

THE NORMAL DISTRIBUTION IS \boxplus -INFINITELY DIVISIBLE

SERBAN T. BELINSCHI, MAREK BOŹEJKO, FRANZ LEHNER, AND ROLAND SPEICHER

ABSTRACT. We prove that the classical normal distribution is infinitely divisible with respect to the free additive convolution. We study the Voiculescu transform first by giving a survey of its combinatorial implications and then analytically, including a proof of free infinite divisibility. In fact we prove that a subfamily Askey-Wimp-Kerov distributions are freely infinitely divisible, of which the normal distribution is a special case. At the time of this writing this is only the third example known to us of a nontrivial distribution that is infinitely divisible with respect to both classical and free convolution, the others being the Cauchy distribution and the free $1/2$ -stable distribution.

1. INTRODUCTION

We will prove that the classical normal distribution is infinitely divisible with respect to free additive convolution.

This fact might come as a surprise, since the classical Gaussian distribution has no special role in free probability theory. The first known explicit mentioning of that possibility to one of us was by Perez-Abreu at a meeting in Guanajuato in 2007. This conjecture had arisen out of joint work with Arizmendi [3].

Later when the last three of the present authors met in Bielefeld in the fall of 2008 they were led to reconsider this question in the context of investigations about general Brownian motions. We want to give in the following some kind of context for this.

In [14] two of the present authors were introducing the class of generalized Brownian Motions (GBM), i.e., families of self-adjoint operators $G(f)$ ($f \in \mathcal{H}$, for some real Hilbert space \mathcal{H}) and a state φ on the algebra generated by the $G(f)$, given by

$$\varphi(G(f_1)\dots G(f_{2n})) = \sum_{\pi \in \mathcal{P}_2(2n)} t(\pi) \prod_{(i,j) \in \pi} \langle f_i, f_j \rangle.$$

Here $\mathcal{P}_2(2n)$ denotes the set of pairings of $2n$ elements, and t is a weight function for such pairings. The concrete form of t determines the specific Brownian motion.

Date: January 29, 2010.

2000 Mathematics Subject Classification. Primary 46L54; Secondary 05C30.

Key words and phrases. normal distribution, infinite divisibility, free probability, connected matchings, Loday-Ronco Hopf algebra, tree factorial.

Work of S. Belinschi was supported in part by a Discovery Grant of the Natural Sciences and Engineering Research Council of Canada and a University of Saskatchewan start-up grant.

Work of M.Bożejko was partially supported by Grant N N 201 364436 of Polish Ministry of Science .

Work of R. Speicher was supported by a Discovery Grant from NSERC.

The most natural example for such a GBM is classical Brownian motion, where $t(\pi) = 1$ for all pairings π ; in this case one gets the normal law for $G(f)$. The q -Brownian motion fits into this frame by putting $t_q(\pi) = q^{cr(\pi)}$ (where $cr(\pi)$ denotes the number of crossings of the pairing π); in this case the law of the random variable $G(f)$ is related with the theta function of Jacobi, and called q -Gaussian distribution γ_q , see [2, 13, 12]. If the parameter q changes from -1 to 1, one gets an interpolation between the fermionic Brownian Motion ($q = -1$), the free Brownian Motion ($q = 0$), and the classical Brownian Motion ($q = 1$).

In [14], the model of the free product of classical Brownian motions resulted in a new class of GBM, with the function t given by

$$(1.1) \quad t_s(\pi) = s^{cc(\pi)},$$

where $cc(\pi)$ is the number of connected components of the pairing π . Here s has to be bigger than 1. This contains as a special case the result: The $2n$ -th moment of the free additive power of the normal law γ_1 is given as follows:

$$m_{2n}(\gamma_1^{\boxplus s}) = \sum_{\pi \in \mathcal{P}_2(2n)} s^{cc(\pi)},$$

for $s > 1$.

In the light of earlier examples where similar combinatorial identities could be extended beyond their primary domain of applicability (see, e.g., [10]), it was natural to ask whether this relation could also make any sense for $s < 1$. So, in this context, a natural problem is whether the sequence on the right side of the above formula is a moment sequence for all $s > 0$? This question is equivalent to the free infinite divisibility of the normal law! One can check easily that the corresponding more general question on generalized Brownian motions (i.e., whether the t_s from equation (1.1) is still positive for $s < 1$) has a negative answer. From this point of view, the free infinite divisibility of the classical Gauss seemed quite unlikely. However, numerical evidence suggested the validity of that conjecture. In this paper we will give an analytical proof for this conjecture. We want to point out that it still remains somehow a mystery whether the \boxplus -infinite divisibility of the Gauss distribution is a singular result or whether there is a more conceptual broader theory behind this.

