EXPLICIT RATES OF APPROXIMATION IN THE CLT FOR QUADRATIC FORMS

FRIEDRICH GÖTZE[1] AND ANDREI YU. ZAITSEV[1,2]

Abstract. Let $X, X_1, X_2, \dots$ be i.i.d. $\mathbb{R}^d$-valued real random vectors. Assume that $\mathbf{E}\, X = 0$, $\operatorname{cov} X = \mathbb{C}$, $\mathbf{E}\, \|X\|^2 = \sigma^2$ and that X is not concentrated in a proper subspace of $\mathbb{R}^d$. Let G be a mean zero Gaussian random vector with the same covariance operator as that of X. We study the distributions of non-degenerate quadratic forms $\mathbb{Q}[S_N]$ of the normalized sums $S_N = N^{-1/2}\,(X_1 + \dots + X_N)$ and show that, without any additional conditions,

$$\Delta_N \overset{\text{def}}{=} \sup_x \Big| \mathbf{P}\big\{\mathbb{Q}[S_N] \le x\big\} - \mathbf{P}\big\{\mathbb{Q}[G] \le x\big\} \Big| = \mathcal{O}\big(N^{-1}\big),$$

provided that $d \ge 5$ and the fourth moment of X exists. Furthermore, we provide explicit bounds of order $\mathcal{O}\big(N^{-1}\big)$ for Δ_N, for the rate of approximation by short asymptotic expansions and for the concentration functions of the random variables $\mathbb{Q}[S_N + a]$, $a \in \mathbb{R}^d$. Our results extend the corresponding results of Bentkus and Götze (1997a) ($d \ge 9$) to the case $d \ge 5$ which provides the optimal dependence on the dimension. Moreover, we show that, in finite dimensional case and for isometric $\mathbb{Q}$, the constant in $\mathcal{O}\big(N^{-1}\big)$ may be taken in the form $c_d\, \sigma^d\, (\det \mathbb{C})^{-1/2}\, \mathbf{E}\, \|\mathbb{C}^{-1/2}\, X\|^4$ with some c_d depending on d only.

1. Introduction

Let $\mathbb{R}^d$ be the d-dimensional space of real vectors $x = (x_1, \dots, x_d)$ with scalar product $\langle x, y\rangle = x_1 y_1 + \dots + x_d y_d$ and norm $\|x\| = \langle x, x\rangle^{1/2}$. We also denote by $\mathbb{R}^\infty$ a real separable Hilbert space consisting of all real sequences $x = (x_1, x_2, \dots)$ such that $\|x\|^2 = x_1^2 + x_2^2 + \dots < \infty$.

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. $\mathbb{R}^d$-valued random vectors. Assume that $\mathbf{E}\, X = 0$ and $\sigma^2 \overset{\text{def}}{=} \mathbf{E}\, \|X\|^2 < \infty$. Let G be a mean zero Gaussian random vector such that its covariance operator $\mathbb{C} = \operatorname{cov} G : \mathbb{R}^d \to \mathbb{R}^d$ is equal to $\operatorname{cov} X$. It is well-known that

---

Date: July 12, 2010.

1991 *Mathematics Subject Classification.* Primary 60F05; secondary 62E20.

Key words and phrases. Central Limit Theorem, concentration functions, convergence rates, multidimensional spaces, quadratic forms, ellipsoids, hyperboloids, lattice point problem, theta-series.

[1] Research supported by the SFB 701 in Bielefeld and by grant RFBR-DFG 09-01-91331.

[2] Research supported by grant RFBR 09-01-12180, by the Alexander von Humboldt Foundation and by a program of fundamental researches of Russian Academy of Sciences "Modern problems of fundamental mathematics".

the distributions $\mathcal{L}(S_N)$ of sums

$$S_N \overset{\text{def}}{=} N^{-1/2}\,(X_1 + \cdots + X_N) \tag{1.1}$$

converge weakly to $\mathcal{L}(G)$.

Let $\mathbb{Q} : \mathbb{R}^d \to \mathbb{R}^d$ be a linear symmetric bounded operator and let $\mathbb{Q}[x] = \langle \mathbb{Q}x, x\rangle$ be the corresponding quadratic form. We shall say that $\mathbb{Q}$ is non-degenerate if $\ker \mathbb{Q} = \big\{0\big\}$.

Denote, for $q > 0$,

$$\beta_q \overset{\text{def}}{=} \mathbf{E}\,\|X\|^q, \qquad \beta \overset{\text{def}}{=} \beta_4.$$

Introduce the distribution functions

$$F(x) \overset{\text{def}}{=} \mathbf{P}\big\{\mathbb{Q}[S_N] \le x\big\}, \qquad H(x) \overset{\text{def}}{=} \mathbf{P}\big\{\mathbb{Q}[G] \le x\big\}. \tag{1.2}$$

Write

$$\Delta_N \overset{\text{def}}{=} \sup_{x\in\mathbb{R}} \big|F(x) - H(x)\big|. \tag{1.3}$$

Theorem 1.1. *Assume that $\mathbb{Q}$ and $\mathbb{C}$ are non-degenerate and that $d \ge 5$ or $d = \infty$. Then*

$$\Delta_N \le c(\mathbb{Q}, \mathbb{C})\,\beta/N.$$

The constant $c(\mathbb{Q}, \mathbb{C})$ in this bound depends on $\mathbb{Q}$ and $\mathbb{C}$ only.

Theorem 1.2. *Let the conditions of Theorem* 1.1 *be satisfied and let $5 \le d < \infty$. Assume that the operator $\mathbb{Q}$ is isometric. Then*

$$\Delta_N \le c_d\,\sigma^d\,(\det \mathbb{C})^{-1/2}\,\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X\|^4/N.$$

The constant c_d in this bound depends on d only.

Theorems 1.1 and 1.2 are simple consequences of the main result of this paper, Theorem 2.3 (see also Theorem 2.1). Theorem 1.1 was proved in Götze and Zaitsev (2008). It confirms a conjecture of Bentkus and Götze (1997a) (below BG (1997a)). It generalizes to the case $d \ge 5$ the corresponding result of BG (1997a). In their Theorem 1.1, it was assumed that $d \ge 9$, while our Theorem 1.1 is proved for $d \ge 5$. Theorem 1.2 yields an explicit bound in terms of the distribution $\mathcal{L}(X)$.

The distribution function of $\|S_N\|^2$ (for bounded X with values in $\mathbb{R}^d$) may have jumps of order $\mathcal{O}\big(N^{-1}\big)$, for all $1 \le d \le \infty$. See, e.g., BG (1996, p. 468). Therefore, the bounds of Theorems 1.1 and 1.2 are optimal with respect to the order in N.

Theorems 1.1, 1.2 and the method of their proof are closely related to the lattice point problem in number theory. Suppose that $d < \infty$ and that $\langle \mathbb{Q}x, x\rangle > 0$, for $x \ne 0$. Let $\operatorname{vol} E_r$ be the volume of the ellipsoid

$$E_r = \big\{x \in \mathbb{R}^d : \ \mathbb{Q}[x] \le r^2\big\}, \qquad r \ge 0.$$

Write $\operatorname{vol}_{\mathbb{Z}} E_r$ for the number of points in $E_r \cap \mathbb{Z}^d$, where $\mathbb{Z}^d \subset \mathbb{R}^d$ is the standard lattice of points with integer coordinates.

The following result due to Götze (2004) is related to Theorems 1.1 and 1.2 (see also BG (1995a, 1997b)).

Theorem 1.3. *For all dimensions* $d \geq 5$,

$$\sup_{a\in\mathbb{R}^d} \left| \frac{\operatorname{vol}_{\mathbb{Z}}(E_r + a) - \operatorname{vol} E_r}{\operatorname{vol} E_r} \right| = \mathcal{O}(r^{-2}), \qquad \text{for} \quad r \geq 1,$$

where the constant in $\mathcal{O}(r^{-2})$ *depends on the dimension* d *and on the lengths of axes of the ellipsoid* E_1 *only.*

Theorem 1.3 solves the lattice point problem for $d \geq 5$. It improves the classical estimate $\mathcal{O}(r^{-2d/(d+1)})$ due to Landau (1915), just as Theorem 1.1 improves the bound $\mathcal{O}(N^{-d/(d+1)})$ by Esseen (1945) in the CLT for ellipsoids with axes parallel to coordinate axes. A related result for indefinite forms may be found in Götze and Margulis (2010).

The history of estimation of the rate of approximation under the conditions of Theorem 1.1 for Hilbert spaces was started in the second half of the last century. See Zalesskiĭ, Sazonov and Ulyanov (1988) and Nagaev (1989) for the optimal (with respect to eigenvalues of $\mathbb{C}$) bound of order $\mathcal{O}(N^{-1/2})$ under the assumption of finiteness of the third moment. For a more detailed discussion see Yurinskii (1982), Bentkus, Götze, Paulauskas and Račkauskas (1990), BG (1995b, 1996, 1997a) and Senatov (1997, 1998).

Under some more restrictive moment and dimension conditions the estimate of order $\mathcal{O}(N^{-1+\varepsilon})$, with $\varepsilon \downarrow 0$ as $d \uparrow \infty$, was obtained by Götze (1979). The proof in Götze (1979) was based on a new symmetrization inequality for characteristic functions of quadratic forms. This inequality is related to Weyl's (1915/16) inequality for trigonometric sums. This inequality and its extensions (see Lemma 6.1) play the crucial role in the proofs of bounds in the CLT on ellipsoids and hyperboloids in finite and infinite dimensional cases. Under some additional smoothness assumptions, error bounds $\mathcal{O}(N^{-1})$ (and, moreover, Edgeworth type expansions) were obtained in Götze (1979), Bentkus (1984), Bentkus, Götze and Zitikis (1993). BG (1995b, 1996, 1997a) established the bound of order $\mathcal{O}(N^{-1})$ without smoothness-type conditions. Similar bounds for the rate of infinitely divisible approximations were obtained by Bentkus, Götze and Zaitsev (1997). Among recent publications, we should mention the papers of Nagaev and Chebotarev (1999), (2005) ($d \geq 9$, a more precise dependence of constants on the eigenvalues of $\mathbb{C}$) and Bogatyrev, Götze and Ulyanov (2006) (non-uniform bounds for $d \geq 12$), see also Götze and Ulyanov (2000). The proofs of bounds of order $\mathcal{O}(N^{-1})$ are based on discretization (i.e., a reduction to lattice valued random vectors) and the symmetrization techniques mentioned above.

Assuming the matrices $\mathbb{Q}$ and $\mathbb{C}$ to be diagonal, and the independence of first five coordinates of X, Bentkus and Götze (1996) have already reduced the dimension requirement for the bound $\mathcal{O}(N^{-1})$ to $d \geq 5$. The independence assumption in BG (1996) allowed to apply an adaption of the Hardy–Littlewood circle method. For the general case described in Theorem 1.1, we have to develop a new tool. Some yet unpublished results of Götze (1994) provide the rate $\mathcal{O}(N^{-1})$ for sums of two independent *arbitrary* quadratic forms (each of rank $d \geq 3$). Götze and Ulyanov (2003) obtained bounds of order $\mathcal{O}(N^{-1})$ for some ellipsoids in $\mathbb{R}^d$ with $d \geq 5$ in the case of lattice distributions of X.

The optimal possible dimension condition for this rate is just $d \geq 5$, due to the lower bounds of order $\mathcal{O}(N^{-1} \log N)$ for dimension $d = 4$ in the corresponding lattice point problem. The question about precise convergence rates in dimensions $2 \leq d \leq 4$ still remains completely open (even in the simplest case where $\mathbb{Q}$ is the identity operator $\mathbb{I}_d$, and for random vectors with independent Rademacher coordinates). It should be mentioned that, in the case $d = 2$, a precise convergence rate would imply a solution of the famous circle problem. Known lower bounds in the circle problem correspond to the bound of order $\mathcal{O}(N^{-3/4} \log^{\delta} N)$, $\delta > 0$, for Δ_N. Hardy (1916) conjectured that up to logarithmic factors this is the optimal order.

Now we describe the most important elements of the proof. We have to mention that a big part of the proof repeats the arguments of BG (1997a), see BG (1997a) for the description and application of the symmetrization inequality and the discretization procedure. We do not use the multiplicative inequalities of BG (1997a). Here we replace their application by some arguments coming from number theory. The new part of our proof is concentrated in Sections 5–8.

Using the Fourier inversion formula (see (4.3) and (4.4)), we have to estimate some integrals of the absolute values of differences of characteristic functions of quadratic forms. In Section 6, we reduce the estimation of characteristic functions to the estimation of a theta-series (see Lemma 6.5 and inequality (6.27)). To this end, we write the expectation with respect to Rademacher random variables as a sum with binomial weights $p(m)$ and $p(\overline{m})$. Then we estimate $p(m)$ and $p(\overline{m})$ from above by discrete Gaussian exponential weights $c_s\, q(m)$ and $c_s\, q(\overline{m})$, see (6.15), (6.18), (6.20) and (6.21). Together with the non-negativity of some characteristic functions (see (6.19) and (6.23)), this allows us to apply then the Poisson summation formula from Lemma 6.4. This formula reduces the problem to an estimation of integrals of theta-series. Section 7 is devoted to some facts from Number Theory. We consider the lattices, their α-characteristics (which are defined in (7.11) and (7.12)) and Minkowski's successive minima. In Section 8, we reduce the estimation of integrals of theta-series to some integrals of α-characteristics. An application of a new Lemma 8.2 proved by Götze and Margulis (2010) ends the proof.

2. Results

To formulate the results we need more notation repeating most part of the notation used in BG (1997a). Let $\sigma_1^2 \geq \sigma_2^2 \geq \dots$ be the eigenvalues of $\mathbb{C}$, counting their multiplicities. We have $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots$.

We shall identify the linear operators and corresponding matrices. By $\mathbb{I}_d : \mathbb{R}^d \to \mathbb{R}^d$ we denote the identity operator and, simultaneously, the diagonal matrix with entries 1 on the diagonal. By $\mathbb{O}_d$ we denote the $(d \times d)$ matrix with zero entries.

Throughout $\mathcal{S} = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$ denotes a finite set of cardinality s. We shall write $\mathcal{S}_o$ instead of $\mathcal{S}$ if the system $\{e_1, \dots, e_s\}$ is orthonormal.

Let $p>0$ and $\delta\geq 0$. Following BG (1997a), we introduce a somewhat modified non-degeneracy condition for the distribution of a d-dimensional vector Y:

$$\mathcal{N}(p,\delta,\mathcal{S},Y): \qquad \mathbf{P}\big\{\,\|Y-e\|\leq\delta\big\}\geq p, \qquad \text{for all } e\in\mathcal{S}. \tag{2.1}$$

We shall refer to condition (2.1) as condition $\mathcal{N}(p,\delta,\mathcal{S},Y)$. We shall write

$$\mathcal{N}_{\mathbb{Q}}(p,\delta,\mathcal{S},Y)=\mathcal{N}(p,\delta,\mathcal{S},Y)\cup\mathcal{N}(p,\delta,\mathbb{Q}\,\mathcal{S},Y).$$

Just condition $\mathcal{N}_{\mathbb{Q}}(p,\delta,\mathcal{S},Y)$ was used in BG (1997a). Note that

$$\mathcal{N}(p,\delta,\mathcal{S},Y)=\mathcal{N}_{\mathbb{I}_d}(p,\delta,\mathcal{S},Y). \tag{2.2}$$

Introduce truncated random vectors

$$X^{\diamond}=X\,\mathbf{I}\big\{\,\|X\|\leq\sigma\sqrt{N}\big\}, \qquad X_{\diamond}=X\,\mathbf{I}\big\{\,\|X\|>\sigma\sqrt{N}\big\}, \tag{2.3}$$

$$X^{\bullet}=X\,\mathbf{I}\big\{\,\|\mathbb{C}^{-1/2}\,X\|\leq\sqrt{dN}\big\}, \qquad X_{\bullet}=X\,\mathbf{I}\big\{\,\|\mathbb{C}^{-1/2}\,X\|>\sqrt{dN}\big\}, \tag{2.4}$$

and their moments (for $q>0$)

$$\Lambda_4^{\diamond}=\frac{1}{\sigma^4\,N}\,\mathbf{E}\,\|X^{\diamond}\|^4, \qquad \Pi_q^{\diamond}=\frac{N}{(\sigma\sqrt{N})^q}\,\mathbf{E}\,\|X_{\diamond}\|^q, \tag{2.5}$$

$$\Lambda_4^{\bullet}=\frac{1}{d^2\,N}\,\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X^{\bullet}\|^4, \qquad \Pi_q^{\bullet}=\frac{N}{(\sqrt{dN})^q}\,\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X_{\bullet}\|^q. \tag{2.6}$$

Here and below $\mathbf{I}\big\{A\big\}$ denotes the indicator of an event A. Of course, definitions (2.4) and (2.6) had sense if $d<\infty$ and the covariance operator $\mathbb{C}$ is non-degenerate.

Clearly, we have

$$X^{\diamond}+X_{\diamond}=X^{\bullet}+X_{\bullet}=X, \quad \|X^{\diamond}\|\,\|X_{\diamond}\|=\|X^{\bullet}\|\,\|X_{\bullet}\|=0. \tag{2.7}$$

Generally speaking, $X^{\bullet}$ and $X^{\diamond}$ are different truncated vectors. In BG (1997a) the i.i.d. copies of the vectors $X^{\diamond}$ and $X_{\diamond}$ only were involved. Truncation (2.4) was there applied to the vector $X^{\diamond}$. The use of $X^{\bullet}$ is more natural for the estimation of constants in the case $d<\infty$. It is easy to see that

$$\big(\mathbb{C}^{-1/2}\,X\big)^{\diamond}=\big(\mathbb{C}^{-1/2}\,X\big)^{\bullet}=\mathbb{C}^{-1/2}\,X^{\bullet}, \tag{2.8}$$

and

$$\big(\mathbb{C}^{-1/2}\,X\big)_{\diamond}=\big(\mathbb{C}^{-1/2}\,X\big)_{\bullet}=\mathbb{C}^{-1/2}\,X_{\bullet}\,. \tag{2.9}$$

Equalities (2.8) and (2.9) provides a possibility to apply auxiliary results obtained in BG (1997a) for truncated vectors $X^{\diamond}$ and $X_{\diamond}$ to truncated vectors $\mathbb{C}^{-1/2}\,X^{\bullet}$ and $\mathbb{C}^{-1/2}\,X_{\bullet}$. However, one should take into account that σ^2, $\Lambda_4^{\diamond}$, $\Pi_q^{\diamond}$, G, … have to be replaced by d, $\Lambda_4^{\bullet}$, $\Pi_q^{\bullet}$, $\mathbb{C}^{-1/2}\,G$, … .

In Sections 4 and 5 we shall denote

$$X'=X^{\bullet}-\mathbf{E}\,X^{\bullet}+W, \tag{2.10}$$

where W is a centered Gaussian random vector which is independent of all other random vectors and variables and is chosen so that $\operatorname{cov}X'=\operatorname{cov}G$. Such a vector W exists by Lemma 3.2.

By $c, c_1, c_2, \dots$ we shall denote absolute positive constants. If a constant depends on, say, s, then we shall point out the dependence writing c_s or $c(s)$. We denote by c universal constants which might be different in different places of the text. Furthermore, in the conditions of theorems and lemmas (see, e.g., Theorem 2.1, 2.2 and the proofs of Theorems 2.3, 2.6 and 2.7) we write c_0 for an *arbitrary* positive absolute constant, for example one may choose $c_0 = 1$. We shall write $A \ll B$, if there exists an absolute constant c such that $A \leq cB$. Similarly, $A \ll_s B$, if $A \leq c(s)B$. We shall also write $A \asymp_s B$ if $A \ll_s B \ll_s A$. By $\lceil \alpha \rceil$ we shall denote the integer part of a number α.

Throughout we assume that all random vectors and variables are independent in aggregate, if the contrary is not clear from the context. By $X_1, X_2, \dots$ we shall denote independent copies of a random vector X. Similarly, $G_1, G_2, \dots$ are independent copies of G and so on. By $\mathcal{L}(X)$ we shall denote the distribution of X. Define the symmetrization $\widetilde{X}$ of a random vector X as a random vector with distribution $\mathcal{L}(\widetilde{X}) = \mathcal{L}(X_1 - X_2)$.

Instead of normalized sums S_N, it is sometimes more convenient to consider the sums $Z_N = X_1 + \dots + X_N$. Then $S_N = N^{-1/2} Z_N$. Similarly, by $Z_N^\diamond$ (resp. $Z_N^\bullet$ and Z_N') we shall denote sums of N independent copies of $X^\diamond$ (resp. $X^\bullet$ and X'). For example, $Z_N' = X_1' + \dots + X_N'$.

The expectation $\mathbf{E}_Y$ with respect to a random vector Y we define as the conditional expectation

$$\mathbf{E}_Y\, f(X, Y, Z, \dots) = \mathbf{E}\,\big(f(X, Y, Z \dots) \,\big|\, X, Z, \dots\big)$$

given all random vectors but Y.

Throughout we write $\mathrm{e}\{x\} \overset{\text{def}}{=} \exp\{ix\}$. By

$$\widehat{F}(t) = \int_{-\infty}^{\infty} \mathrm{e}\{tx\}\, dF(x) \tag{2.11}$$

we denote the Fourier–Stieltjes transform of a function F of bounded variation or, in other words, the Fourier transform of the measure which has the distribution function F.

