

On homogeneity of embedded submanifolds

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Introduction

We discuss in the paper the problem that is connected in their roots with the several variables complex analysis. But its study shows the actuality of the problem itself as well as the questions near to it for the many sections of modern mathematics, using or touching the notion of homogeneity.

To explain the interest of complex analysis to last term recall for example the classical Riemann theorem that asserts the holomorphic equivalence of unit circle to any simply connected domain (with "large" boundary) of complex plane. In the case of several complex variables this theorem is not valid.

One of the reasons of such phenomenon is the holomorphic distinction between the arbitrary domain boundary and the real sphere in \mathbb{C}^n ($n > 1$). Moreover two arbitrary real hypersurfaces of the space \mathbb{C}^n cannot be reduced one to another by biholomorphic transformation. This principle is valid even in local situation. As a consequence two germs of any surface (connected even with near its points) turn as a rule to be inequivalent from the holomorphic point of view.

In this situation the interest to the "exclusive" hypersurfaces is natural, that are "the same" in all its points or (in strong terms) are homogeneous according to the holomorphic transformations.

The problem of description of holomorphically homogeneous real hypersurfaces in 2-dimensional complex spaces had completely solved by E. Cartan in his work [1].

One can note that similar problems (although having no direct connections with the complex analysis) were settled and solved by mathematicians of the end of 19-th and beginning of 20-th centuries. For instance, by Blaschke and his school the description had been obtained [2] for affinely homogeneous plane curves. Surfaces of the 3-dimensional real space that are homogeneous according to the different subgroups of the affine group were described in the middle of the 20-th century. At

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this time these descriptions were involved in the differential geometry handbooks. So the "complete" list of the equiaffine homogeneous surfaces in space \mathbb{R}^3 had published in the book [3].

Note that in holomorphic geometry unlike the affine case the notion of homogeneity turns to be localized in more essential power. Here we deal only with the pseudogroup of locally defined mappings instead the group of global transformations. However one can restrict himself to the compact manifolds and corresponding groups of holomorphic mappings considerations to obtaine global classification results (see, e.g., [4]).

But in a hall we can consider the complex affine homogeneity problem as a partial case of more complicated holomorphic one because any affinely homogeneous submanifold, is, of course, homogeneous in holomorphic sence too.

The interest to the homogeneity questions for the embedded manifolds had revived in the end of the 20-th century. First of all this interest appears in a publication of a lot of differential geometry works dealing with the homogeneity in a "simplest" forms and corresponding study of embedded manifolds invariant structures.

We can call in this connection the papers on affine homogeneity and (in more wide sense) on the affine differential geometry works of such known authors as Shirokov A.P. and Shirokov P.A. [5], Nomizu and Sasaki [6], [7], Opozda B. [8], Simon U. [9], representatives og the Belgium differential geometry school [10], [11].

One understood after publication of these works that the "simple" questions and their "known" answers need be reinterpreted and cheked. So the list [3] of the space \mathbb{R}^3 surfaces that are homogeneous accordingly to the equiaffine transformations turns [6] to be incolmplete. However the publication flow of that time connected (directly or indirectly) with the problem of comlete description of affinely homogeneous surfaces of 3-dimensional real space and written from differential geometry position did not solve this problem.

Apart from the affine geometry direction there were anoter works in homogeneity subjects at that time. It was an algebraic (not geometric !) approach that allows to obtain in [12] the complete description of affinely homogeneous surfaces of 3-dimensional real space. The solution of this problem is based on the description of all the matrix algebras consisting of the 3-d order real square matrix. The work [13] was published devoted to the projective homogeneity of embedded manifolds. In the interesting paper [14] of topologists Schchepin, Skopenkov and Repovsh the smoothness is established for the submanifold posessing initially only "weak" type of homogeneity.

At that time the common analytic approach was proposed by the author of this survey to the homogeneity study of smooth embedded submanifolds. This approach is connected with the canonical (accordingly to given class of transformations) equations of the objects under consideration. As the results show discussed below, it turns to be effective in the considerations via common sheme of affine homogeneity as well as the holomorphic one. One of the peculiarities of its using in different situations consists in the need of tedious preliminary elaboration of detailes. But generally this approach can be usefull under homogeneity study accordingly to the other transformation classes and in another dimensions.

The survey presented below is mainly devoted to the discussion of the results that were obtained by this method in the homogeneity problems in complex spaces of dimensions 2 and 3. At the same time we give a great attention to the Lie algebras techniques; it is just impossible to study the settled problems without it.

The need of Lie algebras techniques use in the homogeneity problems is nowadays evident. But it is clear that the growth of dimension of the manifolds under consideration leads to the technical difficulties increasing.

Note for example that the case of 5-dimensional real hypersurfaces studied below is apparently a "boundary" one between the homogeneity problems having the simply constructing complete set of solutions, and the analogous problems with the "practically boundless" solution sets.

Such location of the problem under consideration does not allow to get its complete solution. However, on the author opinion, the comprehensive description of possible homogeneity situations is presented in the survey, that allows to hope on the getting (in near future) of complete solutions in these situations.

Besides the mentioned goals of this article some more simple models are considered below that have the natural connections with the main topic of the survey.

1 The notion of homogeneity for embedded manifolds

Definition 1. Manifold M is called *homogeneous accordingly to some group* (of transformations) G , if this group acts transitively on M , i.e. every point of M can be transformed into each other point of manifold M by suitable transformation from a group G .

Note that there are different modifications of presented definition in mathematical literature. For example the group G can be initially a Lie group [15]. This condition is convenient from the technical point of view. Another extreme point of view conversely do not demands such rigid structure on G and allows to consider instead of the group G some family (pseudogroup) of transformations. Discussion of the difference in such definitions (or more precisely, their coincidence) in the case of real submanifolds of complex spaces is the subject of interesting work [16].