Another example of this phenomenon was found in [35], namely that the 1/2-free stable law [7] is also classically infinitely divisible, being a β -distribution of the second kind with density $(4x - 1)^{1/2}/x^2$.

1.1. Related Questions. One way to describe certain probability distributions on \mathbb{R} is by specifying their orthogonal polynomials. The orthogonal polynomials of the classical Gaussian distribution are the so-called Hermite polynomials [27]. In [4], Askey and Wimp describe a family of deformations, indexed by $c \in (-1, +\infty)$, of the Hermite polynomials, called the associated Hermite polynomials. These polynomials are orthogonal with respect to a family of probability measures $\{\mu_c : c \in (-1, +\infty)\}$, which can be described in terms

of a continued fraction expansion of their Cauchy-Stieltjes transform as

$$G_{\mu_c}(z) = \frac{1}{z - \frac{c+1}{z - \frac{c+2}{z - \ddots}}}.$$

The Cauchy-Stieltjes transform of a measure μ on the real line is defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t), \quad z \in \mathbb{C}^+;$$

when μ is a positive measure, this function maps \mathbb{C}^+ into the lower half-plane. For details we refer to the excellent book of Akhiezer [1]. For $c = 0$, we have $\mu_0 = \gamma$, the normal distribution [27], and one can easily check that one can extend this family continuously to $c = -1$ by letting $\mu_{-1} = \delta_0$, the probability giving mass one to $\{0\}$. This family, introduced in [4], plays an important role in [27]; we shall call its members the *Askey-Wimp-Kerov distributions*. It will turn out from our proof that $\{\mu_c : c \in [-1, 0]\}$ are freely infinitely divisible. Numerical computations show that for several values of $c > 0$, μ_c is not freely infinitely divisible. Numerical evidence seems also to indicate that μ_c is classically infinitely divisible only when $c = 0$ or $c = -1$.

An interesting interpolation between the normal and the semicircle law was constructed by Bryc, Dembo and Jiang [18] and further investigated by Buchholz [19]. This leads to a generalized Brownian motion, given by a weight function ($0 < b < 1$)

$$t_b(\pi) = b^{n-h(\pi)},$$

where for a pairing π of $2n$ elements we denote by $h(\pi)$ the number of connected components which have only one block with 2 elements. It was calculated in [18, 19], that the measures satisfy

$$\nu_{b^2} = D_b \gamma_1 \boxplus D_{1-b} \gamma_0,$$

where D_b is the dilation of a measure by parameter $b > 0$. With the results of the present paper it is immediate that these measures are infinitely divisible with respect to free convolution. However a short calculation shows that these measures are not infinitely divisible with respect to classical convolution unless $b = 1$.

One other tempting example is to consider the distribution of $N \times N$ Gaussian random matrices. For $N = 1$, this is the classical Gauss distribution, whereas for $N \rightarrow \infty$ it converges to the semicircle distribution. Both of them are infinitely divisible in the free sense (the prior by our main result here, and the latter because the semicircle is the limit in the free central limit theorem). So one might conjecture that the interpolating distributions, for integer $1 < N < \infty$, are also freely infinitely divisible. However, numerical calculations of the first few Jacobi coefficients of the corresponding moments, using the Harer-Zagier recurrence, show readily that this is not the case.

One may also ask the “opposite” question, whether the Wigner distribution is infinitely divisible with respect to classical convolution. However this is impossible because any non-trivial classically infinitely divisible measure has unbounded support, see [42, Proposition 2.3].

For the same reason the distributions whose density is a power of the Wigner density are not classically infinitely divisible. It is however an open question whether the latter are freely infinitely divisible. Numerical evidence points to a positive answer to this question. This would provide another proof that the normal law is freely infinitely divisible. See [3] for a survey on these questions.

In the next section we will consider the combinatorial aspects of the free infinite divisibility of the classical Gaussian distribution; in particular, we will give some new combinatorial interpretations for the free cumulants of the Gaussian distribution. In Section 3, we will then give an analytical proof of our free infinite divisibility result.

Acknowledgements The last three authors would like to thank Prof. Götze for the kind invitations and hospitality at SFB 701, Bielefeld. STB would also like to thank Michael Anshelevich for many useful discussions, especially regarding the Riccati equation.