Introduce the distribution functions

$$F_a(x) \overset{\text{def}}{=} \mathbf{P}\big\{\mathbb{Q}[S_N - a] \leq x\big\}, \quad H_a(x) \overset{\text{def}}{=} \mathbf{P}\big\{\mathbb{Q}[G - a] \leq x\big\}, \qquad a \in \mathbb{R}^d, \quad x \in \mathbb{R}. \tag{2.12}$$

Furthermore, define, for $d = \infty$ and $a \in \mathbb{R}^d$, the Edgeworth correction

$$E_a(x) = E_a(x; \mathbb{Q}, \mathcal{L}(X), \mathcal{L}(G))$$

as a function of bounded variation such that $E_a(-\infty) = 0$ and its Fourier–Stieltjes transform is given by

$$\widehat{E}_a(t) = \frac{2\,(it)^2}{3\sqrt{N}}\,\mathbf{E}\;\mathrm{e}\big\{t\mathbb{Q}[Y]\big\}\big(3\,\langle \mathbb{Q}X, Y\rangle\,\langle \mathbb{Q}X, X\rangle + 2it\,\langle \mathbb{Q}X, Y\rangle^3\big), \quad Y = G - a. \tag{2.13}$$

In finite dimensional spaces (for $1 \leq d < \infty$) we define the Edgeworth correction as follows (see Bhattacharya and Rao (1986)). Let ϕ denote the standard normal density in $\mathbb{R}^d$. Then $p(y) = \phi(\mathbb{C}^{-1/2}y)/\sqrt{\det \mathbb{C}}$, $y \in \mathbb{R}^d$, is the density of G, and, for $a \in \mathbb{R}^d$, $b = \sqrt{N}\,a$, we have

$$E_a(x) \overset{\text{def}}{=} \Theta_b(Nx) \overset{\text{def}}{=} \frac{1}{6\sqrt{N}}\,\chi(A_x), \qquad A_x = \bigl\{u \in \mathbb{R}^d:\ \mathbb{Q}[u-a] \le x\bigr\}, \tag{2.14}$$

with the signed measure

$$\chi(A) \overset{\text{def}}{=} \int_A \mathbf{E}\, p'''(y) X^3\, dy, \qquad \text{for the Borel sets } A \subset \mathbb{R}^d, \tag{2.15}$$

and where

$$p'''(y)\, u^3 = p(y)\bigl(3\,\langle \mathbb{C}^{-1}u, u\rangle\langle \mathbb{C}^{-1}y, u\rangle - \langle \mathbb{C}^{-1}y, u\rangle^3\bigr) \tag{2.16}$$

denotes the third Frechet derivative of p in the direction u.

Notice that $E_a = 0$ if $a = 0$ or if $\mathbf{E}\,\langle X, y\rangle^3 = 0$, for all $y \in \mathbb{R}^d$. In particular, $E_a = 0$ if X is symmetric (that is, $\mathcal{L}(X) = \mathcal{L}(-X)$).

We can write similar representations for $E_a^{\bullet}(x) = \Theta_b^{\bullet}(Nx)$, $E_a^{\diamond}(x) = \Theta_b^{\diamond}(Nx)$ and $E_a'(x) = \Theta_b'(Nx)$ just replacing X by $X^{\bullet}$, $X^{\diamond}$ and X' in (2.13) or (2.15).

For $b \in \mathbb{R}^d$, introduce the distribution functions

$$\Psi_b(x) \overset{\text{def}}{=} \mathbf{P}\bigl\{\mathbb{Q}[Z_N - b] \le x\bigr\}, \tag{2.17}$$

and

$$\Phi_b(x) \overset{\text{def}}{=} \mathbf{P}\bigl\{\mathbb{Q}[\sqrt{N}\,G - b] \le x\bigr\}. \tag{2.18}$$

Define, for $a \in \mathbb{R}^d$, $b = \sqrt{N}\,a$,

$$\Delta_N^{(a)} \overset{\text{def}}{=} \sup_{x \in \mathbb{R}} \bigl|F_a(x) - H_a(x) - E_a(x)\bigr| = \sup_{x \in \mathbb{R}} \bigl|\Psi_b(x) - \Phi_b(x) - \Theta_b(x)\bigr|, \tag{2.19}$$

where $E_a(x)$ is the Edgeworth correction (see (2.12), (2.14), (2.17) and (2.18) to justify the last equality in (2.19)). We write $\Delta_{N,\bullet}^{(a)}$ and $\Delta_{N,\diamond}^{(a)}$ replacing E_a by $E_a^{\bullet}$ and $E_a^{\diamond}$ in (2.19).

The aim of this paper is to derive for $\Delta_N^{(a)}$ explicit bounds of order $\mathcal{O}(N^{-1})$ without any additional smoothness type assumptions. Theorem 2.1 (which was proved in BG (1997a)) solved this problem in the case $13 \le d \le \infty$.

In Theorems 2.1–2.7 we assume that the symmetric operator $\mathbb{Q}$ is isometric, that is, that $\mathbb{Q}^2$ is the identity operator $\mathbb{I}_d$. This does not restrict generality (see Remark 1.7 in BG (1997a)). Indeed, any symmetric operator $\mathbb{Q}$ may be decomposed as $\mathbb{Q} = \mathbb{Q}_1\mathbb{Q}_0\mathbb{Q}_1$, where $\mathbb{Q}_0$ is symmetric and isometric and $\mathbb{Q}_1$ is symmetric bounded and non-negative, that is, $\langle \mathbb{Q}_1 x, x\rangle \ge 0$, for all $x \in \mathbb{R}^d$. Thus, for any symmetric $\mathbb{Q}$, we can apply all our bounds replacing the random vector X by $\mathbb{Q}_1 X$, the Gaussian random vector G by $\mathbb{Q}_1 G$, the shift a by $\mathbb{Q}_1 a$, etc. In the case of concentration functions (see Theorems 2.6 and 2.7), we have $Q(X;\,\lambda;\,\mathbb{Q}) = Q(\mathbb{Q}_1 X;\,\lambda;\,\mathbb{Q}_0)$, and we may apply the results provided $\mathbb{Q}_1 X$ (instead of X) satisfies the conditions.

Theorem 2.1. (BG (1997a, Theorem 1.3)) *Let* $\delta = 1/300$, $\mathbb{Q}^2 = \mathbb{I}_d$, $s = 13$ *and* $13 \le d \le \infty$. *Let* c_0 *be an arbitrary positive absolute constant. Assume that condition* $\mathcal{N}_{\mathbb{Q}}(p, \delta, \mathcal{S}_o, c_0 G/\sigma)$ *holds. Then we have:*

$$\Delta_N^{(a)} \le C\big(\Pi_3^\diamond + \Lambda_4^\diamond\big)\big(1 + \|a/\sigma\|^6\big) \tag{2.20}$$

and

$$\Delta_{N,\diamond}^{(a)} \le C\big(\Pi_2^\diamond + \Lambda_4^\diamond\big)\big(1 + \|a/\sigma\|^6\big) \tag{2.21}$$

with $C = cp^{-6} + c(\sigma/\theta_8)^8$, *where* $\theta_1^4 \ge \theta_2^4 \ge \dots$ *are the eigenvalues of* $(\mathbb{C}\mathbb{Q})^2$.

Unfortunately, we cannot apply Theorem 2.1 for $d = 5, 6, \dots, 12$. Moreover, the quantity C depends on p which is exponentially small with respect to eigenvalues of $\mathbb{C}$.

In Götze and Zaitsev (2009), the following analogue of Theorem 2.1 is proved with bounds for constants which are not optimal.

Theorem 2.2. *Let* $\delta = 1/300$, $\mathbb{Q}^2 = \mathbb{I}_d$, $s = 5$ *and* $5 \le d < \infty$. *Let* c_0 *be an arbitrary positive absolute constant. Assume that condition* $\mathcal{N}_{\mathbb{Q}}(p, \delta, \mathcal{S}_o, c_0\, G/\sigma)$ *holds. Then*

$$\Delta_N^{(a)} \le C\big(\sigma_d^{-3} N^{-1/2}\,\mathbf{E}\,\|X_\bullet\|^3 + \sigma_d^{-4} N^{-1}\,\mathbf{E}\,\|X^\bullet\|^4\big)\big(1 + \|a/\sigma\|^3\big), \tag{2.22}$$

and

$$\Delta_{N,\bullet}^{(a)} \le C\big(\sigma_d^{-2}\mathbf{E}\,\|X_\bullet\|^2 + \sigma_d^{-4} N^{-1}\,\mathbf{E}\,\|X^\bullet\|^4\big)\big(1 + \|a/\sigma\|^3\big), \tag{2.23}$$

with $C = c_d p^{-3}$.

Theorem 2.2 extends to the case $d \ge 5$ Theorem 1.5 of BG (1997a) which contains the corresponding bounds for $d \ge 9$. Moreover, Theorem 2.2 is a little bit sharper than Theorem 1.5 of BG (1997a) even in the case $9 \le d < \infty$. In BG (1997a), the quantities $\sigma^4\sigma_d^{-4}\,(\Pi_3^\diamond + \Lambda_4^\diamond)$ and $\sigma^4\sigma_d^{-4}\,(\Pi_2^\diamond + \Lambda_4^\diamond)$ were standing in (2.22) and (2.23) instead of $\sigma_d^{-3}N^{-1/2}\,\mathbf{E}\,\|X_\bullet\|^3 + \sigma_d^{-4}N^{-1}\,\mathbf{E}\,\|X^\bullet\|^4$ and $\sigma_d^{-2}\mathbf{E}\,\|X_\bullet\|^2 + \sigma_d^{-4}N^{-1}\,\mathbf{E}\,\|X^\bullet\|^4$ respectively. Unfortunately, in both papers, the quantity C depends on p which is exponentially small with respect to σ_9/σ^2 (in BG (1997a)) and to σ_5/σ^2 (in Götze and Zaitsev (2009)). Under some additional conditions, C may be estimated from above by $c_d\,\exp(c\sigma^2\sigma_9^{-2})$ and by $c_d\,\exp(c\sigma^2\sigma_5^{-2})$ respectively.

In Götze and Zaitsev (2008) we proved Theorem 2.2 in the case $a = 0$ and hence, Theorem 1.1. First direct attempts to prove similar bounds for $a \ne 0$, assuming $d \ge 5$ instead of $d \ge 9$ in Theorems 1.4 and 1.5 of BG (1997a) failed. The main problem was that Lemma 3.2 of BG (1997a) allowed us to integrate the remainder terms of expansions for $\alpha < s/2$ only. Our Lemma 4.2 provides the bounds for $\alpha \ge s/2$ too. Note, however, that this difficulty was already successively avoided in the proof of the main result of BG (1996) (without estimation of constants).

The main result of the paper is Theorem 2.3. It is valid for $5 \le d < \infty$ in finite-dimensional spaces $\mathbb{R}^d$ only. However, the bounds of Theorem 2.3 depend on the smallest σ_j's. This makes them unstable if one or more of coordinates of X degenerates. In our finite dimensional results, Theorems 2.3, 2.6 and 2.7, we always assume that the covariance operator $\mathbb{C}$ is non-degenerate.

Theorem 2.3. *Let* $\mathbb{Q}^2=\mathbb{I}_d$, $5\le d<\infty$. *Then we have*:

$$\Delta_N^{(a)}\le C\left(\Pi_3^{\bullet}+\Lambda_4^{\bullet}\right)\left(1+\|a/\sigma\|^3\right), \tag{2.24}$$

and

$$\Delta_{N,\bullet}^{(a)}\le C\left(\Pi_2^{\bullet}+\Lambda_4^{\bullet}\right)\left(1+\|a/\sigma\|^3\right), \tag{2.25}$$

with $C=c_d\,\sigma^d\,(\det\mathbb{C})^{-1/2}$.

Theorems 2.1 and 2.3 yield Theorems 1.1 and 1.2, using that $\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X\|^4\le\beta/\sigma_d^4$, $E_0(x)\equiv 0$,

$$\Pi_3^{\bullet}+\Lambda_4^{\bullet}\le\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X\|^4/(d^2N),\quad \Pi_3^{\diamond}+\Lambda_4^{\diamond}\le\beta/(\sigma^4N), \tag{2.26}$$

and

$$\Pi_2^{\bullet}+\Lambda_4^{\bullet}\le\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X\|^4/(d^2N),\quad \Pi_2^{\diamond}+\Lambda_4^{\diamond}\le\beta/(\sigma^4N). \tag{2.27}$$

If, in the conditions of Theorem 2.3, the distribution of X is symmetric or $a=0$, then the Edgeworth corrections $E_a(x)$ and $E_a^{\bullet}(x)$ vanish and

$$\Delta_N^{(a)}=\Delta_{N,\bullet}^{(a)}\le C\left(\Pi_2^{\bullet}+\Lambda_4^{\bullet}\right)\left(1+\|a/\sigma\|^3\right),\qquad C=c_d\sigma^d\,(\det\mathbb{C})^{-1/2}. \tag{2.28}$$

The corresponding inequality from Theorem 1.4 of BG (1997a) yields in the case $s=9$ and $9\le d\le\infty$ under the condition $\mathcal{N}_{\mathbb{Q}}(p,\delta,\mathcal{S}_o,c_0G/\sigma)$ with $\delta=1/300$ the bound

$$\Delta_N^{(a)}\le C\left(\Pi_2^{\diamond}+\Lambda_4^{\diamond}\right)\left(1+\|a/\sigma\|^4\right),\qquad C=c\,p^{-4}. \tag{2.29}$$

It is clear that sometimes the bound (2.29) may be sharper than (2.28), but unfortunately, it depends on p which is usually exponentially small with respect to σ_9/σ^2.

One can find more precise estimates of constants in the case of d-dimensional balls with $d\ge 12$ in the papers of Nagaev and Chebotarev (1999), (2005), Götze and Ulyanov (2000), and Bogatyrev, Götze and Ulyanov (2006). In this case $\mathbb{Q}=\mathbb{I}_d$. See also Götze and Ulyanov (2000) for lower bounds for $\Delta_N^{(a)}$ under different conditions on a and $\mathcal{L}(X)$. In the papers mentioned above, the authors have used the aproach of BG (1997a) and obtained bounds with constants depending on $s<d$ largest eigenvalues $\sigma_1^2\ge\sigma_2^2\ge\cdots\ge\sigma_s^2$ of the covariance operator $\mathbb{C}$ (see Nagaev and Chebotarev (1999), (2005), with $d\ge s=13$, and Götze and Ulyanov (2000), and Bogatyrev, Götze and Ulyanov (2006), with $d\ge s=12$). It should be mentioned, that, in a particular case, where $\mathbb{Q}=\mathbb{I}_d$ and $d\ge 12$, these results may be sharper than (2.24), for some covariance operators $\mathbb{C}$.

Thus, we see that the statement of Theorem 2.3 is especially interesting for $d=5,6,\dots,11$. It is new even in the case of d-dimensional balls. It is plausible that the bounds for constants in Theorem 2.3 could be also improved for balls with $d\ge 5$, especially in the case where d is large. It seems however that this is impossible in the case of general $\mathbb{Q}$ even if $\mathbb{Q}^2=\mathbb{I}_d$. For example, we can consider the operator $\mathbb{Q}$ such that $\mathbb{Q}e_j=e_{d-j+1}$, where $\mathbb{C}e_j=\sigma_j^2e_j$, $j=1,2,\dots,d$, are eigenvectors of $\mathbb{C}$. Following the proof of Theorem 2.3, we see that the bounds for the modulus of the characteristic function $\bigl|\widehat{\Psi}_b(t)\bigr|=\bigl|\mathbf{E}\ \mathrm{e}\bigl\{t\,\mathbb{Q}\,[Z_N-b]\bigr\}\bigr|$ behave as the bounds for the modulus of the characteristic function $\bigl|\mathbf{E}\ \mathrm{e}\bigl\{t\,\mathbb{I}_d\,[Z_N-b]\bigr\}\bigr|$ but with eigenvalues of the covariance operator $\sigma_1\sigma_d$, $\sigma_2\sigma_{d-1}$,

$\sigma_3\sigma_{d-2}$, ... which can be essentially smaller than $\sigma_1^2 \ge \sigma_2^2 \ge \sigma_3^2 \ge \cdots$. Therefore, it is natural that the bounds for constants in Theorem 2.3 depends on the smallest eigenvalues of the covariance operator $\mathbb{C}$.

Note that, in the proof of Theorem 2.1 in BG (1997a), inequalities (2.20) and (2.21) were derived for the Edgeworth correction $E_a(x)$ defined by (2.13). However, from Theorems 2.1 and 2.2 or 2.3 it follows that, at least for $13 \le d < \infty$, definitions (2.13) and (2.14) determine the same function $E_a(x)$. Indeed, both functions may be represented as $N^{-1/2}\,K(x)$, where $K(x)$ are some functions of bounded variation which are independent of N. Furthermore, inequalities (2.20) and (2.24) provide both bounds of order $\mathcal{O}(N^{-1})$. This is possible if the Edgeworth corrections $E_a(x)$ are the same in these inequalities.

On the other hand, it is proved (for $d \ge 9$) that definition (2.13) determine a function of bounded variation (see BG (1997a, Lemma 5.7)), while definition (2.14) has no sense for $d = \infty$.

Introduce the concentration function

$$Q(X;\,\lambda) = Q(X;\,\lambda;\,\mathbb{Q}) = \sup_{a,\,x\in\mathbb{R}^d} \mathbf{P}\big\{x \le \mathbb{Q}[X-a] \le x+\lambda\big\}, \quad \text{for } \lambda \ge 0. \tag{2.30}$$

It should be mentioned that the supremum in (2.30) is taken not only over all x, but over all x and $a \in \mathbb{R}^d$. Usually, one defines the concentration function of the random variable $\mathbb{Q}[X-a]$ taking the supremum over all $x \in \mathbb{R}^d$ only. Note that, evidently, $Q(X+Y;\,\lambda) \le Q(X;\,\lambda)$, for any Y which is independent of X.

The following Theorems 2.4 and Theorem 2.5 are Theorems 1.5 and 2.1 from Götze and Zaitsev (2009).

Theorem 2.4. *Let* $\mathbb{Q}^2 = \mathbb{I}_d$, $5 \le s \le d \le \infty$, $s < \infty$ *and* $0 \le \delta \le 1/(5s)$. *Then we have*:

(*i*) *If condition* $\mathcal{N}_{\mathbb{Q}}(p,\delta,\mathcal{S}_o,\widetilde{X})$ *is fulfilled with some* $p > 0$, *then*

$$Q(Z_N;\,\lambda) \ll_s (pN)^{-1}\max\big\{1;\,\lambda\big\}, \qquad \lambda \ge 0. \tag{2.31}$$

(*ii*) *If, for some* m, *condition* $\mathcal{N}_{\mathbb{Q}}(p,\delta,\mathcal{S}_o,m^{-1/2}\widetilde{Z}_m)$ *is fulfilled, then*

$$Q(Z_N;\,\lambda) \ll_s (pN)^{-1}\max\big\{m;\,\lambda\big\}, \qquad \lambda \ge 0. \tag{2.32}$$

Theorem 2.5. *Let* $\mathbb{Q}^2 = \mathbb{I}_d$ *and* $5 \le d \le \infty$. *Let* c_0 *be an arbitrary positive absolute constant. Assume condition* $\mathcal{N}_{\mathbb{Q}}(p,\delta,\mathcal{S}_o,c_0G/\sigma)$ *to be satisfied with* $s = 5$ *and* $\delta = 1/200$. *Then*

$$Q(Z_N;\,\lambda) \ll p^{-2}\max\big\{\Pi_2^\diamond + \Lambda_4^\diamond;\,\lambda\sigma^{-2}N^{-1}\big\}, \qquad \lambda \ge 0. \tag{2.33}$$

In particular, $Q(Z_N;\,\lambda) \ll p^{-2}N^{-1}\max\big\{\beta/\sigma^4;\,\lambda/\sigma^2\big\}$.

Theorems 2.4 and Theorem 2.5 extend to the case $5 \le d \le \infty$ Theorems 1.6 and 2.1 of BG (1997a) which were proved for $9 \le d \le \infty$.

We say that a random vector Y is concentrated in $\mathbb{L} \subset \mathbb{R}^d$ if $\mathbf{P}\{Y \in \mathbb{L}\} = 1$. In BG (1997a, item (*iii*) of Theorem 1.6) it was shown that if $\widetilde{X}$ is not concentrated in a proper closed linear subspace of $\mathbb{R}^d$, $1 \le d \le \infty$, then, for any $\delta > 0$ and $\mathcal{S}$ there exists a natural number m such that the condition $\mathcal{N}_{\mathbb{Q}}(p, \delta, \mathcal{S}, m^{-1/2}\,\widetilde{Z}_m)$ holds with some $p > 0$.

In this paper, we shall prove the following Theorems 2.6 and Theorem 2.7.

Theorem 2.6. *Let $\mathbb{Q}^2 = \mathbb{I}_d$, $5 \le s = d < \infty$ and $0 \le \delta \le 1/(5s)$. Then we have:*

(i) If condition $\mathcal{N}(p, \delta, \mathcal{S}_o, \mathbb{C}^{-1/2}\,\widetilde{X})$ is fulfilled with some $p > 0$, then

$$Q(Z_N;\,\lambda) \ll_d (pN)^{-1}\,\max\big\{1;\,\lambda\,\sigma^{-2}\big\}\,\sigma^d\,(\det\mathbb{C})^{-1/2}, \qquad \lambda \ge 0. \tag{2.34}$$

(ii) If, for some m, condition $\mathcal{N}(p, \delta, \mathcal{S}_o, m^{-1/2}\mathbb{C}^{-1/2}\,\widetilde{Z}_m)$ is fulfilled, then

$$Q(Z_N;\,\lambda) \ll_d (pN)^{-1}\,\max\big\{m;\,\lambda\,\sigma^{-2}\big\}\,\sigma^d\,(\det\mathbb{C})^{-1/2}, \qquad \lambda \ge 0. \tag{2.35}$$

Theorem 2.7. *Assume that $5 \le d < \infty$ and that $\mathbb{Q}^2 = \mathbb{I}_d$. Then*

$$Q(Z_N;\,\lambda) \ll_d \max\big\{\Pi_2^\bullet + \Lambda_4^\bullet;\,\lambda\sigma^{-2}N^{-1}\big\}\,\sigma^d\,(\det\mathbb{C})^{-1/2}, \qquad \lambda \ge 0. \tag{2.36}$$

In particular, $Q(Z_N;\,\lambda) \ll_d N^{-1}\max\big\{\mathbf{E}\,\|\mathbb{C}^{-1/2}X\|^4;\,\lambda/\sigma^2\big\}\,\sigma^d\,(\det\mathbb{C})^{-1/2}$.

Theorem 2.6 and Theorem 2.7 yield more explicit versions of Theorems 2.4 and Theorem 2.5 as well as Theorem 2.3 is in a sense a more explicit version of Theorem 2.2. We should mention that Theorems 2.2, 2.4 and 2.5 do not follow from Theorems 2.3, 2.6 and 2.7. For the proofs of these theorems we refer the reader to the preprint of Götze and Zaitsev (2009) which is available in internet.

For example, the bounds in Theorems 2.2, 2.4 and 2.5 may be sharper than those from Theorems 2.3, 2.6 and 2.7, in a particular case, where $\mathbb{Q} = \mathbb{I}_d$ and $\sigma_5 \asymp_d \sigma$. Under some additional conditions, $\sigma^d\,(\det\mathbb{C})^{-1/2}$ is replaced by $\exp(c\sigma^{-2}\sigma_5^2) \asymp_d 1$. On the other hand, $\sigma^d\,(\det\mathbb{C})^{-1/2}$ provides a power-type dependence on eigenvalues of $\mathbb{C}$ and the results are valid for $\mathbb{Q}$ which might be not positive definite.

