On spatial mutation-selection models

Yuri G. Kondratiev
Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany; BiBoS, Univ. Bielefeld, Germany.
e-mail: kondrat@mathematik.uni-bielefeld.de

Oleksandr V. Kutoviy
Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany; BiBoS, Univ. Bielefeld, Germany.
e-mail: kutoviy@mathematik.uni-bielefeld.de

Robert A. Minlos
IITP, RAS, B. Karetnyi 19, Moscow, Russia
e-mail: minl@iitp.ru

Sergey A. Pirogov
IITP, RAS, B. Karetnyi 19, Moscow, Russia
e-mail: pirogov@proc.ru

1 Introduction

The study of spatial birth-and-death processes has several motivations coming from applications in a wide range of stochastic evolution models. Such processes appear naturally in mathematical models of physics, biology, ecology, economics, see, e.g., [1] and the references therein. A general feature of these models is a possibility to include different types of regulation mechanisms reflecting a competition between elements of considered systems. Depending on particular frameworks, these mechanisms may be realized via death intensities or in the birth rates (as fecundity and establishment phenomena in the ecology). But there exists another concept of regulation for stochastically developing systems based on a transformation of the initial Markov process which includes a cost functional on the phase space of the model. This transformation may be described as a modification of transition probabilities of the process by the cost functional. The similar construction is known in the general theory of Markov processes as a multiplicative functional transformation. Its generalization to random fields is sometimes called the Gibbs modification. The transformation of initial Markov process by cost functional which we consider in the present paper has an analytic realization in terms of a Kimura-Maruyama type equation for time evolution of the process distribution or via corresponding Feynman-Kac formula on the path space (see below). We will discuss mentioned regulation mechanism in a special framework of mutation models where it has an interpretation as the selection procedure.

We start with a heuristic discussion of the following mathematical model, describing the accumulation of mutations in a genome, see [8].

Let M be a (Polish) space, which is the space of loci, i.e., positions of possible mutations. We assume that at each locus at most one mutation can happen, and each genotype, i.e., collection of mutated alleles is some finite (or locally finite) configuration of points in M. For technical reasons we consider $M = \mathbb{R}^d$ (with Euclidean metric and Lebesgue measure dx, $x \in \mathbb{R}^d$) but some particular results may be stated also in the general case of a locally compact M, cf. [8], [2]. The latter paper is based on a preliminary version of the presented work.

Denote the set of all locally finite configurations $\gamma \subset \mathbb{R}^d$ by $\Gamma(\mathbb{R}^d)$. Each locally finite configuration in $\mathbb{R}^d$ is interpreted as a genotype. The coexistence of genotypes is described by a selection cost E, which is a continuous function $E : \Gamma \to \mathbb{R}$. Later in the first considered model one assumes that a mutation, after appearing in some point of $\mathbb{R}^d$, will always stay there. An initial Markov process of mutations is described as the pure birth stochastic process on Γ which has the Markov generator of the form

$$(LF)(\gamma) = \int_{\mathbb{R}^d} [F(\gamma \cup x) - F(\gamma)]dx, \quad \gamma \in \Gamma.$$

In this process, a new mutation appears spontaneously in any point $x \in \mathbb{R}^d$ with constant intensity. Because emergence of mutations is assumed to be a stochastic process, the state of the population of genotypes at each fixed moment of time t is described by a probability measure P_t on $\Gamma(\mathbb{R}^d)$ and the history of the process of mutation is described by a probability distribution μ on the space of "trajectories". Of course, for the pure birth process all these object may be described explicitly. The selection cost E changes the Markov process of mutation in such a way that the evolution law of P_t is described by a Kimura-Maruyama type equation [8]:

$$\frac{d}{dt}P_t(F) = P_t\left(\int_M (F(\cdot \cup x) - F(\cdot))dx\right) - P_t(F \cdot E) + P_t(F)P_t(E),$$

where $P_t(F) = \int_\Gamma F dP_t$, $F : \Gamma \to \mathbb{R}$ is a bounded cylindric function.

Our aim is to study the evolution law for measure P_t and establish its limiting behavior. For the technical purposes we start the process in the remote past. Namely, the initial distribution for us is P_T at the moment of time $-T$, $T > 0$. The present distribution at the moment of time $t = 0$ is described by $P_{0,T}$. The long time behavior of the system will be described by

$$P_0 := \lim_{T \to +\infty} P_{0,T}.$$

The measure P_0 is of primary interest in considered model. It describes the limiting state of genotypes which will appear in the stochastic mutation process with a given selection cost functional. Note that the existence of this limiting measure is not obvious even in the case of a regular functional E. But in our model below we will study the case in which the selection cost itself is a very singular object having a pointwise sense only for finite configurations of mutations. The latter will lead to an additional approximation procedure analogous to the thermodynamic limit transition in models of statistical physics.

Let us point out that the described scheme of the modification of a Markov process by a cost functional looks reasonable also in the case of general Markov processes. Such Markov processes with selection and their limiting behaviors may be subjects of independent interests inside of stochastic analysis.

The limiting distribution heuristically can be represented in the form

$$P_0(f) = \int_{\Omega(\mathbb{R}_- \to \Gamma)} f(\xi(0))d\mu_\infty(\xi(\cdot)),$$

where ξ_τ is a pure birth process on Γ started in the remote past and μ_∞ corresponding measure on the path space $\Omega(\mathbb{R}_- \to \Gamma)$. The aim of the present work is to give rigorous meaning to the measure μ_∞ that gives, in particular, the existence result for the limiting state.

Our considerations will be related with two types of mutation-selection models. In the first one introduced above, we assume that once mutations appear in the past, they will stay alive until time $t = 0$. The second model is more complicated. It admits disappearing of mutations. In the terminology of birth-and-death processes, our starting Markov process is a birth process with independent constant intensity death rate that itself is so-called Surgailis process, see, e.g., [4]. This process has a Poisson measure as a unique invariant distribution and the selection leads to a new limiting state of the system.

2 Description of the model

We will describe first of all a general scheme of the path space measure construction for our models.