We introduce also some corrections in initial definition 1.

Depending from the situation one can interesting by:

- a) different manifolds that are homogeneous accordingly to the fixed group;
- b) different groups under action of which a given manifold is homogeneous.

Example 1. One of the simplest examples of homogeneous manifolds is presented by the unit circumference S^1 in the complex plane \mathbb{C} with the transitively acting on it the group of rotations.

Note that the manifold S^1 from this example is *embedded* into the ambient space, namely in \mathbb{C} ; under this the rotation on the arbitrary value of an angle is defined, as an element of a transformation group, not only for the points of S^1 but also for all points of a plane.

Definition 2. Under homogeneity of embedded manifold we mean the homogeneity accordingly to the (given) group of ambient space transformations.

Rotations are the affine transformations of a plane. For this reason we can consider the circumference S^1 from the mentioned example as *affinely homogeneous curve in a plane* \mathbb{R}^2 .

Remark. The manifold S^1 can be called also homogeneous accordingly to the linear (in real or complex sense) or orthogonal transformations of a plane.

Note that the elements of a fixed transformation group G of the space X possessing the conservation property for some additional structure (for instance, embedded in X submanifold M) form a subgroup in G . So it is natural to modify presented definitions in a following manner.

Definition 3. Let X be a space with the transformation group G acting on it, M - embedded in X manifold. We call M homogeneous according to G , if there exists a subgroup H in G that transitively acts on M .

In our further considerations the main ambient spaces will be (affine) spaces \mathbb{R}^n and \mathbb{C}^n with the affine transformation groups $Aff(n, \mathbb{R})$ and $Aff(n, \mathbb{C})$ respectively. The main question we will discuss be the description of affinely homogeneous (in a sense of definitions 1-3) real hypersurfaces of these spaces (especially for $n = 2$ and $n = 3$).

Note that under this moment all homogeneous manifolds under consideration were understood as a *global* objects. Now we will localize our approach and will consider submanifolds of \mathbb{R}^n and \mathbb{C}^n that are *locally homogeneous* accordingly to the groups $Aff(n, \mathbb{R})$ and $Aff(n, \mathbb{C})$ respectively.

Definition 4. Manifold M embedded in \mathbb{R}^n (respectively in \mathbb{C}^n) will be called *locally homogeneous* (accordingly to $Aff(n, \mathbb{R})$ or $Aff(n, \mathbb{C})$ respectively) in some its point p if there exists some Lie subgroup in $Aff(n, \mathbb{R})$ (respectively in $Aff(n, \mathbb{C})$) acting transitively on M near a point p .

Such localization of homogeneity notion allows easily pass from the Lie groups to the local ones and, consequently, to the Lie algebras connected with the corresponding groups. It is clear that the unique considered example (as well as any "global" manifold) is also a locally homogeneous manifold in every its point.

It is easy to construct the complete list (up to affine equivalence) of all plane curves that are locally homogeneous in a sense of definition 4. For instance the following statement is known (see [5] and [17]).

THEOREM 1. *Every plane affinely homogeneous curve is affinely equivalent near arbitrary its point to one of the following list of affinely different curves:*

- 1) $y = x^s$ ($-1 \leq s < 1$), (1.1)
- 2) $y = \ln x$,
- 3) $y = x \ln x$,
- 4) $r = e^{a\varphi}$ (r is a polar radius, φ is a polar angle, $a \geq 0$).

One can give another reformulations of this statement; the ways leading to such results are also different. For example in [5] this problem is reduced to the description of the ODE linear system solutions of the kind

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.2)$$

(with the constant matrix A).

In accordance with the Jordan normal form of a matrix A every integral curve of the system (1.2) is locally affinely equivalent to one of the curves of the list (1.1).

From the point of view of canonical equations the answer in the mentioned question is looking (see [18], [17]) also elegant. All convex analytical curves (except a parabola) allows the affine equations of the form

$$y = x^2 \pm x^4 + bx^5 + \dots \quad (b \geq 0). \quad (1.3)$$

Under this any affinely homogeneous curve is uniquely determined by a sign of x^4 and a value of parameter b from the equation (1.3). Complete set of all homogeneous curves is straightened in a form of two rays corresponding to the variation of b from zero to infinity.

On the part $0 \leq b < 8/5$ of the first ray the logarithmic spirals are settled and the another interval $(8/5, \infty)$ is covered by the power curves $y = x^s$ ($1/2 < s < 1$). The curve $y = x \ln x$ separates these two parts.

On the second ray (connected with the sign "minus" under x^4) all other power curves are settled and under $b = 2\sqrt{2}/5$ this set is "punctured" by the curve $y = e^x$ (or $y = \ln x$).

However it's clear that the dimension growth leads to the more difficulties in the homogeneous manifolds description. Besides the geometric and analytic methods mentioned above here the Lie algebras technique is especially effective.

Note that the system (1.2) can be considered as the description of the simplest, 1-dimensional, Lie algebras connected with the affinely homogeneous plane curves.

The next situation corresponds to affinely homogeneous 2-dimensional surfaces in 3-dimensional real space, and here the Lie groups and Lie algebras arise having the dimension greater or equal 2. It were these algebras consisting of square matrixes of the 3-d order that have been studied and completely described in a work [12]. As a consequence the complete description of the affinely homogeneous surfaces in this case had been constructed by these authors.

This description is more complicated in comparison with the Theorem 1. It contains (besides the 2-d order surfaces that are automatically homogeneous in all cases) 18 types of affinely different manifolds.