In Theorems 2.3 and Theorem 2.7, we do not assume the fulfilmemt of conditions $\mathcal{N}(\,\cdot\,)$ or $\mathcal{N}_{\mathbb{Q}}(\,\cdot\,)$. In the proofs, we shall use, however, that, for an arbitrary absolute positive constant c_0 and any positive quantity c_d depending on d only, condition $\mathcal{N}(p, \delta, \mathcal{S}_o, c_0\,\mathbb{C}^{-1/2}\,G)$ is fulfilled with $s = d$, $\delta = c_d$ and $p \asymp_d 1$, for any orthonormal system $\mathcal{S}_o$.

Similarly to BG (1997a), in Section 3, we prove bounds for concentration functions. The proof is technically simpler as that of Theorem 2.3, but it shows how to apply the principal ideas. This proof repeats almost literally the corresponding proof of BG (1997a). The only difference consists in the use of new Lemma 8.3 which allows us to estimate characteristic functions of quadratic forms for relatively large values of argument t. In Sections 4 and 5, Theorem 2.3 is proved. We shall replace Lemma 9.4 of BG (1997a) by its improvement, Lemmas 5.1. Another difference is in another choice of k in (5.31) and (5.32) in comparison with that in BG (1997a).

In Sections 6–8 we prove estimates for characteristic functions. Section 6 is started with results from BG (1997a) (Lemmas 6.1–6.2). Their proofs in BG (1997a) are based on conditioning and discretization.

Let $\varepsilon_1, \varepsilon_2, \dots$ denote i.i.d. symmetric Rademacher random variables. Let $\delta > 0$, $\mathcal{S} = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$ and let $\mathbb{D} : \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator. Usually, we shall take

$\mathbb{D} = \mathbb{C}^{-1/2}$. We shall write $\mathcal{L}(Y) \in \mathbf{\Gamma}(\delta; \mathbb{D}, \mathcal{S})$ if a discrete random vector Y is distributed as $\varepsilon_1 z_1 + \dots + \varepsilon_s z_s$, with some (non-random) $z_j \in \mathbb{R}^d$ such that $\|\mathbb{D} z_j - e_j\| \le \delta$, for all $1 \le j \le s$.

In this paper, assuming condition $\mathcal{N}(p, \delta, \mathcal{S}_o, \mathbb{C}^{-1/2}\,\widetilde{X})$ with $0 \le \delta \le 1/(5s)$ and an orthonormal system $\mathcal{S}_o$, we shall use that, by Lemma 6.3, for any $0 < A \le B$, $b \in \mathbb{R}^d$ and $\gamma > 0$,

$$\int_A^B \bigl|\,\widehat{\Psi}_b(t)\,\bigr| \,\frac{dt}{|t|} \le c_\gamma(s)\,(pN)^{-\gamma}\,\log\frac{B}{A} + \sup_{\Gamma} \int_A^B \sqrt{\mathbf{E}\ \mathrm{e}\bigl\{t\,\langle \widetilde{W}, \widetilde{W'}\rangle/2\bigr\}}\,\frac{dt}{|t|}\,, \tag{2.37}$$

where $W = V_1 + \dots + V_n$ and $W' = V_1' + \dots + V_n'$ are independent sums of independent copies of random vectors V and V' respectively, and the supremum $\sup_\Gamma$ is taken over all $\mathcal{L}(V), \mathcal{L}(V') \in \mathbf{\Gamma}(\delta; \mathbb{C}^{-1/2}, \mathcal{S}_o)$.

Comparing inequalities (2.37) and (6.6), we see that inequality (6.6) (which was used in BG (1997a)) is related to sums of *non-i.i.d.* vectors $\{V_j\}$ and $\{V_j'\}$ while inequality (2.37) deals with i.i.d. vectors. Nevertheless, we shall derive (2.37) from (6.6) by an application of Hölder's and the arithmetic-geometric mean inequalities.

Define the function

$$\mathcal{M}(t; N) = 1/\sqrt{|\,t\,|\,N}, \quad \text{for } |\,t\,| \le N^{-1/2}, \qquad \mathcal{M}(t; N) = \sqrt{|\,t\,|}, \quad \text{for } |\,t\,| \ge N^{-1/2}. \tag{2.38}$$

It is easy to see that, for $s > 0$,

$$2^{-1}\bigl(|\,tN|^{-s/2} + |\,t\,|^{s/2}\bigr) \le \mathcal{M}^s(t; N) \le |\,tN|^{-s/2} + |\,t\,|^{s/2}. \tag{2.39}$$

Assuming the condition $\mathcal{N}_{\mathbb{Q}}(p, \delta, \mathcal{S}_o, \widetilde{X})$ with $0 \le \delta \le 1/(5s)$, we could use Theorem 7.1 from BG (1997a) which implies that, for any $b \in \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$\bigl|\,\widehat{\Psi}_b(t)\,\bigr| = \bigl|\,\mathbf{E}\ \mathrm{e}\bigl\{t\,\mathbb{Q}[Z_N - b]\bigr\}\,\bigr| \ll_s \mathcal{M}^s(t; pN). \tag{2.40}$$

Inequality (2.40) was essentially used in BG (1997a) and Götze and Zaitsev (2008, 2009). Applying Lemma 2.5 from Götze and Ulyanov (2000) and techniques from BG (1997a), we could show that conditions $\sigma^2 = 1$ and $\mathcal{N}(p, \delta, \mathcal{S}_o, \mathbb{C}^{-1/2}\,\widetilde{X})$ with $0 \le \delta \le 1/(5s)$, $s = d$, imply that, for any $b \in \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$\bigl|\,\widehat{\Psi}_b(t)\,\bigr| \ll_s \mathcal{M}^s(t; pN)\,(\det \mathbb{C})^{-1/2}. \tag{2.41}$$

Inequalities (2.40) and (2.41) will be not used in this paper. We replace their application by using inequalities (8.32) and (8.34) of Lemma 8.1.

Inequalities of type (2.40) or (2.41) allow to prove Theorem 1.1 with error bounds $\mathcal{O}(N^{-\alpha})$ only, for some $\alpha < 1$. This is due to possible oscillations of $\bigl|\,\widehat{\Psi}_a(t)\,\bigr|$ between 0 and 1, as $|\,t\,| \sim N^{-\varepsilon}$ with small $\varepsilon \ge 0$. In Section 6, we reduce the estimation of $\mathbf{E}\ \mathrm{e}\bigl\{t\,\langle \widetilde{W}, \widetilde{W'}\rangle/2\bigr\}$ to the estimation of a theta-series (see Lemma 6.5 and inequality (6.27)). The rest of the proof is described at the end of Section 1.

Acknowledgment We would like to thank V.V. Ulyanov for helpful discussions.

3. Proofs of bounds for concentration functions

Proof of Theorems 2.6 *and* 2.7. Below we shall prove the assertions (2.34); (2.34) $\Longrightarrow$ (2.35) and (2.35) $\Longrightarrow$ (2.36). The proof repeats almost literally the corresponding proof of BG (1997a). It is given here for the sake of completeness. The only essential difference is in the use of Lemma 8.3 in the proof of Lemma 3.1. We have also to replace everywhere 9 by 5 and $\diamond$ by $\bullet$. $\square$

For $0 \le t_0 \le T$ and $b \in \mathbb{R}^d$, define the integrals

$$I_0 = \int_{-T}^{T} \bigl|\,\widehat{\Psi}_b(t)\,\bigr|\,dt, \qquad\qquad I_1 = \int_{t_0 \le |t| \le T} \bigl|\,\widehat{\Psi}_b(t)\,\bigr|\,\frac{dt}{|t|}\,,$$

where

$$\widehat{\Psi}_b(t) = \mathbf{E}\ \mathrm{e}\bigl\{t\,\mathbb{Q}[Z_N - b]\bigr\} \tag{3.1}$$

denotes the Fourier–Stieltjes transform of the distribution function Ψ_b of $\mathbb{Q}[Z_N - b]$ (see (2.11) and (2.17)). Note that $\bigl|\,\widehat{\Psi}_b(-t)\bigr| = \bigl|\,\widehat{\Psi}_b(t)\bigr|$.

Lemma 3.1. *Assume condition* $\mathcal{N}(p, \delta, \mathcal{S}_o, \mathbb{C}^{-1/2}\,\widetilde{X})$ *with some* $0 \le \delta \le 1/(5s)$ *and* $5 \le s = d < \infty$. *Let* $\sigma^2 = 1$ *and*

$$t_0 = c_1(s)\,\sigma_1^{-2}(pN)^{-1+2/s}, \qquad c_2(s)\,\sigma_1^{-2} \le T \le c_3(s)\,\sigma_1^{-2} \tag{3.2}$$

with some positive constants $c_j(s)$, $1 \le j \le 3$. *Then*

$$I_0 \ll_s (\det \mathbb{C})^{-1/2}\,(pN)^{-1}, \qquad\qquad I_1 \ll_s (\det \mathbb{C})^{-1/2}\,(pN)^{-1}. \tag{3.3}$$

Proof. Note that the condition $\sigma^2 = 1$ implies that

$$T \asymp_s \sigma_1^2 \asymp_s \sigma^2 = 1 \quad \text{and} \quad \det \mathbb{C} \ll_s 1. \tag{3.4}$$

Denote $k = pN$. Without loss of generality we assume that $k \ge c_s$, for a sufficiently large quantity c_s depending on s only. Indeed, if $k \le c_s$, then one can prove (3.3) using (3.4) and $|\widehat{\Psi}_b| \le 1$. Choosing c_s to be large enough, we ensure that $k \ge c_s$ implies $1/k \le t_0 \le T$.

Lemma 8.3 and (3.4) imply now that

$$\int_{c_4(s)k^{-1+2/s}}^{T} \bigl|\,\widehat{\Psi}_b(t)\,\bigr|\,\frac{dt}{t} \ \ll_s \ \frac{(\det \mathbb{C})^{-1/2}}{k}\,, \tag{3.5}$$

for any $c_4(s)$ depending on s only. Inequalities (3.4) and (3.5) imply (3.3) for I_1.

Let us prove inequality (3.3) for I_0. By (3.4) and by Lemma 8.1, for any $\gamma > 0$ and any fixed $t \in \mathbb{R}$ satisfying $k^{1/2}\,|t| \le c_5(s)$, where $c_5(s)$ is an arbitrary quantity depending on s only, we have (taking into account that $|\widehat{\Psi}_b| \le 1$)

$$\bigl|\,\widehat{\Psi}_b(t)\,\bigr| \ll_{\gamma,s} \min\bigl\{1;\ k^{-\gamma} + k^{-s/2}\ |t|^{-s/2}\ (\det \mathbb{C})^{-1/2}\bigr\}, \qquad k = pN. \tag{3.6}$$

Furthermore, choosing an appropriate γ and using (3.4)–(3.6), we obtain

$$(\det\mathbb{C})^{1/2}\,I_0 \ll_s \int_0^{1/k} dt + \frac{1}{k} + \int_{1/k}^{\infty} \frac{dt}{(t\,k)^{s/2}} \ll_s \frac{1}{k}\,, \tag{3.7}$$

proving (3.3) for I_0. $\square$

Proof of (2.34). Let $\sigma^2 = 1$. Using a well-known inequality for concentration functions (see, for example, Petrov (1975, Lemma 3 of Ch. 3)), we have

$$Q(Z_N;\,\lambda) \le 4\,\sup_{b\in\mathbb{R}^d}\,\max\bigl\{\lambda;\,1\bigr\}\int_0^1 \bigl|\,\widehat{\Psi}_b(t)\,\bigr|\,dt. \tag{3.8}$$

To estimate the integral in (3.8) we shall apply Lemma 3.1 which implies that

$$Q(Z_N;\,\lambda) \ll_d \max\bigl\{\lambda;\,1\bigr\}(pN)^{-1}\,(\det\mathbb{C})^{-1/2}, \tag{3.9}$$

proving (2.34) in the case $\sigma^2 = 1$. If $\sigma^2 \ne 1$, we obtain (2.34) applying (3.9) to Z_N/σ. $\square$

Proof of (2.34) $\Longrightarrow$ (2.35). Without loss of generality we can assume that $N/m \ge 2$. Let $Y_1, Y_2, \dots$ be independent copies of $m^{-1/2}\,Z_m$. Denote $W_k = Y_1 + \dots + Y_k$. Then $\mathcal{L}(Z_N) = \mathcal{L}(\sqrt{m}\,W_k + y)$ with $k = \lceil N/m\rceil$ and with some y independent of W_k. Therefore, $Q(Z_N;\lambda) \le Q(W_k;\lambda/m)$. In order to estimate $Q(W_k;\lambda/m)$ we apply (2.34) replacing Z_N by W_k. We have

$$\begin{aligned} Q(W_k;\lambda/m) &\ll_s (pk)^{-1}\max\bigl\{1;\,\lambda\,\sigma^{-2}/m\bigr\}\,\sigma^d\,(\det\mathbb{C})^{-1/2} \\ &\ll_s (pN)^{-1}\max\bigl\{m;\,\lambda\,\sigma^{-2}\bigr\}\,\sigma^d\,(\det\mathbb{C})^{-1/2}. \qquad \square \end{aligned} \tag{3.10}$$

Recall that truncated random vectors and their moments are defined by (2.3)–(2.6) and that $\mathbb{C} = \operatorname{cov} X = \operatorname{cov} G$.

Lemma 3.2. *The random vectors $X^\bullet$, $X_\bullet$ satisfy*

$$\langle\mathbb{C}x, x\rangle = \langle\operatorname{cov} X^\bullet\, x, x\rangle + \mathbf{E}\,\langle X_\bullet, x\rangle^2 + \langle\mathbf{E}\,X^\bullet, x\rangle^2.$$

There exist independent centered Gaussian vectors G_* and W such that

$$\mathcal{L}(G) = \mathcal{L}(G_* + W)$$

and

$$2\operatorname{cov} G_* = 2\operatorname{cov} X^\bullet = \operatorname{cov}\widetilde{X^\bullet}, \qquad \langle\operatorname{cov} W\,x, x\rangle = \mathbf{E}\,\langle X_\bullet, x\rangle^2 + \langle\mathbf{E}\,X^\bullet, x\rangle^2.$$

Furthermore,

$$\mathbf{E}\,\|\mathbb{C}^{-1/2}\,G\|^2 = d = \mathbf{E}\,\|\mathbb{C}^{-1/2}\,G_*\|^2 + \mathbf{E}\,\|\mathbb{C}^{-1/2}\,W\|^2$$

and $\mathbf{E}\,\|\mathbb{C}^{-1/2}\,W\|^2 \le 2\,d\,\Pi_2^\bullet$.

We omit the simple proof of this lemma (see BG (1997a, Lemma 2.4) for the same statement with $\diamond$ instead of $\bullet$). Lemma 3.2 allows us to define the vector X' by (2.10).

Recall that $Z_N^\bullet$ and $Z_N^\diamond$ denote sums of N independent copies of $X^\bullet$ and $X^\diamond$ respectively.

Lemma 3.3. *Let $\varepsilon > 0$. There exist absolute positive constants c and c_1 such that the condition $\Pi_2^\bullet \le c_1 p\,\delta^2/(d\varepsilon^2)$ implies that*

$$\mathcal{N}(p,\delta,\mathcal{S},\varepsilon\,\mathbb{C}^{-1/2}\,G) \Longrightarrow \mathcal{N}(p/4,4\delta,\mathcal{S},\varepsilon\,(2m)^{-1/2}\,\mathbb{C}^{-1/2}\,\widetilde{Z_m^\bullet}),$$

for $m \ge c\varepsilon^4 d^2 N\,\Lambda_4^\bullet/(p\,\delta^4)$.

Lemmas 3.2 and 3.3 are in fact the statements of Lemmas 2.4 and 2.5 from BG (1997a) applied to the vectors $\mathbb{C}^{-1/2}\,X$ instead of the vectors X. We use in this connection equalities (2.2), (2.8) and (2.9) replacing in the formulation σ^2, $\Lambda_4^\diamond$, $\Pi_q^\diamond$, G, $Z_m^\diamond$, … by d, $\Lambda_4^\bullet$, $\Pi_q^\bullet$, $\mathbb{C}^{-1/2}\,G$, $Z_m^\bullet$, … respectively.

Proof of (2.35) $\Longrightarrow$ (2.36). By a standard truncation argument, we have

$$\big|\mathbf{P}\big\{\,Z_N \in A\big\} - \mathbf{P}\big\{\,Z_N^\bullet \in A\big\}\,\big| \le N\,\mathbf{P}\big\{\,\|\mathbb{C}^{-1/2}\,X\| > \sqrt{dN}\,\big\} \le \Pi_2^\bullet, \tag{3.11}$$

for any Borel set A, and

$$Q(Z_N,\lambda) \le \Pi_2^\bullet + Q(Z_N^\bullet,\lambda). \tag{3.12}$$

Recall that we are proving (2.36) assuming that $5 \le d < \infty$. It is easy to see that, for an arbitrary absolute positive constant c_0, condition $N(p,\delta,\mathcal{S}_o,c_0\,\mathbb{C}^{-1/2}\,G)$ with

$$s = d, \quad \delta = 1/(20\,s), \quad p \asymp_d 1 \tag{3.13}$$

is in fact fulfilled automatically for any orthonormal system $\mathcal{S}_o$, since the vector $\mathbb{C}^{-1/2}\,G$ has standard Gaussian distribution in $\mathbb{R}^d$ and $\mathbf{P}\big\{\,\|c_0\,\mathbb{C}^{-1/2}\,G - e\| \le \delta\,\big\} = c(d)$ for any vector $e \in \mathbb{R}^d$ with $\|e\| = 1$. Clearly, $4\delta = 1/(5s)$. Write $K = \varepsilon/\sqrt{2}$ with $\varepsilon = c_0$. Then, by (3.13) and Lemma 3.3, we have

$$\mathcal{N}(p,\delta,\mathcal{S}_o,\varepsilon\mathbb{C}^{-1/2}\,G) \Longrightarrow \mathcal{N}(p/4,4\delta,\mathcal{S}_o,m^{-1/2}\,K\mathbb{C}^{-1/2}\,\widetilde{Z_m^\bullet}), \tag{3.14}$$

provided that

$$\Pi_2^\bullet \le c_1(d), \qquad m \ge c_2(d)\,N\,\Lambda_4^\bullet. \tag{3.15}$$

Without loss of generality we may assume that $\Pi_2^\bullet \le c_1(d)$, since otherwise the result follows easily from the trivial inequality $Q(Z_N;\lambda) \le 1$.

The non-degeneracy condition (3.14) for $K\widetilde{Z_m^\bullet}$ allows to apply (2.35) of Theorem 2.6, and, using (3.13), we obtain

$$Q(Z_N^\bullet,\lambda) = Q(K\,Z_N^\bullet,K^2\,\lambda) \ll_d N^{-1}\max\{m;\ K^2\,\lambda/K^2\,\sigma^2\}\,\sigma^d\,(\det\mathbb{C})^{-1/2}, \tag{3.16}$$

for any m such that (3.15) is fulfilled. Choosing the minimal m in (3.15), we obtain

$$Q(Z_N^\bullet,\lambda) \ll_d \max\{\Lambda_4^\bullet;\ \lambda/(\sigma^2\,N)\}\,\sigma^d\,(\det\mathbb{C})^{-1/2}. \tag{3.17}$$

Combining the estimates (3.12) and (3.17), we conclude the proof. $\square$

4. Auxiliary lemmas

In Sections 4 and 5 we shall prove Theorem 2.3. Therefore, we shall assume that its conditions are satisfied. We consider the case $d < \infty$ assuming that the following conditions are satisfied:

$$\mathbb{Q}^2 = \mathbb{I}_d, \quad \sigma^2 = 1, \quad d \geq 5, \quad b = \sqrt{N}\, a. \tag{4.1}$$

Moreover, it is easy to see that, for any absolute positive constant c_0 and for any orthonormal system $\mathcal{S}_o = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$, condition

$$N(p, \delta, \mathcal{S}_o, c_0\, \mathbb{C}^{-1/2}\, G) \quad \text{with} \quad p \asymp_d 1, \quad 5 \leq s = d < \infty, \quad \delta = 1/(20\, s) \tag{4.2}$$

is in fact fulfilled automatically since the vector $\mathbb{C}^{-1/2}\, G$ has standard Gaussian distribution in $\mathbb{R}^d$ and, therefore, $\mathbf{P}\big\{ \|c_0\, \mathbb{C}^{-1/2}\, G - e\| \leq \delta \big\} = \mathbf{P}\big\{ \|\, \mathbb{C}^{-1/2}\, G - c_0^{-1}\, e\| \leq c_0^{-1}\, \delta \big\} = c(d)$ for any vector $e \in \mathbb{R}^d$ with $\|e\| = 1$.

Notice that the assumption $\sigma^2 = 1$ does not restrict generality since from Theorem 2.3 with $\sigma^2 = 1$ we can derive the general result replacing X, G by X/σ, G/σ, etc. Other assumptions in (4.1) are included as conditions in Theorem 2.3. Section 4 is devoted to some auxiliary lemmas which are similar to corresponding lemmas of BG (1997a).

In several places, the proof of Theorem 2.3 repeats almost literally the proof of Theorem 1.5 in BG (1997a). Note, however, that we shall use truncated vectors $X_j^\bullet$, while in BG (1997a) the vectors $X_j^\diamond$ were involved. We start with an application of the Fourier transform to the functions Ψ_b and Φ_b, where $b = \sqrt{N}\, a$. We shall estimate integrals over the Fourier transforms using results of Sections 3, 6–8 and some technical lemmas of BG (1997a). We shall also apply some methods of estimation of the rate of approximation in the CLT in multidimensional spaces (cf., e.g., Bhattacharya and Rao (1986)).

We shall use the following formulas for the Fourier inversion (see BG (1997a)). A smoothing inequality of Prawitz (1972) implies (see BG (1996, Section 4)) that

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\ \mathrm{V.\,P.} \int_{|t| \leq K} \mathrm{e}\big\{ -xt \big\} \widehat{F}(t)\, \frac{dt}{t} + R, \tag{4.3}$$

for any $K > 0$ and any distribution function F with characteristic function $\widehat{F}$ (see (2.11)), where

$$|R| \leq \frac{1}{K} \int_{|t| \leq K} |\widehat{F}(t)|\, dt. \tag{4.4}$$

Here $\mathrm{V.\,P.} \int f(t)\, dt = \lim_{\varepsilon \to 0} \int_{|t| > \varepsilon} f(t)\, dt$ denotes the Principal Value of the integral.