2.1 Mutation-selection (MS) model

We will assume that in the remote past (at the moment of time $-T$, $T > 0$; further we consider $T \longrightarrow \infty$) there were no mutations and afterwards they gradually appear in points $x_i \in \mathbb{R}^d$ at times t_i, $-T < t_i \leq 0$. The mutations accumulated at time $t = 0$ (together with their history) can be considered as rods located in space-time $\mathbb{R}^d \times \mathbb{R}_-$ and directed along the time axis t, where the points $(x_i, t_i) \in \mathbb{R}^d \times \mathbb{R}_-$ are the starting points of the rods, which go to the time $t = 0$ and end at the points $(x_i, 0) \in \mathbb{R}^d \times \mathbb{R}_-$. These rods can be regarded as "trajectories of mutations" in time. The evolution of rods is considered in the space of marked configurations (see, e.g., [3])

$$V := \hat{\Gamma}(\mathbb{R}^d, \mathbb{R}_-) = \{\eta = (\gamma, t(\gamma)) \,|\, t(\gamma) = \{t_x \,|\, x \in \gamma\},\ t_x \in \mathbb{R}_-\},$$

where $\gamma \subset \mathbb{R}^d$ is a configuration of ends of rods from η (i.e., location of mutations at the moment $t = 0$), and $t_\gamma = \{t(x) \,|\, x \in \gamma\}$ the lengths of rods from η (i.e., $|t_x|$ is a time of existence of mutation until the moment $t = 0$). Sometimes in order to emphasize that γ is related to η we write $\gamma(\eta)$ instead of γ.

The spaces $V_\Lambda := \hat{\Gamma}(\Lambda, \mathbb{R}_-)$, $\Lambda \subset \mathbb{R}^d$, $V_T := \hat{\Gamma}(\mathbb{R}^d,\ [-T, 0])$, $V_{\Lambda,T} := \hat{\Gamma}(\Lambda,\ [-T, 0])$ are defined analogously.

The space of marked finite configuration is defined as

$$\hat{\Gamma}_0 := \hat{\Gamma}_0(\mathbb{R}^d, \mathbb{R}_-) = \bigsqcup_{n \in \mathbb{N}_0} \hat{\Gamma}_0^{(n)}(\mathbb{R}^d, \mathbb{R}_-), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where

$$\hat{\Gamma}_0^{(n)} := \hat{\Gamma}_0^{(n)}(\mathbb{R}^d, \mathbb{R}_-) = \{\eta = (\gamma, t(\gamma)) \,|\, \gamma \subset \mathbb{R}^d,\ |\gamma| = n, t(\gamma) = \{t_x \,|\, x \in \gamma\},\ t_x \in \mathbb{R}_-\}, \quad n \in \mathbb{N}$$

and $\hat{\Gamma}_0^{(0)} = \{\emptyset\}$. The space $\hat{\Gamma}_0(\Lambda, \mathbb{R}_-)$ we denote by $\hat{\Gamma}_0(\Lambda)$.

Next we introduce the so-called "free" probability distribution μ_T^0 on the space V_T of configurations of trajectories. We will assume that the starting points (x_i, t_i) of the rods form in the stripe $\mathbb{R}^d \times [-T, 0] \subset \mathbb{R}^d \times \mathbb{R}_-$ Poissonian $(d+1)-$field with intensity $\lambda > 0$. Because the starting points of the rods $\eta = (\{x_i, t_i\}), |t_i| < T, x_i \in \mathbb{R}^d$ uniquely determine their configuration we assume that the distribution π_λ^T of this field is the "free" distribution μ_T^0. Further we include the mechanism of "mutation selection" $E = \mathcal{H}_{ex}^T + \mathcal{H}_{int}^T$, which becomes apparent in the "cost" of existence of mutations for their life time

$$\mathcal{H}_{ex}^T(\eta) = u \sum_i |t_i|, \quad \eta = (\{x_i, t_i\}) \in V_T, \qquad \text{where } u > 0 \text{ is a parameter,} \tag{2.1}$$

and for their coexistence due to an interaction potential $\phi : \mathbb{R}^d \to \mathbb{R}$

$$\mathcal{H}_{int}^T(\eta) = \beta \sum_{i \neq j} \phi(x_i - x_j) \min\{|t_i|, |t_j|\}, \quad \eta \in V_T, \tag{2.2}$$

where $\beta > 0$ is a parameter. The precise assumptions about the potential ϕ we need for rigorous realization of this scheme will be explained below

1. **Stability**: for any finite configuration $\zeta \subset \mathbb{R}^d$

$$\sum_{\{x_1,x_2\}\in\zeta,\,x_1\neq x_2} \phi(x_1 - x_2) > -B|\zeta|, \tag{2.3}$$

where $B \geq 0$ is some constant.

2. **Positivity**: for some $r_1 \geq 0$

$$\phi(u) > 0, \quad \text{for} \quad |u| < r_1. \tag{2.4}$$

3. **Boundness**:

$$\sup_{u\in\mathbb{R}^d} |\phi(u)| = c_0 < \infty. \tag{2.5}$$

4. **Integrability**: $\phi \in L^1(\mathbb{R}^d)$.

   We denote

$$c_1 := \int_{|u|>r_1} |\phi(u)|du < \infty. \tag{2.6}$$

In the sequel, we will be also interested in one of the following modifications of the Assumption 4:

Assumptions 4′

a) for $|u| > r_1$

$$|\phi(u)| < \frac{C}{(1+|u|)^{2m}}, \tag{2.7}$$

where $C > 0$ and $m > d$.

b) for $|u| > r_1$,

$$|\phi(u)| < \bar{C}\exp\{-\kappa u\}, \tag{2.8}$$

where $\bar{C} > 0$ and $\kappa > 0$.

It is very important to emphasize the technical importance of the stability condition.

Consequences of Assumption 1:

1. for any configuration $\eta = (\gamma,\, t(\gamma))$

$$\mathcal{H}^{\Lambda}_{int}(\eta) > -B\sum_{x\in\gamma} t_x; \tag{2.9}$$

2. there exists point $(x_0, t_0) \in \eta = (\gamma,\, t(\gamma))$, such that

$$\sum_{x\in\gamma\setminus x_0} \phi(x_0 - x)\min\{t_{x_0},\, t_x\} > -2Bt_{x_0}. \tag{2.10}$$

Proof. The second bound is trivial, and the first one follows from the representation

$$\mathcal{H}^{\Lambda}_{int}(\eta) = \int_{-\infty}^{0}\left(\sum_{x,\,y\in\gamma(\tau)} \phi(x-y)\right)d\tau, \tag{2.11}$$

and the identity

$$\int_{-\infty}^{0} |\gamma(\tau)|d\tau = \sum_{x\in\gamma} t_x.$$

In the representation (2.11) $\gamma(\tau) \subset \gamma$ is the sub-configuration of γ consisting of those points $x \in \gamma$ for which $s_\gamma(x) > -\tau$. □

Because the configuration $\eta \in V_T$ contains, in general, infinite number of rods and expressions (2.1), (2.2) are meanless, we will first consider "costs" in a bounded region $\Lambda \subset \mathbb{R}^d$

$$\mathcal{H}_{ex}^{T,\Lambda}(\eta) = u \sum_{i:\, x_i \in \Lambda} |t_i|, \tag{2.12}$$

$$\mathcal{H}_{int}^{T,\Lambda}(\eta) = \beta \sum_{i,j,i\neq j,\, x_i,x_j\in\Lambda} \phi(x_i - x_j)\min\{|t_i|,|t_j|\}, \quad \eta = (\{x_i,t_i\}) \in V_T. \tag{2.13}$$