2. COMBINATORIAL CONSIDERATIONS

2.1. Partitions. First we review a few properties of set partitions which will be needed below. As usual, set partitions will be depicted by diagrams like the ones shown in Figure 2.1.

Definition 2.1. A partition of $[n] = \{1, 2, \dots, n\}$ is called

- (1) *connected* if no proper subinterval of $[n]$ is a union of blocks; this means that any diagram depicting the partition is a connected graph.
- (2) *irreducible* if 1 and n are in the same connected component, i.e., there is only one outer block.
- (3) *noncrossing* if its blocks do not intersect in their graphical representation, i.e., if there are no two distinct blocks B_1 and B_2 and elements $a, c \in B_1$ and $b, d \in B_2$ s.t. $a < b < c < d$. Equivalently one could say that a partition is noncrossing if each of its connected components consists of exactly one block.

Typical examples of these types of partitions are shown in Fig. 2.1.

We denote the lattice of partitions of $[n]$ by \mathcal{P}_n , the irreducible partitions by \mathcal{P}_n^{irr} and the order ideal of connected partitions by \mathcal{P}_n^{conn} ; the lattice of noncrossing partitions will be denoted by NC_n , and the sublattice of irreducible noncrossing partitions by NC_n^{irr} . Finally, let us denote by \mathcal{I}_n the lattice of interval partitions, i.e. the lattice of partitions consisting entirely of intervals.

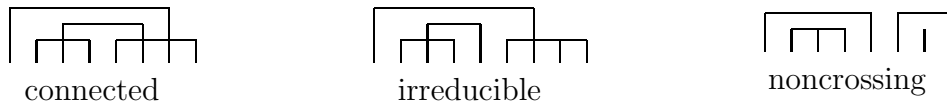


FIGURE 1. Typical partitions

2.2. Cumulants. Cumulants linearize convolution of probability measures coming from various notions of independence.

Definition 2.2. A *non-commutative probability space* is a pair (\mathcal{A}, ϕ) of a (complex) unital algebra \mathcal{A} and a unital linear functional ϕ . The elements of \mathcal{A} are called (*non-commutative*) *random variables*.

Given a notion of independence, convolution is defined as follows. Let a and b be “independent” random variables, then the convolution of the distributions of a and b is defined to be the distribution of the sum $a+b$. In all the examples below, the distribution of the sum of “independent” random variables only depends on the individual distributions of the summands and therefore convolution is well defined on the level of probability measures. Moreover, the n -th moment $m_n(a+b)$ is a polynomial function of the moments of a and b of order less or equal to n . For our purposes it is sufficient to axiomatize cumulants as follows.

Definition 2.3. Given a notion of independence on a noncommutative probability space (\mathcal{A}, ϕ) , a sequence of maps $a \mapsto k_n(a)$, $n = 1, 2, \dots$ is called a *cumulant sequence* if it satisfies the following properties

- (1) $k_n(a)$ is a polynomial in the first n moments of a with leading term $m_n(a)$. This ensures that conversely the moments can be recovered from the cumulants.
- (2) homogeneity: $k_n(\lambda a) = \lambda^n k_n(a)$.
- (3) additivity: if a and b are “independent” random variables, then $k_n(a+b) = k_n(a) + k_n(b)$.

Möbius inversion on the lattice of partitions plays a crucial role in the combinatorial approach to cumulants. We need three kinds of cumulants here, corresponding to classical, free and boolean independence, which involve the three lattices of set partitions, noncrossing partitions and interval partitions, respectively. Let X be a random variable with distribution ψ and moments $m_n = m_n(X) = \int x^n d\psi(x)$

2.3. Classical cumulants. Let

$$\mathcal{F}(z) = \int e^{xz} d\psi(x) = \sum_{n=0}^{\infty} \frac{m_n}{n!} z^n$$

be the formal Laplace transform (or exponential moment generating function). Taking the formal logarithm we can write this series as

$$\mathcal{F}(z) = e^{K(z)}$$

where

$$K(z) = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} z^n$$

is the *cumulant generating function* and the numbers κ_n are called the (*classical*) *cumulants* of the random variable X .