Recall that the random vectors $X^\bullet$, X' are defined in (2.4) and (2.10) and $Z_N^\bullet$, Z_N' are sums of N their independent copies. Note that the Gaussian vector W involved in (2.10) is independent of all other vectors and have properties described in Lemma 3.2. Write $\Psi_b^\bullet$ and Ψ_b' for the distribution function of $\mathbb{Q}[Z_N^\bullet - b]$ and $\mathbb{Q}[Z_N' - b]$ respectively. For $0 \leq k \leq N$ introduce the the distribution function

$$\Psi_b^{(k)}(x) = \mathbf{P}\big\{ \mathbb{Q}\big[G_1 + \dots + G_k + X_{k+1}' + \dots + X_N' - b \big] \leq x \big\}. \tag{4.5}$$

Notice that $\Psi_b^{(0)} = \Psi'_b$, $\Psi_b^{(N)} = \Phi_b$.

The proof of the following lemma repeats the proof of Lemma 3.1 of BG (1997a). The difference is that here we use the truncated vectors $X_j^\bullet$ instead of $X_j^\diamond$.

Lemma 4.1. *Let c_d be a quantity depending on d only. There exist positive quantities $c_1(d)$ and $c_2(d)$ depending on d only such that the following statement is valid. Let $\Pi_2^\bullet \le c_1(d)p$ and let an integer $1 \le m \le N$ satisfy $m \ge c_2(d)\, N\Lambda_4^\bullet/p$, Write*

$$K = c_0^2/(2m), \qquad\qquad t_1 = c_d (pN/m)^{-1+2/d}.$$

Let F denote any of the functions $\Psi_b^\bullet$, Ψ'_b, $\Psi_b^{(k)}$ or Φ_b. Then we have

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\,\mathrm{V.P.}\int_{|t|\le t_1} \mathrm{e}\{-xtK\}\,\widehat{F}(tK)\,\frac{dt}{t} + R_1, \tag{4.6}$$

with $|R_1| \ll_d (pN)^{-1} m\, (\det \mathbb{C})^{-1/2}$.

Proof. We shall assume that $(pN)^{-1}m \le c_3(d)$ with sufficiently small $c_3(d)$ since otherwise the statement of Lemma 4.1 is trivial (see (3.4), (4.3) and (4.4)). Let us prove (4.6). We shall combine (4.3) and Lemma 3.1. Changing the variable $t = \tau K$ in formula (4.3), we obtain

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\,\mathrm{V.P.}\int_{|t|\le 1} \mathrm{e}\{-xtK\}\,\widehat{F}(tK)\,\frac{dt}{t} + R, \tag{4.7}$$

where

$$|R| \le \int_{|t|\le 1} \big|\widehat{F}(tK)\big|\,dt. \tag{4.8}$$

Notice that $\Psi_b^\bullet$, Ψ'_b, $\Psi_b^{(k)}$ and Φ_b are distribution functions of random variables which may be written in the form:

$$\mathbb{Q}[V+T], \qquad\qquad V \overset{\mathrm{def}}{=} G_1 + \dots + G_k + X_{k+1}^\bullet + \dots + X_N^\bullet,$$

with some k, $0 \le k \le N$, and some random vector T which is independent of $X_j^\bullet$ and G_j, for all j. Let us consider separately two possible cases: $k \ge N/2$ and $k < N/2$.

The case $k < N/2$. Let Y denote a sum of m independent copies of $K^{1/2}X^\bullet$. Let $Y_1, Y_2, \dots$ be independent copies of Y. Then we have

$$\mathcal{L}(K^{1/2}V) = \mathcal{L}(Y_1 + \dots + Y_l + T_1) \tag{4.9}$$

with $l = \lceil N/(2m)\rceil$ and some random T_1 independent of $Y_1, \dots, Y_l$. By (4.2) and by Lemma 3.3, we have

$$\mathcal{N}(p, \delta, \mathcal{S}, c_0\mathbb{C}^{-1/2}G) \Longrightarrow \mathcal{N}(p/4, 4\delta, \mathcal{S}, \mathbb{C}^{-1/2}\widetilde{Y}) \tag{4.10}$$

provided that

$$\Pi_2^\bullet \ll p/d^3 \qquad \text{and} \qquad m \gg d^6 N\Lambda_4^\bullet/p. \tag{4.11}$$

The inequalities in (4.11) follow from conditions of Lemma 4.1 if we choose some sufficiently small (resp. large) $c_1(d)$ (resp. $c_2(d)$). Due to (4.1), (4.2), (4.9) and (4.10), we can apply Lemma 3.1 in order to estimate the integrals in (4.7) and (4.8). Replacing in Lemma 3.1 X by Y and N by l, we obtain (4.6) in the case $k < N/2$.

The case $k \geq N/2$. We can argue as in the previous case defining now Y as a sum of m independent copies of $K^{1/2}G$. Condition $\mathcal{N}(p/4, 4\delta, \mathcal{S}_o, \mathbb{C}^{-1/2}\widetilde{Y})$ is satisfied by (4.2), since now $\mathcal{L}(\widetilde{Y}) = \mathcal{L}(c_0 G)$. $\square$

Following BG (1997a), introduce the upper bound $\varkappa\bigl(t; N, \mathcal{L}(X), \mathcal{L}(G)\bigr)$ for the characteristic function of quadratic forms (cf. Bentkus (1984) and Bentkus, Götze and Zitikis (1993)). We define $\varkappa\bigl(t; N, \mathcal{L}(X), \mathcal{L}(G)\bigr) = \varkappa\bigl(t; N, \mathcal{L}(X)\bigr) + \varkappa\bigl(t; N, \mathcal{L}(G)\bigr)$, where

$$\varkappa(t; N, \mathcal{L}(X)) = \sup_{x\in\mathbb{R}^d} \bigl|\mathbf{E}\,\mathrm{e}\bigl\{t\mathbb{Q}[Z_j] + \langle x, Z_j\rangle\bigr\}\bigr|, \qquad Z_j = X_1 + \cdots + X_j, \tag{4.12}$$

with $j = \bigl\lceil (N-2)/14 \bigr\rceil$. In the sequel, we shall use that

$$\varkappa(t; N, \mathcal{L}(X'), \mathcal{L}(G)) \leq \varkappa(t; N, \mathcal{L}(X^\bullet), \mathcal{L}(G)). \tag{4.13}$$

For the proof, it suffices to note that $X' = X^\bullet - \mathbf{E}\,X^\bullet + W$ and W is independent of $X^\bullet$.

Lemma 4.2. *Let the conditions of Lemma* 4.1 *be satisfied. Then*

$$\int_{|t|\leq t_1} \bigl(|t|\,K\bigr)^\alpha \varkappa\bigl(tK; N, \mathcal{L}(X^\bullet), \mathcal{L}(G)\bigr)\,\frac{dt}{|t|}$$
$$\ll_{\alpha,d}\ (\det\mathbb{C})^{-1/2} \begin{cases} (Np)^{-\alpha}, & \textit{for } 0 \leq \alpha < d/2, \\ (Np)^{-\alpha}\bigl(1 + |\log(Np/m)|\bigr), & \textit{for } \alpha = d/2, \\ (Np)^{-\alpha}\bigl(1 + (Np/m)^{(2\alpha-d)/d}\bigr), & \textit{for } \alpha > d/2. \end{cases} \tag{4.14}$$

Lemma 4.2 is a generalization of Lemma 3.2 from BG (1997a) which contains the same bound for $0 \leq \alpha < d/2$. In this paper, we have to estimate the left hand side of (4.14) in the case $d/2 \leq \alpha$ too.

Proof. We shall assume again that $(pN)^{-1}m \leq c_3(d)$ with sufficiently small $c_3(d)$ since otherwise (4.14) is an easy consequence of $|\varkappa| \leq 1$.

Note that $\bigl|\mathbf{E}\,\mathrm{e}\bigl\{t\mathbb{Q}[Z_j] + \langle x, Z_j\rangle\bigr\}\bigr| = \bigl|\mathbf{E}\,\mathrm{e}\bigl\{t\,\mathbb{Q}[Z_j - y]\bigr\}\bigr|$ with $y = -\mathbb{Q}x/2$. By (4.2) and (4.10), the condition $\mathcal{N}(p/4, 4\delta, \mathcal{S}_o, K^{1/2}\mathbb{C}^{-1/2}\,\widetilde{Z^\bullet_m})$ is fulfilled. Therefore, collecting independent copies of $K^{1/2}X^\bullet$ in groups as in (4.9), we can apply Lemma 8.1. By (3.4), (4.2) and this lemma, for any $\gamma > 0$ and $|t| \leq t_1$,

$$\varkappa(tK; N, \mathcal{L}(X^\bullet)) \ll_{\gamma,d} (pN/m)^{-\gamma} + \min\bigl\{1;\ (Np/m)^{-d/2}\,|t|^{-d/2}\,(\det\mathbb{C})^{-1/2}\bigr\}.$$

We have used that $\sigma^2 = 1$ implies $\sigma_1^2 \asymp_d 1$. A similar upper bound is valid for the quantity $\varkappa(tK; N, \mathcal{L}(G))$ (cf. the proof of (4.6) for $k > N/2$). Thus, we get, for any $\gamma > 0$ and $|t| \leq t_1$,

$$\varkappa(tK; N, \mathcal{L}(X^\bullet), \mathcal{L}(G)) \ll_{\gamma,d} (pN/m)^{-\gamma} + \min\bigl\{1;\ (\det\mathbb{C})^{-1/2}\bigl(m/(|t|pN)\bigr)^{d/2}\bigr\}.$$

Integrating this bound (cf. the estimation of I_1 in Lemma 3.1), we obtain (4.14). $\square$

5. Proof of Theorem 2.3

To simplify notation, in Section 5 we write $\Pi=\Pi_2^\bullet$ and $\Lambda=\Lambda_4^\bullet$. The assumption $\sigma^2=1$ and equalities $\mathbf{E}\,\|\mathbb{C}^{-1/2}X\|^2=d$, (2.4) and (2.6) imply

$$\Pi+\Lambda\,N\gg 1,\quad \Pi+\Lambda\le 1,\quad \sigma_j^2\le 1,\quad \det\mathbb{C}\le 1. \tag{5.1}$$

Recall that $\Delta_N^{(a)}$ and functions Ψ_b, Φ_b and Θ_b are defined by (2.14) and (2.17)–(2.19). Note now that $\Theta_b^\bullet(x)=E_a^\bullet(x/N)$ and, according to (2.19),

$$\Delta_N^{(a)}\le\Delta_{N,\bullet}^{(a)}+\sup_{x\in\mathbb{R}}\big|\Theta_b(x)-\Theta_b^\bullet(x)\big|, \tag{5.2}$$

where $b=\sqrt{N}\,a$ and

$$\Delta_{N,\bullet}^{(a)}=\sup_{x\in\mathbb{R}}\big|\Psi_b(x)-\Phi_b(x)-\Theta_b^\bullet(x)\big|. \tag{5.3}$$

Let us verify that

$$\sup_{x\in\mathbb{R}}\big|\Theta_b(x)-\Theta_b^\bullet(x)\big|\ll_d\Pi_3^\bullet. \tag{5.4}$$

To this end we shall apply representation (2.14)–(2.15) of the Edgeworth correction as a signed measure and estimate the variation of that measure. Indeed, using (2.14)–(2.15), we have

$$\sup_{x\in\mathbb{R}}\big|\Theta_b(x)-\Theta_b^\bullet(x)\big|\ll N^{-1/2}I,\qquad I\overset{\text{def}}{=}\int_{\mathbb{R}^d}\big|\mathbf{E}\,p'''(x)X^3-\mathbf{E}\,p'''(x)X^{\bullet 3}\big|\,dx. \tag{5.5}$$

By the explicit formula (2.16), the function $u\mapsto p'''(x)u^3$ is a 3-linear form in the variable u. Therefore, using $X=X^\bullet+X_\bullet$ and $\|X^\bullet\|\,\|X_\bullet\|=0$, we have $p'''(x)X^3-p'''(x)X^{\bullet 3}=p'''(x)X_\bullet^3$, and

$$N^{-1/2}I\le 3d^{3/2}\Pi_3^\bullet\int_{\mathbb{R}^d}\big(\|\mathbb{C}^{-1/2}x\|+\|\mathbb{C}^{-1/2}x\|^3\big)\,p(x)\,dx=c_d\,\Pi_3^\bullet. \tag{5.6}$$

Inequalities (5.5) and (5.6) imply now (5.4).

To prove the statement of Theorem 2.3, we have to derive that

$$\Delta_{N,\bullet}^{(a)}\ll_d(\Pi+\Lambda)(1+\|a\|)^3\,(\det\mathbb{C})^{-1/2}. \tag{5.7}$$

While proving (5.7) we assume that

$$\Pi\le c_d,\qquad\text{and}\qquad\Lambda\le c_d, \tag{5.8}$$

with a sufficiently small positive constant c_d depending on d only. These assumptions do not restrict generality. Indeed, we have $\big|\Psi_b(x)-\Phi_b(x)\big|\le 1$. If conditions (5.8) do not hold, then the estimate

$$\sup_{x\in\mathbb{R}}\big|\Theta_b^\bullet(x)\big|\ll_d N^{-1/2}\mathbf{E}\,\|\mathbb{C}^{-1/2}X^\bullet\|^3\ll_d\Lambda^{1/2} \tag{5.9}$$

immediately implies (5.7). In order to prove (5.9) we can use (2.6) and representation (2.14)–(2.15) of the Edgeworth correction. Estimating the variation of that measure and using

$$\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X^{\bullet}\|^2 \le \mathbf{E}\,\|\mathbb{C}^{-1/2}X\|^2 = d, \tag{5.10}$$

$$(\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X^{\bullet}\|^3)^2 \le \mathbf{E}\,\|\mathbb{C}^{-1/2}\,X^{\bullet}\|^2\,\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X^{\bullet}\|^4, \tag{5.11}$$

we obtain (5.9).

It is clear that

$$\Delta_{N,\bullet}^{(a)} \le \sup_{x\in\mathbb{R}} \Big(\big|\Psi_b(x)-\Psi_b'(x)\big| + \big|\Theta_b^{\bullet}(x)-\Theta_b'(x)\big| + \big|\Psi_b'(x)-\Phi_b(x)-\Theta_b'(x)\big|\Big). \tag{5.12}$$

Similarly to (5.5), we have

$$\sup_{x\in\mathbb{R}} \big|\Theta_b^{\bullet}(x)-\Theta_b'(x)\big| \ll N^{-1/2}\,J, \qquad J \overset{\text{def}}{=} \int_{\mathbb{R}^d} \big|\mathbf{E}\,p'''(x)X^{\bullet 3} - \mathbf{E}\,p'''(x)X'^3\big|\,dx. \tag{5.13}$$

Recall that vector X' is defined in (2.10). By Lemma 3.2, we have $\mathbf{E}\,\|\mathbb{C}^{-1/2}\,W\|^2 \le 2d\,\Pi$ (hence, $\mathbf{E}\,\|\mathbb{C}^{-1/2}\,W\|^q \ll_d \Pi^{q/2}$, for $0 \le q \le 2$). Moreover, representing W as a sum of a large number of i.i.d. Gaussian summands and using the Rosenthal inequality (see BG (1997a, inequality (1.24)), we conclude that

$$\mathbf{E}\,\|\mathbb{C}^{-1/2}\,W\|^q \ll_q \big(\mathbf{E}\,\|\mathbb{C}^{-1/2}\,W\|^2\big)^{q/2} \ll_{q,d} \Pi^{q/2}, \quad q \ge 0. \tag{5.14}$$

Furthermore, according to (2.4), (2.6) and (5.8),

$$\mathbf{E}\,\|\mathbb{C}^{-1/2}\,X_{\bullet}\| \ll_d \Pi\,N^{-1/2} \ll_d \Pi^{1/2}\,N^{-1/2}. \tag{5.15}$$

Hence, by (2.6), (2.10), (5.1), (5.14) and (5.15),

$$\mathbf{E}\,\|X'\|^4 \ll \overline{\beta} \overset{\text{def}}{=} \mathbf{E}\,\|\mathbb{C}^{-1/2}\,X'\|^4 \ll_d N\Lambda + \Pi^2. \tag{5.16}$$

Using (2.16), (5.1), (5.8), (5.10) and (5.13)–(5.15), we get

$$\begin{aligned} N^{-1/2}\,J &\ll_d \Pi^{1/2}(N^{-1/2}\Pi + \Lambda^{1/2}) \int_{\mathbb{R}^d} \big(\|\mathbb{C}^{-1/2}x\| + \|\mathbb{C}^{-1/2}x\|^3\big)\,p(x)\,dx \\ &\ll_d \Pi + \Lambda. \end{aligned} \tag{5.17}$$

Thus, according to (5.13) and (5.17),

$$\sup_{x\in\mathbb{R}} \big|\Theta_b^{\bullet}(x)-\Theta_b'(x)\big| \ll_d \Pi + \Lambda. \tag{5.18}$$

The same approach is applicable for the estimation of $\big|\Theta_b'\big|$. Using (2.10), (2.14)–(2.16), (5.1), (5.10), (5.11), (5.14) and (5.15), we get

$$\begin{aligned} \sup_{x\in\mathbb{R}} \big|\Theta_b'(x)\big| &\ll N^{-1/2} \int_{\mathbb{R}^d} \big|\mathbf{E}\,p'''(x)X'^3\big|\,dx \\ &\ll_d \Lambda^{1/2} + N^{-1/2}\,\Pi^{3/2}. \end{aligned} \tag{5.19}$$

Let us prove that

$$\sup_{x\in\mathbb{R}} \bigl|\Psi_b(x) - \Psi_b'(x)\bigr| \ll (\det \mathbb{C})^{-1/2}\, p^{-2}\, (\Pi + \Lambda)(1 + \|a\|^2). \tag{5.20}$$

Using truncation (see (3.11)), we have $|\Psi_b - \Psi_b^\bullet| \le \Pi$, and

$$\sup_{x\in\mathbb{R}} \bigl|\Psi_b(x) - \Psi_b'(x)\bigr| \le \Pi + \sup_{x\in\mathbb{R}} \bigl|\Psi_b^\bullet(x) - \Psi_b'(x)\bigr|. \tag{5.21}$$

In order to estimate $|\Psi_b^\bullet - \Psi_b'|$, we shall apply Lemmas 4.1 and 4.2. The number m in these Lemmas exists and $N\Lambda/p \gg_d 1$, as it follows from (5.1) and (5.8). Let us choose the minimal m, that is, $m \asymp_d N\Lambda/p$. Then $(pN)^{-1} m \ll_d \Lambda/p^2$ and $m/N \ll_d \Lambda/p$. Therefore, using Lemma 4.1, we have

$$\sup_x \bigl|\Psi_b^\bullet(x) - \Psi_b'(x)\bigr| \ll_d p^{-2}\Lambda\,(\det \mathbb{C})^{-1/2} + \int_{|t|\le t_1} \bigl|\widehat\Psi_b^\bullet(\tau) - \widehat\Psi_b'(\tau)\bigr| \,\frac{dt}{|t|}\,, \qquad \tau = tK. \tag{5.22}$$

We shall prove that

$$\bigl|\widehat\Psi_b^\bullet(\tau) - \widehat\Psi_b'(\tau)\bigr| \ll_d \varkappa\, \Pi\, |\tau|\, N\, (1 + |\tau| N)(1 + \|a\|^2), \tag{5.23}$$

with $\varkappa = \varkappa(\tau; N, \mathcal{L}(X^\bullet))$. Combining (5.21)–(5.23), using $\tau = tK$ and integrating inequality (5.23) with the help of Lemma 4.2, we derive (5.20).

Let us prove (5.23). Recall that $X' = X^\bullet - \mathbf{E}\, X^\bullet + W$, where W denotes a centered Gaussian random vector which is independent of all other random vectors and such that $\operatorname{cov} X' = \operatorname{cov} G$ (see Lemma 3.2). Writing $D = Z_N^\bullet - \mathbf{E}\, Z_N^\bullet - b$, we have

$$Z_N^\bullet - b = D + \mathbf{E}\, Z_N^\bullet, \qquad \mathcal{L}(Z_N' - b) = \mathcal{L}(D + \sqrt{N} W),$$

and

$$\bigl|\widehat\Psi_b^\bullet(\tau) - \widehat\Psi_b'(\tau)\bigr| \le \bigl|f_1(\tau)\bigr| + \bigl|f_2(\tau)\bigr| \tag{5.24}$$

with

$$\begin{aligned} f_1(\tau) &= \mathbf{E}\, \mathrm{e}\bigl\{\tau \mathbb{Q}[D + \sqrt{N} W]\bigr\} - \mathbf{E}\, \mathrm{e}\bigl\{\tau \mathbb{Q}[D]\bigr\},\\ f_2(t) &= \mathbf{E}\, \mathrm{e}\bigl\{\tau \mathbb{Q}[D + \mathbf{E}\, Z_N^\bullet]\bigr\} - \mathbf{E}\, \mathrm{e}\bigl\{\tau \mathbb{Q}[D]\bigr\}. \end{aligned} \tag{5.25}$$

Now we have to prove that both $\bigl|f_1(\tau)\bigr|$ and $\bigl|f_2(\tau)\bigr|$ may be estimated by the right hand side of (5.23).

Let us consider f_1. We can write $\mathbb{Q}[D + \sqrt{N} W] = \mathbb{Q}[D] + A + B$ with $A = 2\sqrt{N}\langle \mathbb{Q} D, W\rangle$ and $B = N\mathbb{Q}[W]\bigr\}$. Taylor's expansions of the exponent in (5.25) in powers of $i\tau B$ and $i\tau A$ with remainders $\mathcal{O}(\tau B)$ and $\mathcal{O}(\tau^2 A^2)$ respectively imply (recall that $\mathbf{E}\, W = 0$ and $\mathbb{Q}^2 = \mathbb{I}_d$)

$$\bigl|f_1(\tau)\bigr| \ll \varkappa |\tau| N\, \mathbf{E}\, \|W\|^2 + \varkappa \tau^2 N\, \mathbf{E}\, \|W\|^2\, \mathbf{E}\, \|D\|^2, \tag{5.26}$$

where $\varkappa = \varkappa(\tau; N, \mathcal{L}(X^\bullet))$. The estimation of the remainders of these expansions is based on the splitting and conditioning techniques described in Section 9 of BG (1997a), see also Bentkus, Götze and Zaitsev (1997). Using the relations $\sigma^2 = 1$, $\mathbf{E}\,\|W\|^2 \ll \mathbf{E}\,\|\mathbb{C}^{-1/2}W\|^2 \ll_d \Pi$ and $\mathbf{E}\,\|D\|^2 \ll N(1+\|a\|^2)$, we derive from (5.26) that

$$\big|\,f_1(\tau)\,\big| \ll_d \varkappa\Pi\,|\tau|\,N\,\big(1+|\tau|\,N)\big)(1+\|a\|^2). \tag{5.27}$$

Note that $\mathbf{E}\,Z_N^\bullet = N\,\mathbf{E}\,X^\bullet = -N\,\mathbf{E}\,X_\bullet$. Expanding the exponent $\mathrm{e}\big\{\tau\mathbb{Q}[D+\mathbf{E}\,Z_N^\bullet]\big\}$, using (5.15) and proceeding similarly to the proof of (5.27), we obtain

$$\big|\,f_2(\tau)\,\big| \ll_d \varkappa\Pi\,|\tau|\,N(1+\|a\|). \tag{5.28}$$

Inequalities (5.24), (5.27) and (5.28) imply now (5.23).