Now the probability distribution $\mu_{\Lambda,T}$ on the space of configurations of rods V_T is defined for each fixed $\Lambda \subset \mathbb{R}^d$ as a so-called Gibbs reconstruction

$$\frac{d\mu_{\Lambda,T}}{d\pi_\lambda^T}(\eta) = \frac{1}{Z_{\Lambda,T}} \exp\{-\mathcal{H}_{ex}^{T,\Lambda}(\eta) - \mathcal{H}_{int}^{T,\Lambda}(\eta)\}, \tag{2.14}$$

where $Z_{\Lambda,T}$ is the normalising constant

$$Z_{\Lambda,T} = \int_{V_T} \exp\{-\mathcal{H}_{ex}^{T,\Lambda}(\eta) - \mathcal{H}_{int}^{T,\Lambda}(\eta)\} d\pi_\lambda^T. \tag{2.15}$$

Observe that each configuration of rods $\eta = (\{x_i,t_i\})$ can be considered as a union of two subconfigurations $\eta = \eta_\Lambda \cup \eta_{\Lambda^c}, \Lambda^c = \mathbb{R}^d \setminus \Lambda$, where

$$\eta_\Lambda = \{\{x_i,t_i\} \in \eta : x_i \in \Lambda\}, \ \ \eta_{\Lambda^c} = \{\{x_i,t_i\} \in \eta : x_i \in \Lambda^c\},$$

where these subconfigurations are independent w.r.t distribution π_λ^T :

$$\pi_\lambda^T = \pi_\lambda^{T,\Lambda} \times \pi_\lambda^{T,\Lambda^c}. \tag{2.16}$$

The $\pi_\lambda^{T,\Lambda}$ and $\pi_\lambda^{T,\Lambda^c}$ are distributions of Poisson field in $\Lambda \times [-T,0]$ and $\Lambda^c \times [-T,0]$ respectively. Because the reconstruction affects only the distribution $\pi_\lambda^{T,\Lambda}$ for the component η_Λ we will consider the restriction of the measure $\mu_{\Lambda,T}$ on the space $V_{\Lambda,T}$ of configurations of rods which lie inside of $\Lambda \times [-T,0]$. This measure is still given by the formulas (2.14) and (2.15) if we write $\pi_\lambda^{T,\Lambda}$ instead of π_λ^T and space $V_{\Lambda,T}$ instead of V_T. For convenience introduce the measure $\mu_{\Lambda,T}$ in two steps:

The first step: we consider Gibbs reconstruction $\tilde{\mu}_{\Lambda,T}$ on $V_{\Lambda,T}$ which is determined only by "life costs":

$$\frac{d\tilde{\mu}_{\Lambda,T}}{d\pi_\lambda^{T,\lambda}}(\eta_\Lambda) = \frac{1}{\tilde{Z}_{\Lambda,T}} \exp\{-\mathcal{H}_{ex}^{T,\Lambda}(\eta_\Lambda)\}. \tag{2.17}$$

It is easy to check that the distribution $\tilde{\mu}_{\Lambda,T}$ is to be considered as distribution of configurations of rods' beginnings from $V_{\Lambda,T}$, which is a non-uniform Poisson field in space $\Lambda\times[-T,0] \subset \mathbb{R}^d\times\mathbb{R}_-$ with changing intensity (see [8])

$$z = z(x,t) = \lambda e^{-u|t|}, \quad (x,t) \in \Lambda \times [-T,0]. \tag{2.18}$$

Note that the intensity of this field does not depend on T and one can consider it as a restriction on $\Lambda \times [-T,0]$ of Poisson field in $\Lambda \times [-\infty,0]$ with the same intensity z; the distribution of this field we denote by $\tilde{\mu}_{\Lambda,\infty}$. Note that the ends of rods $\{x_i,0\} = \gamma$ form a Poisson field in Λ with intensity λ/u with respect to the distribution $\tilde{\mu}_{\Lambda,\infty}$. If $\gamma \subset \Lambda$ is fixed, then the lengths of rods s_i are independent and distributed identically with densities $p(s) = ue^{-us}$.

The second step: we introduce the "influence cost" of mutation and measure $\mu_{\Lambda,T}$ by the density

$$\frac{d\mu_{\Lambda,T}}{d\tilde{\mu}_{\Lambda,T}}(\eta_\Lambda) = \frac{1}{\hat{Z}_{\Lambda,T}} \exp\{-\mathcal{H}_{int}^{T,\Lambda}(\eta_\Lambda)\}. \tag{2.19}$$

In order to pass to the limit $T \longrightarrow \infty$ we consider the following measures on the space V_Λ (configurations of rods which are in the "tube" $\Lambda \times [-\infty, 0]$): the introduced above measure $\tilde{\mu}_{\Lambda,\infty}$ (with intensity z) and measure $\mu_{\Lambda,\infty}$ which is given by the density

$$\frac{d\mu_{\Lambda,\infty}}{d\tilde{\mu}_{\Lambda,\infty}}(\eta) = \frac{1}{Z_{\Lambda,\infty}} \exp\{-\mathcal{H}^\Lambda_{int}(\eta)\}, \;\; \eta \in V_\Lambda, \tag{2.20}$$

where $\mathcal{H}^\Lambda_{int}(\eta)$ is given by (2.13), but is considered on the space V_Λ. Note, that

$$\mathcal{H}^\Lambda_{int}(\eta) < \infty \quad \text{for} \quad \tilde{\mu}_{\Lambda,\infty} \quad \text{almost all} \quad \eta \in V_\Lambda.$$

Indeed,

$$\left\langle H^\Lambda_{int} \right\rangle_{\tilde{\mu}_{\Lambda,\infty}} = \int_{-\infty}^0 \int_{-\infty}^0 dt_1 dt_2 \int_\Lambda \int_\Lambda dx_1 dx_2 \phi(x_1 - x_2) \min\{|t_1|, \, |t_2|\} \rho_2(x_1, t_1; \, x_2, t_2), \tag{2.21}$$

where $\rho_2(x_1, t_1; \, x_2, t_2)$ is a second correlation function of Poisson field with intensity (2.18).

Therefore,

$$\left\langle H^\Lambda_{int} \right\rangle_{\tilde{\mu}_{\Lambda,\infty}} \leq \frac{\lambda^2}{u^2} \int_\Lambda \int_\Lambda |\phi(x_1 - x_2)| dx_1 dx_2 \int_{-\infty}^0 \int_{-\infty}^0 (|t_1| + |t_2|) e^{-u(|t_1|+|t_2|)} u^2 dt_1 dt_2 \leq$$

$$2 \frac{\lambda^2}{u^3} |\Lambda| \int_{\mathbb{R}^d} |\phi(u)| du < \infty.$$

The last bound follows from the assumptions (3)-(4).