It is interesting to mention the generality of the homogeneity property in two simplest cases. For instance, analogously with the power curves $y = x^\alpha$ from the theorem 1 the family of power affinely homogeneous surfaces

$$z = x^\alpha y^\beta, \quad (\alpha, \beta \in \mathbb{R})$$

there exists in the space \mathbb{R}^3 . To the family of plane logarithmic spirals the affinely homogeneous surfaces

$$z = (x^2 + y^2)^\alpha e^{i\beta \arg(x+iy)}, \quad (\alpha, \beta \in \mathbb{R})$$

of the space \mathbb{R}^3 correspond.

Note in addition that the list of [12] contains also 1-parameter and discrete (pointwise) subsets besides the mentioned examples of 2-parameter subfamilies of homogeneous surfaces. The "quantity"-question for homogeneous manifolds is also interesting and rather difficult one in common problem of their description in the another situations.

Coming to the examples of the homogeneous submanifolds of the complex spaces it is natural to recall firstly *tube manifolds*. The submanifold M of the space \mathbb{C}^n of arbitrary dimension n is called a tube or tubular manifold if it presents in a form

$$M = \Gamma + i\mathbb{R}^n$$

where Γ is a submanifold of \mathbb{R}^n .

Large class of affinely homogeneous real hypersurfaces in complex space \mathbb{C}^n is formed by tube manifolds with the real affinely homogeneous bases Γ .

Under this the lists of all affinely homogeneous hypersurfaces of the spaces \mathbb{R}^2 and \mathbb{R}^3 are known. So we have a big family of manifolds given in a simply explicit manner and presenting a solutions of our homogeneity problem both in affine and in holomorphic settlements. Let us note however one important moment that decreases the tube family validity in similar considerations.

As it is well known (see, e.g. [19], [17]) the tubes over two different affinely homogeneous plane curves $y = e^x$ and $y = x^2$ are locally holomorphically equivalent to the 3-dimensional sphere $|z_1|^2 + |z_2|^2 = 1$, and therefore one to another. The similar effects there are also for the tubes in 3-dimensional complex space. The possibility or nonexistence of (complex) affine equivalence for the tubes over the affinely different real surfaces did not still checked for anyone.

It means that it is the complex geometry in terms of which the description of the family of affinely homogeneous hypersurfaces of the 3-dimensional complex space must be obtained. The means of the pure real geometry of the space \mathbb{R}^3 are in general insufficient ones for this purpose.

2 Holomorphic homogeneity of real hypersurfaces

In comparison with the affine considerations holomorphic homogeneity case became to be more complicated for study. The fundamental work in this direction is a E. Cartan paper mentioned above that devoted to the description of holomorphically homogeneous real hypersurfaces of 2-dimensional complex spaces.

In spite of big volume of this work its main result can be formulated from the local point of view in a simple form.

THEOREM 2. *Any holomorphically homogeneous real hypersurfaces of 2-dimensional complex space is holomorphically equivalent near each its point either*

to the tube over affinely homogeneous real base (see Theorem 1), or to one of the projectively homogeneous surfaces

$$1 + |z|^2 + |w|^2 = a|1 + z^2 + w^2| \quad (a > 1), \quad (2.1)$$

$$1 + |z|^2 - |w|^2 = a|1 + z^2 - w^2| \quad (a > 1), \quad (2.2)$$

$$|z|^2 + |w|^2 - 1 = a|z^2 + w^2 - 1| \quad (0 < |a| < 1). \quad (2.3)$$

During the end of 1990 still our days essential efforts was made by the author of this work to obtain the analogous classification of holomorphically homogeneous real hypersurfaces of 3-dimensional complex spaces.

It reveals however that the passage from the studied case to the 3-dimensional ambient complex space is connected with the essential growth of the difficulties. Now the lot of partial results is obtained in connection of the stated problem. These results are interesting for the waited (in the near future) solution of the problem and for the understanding of different approaches to this problem and similar ones.

The author is working in the frame of coefficients approach to the embedded submanifolds homogeneity problem connected with the using of canonical equations of manifolds under consideration. This method is applicable to the different situations (see, for instance §1). In the work [] the Cartan's classification result (Theorem 2) is obtained via canonical equations study for the real hypersurfaces in space \mathbb{C}^2 .

Using of this approach in 3-dimensional case allows to describe all holomorphically homogeneous real hypersurfaces with reach symmetry groups. The defining equations for such surfaces turn to be more symmetric than the common ones and this fact enables to get their constructive description.

We show here some examples of classification results related with the *strictly pseudo convex* (SPC) hypersurfaces of the space \mathbb{C}^3 .

Proposition 1. ([21]) *If real analytic strictly pseudo convex hypersurface M of the space \mathbb{C}^3 is holomorphically inequivalent to the sphere then the dimension of its local holomorphic transformation group $Aut(M)$ does not exceed 7.*

Remark 1. Dimension of real hypersurface M of the space \mathbb{C}^3 equals 5. Hence the isotropy group $Aut_0(M)$, i.e. the subgroup of $Aut(M)$ preserving some fixed point of M , has in considered analytic SPC-case dimension that is less or equal to 2.

Remark 2. In 2-dimensional complex space every hypersurface that is non-degenerate in Levi sense becomes to be automatically strictly pseudo convex. In 3-dimensional case there is a big family of the surfaces, for which the Levi form is nondegenerate sign-indefinite Hermitian form. It is natural to call such surfaces *indefinite* ones unlike the SPC-case.

Remark 3. For the indefinite surfaces of the space \mathbb{C}^3 proposition 1 is not valid: there is (unique up to holomorphic transformations) analytic surface

$$Imw = z_1\bar{z}_2 + z_2\bar{z}_1 + |z_1|^4 \quad (2.4)$$

with the 8-dimensional group $Aut(M)$.

The surface (10) was constructed by Winkelmann [22] in connection with the study of homogeneous domains in 3-dimensional complex spaces. Accordingly to holomorphic transformation of the space \mathbb{C}^3 this surface is homogeneous itself and separates this space on two parts that are holomorphically homogeneous too.