It remains to estimate $\big|\,\Psi_b'-\Phi_b-\Theta_b'\,\big|$. Recall that the distribution functions $\Psi_b^{(l)}(x)$, for $0\le l\le N$, are defined in (4.5).

Fix an integer k, $1\le k\le N$. Clearly, we have

$$\sup_{x\in\mathbb{R}}\big|\,\Psi_b'(x)-\Phi_b(x)-\Theta_b'(x)\,\big| \le I_1+I_2+I_3, \tag{5.29}$$

where

$$I_1 = \sup_{x\in\mathbb{R}}\big|\,\Psi_b^{(k)}(x)-\Phi_b(x)-(N-k)\,\Theta_b'(x)/N\,\big|, \tag{5.30}$$

$$I_2 = \sup_{x\in\mathbb{R}}\big|\,\Psi_b'(x)-\Psi_b^{(k)}(x)\,\big|, \tag{5.31}$$

and

$$I_3 = \sup_{x\in\mathbb{R}}\ k\,N^{-1}\,\big|\,\Theta_b'(x)\,\big|. \tag{5.32}$$

Let estimate I_1. Define the distributions

$$\mu(A) = \mathbf{P}\Big\{U_k+\sum_{j=k+1}^{N}X_j'\in\sqrt{N}\,A\Big\}, \qquad \mu_0(A) = \mathbf{P}\big\{U_N\in\sqrt{N}\,A\big\} = \mathbf{P}\big\{G\in A\big\}, \tag{5.33}$$

where $U_l = G_1+\dots+G_l$. Introduce the measure χ' replacing X by X' in (2.15). For the Borel sets $A\subset\mathbb{R}^d$ define the Edgeworth correction (to the distribution μ) as

$$\mu_1^{(k)}(A) = (N-k)\,N^{-3/2}\chi'(A)/6. \tag{5.34}$$

Introduce the signed measure

$$\nu = \mu-\mu_0-\mu_1^{(k)}. \tag{5.35}$$

It is easy to see that a re-normalization of random vectors implies (see relations (2.14), (2.17)– (2.19), (4.5) and (5.33)–(5.35))

$$\begin{aligned}\big|\,\Psi_b^{(k)}(x)-\Phi_b(x)-(N-k)\,\Theta_b'(x)/N\,\big| &= \nu\big(\big\{u\in\mathbb{R}^d:\mathbb{Q}[u-a]\le x/N\big\}\big)\\ &\le \delta_N \overset{\mathrm{def}}{=} \sup_{A\subset\mathbb{R}^d}\big|\,\nu(A)\,\big|.\end{aligned} \tag{5.36}$$

Lemma 5.1. *Assume that $d < \infty$ and $1 \le k \le N$. Then there exists a $c(d)$ depending on d only and such that δ_N defined in* (5.36) *satisfies the inequality*

$$\delta_N \ll_d \frac{\overline{\beta}}{N} + \frac{N^{d/2}}{k^{d/2}} \exp\bigl\{-c(d)\, k/\overline{\beta}\bigr\} \tag{5.37}$$

with $\overline{\beta} = \mathbf{E}\,\|\mathbb{C}^{-1/2}X'\|^4$.

An outline of the proof. We repeat and slightly improve the proof of Lemma 9.4 in BG (1997a) (cf. the proof of Lemma 2.5 in BG (1996)). We shall prove (5.37) assuming that $\operatorname{cov} X = \operatorname{cov} X' = \operatorname{cov} G = \mathbb{I}_d$, Applying it to $\mathbb{C}^{-1/2}X'$ and $\mathbb{C}^{-1/2}G$, we obtain (5.37) in general case.

While proving (5.37) we assume that $\overline{\beta}/N \le c_d$ and $N \ge 1/c_d$ with a sufficiently small positive constant c_d. Otherwise (5.37) follows from the obvious bounds $\overline{\beta} \ge \sigma^4 = d^2$ and

$$\delta_N \ll_d 1 + (\overline{\beta}/N)^{1/2} \int_{\mathbb{R}^d} \|x\|^3 p(x)\, dx \ll_d 1 + (\overline{\beta}/N)^{1/2}.$$

Set $n = N - k$. Denoting by Z'_j and U'_j sums of j independent copies of X' and G' respectively, introduce the multidimensional characteristic functions

$$g(t) = \mathbf{E}\ \mathrm{e}\bigl\{\langle N^{-1/2}t, G\rangle\bigr\}, \quad h(t) = \mathbf{E}\ \mathrm{e}\bigl\{\langle N^{-1/2}t, X'\rangle\bigr\}, \tag{5.38}$$

$$f(t) = \mathbf{E}\ \mathrm{e}\bigl\{\langle N^{-1/2}t, Z'_n\rangle\bigr\} = h^n(t), \qquad f_0(t) = \mathbf{E}\ \mathrm{e}\bigl\{\langle N^{-1/2}t, U'_n\rangle\bigr\} = g^n(t), \tag{5.39}$$

$$f_1(t) = n\, m(t)\, f_0(t), \quad \text{where} \quad m(t) = \frac{1}{6\, N^{3/2}}\, \mathbf{E}\, \langle it, X'\rangle^3, \tag{5.40}$$

$$\widehat{\nu}(t) = (f(t) - f_0(t) - f_1(t))\, g(\rho t), \quad \rho^2 = k. \tag{5.41}$$

It is easy to see that

$$\widehat{\nu}(t) = \int_{\mathbb{R}^d} \mathrm{e}\{\langle t, x\rangle\}\, \nu(dx). \tag{5.42}$$

Using the truncation, we obtain

$$\mathbf{E}\, \bigl\| Z'_l/\sqrt{N}\bigr\|^{\gamma} \ll_{\gamma,d} 1, \qquad \gamma > 0, \quad 1 \le l \le N. \tag{5.43}$$

By an extension of the proof of Lemma 11.6 in Bhattacharya and Rao (1986), see also the proof of Lemma 2.5 in BG (1996), we obtain

$$\delta_N \ll_d \max_{|\alpha| \le 2d} \int_{t\in\mathbb{R}^d} \bigl|\partial^{\alpha}\widehat{\nu}(t)\bigr|\, dt. \tag{5.44}$$

Here $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_j \in \mathbb{Z}$, $\alpha_j \ge 0$. In order to derive (5.37) from (5.44), it suffices to prove that, for $|\alpha| \le 2d$,

$$\bigl|\partial^{\alpha}\widehat{\nu}(t)\bigr| \ll_d g(c_1 \rho t), \tag{5.45}$$

$$\bigl|\partial^{\alpha}\widehat{\nu}(t)\bigr| \ll_d \overline{\beta} N^{-1} (1 + \|t\|^6) \exp\{-c_2\, \|t\|^2\}, \quad \text{for } \|t\|^2 \le c_3(d) N/\overline{\beta}. \tag{5.46}$$

Indeed, using (5.45) and denoting $T=\sqrt{c_3(d)\,N/\overline{\beta}}$, we obtain

$$\int\limits_{\|t\|\geq T}\big|\,\partial^\alpha\widehat{\nu}(t)\,\big|\,dt\ll_d\int\limits_{\|t\|\geq T}g(c_1\,\rho t)\,dt\ll_d\frac{N^{d/2}}{\rho^d}\,\exp\Big\{-\frac{c_1^2\rho^2T^2}{8N}\Big\}\int\limits_{\mathbb{R}^d}\exp\{-c_1^2\,\|t\|^2/8\}\,dt,\tag{5.47}$$

and it is easy to see that the right hand side of (5.47) is bounded from above by the second summand in the right hand side of (5.37). Similarly, using (5.46), we can integrate $\big|\,\partial^\alpha\widehat{\nu}(t)\,\big|$ over $\|t\|\leq T$, and the integral is bounded from above by $c_d\,\overline{\beta}/N$.

In the proof of (5.45)–(5.47) we applied standard methods of estimation which are provided in Bhattacharya and Rao (1986). In particular, we used a Bergström type identity

$$f-f_0-f_1=\sum_{j=0}^{n-1}(h-g-m)\,h^j\,g^{n-j-1}+\sum_{j=0}^{n-1}m\sum_{l=0}^{j-1}(h-g)\,h^l\,g^{n-l-1},\tag{5.48}$$

relations (5.38)–(5.43), $1\leq k\leq N$, $\big|\,\partial^\alpha\exp\{-c_4\,\|t\|^2\}\,\big|\ll_\alpha\exp\{-c_5\,\|t\|^2\}$, $\sqrt{N}/\overline{\beta}^{1/2}\gg_d 1$ and $y^{c_d}\exp\{-y\}\ll_d 1$, for $y>0$. $\square$

Applying (5.30), (5.36) and Lemma 5.1, we get

$$I_1\ll_d\frac{\overline{\beta}}{N}+\frac{N^{d/2}}{k^{d/2}}\exp\big\{-c(d)\,k/\overline{\beta}\big\}.\tag{5.49}$$

For the estimation of I_2 we shall use Lemma 5.2 which is an easy consequence of BG (1997a, Lemma 9.3), (4.13) and (5.16).

Lemma 5.2. *We have*

$$\big|\,\widehat{\Psi}'_b(t)-\widehat{\Psi}_b^{(l)}(t)\,\big|\ll\varkappa t^2\,l\,\big(\overline{\beta}+|t|\,N\,\overline{\beta}+|t|\,N\sqrt{N\,\overline{\beta}}\big)(1+\|a\|^3),\quad\textit{for}\ \ 0\leq l\leq N,$$

where $\varkappa=\varkappa(t;N,\mathcal{L}(X^\bullet),\mathcal{L}(G))$ (*cf.* (4.12)).

As in the proof of (5.22), applying Lemma 4.1 (choosing $m\asymp_d N(\Lambda+\Pi)/p$) and using (4.2), we obtain

$$I_2\ll_d(\Lambda+\Pi)\,(\det\mathbb{C})^{-1/2}+\int_{|t|\leq t_1}\big|\,\widehat{\Psi}'_b(\tau)-\widehat{\Psi}_b^{(k)}(\tau)\,\big|\,dt/|t|,\qquad\tau=t\,K.$$

The existence of such an m is ensured by (4.2), (5.1) and (5.8), Applying Lemma 5.2 and replacing in that Lemma t by τ, we have

$$\big|\,\widehat{\Psi}'_b(\tau)-\widehat{\Psi}_b^{(k)}(\tau)\,\big|\ll\varkappa\tau^2\,k\,\big(\overline{\beta}+|\tau|\,N\,\overline{\beta}+|\tau|\,N\sqrt{N\,\overline{\beta}}\big)(1+\|a\|^3).\tag{5.50}$$

Integrating with the help of Lemma 4.2 and using (4.2), we obtain

$$I_2\ll_d(\det\mathbb{C})^{-1/2}\big(\Pi+\Lambda+k\,N^{-2}\big(\overline{\beta}+\sqrt{N\,\overline{\beta}}\big)\big(1+(\Pi+\Lambda)^{-1/d}\big)(1+\|a\|^3)\big).\tag{5.51}$$

Let us choose $k \asymp_d N^{1/4}\overline{\beta}^{3/4}$. Such $k \le N$ exists by $\overline{\beta} \gg_d \sigma^4 = 1$, by (5.16) and by assumption (5.8). Then (5.49) and (5.51) turn into

$$I_1 \ll_d \frac{\overline{\beta}}{N} + \Big(\frac{N}{\overline{\beta}}\Big)^{3d/8} \exp\Big\{-c_d\Big(\frac{N}{\overline{\beta}}\Big)^{1/4}\Big\} \ll_d \frac{\overline{\beta}}{N}, \tag{5.52}$$

and

$$I_2 \ll_d (\det\mathbb{C})^{-1/2}\Big(\Pi + \Lambda + \Big(\Big(\frac{\overline{\beta}}{N}\Big)^{5/4} + \Big(\frac{\overline{\beta}}{N}\Big)^{7/4}\Big)\big(1 + (\Pi+\Lambda)^{-1/d}\big)(1 + \|a\|^3)\Big). \tag{5.53}$$

Using (4.2), (5.8), (5.16) and (5.53), we get

$$I_2 \ll_d (\det\mathbb{C})^{-1/2}\Big(\Pi + \Lambda + \frac{\overline{\beta}}{N}(1 + \|a\|^3)\Big). \tag{5.54}$$

Finally, by (5.8), (5.16), (5.19) and (5.32),

$$I_3 \ll_d \frac{k}{N}\big(\Lambda^{1/2} + N^{-1/2}\Pi^{3/2}\big) \ll \Lambda + \Pi. \tag{5.55}$$

Inequalities (5.8), (5.12), (5.16), (5.18), (5.20), (5.29), (5.52), (5.54) and (5.55) imply now (5.7) (and, hence, (2.25)) by an application of $\Pi + \Lambda \le 1$. Note that, by (2.6), we have $\Pi \le \Pi_3^\bullet$. Together with (5.2) and (5.4), inequality (5.7) yields (2.24). The statement of Theorem 2.3 is proved. $\square$

6. From Probability to Number Theory

In Section 6 we shall reduce the estimation of the integrals of the modulus of characteristic functions $\widehat{\Psi}_b(t)$ to the estimation the integrals of some theta-series. We shall use the following lemmas.

Lemma 6.1. (BG (1997a, Lemma 5.1)) *Let $L, C \in \mathbb{R}^d$ and let $\mathbb{Q} : \mathbb{R}^d \to \mathbb{R}^d$ be a symmetric linear operator. Let Z, U, V and W denote independent random vectors taking values in $\mathbb{R}^d$. Denote by*

$$P(x) = \langle \mathbb{Q}x, x\rangle + \langle L, x\rangle + C, \qquad x \in \mathbb{R}^d,$$

a real-valued polynomial of second order. Then

$$2\Big|\mathbf{E}\,\mathrm{e}\big\{t\,P(Z+U+V+W)\big\}\Big|^2 \le \mathbf{E}\,\mathrm{e}\big\{2\,t\,\langle\mathbb{Q}\widetilde{Z}, \widetilde{U}\rangle\big\} + \mathbf{E}\,\mathrm{e}\big\{2\,t\,\langle\mathbb{Q}\widetilde{Z}, \widetilde{V}\rangle\big\}.$$

Let $\varepsilon_1, \varepsilon_2, \dots$ denote i.i.d. symmetric Rademacher random variables. Let $\delta > 0$, $\mathcal{S} = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$ and let $\mathbb{D} : \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator. Usually, we shall take $\mathbb{D} = \mathbb{C}^{-1/2}$. We shall write $\mathcal{L}(Y) \in \mathbf{\Gamma}(\delta; \mathbb{D}, \mathcal{S})$ if a discrete random vector Y is distributed as $\varepsilon_1 z_1 + \dots + \varepsilon_s z_s$, with some (non-random) $z_j \in \mathbb{R}^d$ such that $\|\mathbb{D}z_j - e_j\| \le \delta$, for all $1 \le j \le s$. Recall that $\mathcal{S}_o = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$ denotes an orthonormal system.

Lemma 6.2. *Assume that* $\mathbb{Q}^2 = \mathbb{I}_d$ *and that the condition* $\mathcal{N}(p, \delta, \mathcal{S}, \mathbb{D}\widetilde{X})$ *holds with some* $0 < p \leq 1$ *and* $\delta > 0$. *Write* $m = \lceil pN/(5s) \rceil$. *Then, for any* $0 < A \leq B$, $b \in \mathbb{R}^d$ *and* $\gamma > 0$, *we have*

$$\int_A^B \bigl| \widehat{\Psi}_b(t) \bigr| \, \frac{dt}{|t|} \leq I + c_\gamma(s)\,(pN)^{-\gamma} \log \frac{B}{A}\,, \tag{6.1}$$

with

$$I = \sup_\Gamma \sup_{b \in \mathbb{R}^d} \int_A^B \sqrt{\varphi(t/4)}\, \frac{dt}{|t|}\,, \qquad \varphi(t) \overset{\mathrm{def}}{=} \Bigl| \mathbf{E}\, \mathrm{e}\bigl\{ t\, \mathbb{Q}[Y + b] \bigr\} \Bigr|^2, \tag{6.2}$$

where $Y = U_1 + \cdots + U_m$ *denote a sum of independent* (*non-i.i.d.*) *vectors, and* $\sup\limits_\Gamma$ *is taken over all* $\bigl\{ \mathcal{L}(U_j) : \ 1 \leq j \leq m \bigr\} \subset \mathbf{\Gamma}(\delta; \mathbb{D}, \mathcal{S})$.

Lemma 6.2 is an analogue of Corollary 6.3 from BG (1997a). Its proof is even simpler than that in BG (1997a). Therefore it is omitted.

Lemma 6.3. *Assume that* $\mathbb{Q}^2 = \mathbb{I}_d$ *and that the condition* $\mathcal{N}(p, \delta, \mathcal{S}, \mathbb{D}\widetilde{X})$ *holds with some* $0 < p \leq 1$ *and* $\delta > 0$. *Let*

$$n \overset{\mathrm{def}}{=} \lceil pN/(16s) \rceil \geq 1. \tag{6.3}$$

Then, for any $0 < A \leq B$, $b \in \mathbb{R}^d$ *and* $\gamma > 0$,

$$\int_A^B \bigl| \widehat{\Psi}_b(t) \bigr| \, \frac{dt}{|t|} \leq c_\gamma(s)\,(pN)^{-\gamma} \log \frac{B}{A} + \sup_\Gamma \int_A^B \sqrt{\mathbf{E}\, \mathrm{e}\bigl\{ t\, \langle \mathbb{Q}\widetilde{W}, \widetilde{W}' \rangle /2 \bigr\}}\, \frac{dt}{|t|}\,, \tag{6.4}$$

and for any fixed $t \in \mathbb{R}$,

$$\bigl| \widehat{\Psi}_b(t) \bigr| \leq c_\gamma(s)\,(pN)^{-\gamma} + \sup_\Gamma \sqrt{\mathbf{E}\, \mathrm{e}\bigl\{ t\, \langle \mathbb{Q}\widetilde{W}, \widetilde{W}' \rangle /2 \bigr\}}, \tag{6.5}$$

where $W = V_1 + \cdots + V_n$ *and* $W' = V_1' + \cdots + V_n'$ *are independent sums of independent copies of random vectors* V *and* V' *respectively, and the supremum* $\sup_\Gamma$ *is taken over all* $\mathcal{L}(V), \mathcal{L}(V') \in \mathbf{\Gamma}(\delta; \mathbb{D}, \mathcal{S})$.

Note that this lemma will be proved for general $\mathcal{S}$, but in this paper we need $\mathcal{S} = \mathcal{S}_o$ only. Moreover, a more careful estimation of binomial probabilities could allow us to replace $c_\gamma(s)\,(pN)^{-\gamma}$ in (6.1), (6.4) and (6.5) by $c(s)\,\exp\bigl\{-cpN\bigr\}$ (see e.g. Nagaev and Chebotarev (2005)). However, we do not need to use this improvement.

Proof of Lemma 6.3. Inequality (6.5) is an analogue of the statement of Lemma 7.3 from BG (1997a). Its proof is even simpler than that in BG (1997a). Therefore it is omitted.

Let us show that

$$\int_A^B \big|\,\widehat{\Psi}_b(t)\big|\,\frac{dt}{|t|} \le c_\gamma(s)\,(pN)^{-\gamma}\,\log\frac{B}{A} + \sup_{\Gamma}\int_A^B \sqrt{\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{W},\widetilde{W'}\rangle/2\big\}}\,\frac{dt}{|t|}\,, \tag{6.6}$$

where $W = V_1 + \dots + V_n$ and $W' = V_1' + \dots + V_n'$ are independent sums of of independent (*non-i.i.d.*) vectors, and $\sup\limits_{\Gamma}$ is taken over all $\big\{\mathcal{L}(V_j), \mathcal{L}(V_j') :\ 1 \le j \le n\big\} \subset \mathbf{\Gamma}(\delta; \mathbb{D}, \mathcal{S})$. Comparing (6.4) and (6.6), we see that inequality (6.6) is related to sums of *non-i.i.d.* vectors $\{V_j\}$ and $\{V_j'\}$ while inequality (6.4) deals with i.i.d. vectors. Nevertheless, we shall derive (6.4) from (6.6).

While proving (6.6) we can assume that $pN \ge c_s$ with a sufficiently large constant c_s, since otherwise (6.6) is obviously valid.

Let $\varphi(t)$ be defined in (6.2), where $Y = U_1 + \dots + U_m$ denote a sum of independent (non-i.i.d.) vectors with $\big\{\mathcal{L}(U_j) :\ 1 \le j \le m\big\} \subset \mathbf{\Gamma}(\delta; \mathbb{D}, \mathcal{S})$, $m = \big\lceil pN/(5\,s)\big\rceil$.

We shall apply the symmetrization Lemma 6.1. Split $Y = T + T_1 + T_2$ into sums of independent sums of independent summands so that each of the sums T, T_1 and T_2 contains $n = \lceil pN/(16\,s)\rceil$ independent summands U_j. Such an n exists since $pN \ge c_s$ with a sufficiently large c_s. Lemma 6.1 implies that

$$2\,\varphi(t) \le \mathbf{E}\,\mathrm{e}\big\{2\,t\,\langle\mathbb{Q}\,\widetilde{T}, \widetilde{T}_1\rangle\big\} + \mathbf{E}\,\mathrm{e}\big\{2\,t\,\langle\mathbb{Q}\,\widetilde{T}, \widetilde{T}_2\rangle\big\}. \tag{6.7}$$

Inequality (6.6) follows now from (6.7) and Lemma 6.2.

