\pi(c_{\alpha-1, \beta} + c_{\alpha, \beta-1}) a t^{(2-\alpha-\beta)/2} \\ &\quad + 4\pi c_{\alpha-1, \beta-1} t^{(3-\alpha-\beta)/2} + \sum_{j=0}^J b_j a^{3-j-\alpha-\beta} t^{j/2} + O(t^{(J+1)/2}), \end{aligned} \quad (51)$$

where

$$\begin{aligned} c_{\alpha, \beta} &= 2^{-\alpha-\beta} \pi^{-1/2} \Gamma((2-\alpha-\beta)/2) \\ &\quad \times \int_0^1 d\rho (\rho^{-\alpha} + \rho^{-\beta}) ((1-\rho)^{\alpha+\beta-2} - (1+\rho)^{\alpha+\beta-2}), \end{aligned} \quad (52)$$

and

$$\begin{aligned} b_0 &= -8\pi((\alpha + \beta - 1)(\alpha + \beta - 2)(\alpha + \beta - 3))^{-1}, \\ b_1 &= 0, \\ b_2 &= 8\pi\alpha\beta((\alpha + \beta + 1)(\alpha + \beta)(\alpha + \beta - 1))^{-1}, \\ b_3 &= 0. \end{aligned} \quad (53)$$

We see that the leading term in (51) jibes with (33) since (9) holds for some  $c \geq 2$  and (32) holds with  $d = m - 1$ .

We conjecture that for any precompact  $D$  with smooth  $\partial D$  in  $M$ , and for  $\alpha < 2, \beta < 2, \alpha + \beta > 3$

$$\begin{aligned} Q_{\psi_\alpha, \psi_\beta}(t) = & c_{\alpha, \beta} \int_{\partial D} t^{(1-\alpha-\beta)/2} - 2^{-1}(c_{\alpha-1, \beta} + c_{\alpha, \beta-1}) \int_{\partial D} L_{gg} t^{(2-\alpha-\beta)/2} \\ & + \int_{\partial D} (c_1 L_{gg} L_{hh} + c_2 L_{gh} L_{gh}) t^{(3-\alpha-\beta)/2} + O(1), \end{aligned} \quad (54)$$

where  $c_1$  and  $c_2$  are constants depending on  $\alpha$  and  $\beta$  only, and which satisfy

$$4c_1 + 2c_2 = c_{\alpha-1, \beta-1},$$

and where  $L_{gg}$  be the trace of the second fundamental form on the boundary of  $\partial D$  oriented by an inward unit vector field. Since  $\int_{\partial B_a} 1 = 4\pi a^2$ ,  $\int_{\partial B_a} L_{gg} = 8\pi a$  and  $\int_{\partial B_a} (c_1 L_{gg} L_{hh} + c_2 L_{gh} L_{gh}) = 16\pi c_1 + 8\pi c_2$  we see that (54) holds for the ball in  $\mathbb{R}^3$ .

The proof of Theorem 7 rests on the following result (pp.237, 367-368 in [8]).

**Lemma 8.** *Let  $B_a$  as in Theorem 7, and let the initial datum be radially symmetric i.e.  $\psi_1(x) = f(r)$ , where  $r = |x|$ . Then the solution of (1), (3), (5) is given by*

$$u(x; t) = (4\pi t r^2)^{-1/2} \int_0^a dr' r' f(r') \sum_{n \in \mathbb{Z}} (e^{-(2na-r+r')^2/(4t)} - e^{-(2na+r+r')^2/(4t)}).$$

To prove Theorem 7 we have by Lemma 8 that

$$\begin{aligned} Q_{\psi_\alpha, \psi_\beta}(t) = & (4\pi/t)^{1/2} \int_0^a \int_0^a dr dr' r r' (a-r)^{-\alpha} (a-r')^{-\beta} \\ & \times \sum_{n \in \mathbb{Z}} (e^{-(2na-r+r')^2/(4t)} - e^{-(2na+r+r')^2/(4t)}). \end{aligned} \quad (55)$$

Substitution of  $a-r=p$  and  $a-r'=q$  in (55) gives that

$$Q_{\psi_\alpha, \psi_\beta}(t) = A_0 + A_1 + A_2 + B,$$

where

$$\begin{aligned} A_0 = & (4\pi/t)^{1/2} a^2 \int_0^a \int_0^a dp dq p^{-\alpha} q^{-\beta} (e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)}), \\ A_1 = & -(4\pi/t)^{1/2} a \int_0^a \int_0^a dp dq (p+q) p^{-\alpha} q^{-\beta} (e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)}), \\ A_2 = & (4\pi/t)^{1/2} \int_0^a \int_0^a dp dq p^{1-\alpha} q^{1-\beta} (e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)}), \end{aligned}$$

and

$$\begin{aligned} B = & (4\pi/t)^{1/2} \int_0^a \int_0^a dp dq (a-p)(a-q) p^{-\alpha} q^{-\beta} \sum_{n \geq 1} (e^{-(2na+p-q)^2/(4t)} \\ & + e^{-(2na+q-p)^2/(4t)} - e^{-(2na+q+p)^2/(4t)} - e^{-(2na-q-p)^2/(4t)}). \end{aligned} \quad (56)$$

We have the following.