THEOREM 3 ([23]). *Near each its point every homogeneous real hypersurface of the space \mathbb{C}^3 with positively defined Levi form and 2-dimensional isotropy group is holomorphically equivalent to one of the following pairwise inequivalent homogeneous manifolds:*

$$v = \ln(1 + |z_1|^2) + b \ln(1 + |z_2|^2), \quad b \in (0, 1]; \quad (2.5)$$

$$v = \ln(1 + |z_1|^2) - b \ln(1 - |z_2|^2), \quad b \in (0, 1) \cup (1, \infty); \quad (2.6)$$

$$v = \ln(1 - |z_1|^2) + b \ln(1 - |z_2|^2), \quad b \in (0, 1]; \quad (2.7)$$

$$v = |z_2|^2 + \varepsilon \ln(1 + \varepsilon |z_1|^2), \quad \varepsilon = \pm 1. \quad (2.8)$$

This theorem is proved by using of the canonical (normal in the sense of Moser [24]) equations of homogeneous surfaces. As the first step of the proof an important preliminary result must be obtained about the quantity of holomorphically homogeneous surfaces. This result is formulated in terms of Taylor coefficients of the canonical equations.

THEOREM 4 ([25]). *The germ of every strictly pseudo convex homogeneous hypersurface of 3-dimensional complex space is uniquely determined by the Taylor coefficients not more than 7-th order of its normal equation.*

Result similar to the theorem 3 had obtained under description of holomorphically homogeneous SPC-hypersurfaces of 3-dimensional complex space having 1-dimensional isotropy groups. The quantity of distinct holomorphically homogeneous manifolds in this case turns some greater than in case of 2-dimensional groups.

As it supposed the largest class in quantitative sense must be in the case of homogeneous hypersurfaces with discrete isotropy groups. Situation with the homogeneity of indefinite surfaces is near to the SPC-case. Here in addition to the Winkelmann example (2.4) the complete description was constructed [26] of the homogeneous hypersurfaces having 2-dimensional isotropy groups. Also some families had obtained of the homogeneous manifolds with 1-dimensional groups $Aut(M)$.

To construct the full description of holomorphically homogeneous hypersurfaces of 3-dimensional complex spaces one need in essence to understand the structure of the family of *common position* homogeneous surfaces having discrete isotropy groups.

Taking in mind all the facts mentioned above it is natural to pose the problem of the description of all affinely homogeneous real hypersurfaces of the space \mathbb{C}^3 (having for instance exactly 5-dimensional groups $Aff(M)$).

In the process of this problem solving a lot of important results was obtained. Simultaneously the following moment was recognized: despite of the holomorphic (Cartan's) classification existence for the case of 2-dimensional complex spaces the similar one for the affinely homogeneous real hypersurfaces of the space \mathbb{C}^2 did not still constructed. Scarcely the solving is possible of analogous but more complicated problem for 3-dimensional case is in this situation.

After this conclusion the last years efforts were concentrated on the parallel consideration of two questions connected with the affine homogeneity of real hypersurfaces of the space \mathbb{C}^2 and \mathbb{C}^3 .

3 Affinely homogeneous real hypersurfaces in \mathbb{C}^2

At this moment the description problem for affinely homogeneous real hypersurfaces of the space \mathbb{C}^2 is practically solved. Preliminary classification results were presented at two international conferences: "Almost Complex Geometry and Foliations" (Lille, May-2010) and "Metric geometry of surfaces and polyhedra" (Moscow, August-2010).

Note that degeneracy or nondegeneracy condition for the surface mentioned above has in affine geometry lesser significance in comparison with the holomorphic geometry. However approach we use differs two these big families of real submanifolds of the space \mathbb{C}^2 . It is this reason that explains the separation of results in this section of our paper. We present here complete classification of nondegenerate homogeneous real hypersurfaces and give some results related with the classification of degenerate homogeneous manifolds.

THEOREM 5. *Every Levi nondegenerate affinely homogeneous real hypersurface of the space \mathbb{C}^2 is affinely equivalent (near each its point) to one of the homogeneous surfaces of the following list:*

- 1) $|z|^2 \pm |w|^2 = 1$,
- 2) $Im w = |z|^2 + \varepsilon(z^2 + \bar{z}^2)$, $0 < \varepsilon \neq \frac{1}{2}$,
- 3) $Im w = |z|^A e^{B \arg z}$, $A, B \in \mathbb{R}, z \neq 0$,
- 4) $Im(z\bar{w}) = |z|^A e^{B \arg z}$, $A, B \in \mathbb{R}, z \neq 0$,
- 5) $Re w = (Rez)^s$, $-1 \leq s < 1$,
- 6) $Re w = \ln(Re z)$,
- 7) $Re w = (Re z) \ln(Re z)$,
- 8) $\ln((Re z)^2 + (Re w)^2) = B \cdot arctg\left(\frac{Re w}{Re z}\right)$, $B \geq 0$.

Note for the comparison that the set of Levi-degenerate affinely homogeneous hypersurfaces of the space \mathbb{C}^2 contains (see Theorem 6 below) 3-parameter family

$$|z|^{A_1} |w|^{A_2} = e^{\arg(z^{B_1} w^{B_2})}, \quad (A_1, A_2, B_1, B_2) \in \mathbb{RP}^3, \quad (3.1)$$

and turns greater than the set of nondegenerate manifolds.

Underline in this connection that our results can not be obtained from the Cartan's classification [1] of holomorphically homogeneous real hypersurfaces of 2-dimensional complex spaces that contains only 1-parameter and discrete families of surfaces.