Let now $W = V_1 + \dots + V_n$ and $W' = V_1' + \dots + V_n'$ be independent sums of of independent (non-i.i.d.) vectors with $\big\{\mathcal{L}(V_j), \mathcal{L}(V_j') :\ 1 \le j \le n\big\} \subset \mathbf{\Gamma}(\delta; \mathbb{D}, \mathcal{S})$. Using that all random vectors $\widetilde{V}_j$ are symmetrized and have non-negative characteristic functions and applying Hölder's inequality, we obtain, for each t,

$$\begin{aligned}
\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{W},\widetilde{W'}\rangle\big\} &= \mathbf{E}_{\widetilde{W'}}\Big(\prod_{j=1}^{n}\mathbf{E}_{\widetilde{V}_j}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{V}_j,\widetilde{W'}\rangle\,\big\}\Big) && (6.8)\\
&\le \Big(\prod_{j=1}^{n}\mathbf{E}_{\widetilde{W'}}\big(\mathbf{E}_{\widetilde{V}_j}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{V}_j,\widetilde{W'}\rangle\big\}\big)^{n}\Big)^{1/n} && (6.9)\\
&= \Big(\prod_{j=1}^{n}\mathbf{E}_{\widetilde{W'}}\big(\mathbf{E}_{\widetilde{T}_j}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{T}_j,\widetilde{W'}\rangle\big\}\big)\Big)^{1/n} && (6.10)\\
&= \Big(\prod_{j=1}^{n}\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{T}_j,\widetilde{W'}\rangle\big\}\Big)^{1/n}, && (6.11)
\end{aligned}$$

where $\widetilde{T}_j \overset{\mathrm{def}}{=} \sum_{l=1}^{n}\widetilde{V}_{jl}$ denotes a sum of i.i.d. copies $\widetilde{V}_{jl}$ of $\widetilde{V}_j$ which are independent of all other random vectors and variables.

Repeating the steps (6.8)–(6.11) for each factor $\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{T}_j,\widetilde{W'}\rangle\big\}$ instead of the expectation $\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\,\widetilde{W},\widetilde{W'}\rangle\big\}$ on the right hand side separately, we get (with $\widetilde{T}_k' \overset{\mathrm{def}}{=} \sum_{l=1}^{n}\widetilde{V}_{kl}'$, where $\widetilde{V}'_{kl}$ are i.i.d. copies of $\widetilde{V}'_k$ independent of all other random vectors)

$$\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\widetilde{W},\widetilde{W}'\rangle\big\} \le \Big(\prod_{j=1}^{n}\prod_{k=1}^{n}\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\widetilde{T}_j,\widetilde{T}'_k\rangle\big\}\Big)^{1/n^2}. \tag{6.12}$$

Thus, using (6.12) and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}\int_A^B \sqrt{\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\widetilde{W},\widetilde{W}'\rangle/2\big\}}\,\frac{dt}{|t|} &\le \int_A^B \Big(\prod_{j=1}^{n}\prod_{k=1}^{n}\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\widetilde{T}_j,\widetilde{T}'_k\rangle/2\big\}\Big)^{1/2n^2}\,\frac{dt}{|t|}\\ &\le \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_A^B\Big(\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\widetilde{T}_j,\widetilde{T}'_k\rangle/2\big\}\Big)^{1/2}\,\frac{dt}{|t|}\\ &\le \sup_{\Gamma}\int_A^B\sqrt{\mathbf{E}\,\mathrm{e}\big\{t\,\langle\mathbb{Q}\widetilde{T},\widetilde{T}'\rangle/2\big\}}\,\frac{dt}{|t|},\end{aligned} \tag{6.13}$$

where $T = U_1+\cdots+U_n$ and $T' = U'_1+\cdots+U'_n$ are independent sums of independent copies of random vectors U and U' respectively, and the supremum $\sup_\Gamma$ is taken over all $\mathcal{L}(U),\mathcal{L}(U')\in\mathbf{\Gamma}(\delta;\mathbb{D},\mathcal{S})$. Inequalities (6.6) and (6.13) imply now the statement of the lemma. □

The following Lemma 6.4 provides a Poisson summation formula.

Lemma 6.4. *Let* $\operatorname{Re} z>0$, $a,b\in\mathbb{R}^s$ *and* $\mathbb{S}:\mathbb{R}^s\to\mathbb{R}^s$ *be a positive definite symmetric non-degenerate linear operator. Then*

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^s}\exp\big\{-z\,\mathbb{S}[m+a]+2\pi i\,\langle m,b\rangle\big\}\\ &= \big(\det(\mathbb{S}/\pi)\big)^{-1/2}z^{-s/2}\exp\big\{-2\pi i\,\langle a,b\rangle\big\}\sum_{l\in\mathbb{Z}^s}\exp\Big\{-\frac{\pi^2}{z}\,\mathbb{S}^{-1}[l+b]-2\pi i\,\langle a,l\rangle\Big\},\end{aligned}$$

where $\mathbb{S}^{-1}:\mathbb{R}^s\to\mathbb{R}^s$ *denotes the inverse positive definite operator for* $\mathbb{S}$.

Proof. See, for example, Fricker (1982), p. 116, or Mumford (1983), p. 189, formula (5.1); and p. 197, formula (5.9). □

Let the conditions of Lemma 6.3 be satisfied. Introduce one-dimensional lattice probability distributions $H_n=\mathcal{L}(\xi_n)$ with integer valued ξ_n setting

$$\mathbf{P}\big\{\xi_n=k\big\}=A_n\,n^{-1/2}\,\exp\big\{-k^2/2n\big\},\quad\text{for } k\in\mathbb{Z}.$$

It is easy to see that $A_n\asymp 1$. Moreover, by Lemma 6.4,

$$\widehat{H}_n(t)\ge 0,\qquad\text{for all } t\in\mathbb{R}. \tag{6.14}$$

Introduce the s-dimensional random vector ζ_n having as coordinates independent copies of ξ_n. Then, for $m = (m_1, \ldots, m_s) \in \mathbb{Z}^s$, we have

$$q(m) \overset{\text{def}}{=} \mathbf{P}\big\{ \zeta_n = m \big\} = A_n^s \, n^{-s/2} \exp\big\{ -\|m\|^2/2n \big\} . \tag{6.15}$$

Lemma 6.5. *Let $W = V_1 + \cdots + V_n$ and $W' = V_1' + \cdots + V_n'$ denote independent sums of independent copies of random vectors V and V' such that*

$$V = \varepsilon_1 z_1 + \cdots + \varepsilon_s z_s, \qquad V' = \varepsilon_{s+1} z_1' + \cdots + \varepsilon_{2s} z_s',$$

with some $z_j, z_j' \in \mathbb{R}^d$. Introduce the matrix $\mathbb{B}_t = \{ b_{ij}(t) : 1 \le i, j \le s \}$ with $b_{ij}(t) = t \, \langle \mathbb{Q} z_i, z_j' \rangle$. Then

$$\mathbf{E} \, \mathrm{e}\big\{ t \, \langle \mathbb{Q} \widetilde{W}, \widetilde{W}' \rangle / 4 \big\} \ll_s \mathbf{E} \, \mathrm{e}\big\{ \langle \mathbb{B}_t \zeta_n, \zeta_n' \rangle \big\} + \exp\{ -cn \} , \qquad \textit{for all } t \in \mathbb{R},$$

where ζ_n' are independent copies of ζ_n and c is an absolute constant.

Proof. Without loss of generality, we shall assume that $n \ge c_1$, with a sufficiently large absolute constant c_1. Consider the random vector $Y = (\widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_s) \in \mathbb{R}^s$ with coordinates which are symmetrizations of i.i.d. Rademacher random variables. Let $R = (R_1, \ldots, R_s)$ and T denote independent sums of n independent copies of $Y/2$. Then we can write

$$\mathbf{E} \, \mathrm{e}\big\{ t \, \langle \mathbb{Q} \widetilde{W}, \widetilde{W}' \rangle / 4 \big\} = \mathbf{E} \, \mathrm{e}\big\{ \langle \mathbb{B}_t R, T \rangle \big\}, \qquad \text{for all } t \in \mathbb{R}, \tag{6.16}$$

Note that the scalar product $\langle \cdot, \cdot \rangle$ in $\mathbf{E} \, \mathrm{e}\big\{ \langle \mathbb{B}_t R, T \rangle \big\}$ means the scalar product of vectors in $\mathbb{R}^s$. In order to estimate this expectation, we write it in the form

$$\begin{aligned} \mathbf{E} \, \mathrm{e}\big\{ \langle \mathbb{B}_t R, T \rangle \big\} &= \mathbf{E} \, \mathbf{E}_R \, \mathrm{e}\big\{ \langle \mathbb{B}_t R, T \rangle \big\} \\ &= \sum_{\overline{m} \in \mathbb{Z}^s} p(\overline{m}) \sum_{m \in \mathbb{Z}^s} p(m) \, \mathrm{e}\big\{ \langle \mathbb{B}_t m, \overline{m} \rangle \big\}, \end{aligned} \tag{6.17}$$

with summing over $m = (m_1, \ldots, m_s) \in \mathbb{Z}^s$, $\overline{m} = (\overline{m}_1, \ldots, \overline{m}_s) \in \mathbb{Z}^s$ and

$$p(m) = \mathbf{P}\big\{ R = m \big\} = \prod_{j=1}^{s} \mathbf{P}\big\{ R_j = m_j \big\} = \prod_{j=1}^{s} 2^{-2n} \binom{2n}{m_j + n}, \tag{6.18}$$

if $\max\limits_{1 \le j \le s} |m_j| \le n$ and $p(m) = 0$ otherwise. Clearly, for fixed $T = \overline{m}$,

$$\mathbf{E}_R \, \mathrm{e}\big\{ \langle \mathbb{B}_t R, T \rangle \big\} = \sum_{m \in \mathbb{Z}^s} p(m) \, \mathrm{e}\big\{ \langle \mathbb{B}_t m, \overline{m} \rangle \big\} \ge 0 \tag{6.19}$$

is a value of the characteristic function of symmetrized random vector $\mathbb{B}_t R$. Using Stirling's formula, it is easy to show that there exist absolute constants c_2 and c_3 such that

$$\mathbf{P}\big\{ R_j = m_j \big\} \ll n^{-1/2} \exp\big\{ -m_j^2/2n \big\} , \qquad \text{for } |m_j| \le c_2 n, \tag{6.20}$$

and

$$\mathbf{P}\big\{ |R_j| \ge c_2 n \big\} \ll \exp\{ -c_3 n \} . \tag{6.21}$$

Using (6.17)–(6.21), we obtain

$$\begin{aligned}\mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t R, T\rangle\,\big\} &\ll_s \sum_{\overline{m}\in\mathbb{Z}^s} q(\overline{m}) \sum_{m\in\mathbb{Z}^s} p(m)\,\mathrm{e}\big\{\,\langle\mathbb{B}_t m, \overline{m}\rangle\,\big\} + \exp\{-c_3 n\}\\ &= \sum_{m\in\mathbb{Z}^s} p(m) \sum_{\overline{m}\in\mathbb{Z}^s} q(\overline{m})\,\mathrm{e}\big\{\,\langle\mathbb{B}_t m, \overline{m}\rangle\,\big\} + \exp\{-c_3 n\}\\ &= \mathbf{E}\,\mathbf{E}_{\zeta_n}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t R, \zeta_n\rangle\,\big\} + \exp\{-c_3 n\}\\ &= \mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t R, \zeta_n\rangle\,\big\} + \exp\{-c_3 n\}\,. \qquad (6.22)\end{aligned}$$

Now we repeat our previous arguments, noting that

$$\mathbf{E}_{\zeta_n}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t R, \zeta_n\rangle\,\big\} = \sum_{\overline{m}\in\mathbb{Z}^s} q(\overline{m})\;\mathrm{e}\big\{\,\langle\mathbb{B}_t R, \overline{m}\rangle\,\big\} \ge 0 \qquad (6.23)$$

is a value of the non-negative characteristic function of the random vector ζ_n (see (6.14)). Using again (6.20) and (6.21), we obtain

$$\mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t R, \zeta_n\rangle\,\big\} \ll_s \mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t \zeta_n, \zeta_n'\rangle\,\big\} + \exp\{-c_3 n\}\,. \qquad (6.24)$$

Relations (6.16), (6.22) and (6.24) imply the statement of the lemma. □

Let us estimate the expectation $\mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t \zeta_n, \zeta_n'\rangle\,\big\}$ under the conditions of Lemmas 6.3 and 6.5, assuming that $s = d$, $\mathbb{D} = \mathbb{C}^{-1/2}$, $\delta \le 1/(5s)$, $n \ge c_4$, where c_4 is a sufficiently large absolute constant, and

$$\|\mathbb{C}^{-1/2} z_j - e_j\| \le \delta, \qquad \|\mathbb{C}^{-1/2} z_j' - e_j\| \le \delta, \qquad \text{for } 1 \le j \le s, \qquad (6.25)$$

with an orthonormal system $\mathcal{S} = \mathcal{S}_o = \big\{e_1, e_2, \dots, e_s\big\}$ involved in the conditions of Lemma 6.3. We can rewrite $\mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t \zeta_n, \zeta_n'\rangle\,\big\}$ as

$$\mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t \zeta_n, \zeta_n'\rangle\,\big\} = \sum_{\overline{m}\in\mathbb{Z}^s} q(\overline{m}) \sum_{m\in\mathbb{Z}^s} q(m)\,\mathrm{e}\big\{\,\langle\mathbb{B}_t \overline{m}, m\rangle\,\big\}.$$

Thus, by (6.15),

$$\mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t \zeta_n, \zeta_n'\rangle\,\big\} = A_n^{2s}\, n^{-s} \sum_{\overline{m}\in\mathbb{Z}^s} \sum_{m\in\mathbb{Z}^s} \exp\big\{i\,\langle\mathbb{B}_t \overline{m}, m\rangle - \|m\|^2/2n - \|\overline{m}\|^2/2n\big\}.$$

Denote

$$r = \sqrt{2\pi^2 n}. \qquad (6.26)$$

Applying Lemma 6.4 with $\mathbb{S} = \mathbb{I}_s$, $z = 1/2n$, $a = 0$, $b = (2\pi)^{-1}\,\mathbb{B}_t\,\overline{m}$ and using that $A_n \asymp 1$, we obtain

$$\begin{aligned}\mathbf{E}\,\mathrm{e}\big\{\,\langle\mathbb{B}_t \zeta_n, \zeta_n'\rangle\,\big\} &\ll_s n^{-s/2} \sum_{l,m\in\mathbb{Z}^s} \exp\big\{-2\pi^2 n\,\|l + (2\pi)^{-1}\mathbb{B}_t m\|^2 - \|m\|^2/2n\big\}\\ &\ll_s r^{-s} \sum_{m,\overline{m}\in\mathbb{Z}^s} \exp\big\{-r^2\,\|m - t\,\mathbb{V}\overline{m}\|^2 - \|\overline{m}\|^2/r^2\big\}, \qquad (6.27)\end{aligned}$$

where $\mathbb{V} : \mathbb{R}^s \to \mathbb{R}^s$ is the operator with matrix

$$\mathbb{V} = (2\pi)^{-1}\mathbb{B}_1. \tag{6.28}$$

Note that the right-hand side of (6.27) may be considered as a theta-series.

Denote $y_k = \mathbb{C}^{-1/2} z_k$, $1 \le k \le s$. Let $\mathbb{Y}$ be the $(s \times s)$-matrix with entries $\langle e_j, y_k\rangle$, where index j is the number of the row, while k is the number of the column. Then the matrix $\mathbb{F} \overset{\text{def}}{=} \mathbb{Y}^*\mathbb{Y}$ has entries $\langle y_j, y_k\rangle$. Here $\mathbb{Y}^*$ is the transposed matrix for $\mathbb{Y}$. According to (6.25), we have

$$\|y_j - e_j\| \le \delta, \qquad \text{for } 1 \le j \le s, \tag{6.29}$$

Let us show that (cf. BG (1997a, proof of Lemma 7.4))

$$\|\mathbb{Y}\| \le 3/2 \quad \text{and} \quad \|\mathbb{Y}^{-1}\| \le 2. \tag{6.30}$$

Since $\mathcal{S}_o = \big\{e_1, e_2, \dots, e_s\big\}$ is an orthonormal system, inequalities (6.29) imply that $\mathbb{Y} = \mathbb{I}_s + \mathbb{A}$ with some matrix $\mathbb{A} = \{a_{ij}\}$ such that $|a_{ij}| \le \delta$. Thus, we have $\|\mathbb{A}\| \le \|\mathbb{A}\|_2 \le s\delta$, where $\|\mathbb{A}\|_2$ denotes the Hilbert–Schmidt norm of the matrix $\mathbb{A}$. Therefore, the condition $\delta \le 1/(5s)$ implies $\|\mathbb{A}\| \le 1/2$ and inequalities (6.30).

The matrix $\mathbb{F}$ is symmetric and positive definite. Its determinant is the product of eigenvalues which (by (6.30)) are bounded from above and from below by some absolute positive constants. Moreover,

$$(\det \mathbb{Y})^2 = (\det \mathbb{Y}^*)^2 = \det \mathbb{F} \asymp_s 1 \asymp \|\mathbb{F}\| \asymp \|\mathbb{Y}\|. \tag{6.31}$$

Define the matrices $\overline{\mathbb{Y}}$ and $\overline{\mathbb{F}}$, replacing z_j by z_j' in the definition of $\mathbb{Y}$ and $\mathbb{F}$. Similarly to (6.31), one can show that

$$\big(\det \overline{\mathbb{Y}}\big)^2 = \big(\det \overline{\mathbb{Y}}^*\big)^2 = \det \overline{\mathbb{F}} \asymp_s 1 \asymp \|\overline{\mathbb{F}}\| \asymp \|\overline{\mathbb{Y}}\|. \tag{6.32}$$

Let $\mathbb{G}$ and $\overline{\mathbb{G}}$ be the $(s \times s)$-matrices with entries $\langle e_j, \mathbb{Q} z_k\rangle$ and $\langle e_j, z_k'\rangle$ respectively. Then, clearly, $\mathbb{G} = \mathbb{Q}\mathbb{C}^{1/2}\mathbb{Y}$ and $\overline{\mathbb{G}} = \mathbb{C}^{1/2}\overline{\mathbb{Y}}$. Therefore,

$$\mathbb{B}_1 = \mathbb{G}^*\overline{\mathbb{G}} = \mathbb{Y}^*\mathbb{C}^{1/2}\mathbb{Q}\mathbb{C}^{1/2}\overline{\mathbb{Y}}. \tag{6.33}$$

Moreover, $\mathbb{Q}^2 = \mathbb{I}_d$ implies that $\big|\det \mathbb{Q}\big| = 1$ and $\|\mathbb{Q}\| = 1$. Using relations (6.28) and (6.31)–(6.33), we obtain

$$\big|\det \mathbb{V}\big| \asymp_s \big|\det \mathbb{B}_1\big| \asymp_s \det \mathbb{C}, \tag{6.34}$$

and

$$\|\mathbb{V}\| \ll \|\mathbb{B}_1\| \ll \|\mathbb{C}\| \ll \sigma_1^2. \tag{6.35}$$

7. Some facts from Number Theory

In Section 7, we consider some facts of the geometry of numbers (see Davenport (1958) or Cassels (1959)). They will help us to estimate the integrals of the right-hand side of inequality (6.27).

Let $e_1, e_2, \ldots, e_d$ be linearly independent vectors in $\mathbb{R}^d$. The set

$$\Lambda = \Big\{ \sum_{j=1}^{d} n_j e_j : n_j \in \mathbb{Z},\ j = 1, 2, \ldots, d \Big\} \tag{7.1}$$

is called the lattice with basis $e_1, e_2, \ldots, e_d$. The determinant $\det(\Lambda)$ of a lattice Λ is the modulus of the determinant of the matrix formed from the vectors $e_1, e_2, \ldots, e_d$:

$$\det(\Lambda) \overset{\text{def}}{=} \big| \det(e_1, e_2, \ldots, e_d) \big|. \tag{7.2}$$

The determinant of a lattice does not depend on the choice of basis. Any lattice $\Lambda \subset \mathbb{R}^d$ can be represented as $\Lambda = \mathbb{A}\,\mathbb{Z}^d$, where $\mathbb{A}$ is a non-degenerate linear operator. Clearly, $\det(\Lambda) = |\det \mathbb{A}|$.

Let $m_1, \ldots, m_l \in \Lambda$ be linearly independent vectors belonging to a lattice Λ. Then the set

$$\Lambda' = \Big\{ \sum_{j=1}^{l} n_j m_j : n_j \in \mathbb{Z},\ j = 1, 2, \ldots, l \Big\} \tag{7.3}$$

is an l-dimensional sublattice of the lattice Λ. Its determinant $\det(\Lambda')$ is the modulus of the determinant of the matrix formed from the coordinates of the vectors $m_1, m_2, \ldots, m_l$ with respect to an orthonormal basis of the linear span of the vectors $m_1, m_2, \ldots, m_l$. The determinant $\det(\Lambda')$ could be also defined as $\det\big(\langle m_i, m_j\rangle, i, j = 1, \ldots, l\big)^{1/2}$.

Let $F : \mathbb{R}^d \to [0, \infty]$ denote a norm on $\mathbb{R}^d$, that is $F(\alpha x) = |\alpha|\, F(x)$, for $\alpha \in \mathbb{R}$, and $F(x + y) \le F(x) + F(y)$. The successive minima $M_1 \le \cdots \le M_d$ of F with respect to a lattice Λ are defined as follows: Let $M_1 = \inf\big\{F(m) : m \ne 0,\ m \in \Lambda\big\}$ and define M_j as the infimum of $\lambda > 0$ such that the set $\big\{m \in \Lambda : F(m) < \lambda\big\}$ contains j linearly independent vectors. It is easy to see that these infima are attained, that is there exist linearly independent vectors $b_1, \ldots, b_d \in \Lambda$ such that $F(b_j) = M_j$, $j = 1, \ldots, d$. The following Lemma 7.1 is proved by Davenport (1958, Lemma 1), see also Götze and Margulis (2010).

Lemma 7.1. *Let $M_1 \le \cdots \le M_d$ be the successive minima of a norm F with respect to the lattice $\mathbb{Z}^d$. Denote $M_{d+1} = \infty$. Suppose that $1 \le j \le d$ and $M_j \le b \le M_{j+1}$, for some $b > 0$. Then*

$$\#\big\{ m = (m_1, \ldots, m_d) \in \mathbb{Z}^d :\ F(m) < b \big\} \asymp_d b^j (M_1 \cdot M_2 \cdots M_j)^{-1}. \tag{7.4}$$

Representing $\Lambda = \mathbb{A}\,\mathbb{Z}^d$, we see that the lattice $\mathbb{Z}^d$ may be replaced in Lemma 7.1 by any lattice $\Lambda \subset \mathbb{R}^d$. It suffices to apply this lemma to the norm $G(m) = F(\mathbb{A}m)$, $m \in \mathbb{Z}^d$.