Lemma 2.1. *For small enough β we have:*

$$\lim_{T \longrightarrow \infty} \mu_{\Lambda,T} = \mu_{\Lambda,\infty}, \tag{2.22}$$

where the lim *is in weak sense, i.e. convergence of*

$$\langle F \rangle_{\tilde{\mu}_{\Lambda,T}} \longrightarrow \langle F \rangle_{\tilde{\mu}_{\Lambda,\infty}}, \quad \textit{for} \quad T \longrightarrow \infty, \tag{2.23}$$

where F is any bounded, local function on V_Λ (i.e., there exists such $T_0 = T_0(F)$, such that $F(\eta) = F(\eta|_{\Lambda \times [-T_0, 0]})$ where $\eta|_{\Lambda \times [-T_0, 0]} \in V_{\Lambda, T_0}$ – restriction of the configuration $\eta_\Lambda \in V_\Lambda$ onto $\Lambda \times [-T_0, 0] : \{(x,t) \in \eta \, : \, |t| < T_0\}$).

Proof. In order to prove (2.22) we consider the integral

$$G_T(F) = \int_{V_{\Lambda,T}} F(\eta) e^{-H^{\Lambda,T}_{int}(\eta)} d\tilde{\mu}_{\Lambda,T} = \int_{V_{\Lambda,T}} F(\eta) e^{-H^{\Lambda,T}_{int}(\eta)} \prod_{t_i} e^{-u|t_i|} d\pi^{\Lambda,T}_\lambda, \quad \eta = \{x_i, t_i\}$$

where $T > T_0(F)$ and $\pi^{\Lambda,T}_\lambda$ is a Poisson field in $\Lambda \times [-T, 0]$ with intensity λ. Since for every configuration $\eta \in V_\Lambda$ the following convergence takes place

$$\mathcal{H}^{\Lambda,T}_{int}(\eta) \rightarrow \mathcal{H}^\Lambda_{int}(\eta), \quad as \quad T \rightarrow \infty, \tag{2.24}$$

the measure $\pi^{\Lambda,T}_\lambda$ is the restriction of measure π^Λ_λ on $V_{\Lambda,T}$, and, moreover, the following estimate is fulfilled

$$e^{-\mathcal{H}^{\Lambda,T}_{int}(\eta)} < e^{2\beta B \sum_i |t_i|}.$$

The integral $G_T(F)$ converges to $G_\infty(F)$ as $T \rightarrow \infty$ provided $u > 2\beta B$ is fulfilled. Since,

$$\langle F \rangle_{\mu_{\Lambda,T}} = \frac{G_T(F)}{G_T(1)},$$

the limit of $\langle F\rangle_{\mu_{\Lambda,T}}$ as $T\to\infty$ exists and

$$\lim_{T\to\infty}\langle F\rangle_{\mu_{\Lambda,T}}=\frac{G_\infty(F)}{G_\infty(1)}.$$

The later proves Lemma 2.1. □

In Section 3 we will study the thermodynamic limit $\Lambda\nearrow\mathbb{R}^d$ for measures $\mu_{\Lambda,\infty}$ constructed in the present subsection. The limiting measure μ_∞ will be considered on the space V.

2.2 Mutation-selection model with resumption (MSR)

For this model it is supposed that mutation, which randomly appears at the point $(x,t)\in\mathbb{R}^d\times(-\infty,0)$, will disappear after the random moment of time $s=s(x,t)$. By $q=\{(x,t),s\}$ we denote the history of each mutation. As result the history of all mutations, which appeared in the past, one may introduce as a marked field (the field of rods):

$$\xi=\{\eta,s(\eta)\}=\{q_x\}_{x\in\gamma},\quad q_x=\{(x,\,t_x),s_x\},\quad x\in\gamma,\quad \eta=(\gamma,t(\gamma))\in V, \tag{2.25}$$

where η stands for positions and moment of appearing of mutations (beginning of the rods), and $s(\eta)=\{s(x,t_x),\,(x,t_x)\in\eta\}$ the lifetime of mutations (the lengths of rods q_x). We note also that rods can have their ends in the "future" (i.e., it may happens that $s(x,t_x)>|t_x|$).

Let us denote by Ω the space of all configurations of the field (2.25). As in the previous case let us consider the finite interval $[-T,0]$, $0<T<\infty$ in the past and bounded region $\Lambda\subset\mathbb{R}^d$. By $\Omega_{\Lambda,T}$ we denote the space of all configurations $\xi_{\Lambda,T}=\{\eta_{\Lambda,T},s(\eta_{\Lambda,T})\}$. Note that the beginnings of each $\eta_{\Lambda,T}$ lie in the space $\Lambda\times[-T,0]$. As a reference measure in the space $\Omega_{\Lambda,T}$ we take measure

$$d\bar\mu^0_{\Lambda,T}(\xi_{\Lambda,T})=d\bar\pi^\lambda_{\Lambda,T}(\eta_{\Lambda,T})\prod_{(x,t)\in\eta_{\Lambda,T}}ds(x,t),$$

where $\bar\pi^\lambda_{\Lambda,T}$ is a Poisson distribution on $V_{\Lambda,T}$ with the intensity λ.

As in the previous subsection we introduce the following "penalty" functionals which are analogous to (2.1) and (2.2)

$$G^{\Lambda,T}_{ex}(\xi_{\Lambda,T})=u\sum_{(x,t)\in\eta_{\Lambda,T}}s(x,t),\quad \xi_{\Lambda,T}=(\eta_{\Lambda,T},s(\eta_{\Lambda,T})) \tag{2.26}$$

and

$$G^{\Lambda,T}_{int}(\xi_{\Lambda,T})=\beta\sum_{\{(x,t),s\},\,\{(x',t'),s'\}\in\xi_{\Lambda,T};\,x\neq x'}\phi(x-x')\triangle(t,s;t',s'), \tag{2.27}$$

where $\triangle(t,s;t',s')=\min\{|(t,t+s)|,\,|(t',t'+s')|\}$ is an interval of joint life for both mutations, and ϕ is the same potential as in the previous model.

The stability condition for the potential ϕ implies the following bounds in the case of "penalty" functionals $G^{\Lambda,T}_{ex}$ and $G^{\Lambda,T}_{int}$.