The testing of homogeneity for each surface from the theorem 5 presents no difficulties. So the main matter of this theorem is the completeness statement for the presented list. Proving this completeness we essentially use like the authors of many similar works (see e.g. [4], [12]) Lie algebras technique. But the main pivot of our work is the using of the canonical equations for the homogeneous surfaces under consideration.

Let us consider the notion of affine canonical equation for the real analytic hypersurface of complex space \mathbb{C}^2 . Denoting the coordinates in this space via z, w we will define such surfaces (near its nonsingular points) by the equations of the form

$$Im w = F(z, \bar{z}, Re w),$$

solved with respect to one of the real variables. Using the additional notations $Re w = u, Im w = v$ we can simplify it to

$$v = F(z, \bar{z}, u). \quad (3.2)$$

Analytic function

$$F(z, \bar{z}, u) = \sum_{k,l,m \geq 0} f_{klm} z^k \bar{z}^l u^m \quad (3.3)$$

from the right hand side of equation (3.2) can be "simplified" by virtue of affine transformations of the space \mathbb{C}^2 .

For instance it is easy to remove the linear terms from it so that equation (3.2) goes to the *reduced* form

$$v = f_{110}|z|^2 + f_{200}z^2 + f_{020}\bar{z}^2 + f_{101}zu + f_{011}\bar{z}u + \dots, \quad (3.4)$$

and the fixed point of a surface moves to the origin.

From the point of view of complex geometry real coefficient f_{110} and Hermitian form $f_{110}|z|^2$ are the main issues in the equation (3.4). This form is called (see [27]) a *Levi form* of a surface (3.4). If the coefficient f_{110} vanishes the surface under consideration is called *Levi-degenerate (Levi-flat)* at the origin, in the opposite case *Levi-nondegenerate*.

In the situation of Levi-nondegeneracy the coefficient f_{110} from (3.4) can be transformed into unity by means of scaling $z \rightarrow tz$ ($t > 0$) and (possibly) of symmetry $w \rightarrow -w$. The remaining terms of equation (3.4) having the second order with respect to z, \bar{z} also can be simplified (by rotation in \mathbb{C}_z -plane) so the coefficient f_{110} will be nonnegative.

The following change of variables $z \rightarrow z + Aw$ with suitable value of parameter A simplifies essentially the coefficients f_{101} and f_{011} of the discussed equation (3.4). As the base for further discussion that summarized partial simplifications of this equation we will use the following statement and remark to it.

Proposition 2. *Equation (3.4) of Levi nondegenerate hypersurface can be transformed by affine mapping to the form*

$$v = |z|^2 + \varepsilon(z^2 + \bar{z}^2) + i\alpha(z - \bar{z})u + (f_{300}z^3 + f_{210}z^2\bar{z} + f_{120}z\bar{z}^2 + f_{030}\bar{z}^3) + \sum_{k+l+2m \geq 4} F_k(z, \bar{z}, u), \quad (3.5)$$

where

$$\varepsilon \geq 0, \quad \alpha \in \{0, 1\}. \quad (3.6)$$

Remark. If a) $0 \leq \varepsilon \neq 1/2$, then one can use additional restriction $\alpha = 0$; if b) $\varepsilon = 1/2$, then in addition to (3.6)-condition on α the restriction

$$if_{210} \in \mathbb{R}. \quad (3.8)$$

holds.

Note that nonnegative coefficients ε and α from the equation (3.5) are affine invariants of the surface. We will call in further such equation a *canonical* one. At the same time revise that by using of the scaling

$$z \rightarrow tz, \quad w \rightarrow t^2w, \quad (t > 0), \quad (3.9)$$

that preserves the form (3.5) and condition $\alpha = 0$ some additional restrictions on the coefficients of this equations are possible.

Now discuss the Lie algebras of linear vector fields tangent to the homogeneous hypersurfaces under consideration. Every such algebra consists from the infinitesimal transformations corresponding to the transitive action of some affine subgroup on our surface.

Let the linear vector field

$$Z = (A_1z + A_2w + p) \frac{\partial}{\partial z} + (B_1z + B_2w + q) \frac{\partial}{\partial w}, \quad (3.10)$$

in the space \mathbb{C}^2 is given tangent to the surface M of the form (3.5).

The fact of tangency can be written as a standart condition (*base identity*)

$$Re\{Z(\Phi)\}_{|M} \equiv 0, \quad (3.11)$$

where Φ is a defining function of a surface.

If this function has a (canonical) type

$$\Phi = -v + F(z, \bar{z}, u)$$

then we get from (3.11) the following formula

$$Re \left\{ p + (A_1z + A_2(u + iF)) \frac{\partial F}{\partial z} + (q + B_1z + B_2(u + iF)) \frac{1}{2} \left(i + \frac{\partial F}{\partial u} \right) \right\} \equiv 0 \quad (3.12)$$

Left hand side of the base identity depends analytically from the canonical equation coefficients and parameter of the field (3.10). Under this parameter p may take any complex and parameter q - any real values.

In other words p and q are *free parameters* of Lie algebra $g(M)$ for every homogeneous surface. In a whole the number of such parameters is equal to real dimension of the algebra $g(M)$ that is an important characteristic of the homogeneous surface M .

From the vanishing conditions for the lower Taylor coefficients of Left hand side of the identity (3.12) one can get preliminary informanion about studied algebra. This algebra can be considered as a matrix one consisting of complex-valued square matrix of 3-d order. The checking of the closedness conditions for the matrix brackets in algebra under consideration leads to some possible partial cases.

On the final step one need integrate obtained algebras and this step leads to the surfaces from the Theorem 5. Using of canonical equations posessing uniqueness property with respect to the affine transformations guarantees affine inequivalence all the surfaces obtained by this method.