Lemma 7.2. *Let* $F_j(m)$, $j = 1, 2$, *be some norms in* $\mathbb{R}^d$ *and* $M_1 \le \cdots \le M_d$ *and* $N_1 \le \cdots \le N_d$ *be the successive minima of* F_1 *with respect to a lattice* Λ_1 *and of* F_2 *with respect to a lattice* Λ_2 *respectively. Let* $C > 0$. *Assume that* $M_k \gg_d C\, F_2(n_k)$, $k = 1, 2, \dots, d$, *for some linearly independent vectors* $n_1, n_2, \dots, n_d \in \Lambda_2$. *Then*

$$M_k \gg_d C\, N_k, \quad k = 1, \dots, d. \tag{7.5}$$

The proof of this lemma is elementary and therefore omitted.

Let $\|x\|_\infty = \max_{1\le j\le d} |x_j|$, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.

Lemma 7.3. *Let* Λ *be a lattice in* $\mathbb{R}^d$ *and let* $0 < \varepsilon \le 1$. *Then*

$$e^{-\varepsilon}\, \#H \le \sum_{v\in\Lambda} \exp\bigl\{-\varepsilon\, \|v\|^2\bigr\} \ll_d \varepsilon^{-d/2} \#H, \tag{7.6}$$

where $H \overset{\text{def}}{=} \bigl\{v \in \Lambda : \|v\|_\infty < 1\bigr\}$.

Proof. The lower bound in (7.6) is almost evident by restricting summation to the set H. Introduce for $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{Z}^d$ the sets

$$B_\mu \overset{\text{def}}{=} \Bigl[\mu_1 - \frac{1}{2}\,,\, \mu_1 + \frac{1}{2}\Bigr) \times \cdots \times \Bigl[\mu_d - \frac{1}{2}\,,\, \mu_d + \frac{1}{2}\Bigr)$$

such that $\mathbb{R}^d = \bigcup_\mu B_\mu$. For any fixed $w^* \in H_\mu \overset{\text{def}}{=} \bigl\{w \in \Lambda \cap B_\mu\bigr\}$ we have

$$w - w^* \in H, \qquad \text{for any } w \in H_\mu.$$

Hence we conclude for any $\mu \in \mathbb{Z}^d$

$$\#H_\mu \le \#H. \tag{7.7}$$

Since $x \in B_\mu$ implies $\|x\|_\infty \ge \|\mu\|_\infty/2$, we obtain by (7.7)

$$\begin{aligned}
\sum_{v\in\Lambda} \exp\bigl\{-\varepsilon\, \|v\|^2\bigr\} \quad &\le \quad \sum_{v\in\Lambda} \exp\bigl\{-\varepsilon\, \|v\|_\infty^2\bigr\} \\
&\ll_d \quad \#H_0 + \sum_{\mu\in\mathbb{Z}^d\setminus 0} \ \sum_{v\in\Lambda} \mathbf{I}\bigl\{\, v \in B_\mu\bigr\} \exp\bigl\{-\varepsilon\, \|\mu\|_\infty^2/4\bigr\} \\
&\ll_d \quad \#H \cdot \sum_{\mu\in\Xi} \exp\bigl\{-\varepsilon\, \|\mu\|_\infty^2/4\bigr\} \\
&\ll_d \quad \varepsilon^{-d/2} \#H.
\end{aligned} \tag{7.8}$$

This conclude the proof of Lemma 7.3. $\square$

It is easy to see that Lemma 7.3 implies the following statement.

Corollary 7.4. *Let Λ be a lattice in $\mathbb{R}^d$ and let $c_j(d)$, $j=1,2,3,4$, be positive quantities depending on d only. Let $F(\cdot)$ be a norm in $\mathbb{R}^d$ such that $F(\cdot)\asymp_d \|\cdot\|$. Then*

$$\begin{aligned}\sum_{v\in\Lambda}\exp\bigl\{-c_1(d)\,\|v\|^2\bigr\} &\asymp_d \sum_{v\in\Lambda}\exp\bigl\{-c_2(d)\,(F(v))^2\bigr\}\\ &\asymp_d \#\bigl\{v\in\Lambda\,:\,\|v\|<c_3(d)\bigr\}\\ &\asymp_d \#\bigl\{v\in\Lambda\,:\,F(v)<c_4(d)\bigr\}. \qquad (7.9)\end{aligned}$$

The proof of Corollary 7.4 is elementary and therefore omitted. Note only that

$$\#\bigl\{v\in\Lambda\,:\,F(v)<\lambda\bigr\}=\#\bigl\{v\in\mu^{-1}\Lambda\,:\,F(v)<\lambda/\mu\bigr\},\quad \text{for } \lambda,\mu>0. \qquad (7.10)$$

For a lattice $\Lambda\subset\mathbb{R}^d$, $\dim\Lambda=d$ and $1\le l\le d$, we define its α_l-characteristics by

$$\alpha_l(\Lambda)\overset{\text{def}}{=}\sup\Bigl\{\,\bigl|\det(\Lambda')\bigr|^{-1}:\Lambda'\subset\Lambda,\ \ l\text{-dimensional sublattice of }\Lambda\Bigr\}. \qquad (7.11)$$

Denote

$$\alpha(\Lambda)\overset{\text{def}}{=}\max_{1\le l\le d}\alpha_l(\Lambda). \qquad (7.12)$$

Lemma 7.5. *Let $F(\cdot)$ be a norm in $\mathbb{R}^d$ such that $F(\cdot)\asymp_d\|\cdot\|$. Let $c(d)$ be a positive quantity depending on d only. Let $M_1\le\cdots\le M_d$ be the successive minima of F with respect to a lattice $\Lambda\subset\mathbb{R}^d$. Then*

$$\alpha_l(\Lambda)\asymp_d (M_1\cdot M_2\cdots M_l)^{-1},\quad l=1,\dots,d. \qquad (7.13)$$

Moreover,

$$\alpha(\Lambda)\asymp_d \bigl\{v\in\Lambda\,:\,\|v\|<c(d)\bigr\}, \qquad (7.14)$$

provided that $M_1\ll_d 1$.

For the proof of Lemma 7.5 we shall use the following lemma formulated in Proposition (p. 517) and Remark (p. 518) in A.K. Lenstra, H.W. Lenstra and Lovász (1982).

Lemma 7.6. *Let $M_1\le\cdots\le M_d$ be the successive minima of the standard Euclidean norm with respect to a lattice $\Lambda\subset\mathbb{R}^d$. Then there exists a basis $e_1,e_2,\dots,e_d$ of Λ such that*

$$M_l\asymp_d\|e_l\|,\quad l=1,\dots,d. \qquad (7.15)$$

Moreover,

$$\det(\Lambda)\asymp_d\prod_{l=1}^d\|e_l\|. \qquad (7.16)$$

Proof of Lemma 7.5. According to Lemma 7.2, we can replace the Euclidean norm $\|\cdot\|$ by the norm $F(\cdot)$, in the formulation of Lemma 7.6. Let $\Lambda'\subset\Lambda$ be an arbitrary l-dimensional sublattice of Λ and $N_1\le\cdots\le N_l$ be the successive minima of the norm $F(\cdot)$ with respect to Λ'. It is clear that $M_j\le N_j$, $j=1,2,\dots,l$. On the other hand, $M_j = F(m_j)$ for some linearly independent vectors $m_1, m_2, \dots, m_l \in \Lambda$. In the case, where

$$\Lambda' = \Big\{\sum_{j=1}^{l} n_j m_j : n_j \in \mathbb{Z}, j = 1, 2, \dots, l\Big\}, \tag{7.17}$$

we have $N_j = M_j$, $j = 1, 2, \dots, l$. In order to justify relation (7.13) it remains to take into account definition (7.11) and to apply Lemma 7.6. Relation (7.14) is an easy consequence of (7.13), Lemma 7.1 and Corollary 7.4. □

8. From Number Theory to Probability

In Section 8, we shall use these number-theoretical results of Section 7 to estimate integrals of the right-hand side of (6.27). Recall that we have assumed the conditions of Lemmas 6.3 and 6.5, $s = d$, $\mathbb{D} = \mathbb{C}^{-1/2}$, $\delta \le 1/(5s)$, $n \ge c$, and (6.25), for an orthonormal system $\mathcal{S} = \mathcal{S}_o$. The notation $\mathrm{SL}(d, \mathbb{R})$ is used below for the set of all $(d \times d)$-matrices with real entries and determinant 1.

Introduce the matrices

$$\mathbb{D}_r \overset{\mathrm{def}}{=} \begin{pmatrix} r\,\mathbb{I}_s & \mathbb{O}_s \\ \mathbb{O}_s & r^{-1}\,\mathbb{I}_s \end{pmatrix} \in \mathrm{SL}(2s, \mathbb{R}), \quad r > 0, \tag{8.1}$$

$$\mathbb{K}_t \overset{\mathrm{def}}{=} \begin{pmatrix} \mathbb{I}_s & -t\,\mathbb{I}_s \\ t\,\mathbb{I}_s & \mathbb{I}_s \end{pmatrix}, \quad t \in \mathbb{R}, \tag{8.2}$$

$$\mathbb{U}_t \overset{\mathrm{def}}{=} \begin{pmatrix} \mathbb{I}_s & -t\,\mathbb{I}_s \\ \mathbb{O}_s & \mathbb{I}_s \end{pmatrix} \in \mathrm{SL}(2s, \mathbb{R}), \quad t \in \mathbb{R}, \tag{8.3}$$

and the lattices

$$\Lambda \overset{\mathrm{def}}{=} \begin{pmatrix} \mathbb{I}_s & \mathbb{O}_s \\ \mathbb{O}_s & \mathbb{V}_0 \end{pmatrix} \mathbb{Z}^{2s}, \tag{8.4}$$

$$\Lambda_j = \mathbb{D}_j\,\mathbb{U}_{j^{-1}}\,\Lambda = \begin{pmatrix} j\,\mathbb{I}_s & -\mathbb{V}_0 \\ \mathbb{O}_s & j^{-1}\,\mathbb{V}_0 \end{pmatrix} \mathbb{Z}^{2s}, \quad j = 1, 2, \dots, \tag{8.5}$$

where

$$\mathbb{V}_0 = \sigma_1^{-2}\,\mathbb{V} \tag{8.6}$$

and the matrix $\mathbb{V}$ is defined in (6.28). Below we shall use the following simplest properties of these matrices:

$$\mathbb{D}_a \mathbb{D}_b = \mathbb{D}_{ab}, \quad \mathbb{U}_a \mathbb{U}_b = \mathbb{U}_{a+b} \quad \text{and} \quad \mathbb{D}_a \mathbb{U}_b = \mathbb{U}_{a^2 b}\,\mathbb{D}_a, \quad \text{for } a, b > 0. \tag{8.7}$$

Let $M_{j,t}$, $j = 1, 2, \dots, 2s$, be the successive minima of the norm $\|\cdot\|_\infty$ with respect to the lattice

$$\Xi_t \overset{\mathrm{def}}{=} \begin{pmatrix} r\,\mathbb{I}_s & -r\,t\,\mathbb{V} \\ \mathbb{O}_s & r^{-1}\,\mathbb{I}_s \end{pmatrix} \mathbb{Z}^{2s}. \tag{8.8}$$

Moreover, simultaneously, $M_{j,t}$ are the successive minima of the norm $F^*(\cdot)$ defined for $(m,\overline{m})\in\mathbb{R}^{2s}$, $m,\overline{m}\in\mathbb{R}^s$, by

$$F^*((m,\overline{m}))\overset{\text{def}}{=}\max\big\{\|m\|_\infty,\,\sigma_1^2\,\|\mathbb{V}^{-1}\overline{m}\|_\infty\big\}\tag{8.9}$$

with respect to the lattice

$$\Omega_t\overset{\text{def}}{=}\begin{pmatrix} r\,\mathbb{I}_s & -r\,t\,\mathbb{V}\\ \mathbb{O}_s & \sigma_1^{-2}\,r^{-1}\,\mathbb{V}\end{pmatrix}\mathbb{Z}^{2s}=\mathbb{D}_r\,\mathbb{U}_u\,\Lambda,\quad\text{where } u\overset{\text{def}}{=}\sigma_1^2\,t.\tag{8.10}$$

Using Lemmas 7.2 and 7.6 and the equality $\det(\Xi_t)=1$, it is easy to show that

$$M_{1,t}\ll_s 1.\tag{8.11}$$

Let $M_{j,t}^*$ be the successive minima of the Euclidean norm with respect to the lattice Ω_t. Note that, according to (6.35) and (8.9),

$$\|\cdot\|\ll_s F^*(\cdot).\tag{8.12}$$

Using (8.12) and Lemma 7.2, we obtain

$$M_{j,t}^*\ll_s M_{j,t},\quad j=1,\dots,2s.\tag{8.13}$$

According to Lemma 7.5,

$$\alpha(\Xi_t)\ll_s\alpha(\Omega_t).\tag{8.14}$$

Let us estimate $\alpha(\Omega_t)$ assuming that $r\ge 1$ and (for a moment) $t=\sigma_1^{-2}\,r^{-1}$. In this case

$$\Omega_t=\begin{pmatrix} r\,\mathbb{I}_s & -\mathbb{V}_0\\ \mathbb{O}_s & r^{-1}\,\mathbb{V}_0\end{pmatrix}\mathbb{Z}^{2s}.\tag{8.15}$$

By relation (7.14) of Lemma 7.5, we have

$$\alpha(\Omega_t)\asymp_s\#\big\{v\in\Omega_t\,:\,\|v\|<1/2\big\}=\#K,\tag{8.16}$$

where

$$K=\big\{v=(m,\overline{m})\in\mathbb{Z}^{2s}:m,\overline{m}\in\mathbb{Z}^s,\ \|rm-\mathbb{V}_0\overline{m}\|^2+\|r^{-1}\mathbb{V}_0\overline{m}\|^2<1/4\big\}.\tag{8.17}$$

Let us estimate from above the right-hand side of (8.16). If $v=(m,\overline{m})\in K$, then

$$r\,\|m\|\le\|rm-\mathbb{V}_0\overline{m}\|+\|\mathbb{V}_0\overline{m}\|<\frac{1}{2}+\frac{r}{2}\le r.\tag{8.18}$$

Hence $m=0$ and $\|\mathbb{V}_0\overline{m}\|\le 1/2$. It remains to estimate the quantity

$$R\overset{\text{def}}{=}\#\big\{\overline{m}\in\mathbb{Z}^s:\ \|\mathbb{V}_0\overline{m}\|<1\big\}\ge\#K.\tag{8.19}$$

Let $N_1\le\dots\le N_s$ be the successive minima of the Euclidean norm with respect to the lattice $\mathbb{V}_0\mathbb{Z}^s$. Let $e_1,e_2,\dots,e_s$ be the standard orthonormal basis of $\mathbb{Z}^s$. By (6.35) and (8.6), we have $\|\mathbb{V}_0e_j\|\le 1$, $j=1,2,\dots,s$. Therefore, using Lemma 7.2, we see that $N_1\le\dots\le N_s\le 1$. By (6.34), (8.6), (8.19) and by Lemmas 7.1, 7.2 and 7.6,

$$R\asymp_s(N_1\cdot N_2\cdots N_s)^{-1}\asymp_s(\det\mathbb{V}_0)^{-1}\asymp_s\sigma_1^{2s}\,(\det\mathbb{C})^{-1}.\tag{8.20}$$

Hence, using (8.16), (8.19) and (8.20) we conclude that

$$\alpha(\Omega_t) \ll_s \sigma_1^{2s}\,(\det\mathbb{C})^{-1}, \quad \text{for } \ r\ge 1 \quad \text{and} \quad t=\sigma_1^{-2}\,r^{-1}. \tag{8.21}$$

Let now $t\in\mathbb{R}$ be arbitrary. By (8.8), (8.11), (8.14), Lemmas 7.1, 7.5 and Corollary 7.4,

$$\begin{aligned}\sum_{m,\overline{m}\in\mathbb{Z}^s}\exp\bigl\{-r^2\,\|m-t\,\mathbb{V}\overline{m}\|^2 \ - \ \|\overline{m}\|^2/r^2\bigr\} &= \sum_{v\in\Xi_t}\exp\bigl\{-\|v\|^2\bigr\}\\ &\ll_s \ R_t \overset{\text{def}}{=} \#\bigl\{v\in\Xi_t\,:\,\|v\|<1\bigr\}\\ &\ll_s \ \alpha(\Xi_t)\ll_s \alpha(\Omega_t).\end{aligned} \tag{8.22}$$

Now, by (6.27), (8.10) and (8.22), we have

$$\mathbf{E}\ \mathrm{e}\bigl\{\langle\mathbb{B}_t\zeta_n,\zeta_n'\rangle\bigr\}\ll_s r^{-s}\,\alpha(\Omega_t)=r^{-s}\,\alpha(\mathbb{D}_r\,\mathbb{U}_u\,\Lambda), \quad \text{where } u=\sigma_1^2\,t. \tag{8.23}$$

Let us estimate the quantity R_t, $t\in\mathbb{R}$, defined in (8.22) assuming that $r\ge 1$ and $|rt|\le c_s^*\,\sigma_1^{-2}$, where $c_s^*\ge 1$ is an arbitrary quantity depending on s only. By Corollary 7.4, we have

$$R_t \asymp_s \#K_0, \tag{8.24}$$

where

$$K_0 \overset{\text{def}}{=} \bigl\{v=(m,\overline{m})\in\mathbb{Z}^{2s}: m,\overline{m}\in\mathbb{Z}^s,\ \|rm-rt\,\mathbb{V}\overline{m}\|^2+\|r^{-1}\overline{m}\|^2<(2c_s^*)^{-2}\bigr\}. \tag{8.25}$$

If $v=(m,\overline{m})\in K_0$, $r\ge 1$ and $|rt|\le c_s^*\,\sigma_1^{-2}$, then, by (6.35) and (8.25),

$$r\,\|m\|\le\|rm-rt\,\mathbb{V}\overline{m}\|+|rt|\ \|\mathbb{V}\overline{m}\|<\ \frac{1}{2}\ +\ \frac{r}{2}\ \le r. \tag{8.26}$$

Hence $m=0$ and $|rt|\ \|\mathbb{V}\overline{m}\|\le(2c_s^*)^{-1}<1$. It remains to estimate the quantity

$$S \overset{\text{def}}{=} \#\bigl\{\overline{m}\in\mathbb{Z}^s:\ |rt|\ \|\mathbb{V}\overline{m}\|<1\bigr\}\ge\#K_0. \tag{8.27}$$

Let $P_1\le\cdots\le P_s$ be the successive minima of the Euclidean norm with respect to the lattice $|rt|\ \mathbb{V}\mathbb{Z}^s$. Let $e_1,e_2,\dots,e_s$ be the standard orthonormal basis of $\mathbb{Z}^s$. By (6.35), we have $\|\,|rt|\ \mathbb{V}e_j\|\ll_s 1$, $j=1,2,\dots,s$. Therefore, using Lemma 7.2, we see that $P_1\le\cdots\le P_s\ll_s 1$. By (6.34), (8.27) and Lemmas 7.1 and 7.6,

$$S\asymp_s(P_1\cdot P_2\cdots P_s)^{-1}\asymp_s(\det(|rt|\ \mathbb{V}))^{-1}\asymp_s|rt|^{-s}\ (\det\mathbb{C})^{-1}. \tag{8.28}$$

Hence, using (8.24), (8.27) and (8.28), we conclude that

$$R_t\ll_s|rt|^{-s}\ (\det\mathbb{C})^{-1}, \quad \text{for } \ r\ge 1 \quad \text{and } \ |rt|\le c_s^*\,\sigma_1^{-2}. \tag{8.29}$$

Now, by (6.27), (8.22) and (8.29), we have

$$\begin{aligned}\mathbf{E}\ \mathrm{e}\bigl\{\langle\mathbb{B}_t\zeta_n,\zeta_n'\rangle\bigr\} &\ll_s \ r^{-s}\,R_t\\ &\ll_s \ r^{-2s}\ |t|^{-s}\ (\det\mathbb{C})^{-1}, \quad \text{for } r\ge 1 \text{ and } |rt|\le c_s^*\,\sigma_1^{-2}.\end{aligned} \tag{8.30}$$

It is easy to verify that

$$\int_{c_s\,\sigma_1^{-2}r^{-2+4/s}}^{\sigma_1^{-2}r^{-1}}\sqrt{r^{-2s}\ |t|^{-s}\ (\det\mathbb{C})^{-1}}\ \frac{dt}{t}\ \ll_s r^{-2}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2}, \tag{8.31}$$

for any c_s depending on s only. Note that $\sigma_1^s\,(\det\mathbb{C})^{-1/2}\ge 1$. Using (8.23), (8.31) and Lemmas 6.3 and 6.5, we derive the following lemma.