Consequences of the Assumption 1:

1. for any configuration $\xi=(\eta,\,s(\eta))$

$$G^{\Lambda,T}_{int}(\xi)>-B\sum_{(x,t)\in\eta}s(x,t); \tag{2.28}$$

2. there exists point $(x_0, t_0) \in \eta = (\gamma, t(\gamma))$, such that

$$\sum_{(x,t)\in\eta\setminus\{x_0,t_0\}} \phi(x - x_0)\triangle(t_0, s_0; t, s) > 2Bs(x_0, t_0).$$

Proof. The proof of these bounds is analogous to the one performed for functionals $\mathcal{H}^T_{ex}$ and $\mathcal{H}^T_{int}$ in the previous subsection. □

Now the Gibbs reconstruction for the measure $\bar{\mu}^0_{\Lambda,T}$ is defined with the help of functional $G^{\Lambda,T}_{ex}$

$$\frac{d\bar{\bar{\mu}}_{\Lambda,T}}{d\bar{\mu}^0_{\Lambda,T}}(\xi_{\Lambda,T}) = \bar{\bar{Z}}_{\Lambda,T}^{\ -1} \exp\{-G^{\Lambda,T}_{ex}(\xi_{\Lambda,T})\}, \tag{2.29}$$

where $\bar{\bar{Z}}_{\Lambda,T}$ is the normalizing constant. It is not difficult to check that such a probability distribution has form

$$d\bar{\bar{\mu}}_{\Lambda,T}(\xi_{\Lambda,T}) = d\bar{\pi}^\lambda_{\Lambda,T}(\eta_{\Lambda,T}) \prod_{(x,t)\in\eta_{\Lambda,T}} ue^{-us(x,t)}ds(x,t). \tag{2.30}$$

The latter means that the configuration $\eta_{\Lambda,T}$ of beginnings of rods is a Poissonian field with the intensity λ, and the lengths of rods for the fixed configuration $\eta_{\Lambda,T}$ are conditionally independent and identically distributed with the intensity

$$p(s) = ue^{-us}.$$

We stress that fields with such distributions one may easily determine on the whole space $\mathbb{R}^d \times (-\infty, 0]$ (on the space of all configurations Ω), Moreover the field $\xi_{\Lambda,T}$ is a restriction of ξ to the region $\Lambda \times [-T, 0] \subset \mathbb{R}^d \times (-\infty, 0]$.

Now we introduce the Gibbs reconstruction for the measure $\bar{\bar{\mu}}_{\Lambda,T}$ with the help of functional $G^{\Lambda,T}_{int}$

$$\frac{d\bar{\mu}_{\Lambda,T}}{d\bar{\bar{\mu}}_{\Lambda,T}}(\xi_{\Lambda,T}) = \bar{Z}^{-1}_{\Lambda,T} \exp\{-G^{\Lambda,T}_{int}(\xi_{\Lambda,T})\}. \tag{2.31}$$

We prove in the next section that there exists region of parameters (λ, u, β) for which the thermodynamic limit (i.e., $\Lambda \nearrow \mathbb{R}^d$, $T \to \infty$) for measures $\bar{\mu}_{\Lambda,T}$ exists provided the potential ϕ satisfies proper conditions.

3 Thermodynamic limit ($\Lambda \nearrow \mathbb{R}^d$). Main results.

3.1 Statement of results

In this subsection we will consider the construction of measures μ_∞ and $\bar{\mu}_\infty$ for both models. These measures appear as a result of thermodynamic limit $\Lambda \nearrow \mathbb{R}^d$ and $T \to \infty$ for measures $\mu_{\Lambda,\infty}$ in the first model and $\bar{\mu}_{\Lambda,T}$ in the second one. The limit $\Lambda \nearrow \mathbb{R}^d$ means that we consider the arbitrary increasing sequence of bounded sets

$$\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_n \subset \ldots$$

such that $\bigcup \Lambda_n = \mathbb{R}^d$. For the next theorem we need the notion of local function on configuration space Ω. A function F on Ω is called local if there exists bounded set $A_0 = A_0(F) \subset \mathbb{R}^d \times \mathbb{R}$, such that F depends only on those rods from the configuration which intersect with A_0. Each such a set is called region of localization for the function F. The space of all bounded local functions on V or Ω we call $C_{b,l}$.

Theorem 3.1. ***1. (MS) model.*** *Under Assumptions 1-4 for the potential ϕ there exists a region in the space of parameters $(\beta,\ \lambda,\ u)$ (see (4.17)) such that the following weak limit exists*

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\mu_{\Lambda,\infty}=\mu_\infty, \tag{3.1}$$

where μ_∞ is the probability measure on the space V. The later means that for any function $F\in C_{b,l}$

$$\lim_{\lambda\nearrow\mathbb{R}^d}\langle F\rangle_{\mu_{\Lambda,\infty}}=\langle F\rangle_{\mu_\infty}, \tag{3.2}$$

where $\langle F\rangle_{\mu_{\Lambda,\infty}}$ and $\langle F\rangle_{\mu_\infty}$ are mean values of function F with respect to the distributions $\mu_{\Lambda,\infty}$ and μ_∞ correspondingly.

2. (MSR) model. *Under Assumptions 1-4 for the potential ϕ there exists a region in the space of parameters $(\beta,\ \lambda,\ u)$ such that the following weak limit exists*

$$\lim_{\Lambda\nearrow\mathbb{R}^d,\,T\to\infty}\bar\mu_{\Lambda,T}=\bar\mu_\infty. \tag{3.3}$$

It turns out that Assumptions 4 implies additionaly the decay correlation property for the limiting measures μ_∞ and $\bar\mu_\infty$ (and also pre-limiting measures $\mu_{\Lambda,\infty}$ and $\bar\mu_{\Lambda,T}$).

Let us introduce for two functions $F_1,\ F_2\in C_{b,l}$ with the regions of localization Λ_1 and Λ_2 and measure μ_∞ the covariation

$$cov(F_1,\ F_2)=\langle F_1\cdot F_2\rangle_{\mu_\infty}-\langle F_1\rangle_{\mu_\infty}\langle F_2\rangle_{\mu_\infty}$$

Analogously we define covariation for the measure $\bar\mu_\infty$. Then the following theorem is true.

Theorem 3.2. *Suppose that conditions of Theorem 1 and one of the conditions of Assumptions 4′ are fulfilled. Then the following bound is true for both models*

$$|cov(F_1,\ F_2|<C||F_1||\,||F_2||e^{\tau(|\Lambda_1|+|\Lambda_2|)}(|\Lambda_1|+|\Lambda_2|)e^{-d(\Lambda_1,\,\Lambda_2)}, \tag{3.4}$$

where $C>0,\ \tau>0$ are some constants,

$$||F_i||=\begin{cases}\sup_\eta|F_i(\eta)|, & \text{(MS) model,}\\ \sup_\xi|F_i(\xi)|, & \text{(MSR) model,}\end{cases}\qquad i=1,\ 2, \tag{3.5}$$

$$d(\Lambda_1,\ \Lambda_2)=\inf_{x_1\in\Lambda_1,\,x_2\in\Lambda_2}\rho(x_1,\ x_2)$$

with metric $\rho(x_1,\ x_2)$ in $\mathbb{R}^d$ defined as

$$\rho(x_1,\ x_2)=m\ln\{1+|x_1-x_2|\}\quad\text{in the case 4}'\text{ a)}$$

and

$$\rho(x_1,\ x_2)=\frac{\kappa}{2}|x_1-x_2|\quad\text{in the case 4}'\text{ b).} \tag{3.6}$$

4 Proofs

4.1 Proof of Theorem 3.1

The complete proof of Theorem 3.1 will be given for (MS) model. Then we will explain the main changes needed for the case of (MSR) model.