If for a partition $\pi = \{\pi_1, \pi_2, \dots, \pi_p\}$ we put $m_\pi = m_{|\pi_1|} m_{|\pi_2|} \cdots m_{|\pi_p|}$ and $\kappa_\pi = \kappa_{|\pi_1|} \kappa_{|\pi_2|} \cdots \kappa_{|\pi_p|}$, then we can express moments and cumulants mutually using the Möbius function μ on the partition lattice Π_n as follows:

$$m_\pi = \sum_{\sigma \leq \pi} \kappa_\sigma \quad \kappa_\pi = \sum_{\sigma \leq \pi} m_\sigma \mu(\sigma, \pi).$$

For example, the standard Gaussian distribution $\gamma = N(0, 1)$ has cumulants

$$\kappa_n(\gamma) = \begin{cases} 1 & n = 2 \\ 0 & n \neq 2 \end{cases}$$

It follows that the even moments $m_{2n} = \frac{2n!}{2^n n!}$ of the standard gaussian distribution count the number of pairings of a set with the corresponding number of elements.

2.4. Free Cumulants. Free cumulants were introduced by Speicher [38] in his combinatorial approach to Voiculescu's free probability theory. Given our random variable X , let

$$(2.1) \quad M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n$$

be its ordinary moment generating function. Define a formal power series

$$C(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

implicitly by the equation

$$C(z) = C(zM(z)).$$

Then the coefficients c_n are called the *free* or *non-crossing cumulants*. The latter name stems from the fact that combinatorially these cumulants are obtained by Möbius inversion on the lattice of non-crossing partitions:

$$(2.2) \quad m_\pi = \sum_{\substack{\sigma \in NC_n \\ \sigma \leq \pi}} c_\sigma \quad c_\pi = \sum_{\substack{\sigma \in NC_n \\ \sigma \leq \pi}} m_\sigma \mu_{NC}(\sigma, \pi)$$

2.5. Boolean cumulants. Boolean cumulants linearize *boolean convolution* [39]. Let again $M(z)$ be the ordinary moment generating function of a random variable X defined by (2.1). It can be written as

$$M(z) = \frac{1}{1 - H(z)}$$

where

$$H(z) = \sum_{n=1}^{\infty} h_n z^n$$

and the coefficients are called *boolean cumulants*. Combinatorially the connection between moments and boolean cumulants is described by Möbius inversion on the lattice of interval partitions:

$$(2.3) \quad m_\pi = \sum_{\substack{\sigma \in \mathcal{I}_n \\ \sigma \leq \pi}} h_\sigma \quad h_\pi = \sum_{\substack{\sigma \in \mathcal{I}_n \\ \sigma \leq \pi}} m_\sigma \mu_{\mathcal{I}}(\sigma, \pi)$$

The connection between these kinds of cumulants is provided by the following theorem (see also [14] for the case of pairings).

Theorem 2.4 ([29]). *Let (m_n) be a (formal) moment sequence with classical cumulants κ_n . Then the free cumulants of m_n are equal to*

$$(2.4) \quad c_n = \sum_{\pi \in \mathcal{P}_n^{\text{conn}}} \kappa_\pi$$

the boolean cumulants are equal to

$$h_n = \sum_{\pi \in \mathcal{P}_n^{\text{irr}}} \kappa_\pi = \sum_{\pi \in NC_n^{\text{irr}}} c_\pi$$

2.6. The normal law and pair partitions. Let $d\gamma(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ be the standard normal (classical Gaussian) distribution. Its moments are given by

$$m_k := \int_{\mathbb{R}} t^k d\gamma(t) = \begin{cases} (k-1)!! := (k-1)(k-3)(k-5)\cdots 5 \cdot 3 \cdot 1, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

A more combinatorial description of this is that the moments count all pairings, i.e.,

$$m_k = \#\mathcal{P}_2(k).$$

From Theorem 2.4 it follows then that the free cumulants c_n of γ are given by the number of connected (or irreducible) pairings,

$$c_n = \#\{\pi \in \mathcal{P}_2(n) \mid \pi \text{ is connected}\}.$$

The question whether γ is infinitely divisible in the free sense is equivalent to the question whether the sequence $(c_n)_{n \in \mathbb{N}}$ is conditionally positive, which is the same as the question whether the shifted sequence $(s_n)_{n \geq 0}$, where $s_n := c_{n+2}$, is positive definite, i.e., the moment sequence of some measure. See Section 13 of [33] for more details on this (note that there only compactly supported measures are considered, but the theory also extends to measures which have a uniquely solvable moment problem).