Lemma 8.1. *Let the conditions of Lemma* 6.3 *be satisfied with* $s=d$, $\mathbb{D}=\mathbb{C}^{-1/2}$, $\delta\le 1/(5s)$ *and with an orthonormal system* $\mathcal{S}=\mathcal{S}_o=\{e_1,\dots,e_s\}\subset\mathbb{R}^d$. *Let* c_s *be an arbitrary quantity depending on* s *only. Then, for any* $b\in\mathbb{R}^d$ *and* $r\ge 1$,

$$\int_{c_s\,\sigma_1^{-2}r^{-2+4/s}}^{\sigma_1^{-2}}\big|\,\widehat{\Psi}_b(t/2)\,\big|\,\frac{dt}{t}\ll_s\,(pN)^{-1}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2}+r^{-s/2}\,\sup_{\Gamma}\,\int_{r^{-1}}^{1}(\alpha(\mathbb{D}_r\,\mathbb{U}_u\,\Lambda))^{1/2}\,\frac{du}{u}\,, \tag{8.32}$$

where r, $\alpha(\cdot)$, $\mathbb{D}_r\,\mathbb{U}_t$ *and the lattice* Λ *are defined in relations* (6.3), (6.26), (6.28), (7.11), (7.12), (8.1), (8.3) *and* (8.4) *and in Lemma* 6.5. *The* $\sup_\Gamma$ *means here the supremum over all possible values of* $z_j,z_j'\in\mathbb{R}^d$ (*involved in the definition of matrices* $\mathbb{B}_t$ *and* $\mathbb{V}$) *such that*

$$\|\mathbb{C}^{-1/2}z_j-e_j\|\le\delta,\qquad \|\mathbb{C}^{-1/2}z_j'-e_j\|\le\delta,\qquad \textit{for }\ 1\le j\le s. \tag{8.33}$$

Moreover, for any $b\in\mathbb{R}^d$, $r\ge 1$ *and* $\gamma>0$ *and any fixed* $t\in\mathbb{R}$ *satisfying* $|rt|\le c_s^*\,\sigma_1^{-2}$, *where* $c_s^*\ge 1$ *is an arbitrary quantity depending on* s *only, we have*

$$\big|\,\widehat{\Psi}_b(t)\,\big|\ll_{\gamma,s}(pN)^{-\gamma}+r^{-s}\;|t|^{-s/2}\;(\det\mathbb{C})^{-1/2}. \tag{8.34}$$

Let $v=(m,\overline{m})\in\mathbb{R}^{2s}$, $m,\overline{m}\in\mathbb{R}^s$ and $t\in\mathbb{R}$. Then

$$\overline{m}+tm=(1+t^2)\,\overline{m}+t\,(m-t\overline{m}). \tag{8.35}$$

Equality (8.35) implies that

$$\|\overline{m}+tm\|\ll_s\|\overline{m}\|+\|m-t\overline{m}\|,\quad \text{for } |t|\ll_s 1. \tag{8.36}$$

Hence,

$$r\,\|m-t\overline{m}\|+r^{-1}\,\|\overline{m}+tm\|\ll_s r\,\|m-t\overline{m}\|+r^{-1}\,\|\overline{m}\|,\quad \text{for } r\gg 1,\ |t|\ll_s 1. \tag{8.37}$$

According to (8.1)–(8.3), we have

$$\mathbb{D}_r\mathbb{U}_t v=(r(m-t\overline{m}),\,r^{-1}\overline{m})\quad\text{and}\quad \mathbb{D}_r\mathbb{K}_t v=(r(m-t\overline{m}),\,r^{-1}(\overline{m}+tm)). \tag{8.38}$$

It is clear that the operators $\mathbb{D}_r\mathbb{U}_t$ and $\mathbb{D}_r\mathbb{K}_t$ are invertible. Therefore, using (8.37) and (8.38) and applying Lemmas 7.2 and 7.5, we derive the inequality

$$\alpha(\mathbb{D}_r\mathbb{U}_t\Omega)\ll_s\alpha(\mathbb{D}_r\mathbb{K}_t\Omega),\quad \text{for } r\gg 1,\ |t|\ll_s 1, \tag{8.39}$$

which is valid for any lattice $\Omega\subset\mathbb{R}^{2s}$.

Let $\mathbb{T}$ be the permutation $(2s\times 2s)$-matrix which permutes the rows of a $(2s\times 2s)$-matrix $\mathbb{A}$ so that the new order (corresponding to the matrix $\mathbb{T}\mathbb{A}$) is:

$$1,s,2,s+1,\dots,s-1,2s.$$

Note that the operator $\mathbb{T}$ is isometric and $\mathbb{A}\mapsto\mathbb{A}\,\mathbb{T}^{-1}$ rearrange the columns of $\mathbb{A}$ in the order mentioned above. It is easy to see that

$$\alpha_j(\mathbb{T}\Omega)=\alpha_j(\Omega),\quad j=1,\dots 2s,\quad\text{and}\quad \alpha(\mathbb{T}\Omega)=\alpha(\Omega), \tag{8.40}$$

for any lattice $\Omega \subset \mathbb{R}^{2s}$.

Note now that

$$\mathbb{T}\mathbb{D}_r\mathbb{K}_t\Lambda_j = \mathbb{T}\mathbb{D}_r\mathbb{K}_t\mathbb{T}^{-1}\mathbb{T}\Lambda_j = \mathbb{W}_t\Delta_j, \tag{8.41}$$

where Δ_j is a lattice defined by

$$\Delta_j = \mathbb{T}\Lambda_j \tag{8.42}$$

and where $\mathbb{W}_t$ is $(2s \times 2s)$-matrix

$$\mathbb{W}_t = \begin{pmatrix} \mathbb{G}_{r,t} & \mathbb{O}_2 & : & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{G}_{r,t} & : & \mathbb{O}_2 \\ \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot \\ \mathbb{O}_2 & \mathbb{O}_2 & : & \mathbb{G}_{r,t} \end{pmatrix} \tag{8.43}$$

constructed of (2×2)-matrices $\mathbb{O}_2$ (with zero entries) and

$$\mathbb{G}_{r,t} \overset{\text{def}}{=} \begin{pmatrix} r & -r\,t \\ r^{-1}t & r^{-1} \end{pmatrix}. \tag{8.44}$$

Let $|t| \leq 2$ and

$$\theta = \arcsin\bigl(t\,(1+t^2)^{-1/2}\bigr) \quad \text{or, equivalently,} \quad t = \tan\theta. \tag{8.45}$$

Then we have

$$|\theta| \leq c^* \overset{\text{def}}{=} \arcsin(2/\sqrt{5}), \quad \cos\theta = (1+t^2)^{-1/2}, \quad \sin\theta = t\,(1+t^2)^{-1/2}. \tag{8.46}$$

It is easy to see that

$$\mathbb{G}_{r,t} = (1+t^2)^{1/2}\,\overline{\mathbb{D}}_r\,\overline{\mathbb{K}}_\theta \tag{8.47}$$

and

$$\mathbb{W}_t = (1+t^2)^{1/2}\,\widetilde{\mathbb{D}}_r\,\widetilde{\mathbb{K}}_\theta, \tag{8.48}$$

where

$$\widetilde{\mathbb{D}}_r = \begin{pmatrix} \overline{\mathbb{D}}_r & \mathbb{O}_2 & : & \mathbb{O}_2 \\ \mathbb{O}_2 & \overline{\mathbb{D}}_r & : & \mathbb{O}_2 \\ \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot \\ \mathbb{O}_2 & \mathbb{O}_2 & : & \overline{\mathbb{D}}_r \end{pmatrix} \quad \text{and} \quad \widetilde{\mathbb{K}}_\theta = \begin{pmatrix} \overline{\mathbb{K}}_\theta & \mathbb{O}_2 & : & \mathbb{O}_2 \\ \mathbb{O}_2 & \overline{\mathbb{K}}_\theta & : & \mathbb{O}_2 \\ \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot \\ \mathbb{O}_2 & \mathbb{O}_2 & : & \overline{\mathbb{K}}_\theta \end{pmatrix} \tag{8.49}$$

are $(2\,s \times 2\,s)$-matrices with

$$\overline{\mathbb{D}}_r \overset{\text{def}}{=} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \quad \text{and} \quad \overline{\mathbb{K}}_\theta \overset{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}(2{,}\mathbb{R}). \tag{8.50}$$

Substituting (8.48) into equality (8.41), we obtain

$$\mathbb{T}\mathbb{D}_r\mathbb{K}_t\Lambda_j = (1+t^2)^{1/2}\,\widetilde{\mathbb{D}}_r\,\widetilde{\mathbb{K}}_\theta\,\Delta_j. \tag{8.51}$$

Below we shall also use the following crucial lemma of Götze and Margulis (2010).

Lemma 8.2. *Let* $\widetilde{\mathbb{K}}_\theta$ *and*

$$\widetilde{\mathbb{H}} = \begin{pmatrix} \overline{\mathbb{H}} & \mathbb{O}_2 & : & \mathbb{O}_2 \\ \mathbb{O}_2 & \overline{\mathbb{H}} & : & \mathbb{O}_2 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \mathbb{O}_2 & \mathbb{O}_2 & : & \overline{\mathbb{H}} \end{pmatrix} \tag{8.52}$$

be $(2d\times 2d)$*-matrices such that* $\overline{\mathbb{H}}\in\mathcal{G}=\mathrm{SL}(2,\mathbb{R})$ *and* $\widetilde{\mathbb{K}}_\theta$ *is defined in* (8.49) *and* (8.50). *Let* β *is a positive number such that* $\beta d>2$. *Then, for any* $\overline{\mathbb{H}}\in\mathcal{G}$ *and any lattice* $\Delta\subset\mathbb{R}^{2d}$,

$$\int_0^{2\pi}\big(\alpha(\widetilde{\mathbb{H}}\,\widetilde{\mathbb{K}}_\theta\,\Delta)\big)^\beta d\theta \ll_{\beta,d} \big(\alpha(\Delta)\big)^\beta\,\|\overline{\mathbb{H}}\|^{\beta d-2}. \tag{8.53}$$

Here $\|\overline{\mathbb{H}}\|$ *is the standard norm of the linear operator* $\mathbb{H}:\mathbb{R}^2\to\mathbb{R}^2$.

Consider, under the conditions of Lemma 8.1,

$$I_0 \overset{\mathrm{def}}{=} \int_{c_s\,\sigma_1^{-2}r^{-2+4/s}/2}^{\sigma_1^{-2}/2}\big|\,\widehat{\Psi}_b(t)\,\big|\,\frac{dt}{t} = \int_{c_s\,\sigma_1^{-2}r^{-2+4/s}}^{\sigma_1^{-2}}\big|\,\widehat{\Psi}_b(t/2)\,\big|\,\frac{dt}{t}\,. \tag{8.54}$$

By Lemma 8.1, we have

$$I_0 \ll_s (pN)^{-1}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2} + r^{-s/2}\sup_\Gamma\, J, \tag{8.55}$$

where

$$J = \int_{r^{-1}}^1\big(\alpha(\mathbb{D}_r\,\mathbb{U}_t\,\Lambda)\big)^{1/2}\,\frac{dt}{t}\,\le\sum_{j=2}^\rho I_j, \tag{8.56}$$

with

$$I_j \overset{\mathrm{def}}{=} \int_{j^{-1}}^{(j-1)^{-1}}\big(\alpha(\mathbb{D}_r\,\mathbb{U}_t\,\Lambda)\big)^{1/2}\,\frac{dt}{t}\,,\quad j=2,3,\dots,\rho\overset{\mathrm{def}}{=}\lceil r\rceil+1. \tag{8.57}$$

Changing variable $t=v\,j^{-2}$ and $v=w+j$ in I_j and using the properties of matrices $\mathbb{D}_r$ and $\mathbb{U}_t$, we have

$$\begin{aligned} I_j &= \int_j^{j^2(j-1)^{-1}}\big(\alpha(\mathbb{D}_r\,\mathbb{U}_{vj^{-2}}\,\Lambda)\big)^{1/2}\,\frac{dv}{v} \\ &\le \int_j^{j+2}\big(\alpha(\mathbb{D}_r\,\mathbb{U}_{vj^{-2}}\,\Lambda)\big)^{1/2}\,\frac{dv}{v} \\ &= \int_0^2\big(\alpha(\mathbb{D}_r\,\mathbb{U}_{wj^{-2}}\,\mathbb{U}_{j^{-1}}\,\Lambda)\big)^{1/2}\,\frac{dw}{w+j}\,. \end{aligned} \tag{8.58}$$

By (8.7),

$$\mathbb{D}_r\,\mathbb{U}_{wj^{-2}} = \mathbb{D}_{rj^{-1}}\,\mathbb{D}_j\,\mathbb{U}_{wj^{-2}} = \mathbb{D}_{rj^{-1}}\,\mathbb{U}_w\,\mathbb{D}_j. \tag{8.59}$$

According to (8.58) and (8.59),

$$I_j \ll \frac{1}{j}\int_0^2\big(\alpha(\mathbb{D}_{rj^{-1}}\mathbb{U}_t\Lambda_j)\big)^{1/2}\,dt, \tag{8.60}$$

where the lattices Λ_j are defined in (8.5) (see also (8.1), (8.3) and (8.4)). Using (8.5), (8.15) and (8.21), we see that

$$\alpha(\Lambda_j) \ll_s \sigma_1^{2s}\,(\det\mathbb{C})^{-1}. \tag{8.61}$$

By (8.39), (8.40) and (8.51), we have

$$\begin{aligned}\alpha(\mathbb{D}_{rj^{-1}}\mathbb{U}_t\Lambda_j) &\ll_s \alpha(\mathbb{D}_{rj^{-1}}\,\mathbb{K}_t\Lambda_j) = \alpha(\mathbb{T}\mathbb{D}_{rj^{-1}}\,\mathbb{K}_t\Lambda_j)\\ &\ll_s \alpha(\widetilde{\mathbb{D}}_{rj^{-1}}\,\widetilde{\mathbb{K}}_\theta\,\Delta_j),\end{aligned} \tag{8.62}$$

for $|t| \ll_s 1$, $r \geq 1$, $j = 2, 3, \dots, \rho$, where Δ_j and θ are defined in (8.42) and (8.45) respectively. Using (8.45), (8.46), (8.49), (8.62) and Lemma 8.2 (with $d = s$), we obtain

$$\begin{aligned}\int_0^2 \big(\alpha(\mathbb{D}_{rj^{-1}}\mathbb{U}_t\Lambda_j)\big)^{1/2}\,dt &\ll_s \int_0^{c^*} \big(\alpha(\widetilde{\mathbb{D}}_{rj^{-1}}\,\widetilde{\mathbb{K}}_\theta\,\Delta_j)\big)^{1/2}\,\frac{d\theta}{\cos^2\theta}\\ &\ll \int_0^{2\pi} \big(\alpha(\widetilde{\mathbb{D}}_{rj^{-1}}\,\widetilde{\mathbb{K}}_\theta\,\Delta_j)\big)^{1/2}\,d\theta\\ &\ll_s \|\overline{\mathbb{D}}_{rj^{-1}}\|^{s/2-2}\,\big(\alpha(\Delta_j)\big)^{1/2},\end{aligned} \tag{8.63}$$

if $s \geq 5$. It is clear that $\|\overline{\mathbb{D}}_{rj^{-1}}\| = rj^{-1}$. Therefore, according to (8.40), (8.42), (8.60) and (8.63),

$$I_j \ll_s \frac{1}{j}\,(rj^{-1})^{s/2-2}\,\big(\alpha(\Lambda_j)\big)^{1/2}. \tag{8.64}$$

By (8.56), (8.61) and (8.64), we obtain, for $s \geq 5$,

$$J \ll_s \sigma_1^s\,(\det\mathbb{C})^{-1/2}\sum_{j=2}^{\rho}\frac{1}{j}\,(rj^{-1})^{s/2-2} \ll_s r^{s/2-2}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2}. \tag{8.65}$$

By (6.3), (6.26), (8.55) and (8.65), we have $r \asymp_s (Np)^{1/2}$ and

$$I_0 \ll_s r^{-2}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2} \ll_s (Np)^{-1}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2}. \tag{8.66}$$

It is clear that in a similar way we can establish that

$$\int_{\sigma_1^{-2}}^{c(s)\sigma_1^{-2}} \big|\,\widehat{\Psi}_b(t/2)\big|\,\frac{dt}{t} \ll_s r^{-2}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2} \ll_s (Np)^{-1}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2}, \tag{8.67}$$

for any quantity $c(s)$ depending on s only. The proof will be easier due to the fact that t cannot be small in this integral.

Thus, we have proved the following lemma.

Lemma 8.3. *Let the conditions of Lemma* 6.3 *be satisfied with* $s = d \geq 5$, $\mathbb{D} = \mathbb{C}^{-1/2}$, $\delta \leq 1/(5s)$ *and with an orthonormal system* $\mathcal{S} = \mathcal{S}_o = \{e_1, \dots, e_s\} \subset \mathbb{R}^d$. *Let* $c_1(s)$ *and* $c_2(s)$ *be some quantities depending on* s *only. Then there exists a* c_s *such that*

$$\int_{c_1(s)\,\sigma_1^{-2}r^{-2+4/s}}^{c_2(s)\,\sigma_1^{-2}} \big|\,\widehat{\Psi}_b(t)\big|\,\frac{dt}{t} \ll_s\ (Np)^{-1}\,\sigma_1^s\,(\det\mathbb{C})^{-1/2}, \tag{8.68}$$

if $Np \gg_s c_s$*, where* r *is defined in* (6.3) *and* (6.26).

References

[1] Bentkus, V., *Asymptotic expansions for distributions of sums of independent random elements in a Hilbert space*, Lithuanian Math. J., **24**, 305–319, 1984

[2] Bentkus, V. and Götze, F., *On the lattice point problem for ellipsoids*, Russian Acad. Sc. Doklady, **343**, 4, 439–440, 1995a

[3] Bentkus, V. and Götze, F., *Optimal rates of convergence in functional limit theorems for quadratic forms*, Preprint 95-091 SFB 343, Universität Bielefeld, 1995b

[4] Bentkus, V. and Götze, F., *Optimal rates of convergence in the CLT for quadratic forms*, Ann. Probab., **24**, 1, 466–490, 1996

[5] Bentkus, V. and Götze, F., *Uniform rates of convergence in the CLT for quadratic forms in multidimensional spaces*, Probab. Theory Rel. Fields, **109**, 3, 367–416, 1997a

[6] Bentkus, V. and Götze, F., *On the lattice point problem for ellipsoids*, Acta Arithmetica, **80**, 2, 101–125, 1997b

[7] Bentkus, V., Götze, F., Paulauskas, V. and Račkauskas, A. *The accuracy of Gaussian approximation in Banach spaces* Preprint 90-100 SFB 343, Universität Bielefeld, 1990 (In Russian: Itogi Nauki i Tehniki, ser. Sovr. Probl. Matem., Moskva, VINITI, **81**, 39–139, 1991)

[8] Bentkus, V., Götze, F. and Zaitsev, A.Yu., *Approximations of quadratic forms of independent random vectors by accompanying laws*, Theory Probab. Appl., **42**, 2, 189–212, 1997

[9] Bentkus, V., Götze, F. and Zitikis, R., *Asymptotic expansions in the integral and local limit theorems in Banach spaces with applications to ω-statistics*, J. Theor. Probab., **6**, 4, 727–780, 1993

[10] Bhattacharya, R.N. and Ranga Rao, R., *Normal Approximation and Asymptotic Expansions*, Wiley New York, 1986

[11] Bogatyrev, S.A., Götze, F. and Ulyanov, V.V., *Non-uniform bounds for short asymptotic expansions in the CLT for balls in a Hilbert space*, J. Multivariate Anal. **97**, 9, 2041–2056, 2006

[12] Cassels, J.W.S., *An introduction to the geometry of numbers*, Springer, Berlin–Göttingen–Heidelberg, 1959

[13] Davenport, H., *Indefinite quadratic forms in many variables* (II), Proc. London Math. Soc., **8**, 3, 109–126, 1958

[14] Esseen, C.-G., *Fourier analysis of distribution functions*, Acta Math., **77**, 1–125, 1945

[15] Fricker, F., *Einführung in die Gitterpunktlehre*, Birkhäuser, Basel–Boston–Stuttgart, 1982

[16] Götze, F., *Asymptotic expansions for bivariate von Mises functionals*, Z. Wahrsch. verw. Geb., **50**, 333–355, 1979

[17] Götze, F., *Lattice point problem and values of quadratic forms*, Invent. Math., **157**, 1, 195–226, 2004

[18] Götze, F. and Margulis G.A., *Distribution of values of quadratic forms at integral points*, Preprint http://arxiv.org/abs/1004.5123, 2010

[19] Götze, F. and Ulyanov, V., *Uniform approximations in the CLT for balls in Euclidian spaces*, Preprint 00-034 SFB 343, Universität Bielefeld, Bielefeld, 2000

[20] Götze, F. and Ulyanov, V., *Asymptotic disrtribution of χ^2-type statistics*, Preprint 03-033, Research group "Spectral analysis, asymptotic distributions and stochastic dynamics", 2003

[21] Götze, F. and Zaitsev, A.Yu., *Uniform rates of convergence in the CLT for quadratic forms*, Preprint 08-119 SFB 701, Universität Bielefeld, Bielefeld, 2008

[22] Götze, F. and Zaitsev, A.Yu., *Uniform rates of approximation by short asymptotic expansions in the CLT for quadratic forms of sums of i.i.d. random vectors*, Preprint 09-073 SFB 701, Universität Bielefeld, Bielefeld, 2009

[23] Hardy, G.H., *The average order of the arithmetical functions $P(x)$ and $\Delta(x)$*, Proc. London Math. Soc. **15**, 2, 192–213, 1916

[24] Landau, E., *Zur analytischen Zahlentheorie der definiten quadratischen Formen*, Sitzber. Preuss. Akad. Wiss. **31**, 458–476, 1915

[25] Lenstra, A.K., Lenstra, H.W.Jr. and Lovász, L., *Factoring polynomials with rational coefficients*, Math. Ann. **261**, 4, 515–534, 1982

[26] Mumford, D., *Tata Lectures on Theta I*, Birkhäuser, Boston–Basel–Stuttgart, 1983

[27] Nagaev S.V., *On a new approach to the study of the distribution of the norm of random element in Hilbert space*, Abstracts of the Fifth Intern. Vilnius Conf. in Probab. Theory and Math. Stat., v. **4**, Mokslas, VSP, Vilnius (in Russian), 77–78, 1989

[28] Nagaev, S.V. and Chebotarev, V.I., *On the accuracy of Gaussian approximation in Hilbert space*, Limit theorems of probability theory (Vilnius, 1999). Acta Appl. Math., **58**, 1-3, 189–215, 1999

[29] Nagaev, S.V. and Chebotarev, V.I., *On the accuracy of Gaussian approximation in Hilbert space*, Siberian Adv. Math., **15**, 1, 11–73, 2005

[30] Petrov, V.V., *Sums of independent random variables*, Springer, Berlin–Heidelberg–New York, 1975

[31] Prawitz, H. *Limits for a distribution, if the characteristic function is given in a finite domain*, Skand. AktuarTidskr., 138–154, 1972

[32] Senatov, V.V. *Qualitative effects in estimates for the rate of convergence in the central limit theorem in multidimensional spaces*, Tr. Mat. Inst. Steklova **215**, 239 pp. 1997; Engl. translation in: Proc. Steklov Inst. Math., **215**, 1–237, 1996

[33] Senatov, V.V. *Normal approximation: new results, methods and problems*, Modern Probability and Statistics. VSP, Utrecht, 1998. viii+363 pp.

[34] Weyl, H., *Über die Gleichverteilung der Zahlen mod-Eins*, Math. Ann., **77**, 313–352, 1915/16

[35] Yurinskii, V.V. *On the accuracy of normal approximation of the probability of hitting a ball*, Theory Probab. Appl., **27**, 2, 270–278, 1982

[36] Zalesskiĭ, B.A., Sazonov, V.V. and Ulyanov, V. V., *Normal approximation in a Hilbert space.* I; II, Theory Probab. Appl. **33**, 2, 207–227; 3, 473–487, 1988

E-mail address: `goetze@math.uni-bielefeld.de`

FAKULTÄT FÜR MATHEMATIK,UNIVERSITÄT BIELEFELD, POSTFACH 100131, D-33501 BIELEFELD, GERMANY

E-mail address: `zaitsevpdmi.ras.ru`

ST. PETERSBURG DEPARTMENT OF STEKLOV MATHEMATICAL INSTITUTE,, FONTANKA 27, ST. PETERSBURG 191023, RUSSIA