Description of the Levi degenerate affinely homogeneous surfaces of the space \mathbb{C}^2 is building by the same scheme. Now the final completeness and uniqueness verifications are fulfilled for the surfaces from this part of classification. For this reason the following statement can be regarded, strictly speacking, only as a big collection of examples.

THEOREM 6. *All Levi degenerate surfaces from the following list are affinely homogeneous hypersurfaces of the space \mathbb{C}^2 :*

- a) *real hyperplane $Im w = 0$,*
-) *cylinder over hyperboloid $Im(w - z^2) = 0$,*
-) *family of "parabolically foliated" spiral surfaces*

$$|w - z^2| = e^{B \arg(w - z^2)}, \quad B \in \mathbb{R}, \quad (3.13)$$

-) *3-parameter family of surfaces*

$$|z|^{A_1} |w|^{A_2} = e^{\arg(z^{B_1} w^{B_2})}, \quad (A_1, A_2, B_1, B_2) \in \mathbb{R}\mathbb{P}^3, \quad (3.14)$$

)

$$v = e^{-2i\theta} \ln(1 + e^{i\theta} z) + e^{2i\theta} \ln(1 + e^{-i\theta} z), \quad \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right]. \quad (3.15)$$

e)

$$Re(\bar{w} e^{i\theta} (z - e^{i\theta} w \ln w)) = 0, \quad \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right]. \quad (3.16)$$

Note that some of the Theorem 6 examples were obtained during the author's visit in Bielefeld at hot July 2010.

Results related to the affine homogeneity in space \mathbb{C}^2 should clarify more complicated situation in 3-dimensional complex spaces. Some statements connected with the description of affinely homogeneous real hypersurfaces of the space \mathbb{C}^3 are listed in the following section of the paper.

4 Affinely homogeneous real hypersurfaces in \mathbb{C}^3

Let us consider real analytic hypersurface M in the space \mathbb{C}^3 with fixed nonsingular point on it. Equation of M can be resolved with respect to one of the real variables and written by using of power series in a form

$$v = F(z, \bar{z}, u) = \sum_{k,l,m \geq 0} F_{klm}(z, \bar{z}) u^m. \quad (4.1)$$

Here $z = (z_1, z_2)$, w are the coordinates in the space \mathbb{C}^3 , $u = \operatorname{Re} w$, $v = \operatorname{Im} w$, F_{klm} is homogeneous polynomial, whose degree in z and \bar{z} variables equals to k and l respectively. The simplest terms of the right hand side of this equation can be easily "killed" by suitable affine transformation (like the case of \mathbb{C}^2).

Note that the power series in traditional "uniform" sense are awkward enough for the construction of canonical equations in spirit of Moser [24]. Everywhere in this paper we use canonical equations that are similar to Moser normal form based on weighted expansions of analytic functions. We assume that the variables $z_1, z_2, \bar{z}_1, \bar{z}_2$ have weight 1 and the variable u , weight 2. The weights of the monomials constructed from the variables $z_1, z_2, \bar{z}_1, \bar{z}_2, u$, are defined by the natural principle of weight addition. For example, the weight of the monomial $z_1 u$ is equal to 3.

Extracting the homogeneous weighted components in the power expansion of the function $F(z, \bar{z}, u)$, we can obtain the following equation of the surface M :

$$v = H(z, \bar{z}) + Q(z) + \overline{Q(z)} + \sum_{k \geq 3} F_k(z, \bar{z}, u), \quad (4.2)$$

where $H(z, \bar{z})$ is a Hermitian form, $Q(z)$ is quadratic form, $F_k(z, \bar{z}, u)$ - homogeneous component of weight k .

Our further discussions are separated on the cases relative to the properties of Hermitian form $H(z, \bar{z})$ from (4.2).

It is natural to extract here 3 cases:

- a) H is positively (or negatively) defined,
- b) H is nondegenerate indefinite form,
- c) H is degenerate form.

The first of this cases corresponds to SPC real hypersurfaces; in the last one we call corresponding surface *Levi degenerate*; surfaces from b) are *indefinite* ones.

Recall that the condition a)-c) are holomorphically (and hence, affinely) invariant.

Homogeneity property in holomorphic sense for Levi degenerate real hypersurfaces of 3-dimensional complex spaces is completely studied in recent big work [28]. In particular, all such surfaces are reducible by (local) holomorphic mappings either to products of Cartan's homogeneous 3-dimensional hypersurfaces (see Theorem 2) on complex plane \mathbb{C} , or to affinely homogeneous (degenerate) hypersurfaces of the space \mathbb{C}^3 .

The last list is (up to holomorphic mappings) brief enough. But in the affine classification problem for such manifolds the appearance is natural of a lot of different representatives, as the previous section considerations show. Question of degenerate affinely homogeneous hypersurfaces of \mathbb{C}^3 description is not still considered.

We consider below only cases a) and b) related with the Hermitian form H . In the case a) the following statement [29] holds.

7. *Equation of the real analytic hypersurface M of the space \mathbb{C}^3 that is strictly pseudo convex in some its point, can be reduced by affine transformations to the form*

$$v = (|z_1|^2 + |z_2|^2) + (\varepsilon_1 z_1^2 + \varepsilon_2 z_2^2) + \overline{(\varepsilon_1 z_1^2 + \varepsilon_2 z_2^2)} + \sum_{k \geq 3} F_k(z, \bar{z}, u). \quad (4.3)$$

The pair $(\varepsilon_1, \varepsilon_2)$ of real nonnegative numbers is here the affine invariant of the surface M .

This theorem contains the transformation of Hermitian form H in canonical coordinates. Such reduction presents a simple exercise from the linear algebra. The main part of the theorem 7 constitutes simultaneous reduction to (some) canonical type for the pair of forms; one of this form is Hermitian and another is quadratic form on two complex variables.

The problem of such reduction had been studied in work [30]. However the priority in this work had quadratic form whereas from the point of view of complex geometry Hermitian form is more important one.