The first few values of the free cumulants of the Gaussian distribution are

$$c_2 = 1, \quad c_4 = 1, \quad c_6 = 4, \quad c_8 = 27, \quad c_{10} = 248, \quad c_{12} = 2830, \quad \dots$$

This sequence of the numbers of irreducible diagrams of $2n$ nodes has been well-studied from a combinatorial point of view, see, e.g., [41, 40]; it appears for example as sequence A000699 in Sloane's *Encyclopedia of Integer Sequences* [37]. For a recent bibliography of this sequence see [28] where it is shown that the sequence is not holonomic, i.e., it does not

satisfy a linear recurrence with polynomial coefficients. However, positivity questions for this sequence have never been considered. It might be interesting to point out that these numbers appear also in the perturbation expansion in quantum field theory for the spinor case in 4 spacetime dimensions, see [15] and in renormalization of quantum electrodynamics, see [16]; however, due to the cryptic style of the mentioned papers the meaning of this remains quite mysterious for the present authors.

2.7. A recursive formula. A main result on the numbers of irreducible diagrams is the following recursion formula due to Riordan [36]

$$c_{2n} = (n-1) \sum_{i=1}^{n-1} c_{2i} c_{2(n-i)},$$

a simple bijective proof of which can be found in [34]. In terms of the shifted sequence $(s_n)_{n \geq 0}$ this reads

$$(2.5) \quad s_{2n} = n \sum_{i=0}^{n-1} s_{2i} s_{2(n-i-1)}.$$

Note the similarity to the standard recursion of the Catalan numbers (just remove the factor n before the sum). An equivalent formulation is

$$(2.6) \quad s_{2n} = \sum_{i=0}^{n-1} (2i+1) s_{2i} s_{2(n-i-1)}.$$

Thus the question is whether the sequence $(s_k)_{k \geq 0}$ – defined by $s_{2n+1} = 0$ ($n \in \mathbb{N}$) and by either of the recursions (2.5) or (2.6) and $s_0 = 1$ – is the moment sequence of some measure. Since $s_0 = 1$, this measure must necessarily be a probability measure. The most direct way to prove this would be to find a selfadjoint operator which has these numbers s_n as moments.

There are some immediate combinatorial interpretations of the above recursions. For example, the recursion (2.6) yields

$$s_{2n} = \sum_{\pi \in NC_2(2n)} \prod_{V \in \pi} (\text{ip}(V) + 1).$$

Here we are summing over all non-crossing pairings of $2n$ elements and the contribution of a pairing π is given by a product over the blocks of π , each block contributing the number $\text{ip}(V)$ of its inner points plus one. These inner points have also been counted in [44, 23] in different contexts.

2.8. Tree factorials. The recursion (2.5), on the other hand, can be interpreted in terms of planar rooted binary trees. These are planar rooted trees such that each vertex at most 2 successors, called *children*. A vertex without successors is called a *leaf*. Denote the set of such trees with n vertices by \mathcal{PRBT}_n . The number of these trees is the n -th Catalan number. The *tree-factorial* is defined as follows. For $n = 0$ there is only one binary tree (the empty tree), whose factorial is defined to be 1. Let t be a binary tree with $n > 0$

vertices. Then t can be decomposed into its root vertex, a left branch t_1 with k vertices and a right branch t_2 with $n - 1 - k$ vertices and we define

$$t! = n \cdot t_1! t_2!$$

Then we have the following identity.

Proposition 2.5.

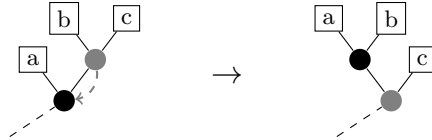
$$(2.7) \quad s_{2n} = \sum_{t \in \mathcal{PRBT}_n} t!$$

Indeed using the above decomposition it is easy to see that the numbers on the right hand side also satisfy the recursion (2.5). For more information on tree factorials see, e.g., [31, Section 2] and [30, Section 2] and section 2.10 below.

Note that these interpretations are canonical for the shifted sequence (s_n) and not for the original sequence (c_n) ; for example, $c_8 = 27$ is the number of irreducible pairings of 8 points, but $s_6 = 27$ is given in terms of non-crossing pairings of 6 points or, equivalently, in terms of planar binary trees with $3 = 6/2$ nodes.

2.9. Two Markov chains.

2.9.1. *MTR on binary search trees.* The tree factorial appears in the stationary distribution of the move-to-root Markov chain on binary trees [22]. Binary trees are used in computer science to arrange data such that it can be accessed using binary search. To reduce search time, every time an entry is searched it is moved to the root of the tree by repeating the so called *simple exchange* shown in the following picture



until the root position is reached. Choosing a vertex randomly (each with probability $1/n$), this induces a Markov chain on the state space \mathcal{PRBT}_n . By Perron-Frobenius theory there is a unique stationary distribution π for this Markov chain and it is shown in [22] that it is given by $\pi(t) = 1/t!$. Equivalently, it describes the distribution of a randomly grown tree.