Let $\eta = (\hat{x}_1, \dots, \hat{x}_n) \in \hat{\Gamma}_0$, $\hat{x}_i = (x_i, t_i)$, $i = 1, \dots, n$, $n \in \mathbb{N}$. The Lebesgue–Poisson measure for each $\hat{\Gamma}_0^{(n)} \subset \hat{\Gamma}_0$ is given by

$$d\nu(\eta) = \frac{z^n}{n!} d\hat{x}_1 \dots d\hat{x}_n, \quad \nu(\emptyset) = 1, \tag{4.1}$$

where $d\hat{x} = ue^{-us} ds dx$ for the element $\hat{x} = (x, s)$, $z = \frac{\lambda}{u}$. It is easy to see that

$$d\tilde{\mu}_{\Lambda,\infty}(\eta_\Lambda) = e^{-z|\Lambda|} d\nu(\eta_\Lambda) \tag{4.2}$$

and, therefore, the density $\hat{\hat{p}}_{\Lambda,\beta}(\eta_\Lambda) = \frac{d\mu_{\Lambda,\infty}}{d\nu}(\eta_\Lambda)$ is again given by formula (2.14) with changing norm factor

$$\hat{\hat{Z}}_{\Lambda,\beta} = e^{z|\Lambda|} Z_{\Lambda,\beta}. \tag{4.3}$$

Let us use now the cluster expansion for the density $\hat{\hat{p}}_{\Lambda,\beta}(\eta_\Lambda)$. The way we obtain this expansion is almost the same as for the analogous cluster expansion in the case of pure point Gibbsian field (see. [5]):

$$\begin{aligned} \hat{\hat{p}}_{\Lambda,\beta}(\eta_\Lambda) &= \hat{\hat{Z}}^{-1}_{\Lambda,\beta} \sum_{\{\eta_1,\dots,\eta_m\}}^{(\eta_\Lambda)} k(\eta_1) \dots k(\eta_m), \quad \eta_\Lambda \neq \emptyset, \\ \hat{\hat{p}}_{\Lambda,\beta}(\emptyset) &= \hat{\hat{Z}}^{-1}_{\Lambda,\beta}, \end{aligned} \tag{4.4}$$

where sum $\sum\limits_{\{\eta_1,\dots,\eta_m\}}^{(\eta_\Lambda)}$ means the summation over all partitions of configuration $\eta_\Lambda = (\gamma_\Lambda, t(\gamma_\Lambda))$ by non-empty sub-configurations $\eta_i \subset \eta_\Lambda$, i.e., over all non-ordered sets $\{\eta_1, \dots, \eta_m\}$, $m = 1, 2, \dots$ of sub-configurations of η_Λ with mutually disjoint supports $\gamma_i \subseteq \gamma$ such that $\bigcup_{i=1}^m \gamma_i = \gamma$.

The values $k(\eta)$ for the finite non-empty configuration η are defined as

$$k(\eta) = \begin{cases} z, & |\eta| = 1 \\ \sum\limits_\sigma^{(\eta)} \varkappa_\sigma, & |\eta| > 1, \end{cases} \tag{4.5}$$

where $\sum\limits_\sigma^{(\eta)}$ means the summation over all connected graphs σ with set of vertices $V(\sigma) = \gamma(\eta)$, and

$$\varkappa_\sigma = z^{|\gamma|} \prod_{x,x' \in \sigma} \left(e^{-\beta\phi(x-x')\min(t,t')} - 1 \right). \tag{4.6}$$

In the last formula $\prod_{x,x' \in \sigma}$ means the product over all edges (x, x') of the graph σ.

Moreover, for the values $k(\eta)$, $\eta = (\gamma, t(\gamma))$, the following estimate holds, see [5].

$$|k(\eta)| < \prod_{x \in \gamma} z e^{2\beta B t_x} \sum_{\mathcal{T}}^{(\eta)} \prod_{(x',x) \in \mathcal{T}} \left| e^{-\beta\phi(x-x')\min(t_x,t'_x)-1} \right|, \tag{4.7}$$

where $\sum_{\mathcal{T}}^{(\eta)}$ means the summation over all trees $\mathcal{T}$ with the set of vertices $V(\mathcal{T}) = \gamma$.

Next, we will use the following criteria to show the existence of the limit (3.2). The proof of that is the same as for the analogous criterion in the pure point field case, see [5].

Proposition 4.1. *Let the following bound*

$$\int_{\hat{\Gamma}_0(\Lambda)\setminus\{\emptyset\}} d\nu(\eta_2) \int_{\hat{\Gamma}_0} d\nu(\eta_1) k(\eta_1 \cup \eta_2) < \infty, \tag{4.8}$$

be satisfied for any bounded $\Lambda \subset \mathbb{R}^d$. Then the limiting measure μ_∞ exists and for any bounded domain $\Lambda_0 \subset \mathbb{R}^d$ the restriction $\mu_\infty^{\Lambda_0} \equiv \mu_\infty|_{V_{\Lambda_0}}$ of the measure μ_∞ on the set V_{Λ_0} is defined by the density $p_\infty^{\Lambda_0} = \frac{d\mu_\infty^{\Lambda_0}}{d\nu}(\eta_{\Lambda_0})$, which is equal to

$$p_\infty^{\Lambda_0}(\eta_{\Lambda_0}) = \left(\tilde{Z}^{(\Lambda_0)}\right)^{-1} \sum_{\{\eta_1,\ldots,\eta_m\}}^{(\eta_{\Lambda_0})} \prod_{i=1}^{m} r^{\Lambda_0}(\eta_i), \tag{4.9}$$

where

$$r^{\Lambda_0}(\eta) = \int_{\hat{\Gamma}_0(\mathbb{R}^d\setminus\Lambda_0)} k(\eta \cup \bar{\eta}) d\nu(\bar{\eta}), \tag{4.10}$$

$$\tilde{Z}^{(\Lambda_0)} = \exp\left\{ \int_{\hat{\Gamma}(\Lambda_0)} r^{\Lambda_0}(\eta) d\nu(\eta) \right\}. \tag{4.11}$$