The case b) of indefinite real hypersurfaces turns to be more complicated. Here the following recently obtained theorem [31] is valid.

THEOREM 8. *Let M be a real analytic indefinite hypersurface of the space \mathbb{C}^3 given by the equation*

$$v = (|z_1|^2 - |z_2|^2) + (Q(z) + \overline{Q(z)}) + \sum_{k \geq 3} F_k(z, \bar{z}, u). \quad (4.4)$$

The form $Q(z)$ can be transformed by affine mapping preserving the origin and equation type (4.4), to the one and only one of the following types:

$$Q(z) \equiv 0; \quad (4.5)$$

$$Q(z) = \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2, \quad \varepsilon_1 \geq \varepsilon_2 \geq 0, \varepsilon_1^2 + \varepsilon_2^2 \neq 0; \quad (4.6)$$

$$Q(z) = \varepsilon z_1(z_1 + z_2), \quad \varepsilon > 0 \quad Q(z) = (z_1 + z_2)^2 \quad (4.7)$$

$$Q(z) = \varepsilon_1(z_1^2 - z_2^2) + \varepsilon_2 z_1 z_2, \quad \varepsilon_1 \geq 0, \varepsilon_2 > 0. \quad (4.8)$$

The main idea of canonical equations using for the description of homogeneous manifolds is the same everywhere in this paper: any such manifold is uniquely determined by the finite collection of the lower Taylor coefficients of its canonical equation.

From the practical point of view such reconstruction of the surface via *base* coefficient collection is, of course, a difficult problem. As an important intermediate step we use here (and everywhere) the study of Lie algebra of vector fields (linear in this case), tangent to the homogeneous surface under consideration. Such algebra can be presented in matrix form if we replace a vector field

$$Z = (a_{11}z_1 + a_{12}z_2 + a_{13}z_3 + p) \frac{\partial}{\partial z_1} + (b_{11}z_1 + b_{12}z_2 + b_{13}z_3 + s) \frac{\partial}{\partial z_2} + (c_{11}z_1 + c_{12}z_2 + c_{13}z_3 + q) \frac{\partial}{\partial z_3}. \quad (4.9)$$

by a square complex-valued matrix

$$Z = \begin{pmatrix} a_{11} & a_{12} & a_{13} & p \\ a_{21} & a_{22} & a_{23} & s \\ a_{31} & a_{32} & a_{33} & q \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.10)$$

Under this change the matrix bracket (commutator)

$$[Z_1, Z_2] = Z_1 Z_2 - Z_2 Z_1,$$

corresponds to the bracket of vector fields, and dimension of the algebra is preserved.

The basis for the getting of the results described in this paper section is again the study of matrix Lie algebras satisfying the base relation

$$Re\{Z(\Phi)\}_{|M} = 0. \quad (4.11)$$

Let, for instance, homogeneous surface M is given by canonical equation (4.3) or (4.4)

$$v = F_2(z, \bar{z}) + \sum_{k \geq 3} F_k(z, \bar{z}, u), \quad \text{where} \quad F_2(z, \bar{z}) = H(z, \bar{z}) + Q(z) + \overline{Q(z)}.$$

Extracting in the identity (4.11) the lower weights components, one can obtain the following conditions:

$$\text{weight } 0 : Re\left(\frac{i}{2}q\right) = 0; \quad (4.12)$$

$$\text{weight } 1 : Re\left(p\frac{\partial F_2}{\partial z_1} + s\frac{\partial F_2}{\partial z_2} + \frac{i}{2}(a_{31}z_1 + a_{32}z_2)\right) = 0. \quad (4.13)$$

Thus the study of homogeneous surfaces can be reduced to the description problem for the matrix algebras consisting of the matrixes of given size and satisfying some additional restrictions, for instance, of the kind (4.12) - (4.13).

The special structure of matrixes, constituting the discussed algebras, is one of this restrictions if we use the canonical equations. Because the set of all affinely homogeneous real hypersurfaces of the space \mathbb{C}^3 is very unambiguous, one can start a considerations from the family of *rigid* homogeneous surfaces (the equation of rigid surface does not depend from one of the real variables). As it was shown in [29], [31] in this case the elements of algebras under consideration are simplifying to the form

$$\begin{pmatrix} A_1 & A_2 & 0 & p \\ B_1 & B_2 & 0 & s \\ a & b & c & q \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.14)$$

The study of matrix algebras with elements of such kind allows the interesting geometric interpretation. Two-dimensional complex space $\mathbb{C}_{z_1, z_2}^2 = \{w = 0\}$ is a complex tangent space at the origin to the surface given by the canonical equation (4.2).

In the case of homogeneity near the origin every surface with such equation has the algebra of tangent vector fields, defined in a full-dimensional neighborhood of this point. Left upper 2x2 blocks of the matrixes, corresponding to the fields from this algebra, form themselves a Lie algebra. One can assume that it acts in complex tangent space to the homogeneous surface. In such a way the idea appears

to study the matrix Lie algebras continuation from 2-dimensional complex space to the algebras of matrix of a kind (4.14).

This proposal can be regarded as a natural one if we want to simplify numerous computations related with the multidimensional problem. A new, sometimes unexpected, tools are using for its solving.

For realization of this idea the classification had been constructed [32] of the matrix (real) Lie algebras, consisting of the complex matrixes of 2-d order (with real dimension from 0 to 4, interesting in the homogeneity problem). Mention here one of the results of cited work.