2.9.2. *The Naimi-Trehel algorithm on planar rooted trees.* The tree factorial also appears in the so-called *Naimi-Trehel* algorithm [43, 32]. This is a queuing model based on yet another Catalan family, namely planar rooted trees \mathcal{PRT} . It solves a scheduling problem for n clients (e.g., computers) who access some resource (e.g., a printer) which can serve at most one client at a time. In order to reduce the number of messages needed to schedule the printer jobs, the queue is arranged as a planar rooted tree and each time a request is sent, the queue is rearranged. The average number of messages is then a certain statistic on these trees. This can be modeled as a Markov chain on labeled rooted trees where at each step a random client sends a request and the tree is transformed accordingly. In the end only the shape of the tree matters and it suffices to consider the corresponding Markov

chain on the unlabeled planar rooted trees. By means of bijection this can be transformed into a Markov chain on Dyck paths where we have the following algebraic rule for the transition probabilities [32].

Let us consider words in the two letter alphabet $\{x, x^*\}$ where x is an *upstep* or *NE* step and x^* is a *downstep* or *SE* step. A *Dyck word* is a word in x and x^* such that each left subword contains not more downsteps than upsteps and the whole word contains an equal number of up- and downsteps. Dyck words can be visualized by Dyck paths, see fig. 2. These are lattice paths which do not descend below the x -axis. We denote by 1 the Dyck



FIGURE 2. Dyck paths of length 6.

word of length 0 and D_n the set of Dyck words of length $2n$.

The recursive structure of rooted planar binary trees has a counterpart in the unique decomposition of a Dyck word w as concatenation

$$w = u v x^*$$

such that both u and v are again Dyck words. Using this recursive structure we get a natural bijection α from planar rooted binary trees to Dyck words by setting recursively $\alpha(t) = \alpha(t_1)x\alpha(t_2)x^*$ if t has left and right subtrees t_1 and t_2 . Under this bijection the tree factorial on Dyck words can be recursively computed as

$$1! = 1 \quad (u v x^*)! = n \cdot u! v!$$

Dyck words form a monoid with the concatenation product and following [32] we recursively define a linear operator on the monoid algebra of formal linear combinations of Dyck words by letting

$$\mu(w) = w + \nu(w)$$

where

$$\nu(1) = 0 \quad \nu(u v x^*) = \nu(u) \otimes (v x^*) + \mu(v) u x^*$$

and the operation \otimes is defined as

$$u v x^* \otimes w = u v w x^*$$

For example, using these rules we have

$$\mu(\text{Dyck path with two peaks of height 2}) = \text{Dyck path with two peaks of height 2} + 2 \cdot \text{Dyck path with one peak of height 2 and one peak of height 1} + \text{Dyck path with three peaks of height 1}$$

Then it is easy to see by induction that μ maps a Dyck path w of length $2n$ to a linear combination of Dyck paths of the same length and that the coefficients are nonnegative integers which sum up to $n+1$. We interpret the matrix representation A of this map as a weighted adjacency matrix and obtain a digraph with vertex set D_n . Dividing the matrix by $n+1$ we obtain a stochastic matrix $P = \frac{1}{n+1}A$. It was shown in [32] that this is exactly the transition matrix of the Naimi-Treherl Markov chain discussed above. Figures 3 and 4

show the graphs for $n = 2, 3$ and 4. Again there is a unique stationary distribution π for

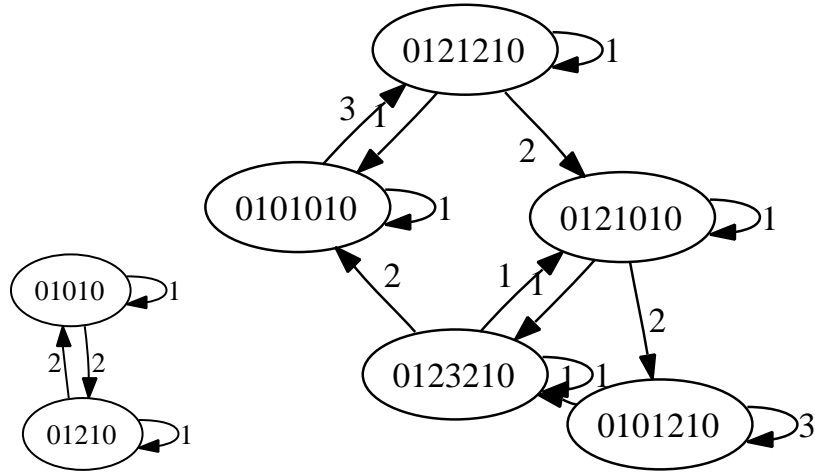


FIGURE 3. Naimi-Trehel graphs of order 2 and 3.