**Lemma 9.** *If  $1 < \alpha < 2, 1 < \beta < 2$  then for  $t \rightarrow 0$*

$$B = -8\pi^{1/2} 3^{-1} a^{-\alpha-\beta} t^{3/2} + O(t^2). \quad (57)$$

*Proof.* The integrand in (56) can be rewritten as

$$(a-p)(a-q)p^{-\alpha}q^{-\beta}\sum_{n\geq 1}e^{-(2na-p-q)^2/(4t)} \\ \times ((e^{(p-2na)q/t} + e^{(q-2na)p/t})(1 - e^{-pq/t}) - (1 - e^{-2pna/t})(1 - e^{-2qna/t})). \quad (58)$$

The contribution from the terms with  $n \geq 2$  in (58) is bounded in absolute value by

$$2a^2p^{1-\alpha}q^{1-\beta}t^{-1}\sum_{n\geq 2}e^{-a^2(n-1)^2/t}(1 + 2n^2a^2t^{-1}).$$

After integrating with respect to  $p$  and  $q$  we see that this term contributes at most  $O(e^{-a^2/(2t)})$  to  $B$ . Next we will show that the main contribution from the term with  $n = 1$  in (58) comes from a neighbourhood of the point  $(p, q) = (a, a)$ . Let

$$C_1(a) = \{(p, q) \in \mathbb{R}^2 : a/3 < p < a, a/3 < q < a\},$$

and

$$C_2(a) = ((0, a) \times (0, a)) \setminus C_1(a).$$

On  $C_2(a)$  we have that  $2a - p - q \geq 2a/3$ . Hence the term with  $n = 1$  in (58) is bounded on  $C_2(a)$  in absolute value by

$$2(a-p)(a-q)p^{1-\alpha}q^{1-\beta}t^{-1}e^{-a^2/(9t)}(1 + 2a^2t^{-1}). \quad (59)$$

Integrating (59) over  $C_2(a)$  gives a contribution which is bounded by  $O(e^{-a^2/(18t)})$ . In order to calculate the contribution from the term with  $n = 1$  on  $C_1(a)$  we use the expression under (56) instead. First we note that  $2a + p - q \geq 2a/3$ ,  $2a + q - p \geq 2a/3$ ,  $2a + p + q \geq 8a/3$ . Hence the first three terms in the summand of (56) with  $n = 1$  give after integration over  $C_1(a)$  a contribution  $O(e^{-a^2/(18t)})$ . Putting all this together gives that

$$B = - (4\pi/t)^{1/2} \iint_{C_1(a)} dpdq (a-p)(a-q)p^{-\alpha}q^{-\beta} \\ \times e^{-(2a-q-p)^2/(4t)} + O(e^{-a^2/(18t)}).$$

Noting that

$$p^{-\alpha}q^{-\beta} = a^{-\alpha-\beta} + O(a-p) + O(a-q) \quad (60)$$

uniformly in  $p$  and  $q$  yields after a change of variables that

$$B = - (4\pi/t)^{1/2} a^{-\alpha-\beta} \iint_{(0,a/3) \times (0,a/3)} dpdq p q e^{-(p+q)^2/(4t)} \\ \times (1 + O(p) + O(q)) + O(e^{-a^2/(18t)}),$$

which agrees with the right hand side of (57).  $\square$

By taking higher order terms of the form  $(a-p)^{n_1}(a-q)^{n_2}$  in (60) into account one can determine the coefficient  $t^{(j+3)/2}$ ,  $j = 0, 1, 2, \dots$  in the expansion of  $B$ .