THEOREM 9. *Every real 3-dimensional subalgebra of Lie algebra $M(2, \mathbb{C})$ is similar either to $su(2)$, or to $su(1, 1)$, or to some algebra of upper triangular matrixes. Complete list of all (up to matrix similarity) real 3-dimensional Lie subalgebras of $M(2, \mathbb{C})$ is cited below (α, β, γ are the coordinates in algebra):*

- 1) diagonal algebras;
- 2) algebras, whose all elements have a multiple cpectrum:

$$S^{(1)} = \left\{ \begin{pmatrix} \alpha + i\beta & \gamma \\ 0 & \alpha + i\beta \end{pmatrix} \right\};$$

$$S_{\theta}^{(2)} = \left\{ \begin{pmatrix} \alpha e^{i\theta} & \beta + i\gamma \\ 0 & \alpha e^{i\theta} \end{pmatrix} \right\}; \quad S_{\theta}^{(3)} = \left\{ \begin{pmatrix} (\alpha + i\beta)e^{i\theta} & \beta + i\gamma \\ 0 & (\alpha + i\beta)e^{i\theta} \end{pmatrix} \right\}, \theta \in [0, \pi);$$

- 3) nondiagonalizable algebras containing matrixes with the symple spectrum:

$$f_{\theta}^{(1)} = \left\{ \begin{pmatrix} \alpha e^{i\theta} & \beta \\ 0 & (\alpha e^{i\theta} + \gamma) \end{pmatrix} \right\}; \quad f_{\theta}^{(2)} = \left\{ \begin{pmatrix} 0 & (\alpha + i\beta) \\ 0 & \gamma e^{i\theta} \end{pmatrix} \right\}, \theta \in [0, \pi);$$

$$f_{(\varphi, \lambda)}^{(3)} = \left\{ \begin{pmatrix} \alpha e^{i\varphi} & (\beta + i\gamma) \\ 0 & \alpha \lambda \end{pmatrix} \right\}, \varphi \in [0, \pi), \lambda \neq e^{i\varphi};$$

$$f_{(\mu, \xi)}^{(4)} = \left\{ \begin{pmatrix} (\alpha + i\beta) & \gamma \\ 0 & (\alpha + i\beta) + (\xi\beta - \mu\alpha) \end{pmatrix} \right\}, (\mu, \xi) \in \mathbb{R}^2 \setminus \{0\}.$$

The study of possible continuations for the algebras from above list to the algebras, related with the affinely homogeneous real hypersurfaces of the space \mathbb{C}^3 give the following result [33].

THEOREM 10. *Lie algebra $g(M)$, related with the rigid affinely homogeneous common position SPC-hypersurface M of the space \mathbb{C}^3 , is a continuation of algebra $h \subset M(2, \mathbb{C})$ of one from 3 types:*

- 1) $\dim_{\mathbb{R}} h = 2$, then h is diagonalizable algebra;

$$2) \dim_{\mathbb{R}} h = 3, \text{ then } h \sim \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \right\rangle; \quad (4.15)$$

$$3) \dim_{\mathbb{R}} h = 4, \text{ then } h \sim \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right\rangle. \quad (4.16)$$

All three types of algebras are realizable on homogeneous surfaces.

It follows, in particular, from the theorem 10 that the discrete and 1-dimensional subalgebras of $M(2, \mathbb{C})$ do not allow a continuation, related with the affinely homogeneous SPC-hypersurfaces of common position. In 2-dimensional case only a continuations for diagonal subalgebras are not forbidden.

Note that using a continuation procedure for matrix Lie algebras one can estimate the quantity of different (up to matrix similarity) Lie algebras of demanded structure. As a consequence some estimates for the number of affinely different affinely homogeneous SPC-hypersurfaces of common position can be obtained. For instance, the following statement [34] is valid.

THEOREM 11. *There exists a 3-parameter family of affinely different affinely homogeneous real hypersurfaces of the space \mathbb{C}^3 , whose linear vector field algebras are the continuations of the algebra (4.15).*

The continued algebras themselves have tedious form and, for this reason, are not listed here. Note that theorem 11 had proven initially by the author of this paper for the SPC-hypersurfaces, and after this the analogous continuations for indefinite case have been obtained by Danilov M.S. Under this the analogous 3-parameter surface families appear in elliptic and hyperbolic types. Intermediate parabolic case turns out to be essentially poor on the representative number (see [31]).

Integration of the algebras, obtained by using of this method, has also many technical difficulties. It leads to the families of affinely homogeneous algebraic hypersurfaces of 6-th, 4-th or 3-d order, whose algebras of infinitesimal transformations are the cotinuations of algebra (4.15).

One of the last technical ideas, that permit to simplify obtained results, is the following. For homogeneous surface given by canonical equation, related matrix algebra has a form, that is just tedious. Reduction to the similar algebra may simplify its representation. For instance, matrix algebra family from theorem 11 can be transformed by using of matrix similarity to the family of upper triangular algebras with the basis of special kind.

The number of parameters of new family increases after such transition (because of the refusal from the canonical type restriction for the surface equation). At the same time the algebra families arise here, depending in a simple manner from 3 (for example) real parameters.

The basis of such family of algebras is listed below (m, t, r are the real parameters):

$$E_1 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & mr \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & (im - t) & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & r(im + t) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.17)$$

Integration of such algebras leads, naturally, to the algebraic surfaces mentioned above (up to affine mappings).

It is interesting to note that analogous algebraic surfaces of 4-th and 6-th order were obtained in a work [35], where holomorphically homogeneous real submanifolds of 3-dimensional complex spaces were studied. However the starting point for this work was the holomorphically homogeneous submanifold of codimension 2 with "large" group of holomorphic transformations, embedded in 3-dimensional complex space. The "equating" of the dimensions for the group and its orbits leads in this situation to the surfaces from the theorem 11.

Finishing the paper, underline that complete understanding of the relationships between the homogeneity effects for the embedded manifold is still not reached. However, one can hope that mentioned results allow to approximate such understanding in the full collection of the discussed questions.

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