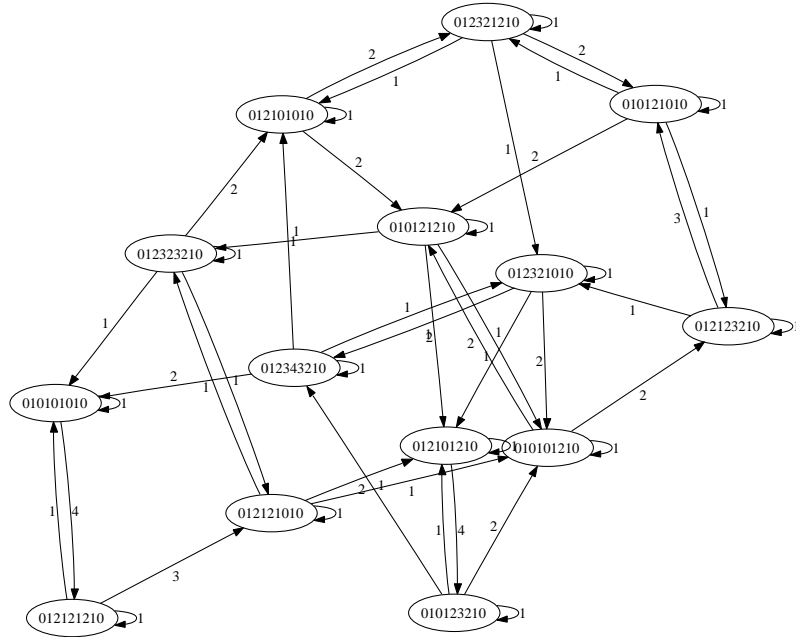


FIGURE 4. Naimi-Trehel graph of order 4.

this Markov chain. It is shown in [32] that $\pi(w) = \frac{1}{w!}$ where $w!$ is the tree factorial defined above.

2.9.3. *A probabilistic interpretation.* For both Markov chains discussed above our sequence appears as

$$(2.8) \quad s_{2n} = \sum_{w \in D_n} \frac{1}{\pi(w)} = \sum \mathbf{E}_w T_w = C_n \mathbf{E} T$$

where by standard Markov chain theory

$$(2.9) \quad \mathbf{E}_w T_w = \frac{1}{\pi(w)}$$

is the expected time of a random walker starting in w to come back to w for the first time and $\mathbf{E} T$ is the expected return time of a random walker starting at a randomly chosen state w , each chosen with probability $1/C_n$ (Catalan number). Although this setting is very close to the Lindström-Gessel-Viennot theory of determinants, we did not manage to exploit it for a combinatorial proof of our theorem.

2.10. **Loday-Ronco Hopf algebra.** Hopf algebras of trees have enjoyed increasing interest recently in renormalization theory and noncommutative geometry [21] and pure algebra (“dendriform algebras”) [30]. Hopf algebras of labeled trees have been studied by Foissy [24, 25].

We provide here yet another Hopf algebra on labeled trees whose Hilbert series is related to our problem.

Definition 2.6. Let t be a planar rooted binary tree. A *labeling* of t is a function from the vertices of t to the integers. A labeling is called *anti-increasing* if the labels are distinct and for every vertex v of t , the labels of the left subtree (with root v) are strictly smaller than the labels of the right subtree. In other words, if we interpret the tree as the Hasse diagram of a poset, every antichain has increasing labels. Two trees with anti-increasing labelings are called *equivalent* if they induce the same linear order on the vertices. The equivalence classes are called *anti-increasingly ordered trees*.

Proposition 2.7. *Let t be a planar rooted binary tree with n vertices. Then the tree factorial $t!$ counts the number of anti-increasing orderings or equivalently the anti-increasing labelings with different numbers $1, 2, \dots, n$.*

The proof is a simple induction using the recursive definition of the tree factorial.