To complete the proof of Theorem 7 we rewrite  $A_0, A_1$  and  $A_2$  respectively as follows.

$$\begin{aligned}
A_0 &= (4\pi/t)^{1/2} a^2 \left( \int_0^a dp \int_0^p dq + \int_0^a dq \int_0^q dp \right) \\
&\quad \times p^{-\alpha} q^{-\beta} (e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)}) \\
&= (4\pi/t)^{1/2} a^2 \int_0^a dpp^{1-\alpha-\beta} \int_0^1 d\rho(\rho^{-\alpha} + \rho^{-\beta}) \\
&\quad \times (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}) \\
&= 4\pi a^2 c_{\alpha,\beta} t^{(1-\alpha-\beta)/2} \\
&\quad - (4\pi/t)^{1/2} a^2 \int_a^\infty dpp^{1-\alpha-\beta} \int_0^1 d\rho(\rho^{-\alpha} + \rho^{-\beta}) \\
&\quad \times (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}), \tag{61}
\end{aligned}$$

$$\begin{aligned}
A_1 &= -4\pi a (c_{\alpha-1,\beta} + c_{\alpha,\beta-1}) t^{(2-\alpha-\beta)/2} + (4\pi/t)^{1/2} a \int_a^\infty dpp^{2-\alpha-\beta} \\
&\quad \times \int_0^1 d\rho(\rho^{1-\alpha} + \rho^{-\alpha} + \rho^{1-\beta} + \rho^{-\beta}) (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}), \tag{62}
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= 4\pi c_{\alpha-1,\beta-1} t^{(3-\alpha-\beta)/2} - (4\pi/t)^{1/2} \int_a^\infty dpp^{3-\alpha-\beta} \\
&\quad \times \int_0^1 d\rho(\rho^{1-\alpha} + \rho^{1-\beta}) (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}). \tag{63}
\end{aligned}$$

The terms to be evaluated in (61), (62) and (63) are all of the form

$$(4\pi/t)^{1/2} a^{2-j} \int_a^\infty dpp^{1+j-\alpha-\beta} \int_0^1 d\rho \rho^{-\gamma} (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}), \tag{64}$$

where  $j = 0, 1, 2$  respectively. Following arguments similar to the proof of Lemma 9 we see that the contribution of the integral with respect to  $\rho \in [0, 1/2)$  in (64) is at most  $O(e^{-a^2/(18t)})$ . Furthermore

$$(4\pi/t)^{1/2} a^{2-j} \int_a^\infty dpp^{1+j-\alpha-\beta} \int_{1/2}^1 d\rho \rho^{-\gamma} e^{-p^2(1+\rho)^2/(4t)} = O(e^{-a^2/(18t)}). \tag{65}$$

Hence the expression under (64) equals

$$(4\pi/t)^{1/2} a^{2-j} \int_a^\infty dpp^{1+j-\alpha-\beta} \int_{1/2}^1 d\rho \rho^{-\gamma} e^{-p^2(1-\rho)^2/(4t)} + O(e^{-a^2/(18t)}). \tag{66}$$

Expanding  $\rho^{-\gamma}$  about  $\rho = 1$  we obtain that

$$\begin{aligned}
&|\rho^{-\gamma} - 1 - \gamma(1-\rho) - 2^{-1}\gamma(\gamma+1)(1-\rho)^2 \\
&\quad - 6^{-1}\gamma(\gamma+1)(\gamma+2)(1-\rho)^3| \leq C(1-\rho)^4, \quad 0 \leq \rho \leq 1/2, \tag{67}
\end{aligned}$$

where  $C$  depends on  $\gamma$  only. By (67) and (66) we obtain that (64) is equal to

$$\begin{aligned} & 2\pi(\alpha + \beta - j - 1)^{-1}a^{3-\alpha-\beta} + 4\pi^{1/2}\gamma(\alpha + \beta - j)^{-1}a^{2-\alpha-\beta}t^{1/2} \\ & + 2\pi\gamma(\gamma + 1)(\alpha + \beta - j + 1)^{-1}a^{1-\alpha-\beta}t \\ & + 8\pi^{1/2}3^{-1}\gamma(\gamma + 1)(\gamma + 2)(\alpha + \beta - j + 2)^{-1}a^{-\alpha-\beta}t^{3/2} + O(t^2). \end{aligned} \quad (68)$$

It remains to compute the coefficients  $b_0, b_1$  and  $b_2$  in Theorem 7. Altogether there are eight terms which contribute to the terms in (68):

$$\begin{aligned} j = 0, & \quad \gamma = \alpha, & \quad \gamma = \beta \\ j = 1, & \quad \gamma = \alpha - 1, & \quad \gamma = \beta - 1, & \quad \gamma = \alpha, & \quad \gamma = \beta \\ j = 2, & \quad \gamma = \alpha - 1, & \quad \gamma = \beta - 1. \end{aligned}$$

Summing these eight terms yield the expressions for  $b_0, b_1$  and  $b_2$  under (53). To calculate  $b_3$  we have that the above eight  $\gamma(\gamma + 1)(\gamma + 2)$  terms in (68) cancel the contribution from (57). This completes the proof of Theorem 7.

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