According to this criterion, we have to check the condition (4.8) only. Using the estimate (4.7) one can write

$$\begin{aligned} I :=& \int_{\hat{\Gamma}_0(\Lambda)\setminus\{\emptyset\}} d\nu(\eta_2) \int_{\hat{\Gamma}_0} d\nu(\eta_1) k(\eta_1 \cup \eta_2) \\ =& \sum_{\substack{m,n\\ m\neq 0}} \frac{z^{m+n}}{m!n!} \sum_{\mathcal{T}} \overbrace{\int_{\Lambda\times\mathbb{R}_+} \cdots \int_{\Lambda\times\mathbb{R}_+}}^{m \text{ times}} \overbrace{\int_{\mathbb{R}^d\times\mathbb{R}_+} \cdots \int_{\mathbb{R}^d\times\mathbb{R}_+}}^{n \text{ times}} \prod_{(i,j)\in\mathcal{T}} \left| e^{-\beta\phi(x_i-x_j)\min(t_i,t_j)} - 1 \right| \\ & \prod_{i=1}^{m} dx_i \prod_{i=m+1}^{m+n} dx_i \prod_{i=1}^{m+n} u e^{-ut_i + 2\beta B t_i} dt_i, \end{aligned} \tag{4.12}$$

where summation is taken over all trees $\mathcal{T}$ with the set of vertices $\{1, 2, \ldots, m+n\}$. The integration in first m variables $x_1, \ldots, x_m$ is taken over sets Λ and in other $x_{m+1}, \ldots, x_{m+n}$ variables it is taken over whole $\mathbb{R}^d$. The estimate of this integral will be based on the following inequality.

Lemma 4.1. *Let us define*

$$C(\beta, u) := \max_t \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}_+} dt' \left| e^{-\beta\phi(x-x')\min(t,t')} - 1 \right| u e^{-ut' + 2\beta B t'}. \tag{4.13}$$

Then, under condition $u - 2\beta B > \beta c_0$, where B, c_0, r_1 and c_1 as in (2.3)*–*(3.6) *one has*

$$C(\beta, u) < \left(c_0 \omega_d r_1^d + c_1\right) \frac{\beta u (1 + e^{-1})}{(u - 2\beta B - c_0\beta)^2}, \tag{4.13$^{\text{a}}$}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

The proof of this lemma will be given bellow. But now we finish the proofs of the both theorems.

Let us consider the integration in (4.12) under fixed tree $\mathcal{T} = \mathcal{T}^0$ and denote the integral in the sum $\sum_{\mathcal{T}}$ by $J(\mathcal{T}^0)$. Next, let $i_1 > 1$ be the number of some end-vertex of the tree $\mathcal{T}^0$. After integration by variables (x_{i_1}, t_{i_1}) in the integral $J(\mathcal{T}^0)$ we may obliterate the vertex i_1 together with emergent edge and then we obtain the new tree $\mathcal{T}^1$, moreover,

$$J(\mathcal{T}^0) < C(\beta) J(\mathcal{T}^1),$$

where $J(\mathcal{T}^1)$ means the integral analogous to the integral $J(\mathcal{T}^0)$ but without integration in (x_{i_1}, t_{i_1}). We will continue this procedure obliterating step-by-step the whole tree $\mathcal{T}^0$ except the vertex with number 1 and, as a result, we obtain the estimate

$$J(\mathcal{T}^0) < \big(C(\beta)\big)^{n+m-1} \int_\Lambda \int_{\mathbb{R}_+^+} dx_1 u e^{-t_1(u-2\beta B)} dt_1 = |\Lambda| \big(C(\beta)\big)^{n+m-1} \frac{u}{u-2\beta B}. \tag{4.14}$$

As result, taking into account that the number of trees with $m+n$ vertices is less or equal than $(m+n)!e^{m+n}$ (see [7]) we obtain

$$\begin{aligned} I \leq \sum_{m>0,n} \frac{z^{m+n}}{m!n!} (m+n)! e^{m+n} \big(C(\beta)\big)^{n+m-1} |\Lambda| \frac{u}{u-2\beta B} \leq \\ \sum_{k=0}^{\infty} z^k (2e)^k C^{k-1}(\beta, u) |\Lambda| \frac{u}{u-2\beta B} < \infty, \end{aligned} \tag{4.15}$$

under condition that

$$2ezC(\beta) < 1. \tag{4.16}$$

The bound (4.13ᵃ) together with

$$\frac{\beta u z}{(u - 2\beta b - c_0\beta)^2} < \frac{1}{2(e+1)} \quad \text{and} \quad u - 2\beta B > c_0 \beta \tag{4.17}$$

yields the integral (4.12) is finite and therefore the limiting measure exists and statements (4.9)–(4.11) are true.

To prove the second part of Theorem 3.1 we introduce the Lebesgue-Poisson measure $\bar{\nu}$ on the space $\hat{\Gamma}_0$ which consists of all finite configurations ξ of rods analogously to (4.1).

$$d\bar{\nu}(\xi) = \frac{\lambda^n}{n!} dx_1 \dots dx_n dt_1 \dots dt_n \prod_{i=1}^n u e^{-us_i} ds_i, \quad \xi = (\{x_i, t_i\}, s_i, i = 1, \dots, n), \ n = 1, 2. \dots . \tag{4.18}$$

In this case

$$d\tilde{\mu}_{\Lambda,T}(\xi_{\Lambda,T}) = d\bar{\nu}(\xi_{\Lambda,T}) e^{-\lambda T |\Lambda|}$$

and therefore the density $\bar{p}_{\Lambda,T}(\xi_{\Lambda,T}) = \frac{d\bar{\mu}_{\Lambda,T}}{d\bar{\nu}}(\xi_{\Lambda,T})$ is equal

$$\bar{p}_{\Lambda,T}(\xi_{\Lambda,T}) = \frac{1}{\tilde{Z}'_{\Lambda,T}} e^{-\beta \sum \phi(x-x') \triangle(t,s;t',s')},$$

where $\tilde{Z}'_{\Lambda,T}$ is equal to $\tilde{Z}_{\Lambda,T}$ multiplied with $e^{-\lambda T |\Lambda|}$.

The density $\bar{p}_{\Lambda,T}(\xi_{\Lambda,T})$ admits the representation analogous to (4.5) with weights $\bar{K}(\xi)$, where

$$\bar{K}(\xi) = \begin{cases} 1, & |\xi| = 1, \ \xi = \{(x,t), s\} \\ \sum_{\mathcal{T}}^{(\xi)} \bar{x}_{\mathcal{T}}, & |\xi| > 1, \end{cases} \tag{4.19}$$

where summation is taken again over connected graphs with set of vertices $V(\mathcal{T}) = \xi$, and

$$\bar{x}_{\mathcal{T}}(\xi) = \prod_{(q,q') \in \mathcal{T}, x \neq x'} (e^{-\phi(x-x')\triangle(t,s;t',s')} - 1),$$

$q = \{(x,t), \ s(x,t)\} \in \xi$, and the product is taken over all edges. Moreover, the following bound is fulfilled

$$|\bar{K}(\xi)| < \prod_{q \in \xi} e^{2\beta B s} \sum_{\mathcal{T}}^{(\xi)} \prod_{(q,q') \in \mathcal{T}} |e^{-\phi(x-x')\triangle(t,s;t',s')} - 1|, \tag{4.20}$$

where summation is taken over all trees $\mathcal{T}$ with the set of vertices $V(\mathcal{T}) = \xi$. The bound (4.20) implies

$$I := \sum_{\substack{m,n\\ m\neq 0}} \frac{\lambda^{m+n}}{m!n!} \sum_{\mathcal{T}} \overbrace{\int_{\Lambda\times\times[-T,0]\times\mathbb{R}_+} \cdots \int_{\Lambda\times[-T,0]\times\mathbb{R}_+}}^{m \text{ times}} \overbrace{\int_{\mathbb{R}^d\times(-\infty,0]\times\mathbb{R}_+} \cdots \int_{\mathbb{R}^d\times(-\infty,0]\times\mathbb{R}_+}}^{n \text{ times}}$$

$$\prod_{(x,t,s),(x',t',s')\in\mathcal{T};\, x\neq x'} \left| e^{-\beta\phi(x-x')\triangle(t,s;t',s')} - 1 \right| \prod_{i=1}^{m} dx_i dt_i \prod_{i=m+1}^{m+n} dx_i dt_i \prod_{i=1}^{m+n} u e^{-us_i+2\beta B s_i} ds_i, \tag{4.21}$$

where the summation $\sum_{\mathcal{T}}$ is taken over all trees with the set of vertices $V(\mathcal{T}) = \{1, 2, \ldots, m+n\}$.

Next we define the number analogous to (4.13):

$$\bar{C}(\beta, u) = \sup_{s,k,t,x} \frac{(u - 2\beta B - c_0\beta)^k}{k!s} \int_{\mathbb{R}^d\times(-\infty,0)} dx'dt' \int_{\mathbb{R}_+} ds' |e^{-\beta\phi(x-x')\triangle(t,s;t',s')} - 1| e^{-us'+2\beta Bs'}$$

and the finitness of I now follows from the bound

$$\frac{2e\lambda C(\beta, u)}{(u - 2\beta B - c_0\beta)^2} < 1 \quad \text{and} \quad u - 2\beta B > c_0\beta.$$

From another side the bound for $\bar{C}(\beta, u)$ has the following form

$$\bar{C}(\beta, u) < \left(c_0\omega_d r_1^d + c_1\right) \frac{\beta u}{(u - 2\beta B - c_0\beta)}.$$

As result, in the case of the (MSR) model, the measure $\bar{\mu}_\infty$ exists if parameters satisfy

$$\left(c_0\omega_d r_1^d + c_1\right) \frac{\lambda\beta u}{(u - 2\beta B - c_0\beta)^3} < \frac{1}{2e}$$

4.2 Proof of Theorem 3.2

For the proof of Theorem 3.2 we use again the cluster expansion (4.4). We should repeat all considerations of the work [6], where analogous bound for decay of correlations for the pure point Gibbs field was obtained.

4.3 Proof of the Lemma 4.1.

Let us rewrite the integral (4.13) in the following form

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}_+} dt' \ldots = \int_{|x-x'|<r_1} dx \int_{\mathbb{R}_+} dt' \ldots + \int_{|x-x'|>r_1} dx \int_{\mathbb{R}_+} dt' \ldots \tag{4.22}$$

The first integral due to the bound

$$\left| e^{-\beta\phi(x-x')\min(t,t')} - 1 \right| < \beta|\phi(x - x')| \min(t, t')$$

may be estimate by

$$\begin{aligned} &\beta c_0 r_1^d \omega_d \left(\int_0^t t' u e^{-t'(u-2\beta B)} dt' + t \int_t^\infty u e^{-t'(u-2\beta B)} dt' \right) \\ &< \beta c_0 r_1^d \omega_d \left(\frac{u}{(u - 2\beta B)^2} + \frac{u}{(u - 2\beta B)} t e^{-t(u-2\beta B} \right) \\ &< \beta c_0 r_1^d \omega_d \frac{u}{(u - 2\beta B)^2} \left(1 + \frac{1}{e} \right). \end{aligned} \tag{4.23}$$

The term in the second integral my be estimated in the following way

$$\int_0^\infty dt' \left(\beta c_1 e^{\beta c_0 \min\{t,t'\}} \min\{t,t'\} e^{-t'(u-2\beta B)}\right) u =$$

$$\int_0^t dt' \left(\beta c_1 e^{\beta c_0 t'} t' e^{-t'(u-2\beta B)}\right) u + \int_t^\infty dt' \left(\beta c_1 e^{\beta c_0 t} t e^{-t'(u-2\beta B)}\right) u \leq$$

$$\beta c_1 u \left[\frac{1}{(u-2\beta B - c_0\beta)^2} + \frac{e^{-1}}{(u-2\beta B)(u-2\beta B-c_0\beta)}\right] \leq$$

$$\beta c_1 u \left[\frac{1}{(u-2\beta B - c_0\beta)^2} + s e^{\beta c_0 s}\frac{e^{-s(u-2\beta B)}}{u-2\beta B}\right] < \frac{\beta c_1 u(1+e^{-1}}{(u-2\beta B-c_0\beta)^2},$$

which gives together with (4.23) the final bound (4.13[a]). Lemma is proved.

References

[1] D.Finkelshtein, Yu.Kondratiev, O.Kutoviy, Individual based model with competition in spatial ecology, *SIAM J. Math. Anal.*, **41**(2009), 297-317

[2] Yu.Kondratiev, T.Kuna, N.Ohlerich, Selection-mutation models with epistatic selection, *Cond. Matter Phys.*, **11**(2008), 283-291

[3] Yu.Kondratiev, T.Kuna, J.L.Da Silva, Marked Gibbs measures, *Methods Funct. Anal. Topology*, **4**(1998), 50-81

[4] Yu. G. Kondratiev, R. A. Minlos, and E. A. Zhizhina, Self-organizing birth-and-death stochastic systems in continuum, *Rev. Math. Phys.*, **20**(2008), 451-492

[5] V. A. Malyshev, R. A. Minlos, *Gibbs Random Fields*, Kluver Academic Publishers (1991)

[6] Yu. G. Kondratiev, R. A. Minlos, M. Röckner, and G. V. Shchepan'uk Exponential mixing for classical continuous systems, Canadian Math Society, Conference Proceeding, Vol. **20** (2000)

[7] O. Ore *Theory of Graphs*, AMS Colloquium Publications 38 (1962)

[8] D. Steinsatz, S. N. Evans, and K. W. Wachter, A generalized model of mutation-selection balance with applications to aging, *Adv. Appl. Math.*, **35**(2005), 16-33