Figure 5 shows an example of an anti-increasing tree. The formal linear combinations of

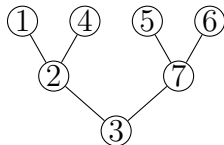
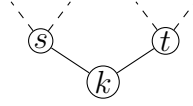


FIGURE 5. An anti-increasingly ordered tree

labeled planar rooted binary trees form a graded vector space, the grading being given by the number of vertices of the trees. It is then straightforward to generalize the coproduct

of Loday and Ronco [30] to labeled trees as follows [24, 25]. Let s and t be labeled binary trees. We define a new labeled binary tree $s \vee_k t$ by grafting them on a new root with label k :



Contrary to the Butcher-Connes-Kreimer Hopf algebra of rooted trees, binary trees cannot be grafted from forests. Therefore the left part of the Loday-Ronco coproduct does not consist of forests, but rather a certain noncommutative “dendriform” product of binary trees. The product of two labeled binary trees $s = s_1 \vee_k s_2$ and $t = t_1 \vee_l t_2$ is recursively defined as

$$(2.10) \quad s * t = s_1 \vee_k (s_2 * t) + (s * t_1) \vee_l t_2$$

with the convention that for the empty tree $|$ the product is

$$| * t = t * | = t.$$

This operation is associative and the coproduct of $t = u \vee_k v$ is recursively defined as

$$\Delta(t) = \Delta(u) \otimes_k \Delta(v) + t \otimes |$$

where in Sweedler’s notation we define for $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$ and $\Delta(v) = \sum v_{(1')} \otimes v_{(2')}$

$$\Delta(u) \otimes_k \Delta(v) = \sum \sum (u_{(1)} * v_{(1')}) \otimes (u_{(2)} \vee_k v_{(2')}).$$

Proposition 2.8. (1) *Let s and t be anti-increasingly labeled trees such that the labels of s are smaller than the labels of t . Then $s * t$ is a sum of anti-increasingly labeled trees.*

(2) *Let t be a anti-increasingly labeled tree. Then all terms of $\Delta(t)$ contain anti-increasingly labeled trees only.*

Proof. (1) By induction, let $s = s_1 \vee_k s_2$ and $t = t_1 \vee_l t_2$, then $s * t = s_1 \vee_k (s_2 * t) + (s * t_1) \vee_l t_2$. By induction hypothesis, the monomials of both $s_2 * t$ and $s * t_1$ are anti-increasingly labeled, the labels of s_1 are smaller than the labels of $s_2 * t$ and the labels of $s * t_1$ are smaller than the labels of t_2 .

(2) Assume that $t = u \vee_k v$ has anti-increasing labels and its children have coproducts $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$ and $\Delta(v) = \sum v_{(1')} \otimes v_{(2')}$, then by the preceding calculation each term $u_{(1)} * v_{(1')}$ is a sum of anti-increasingly labeled trees and each $u_{(2)} \vee_k v_{(2')}$ has anti-increasing labels as well, therefore

$$\Delta(t) = \sum \sum (u_{(1)} * v_{(1')}) \otimes (u_{(2)} \vee_k v_{(2')}) + t \otimes |$$

only contains anti-increasingly labeled terms. □

It follows that the anti-increasingly labeled trees form a graded sub-coalgebra, but not a subalgebra because the product of anti-increasingly labeled trees does not consist of anti-increasingly labeled trees in general. Passing to anti-increasingly ordered trees we obtain a Hopf algebra.

Corollary 2.9. *The anti-increasingly ordered trees span a Hopf algebra whose Hilbert series is the generating series of our sequence (s_n) .*

Proof. We have seen that anti-increasingly labeled trees form a coalgebra. Define the product of anti-increasingly ordered trees s and t as follows: Put arbitrary anti-increasing labelings on s and t such that the labels of s are smaller than the labels of t . Then $s * t$ consists of anti-increasingly labeled trees and the corresponding anti-increasingly ordered equivalence classes do not depend on the choice of the labelings for s and t . Define the coproduct of an anti-increasingly ordered tree by choosing an anti-increasing labeling compatible with the anti-increasing order, compute the coproduct of the obtained anti-increasingly labeled tree and replace the anti-increasingly labeled terms of the result by the corresponding anti-increasingly ordered trees. Again the choice of the initial anti-increasing labeling has no influence on the final result; moreover, the obtained coalgebra is graded and connected, i.e., the first homogeneous component is one-dimensional, and the existence of the antipode follows by standard Hopf algebra theory. \square

Some examples:

$$\begin{aligned} \Delta(|) &= | \otimes | \\ \Delta(\textcircled{1}) &= | \otimes \textcircled{1} + \textcircled{1} \otimes | \\ \Delta(\textcircled{1} \textcircled{2}) &= | \otimes \textcircled{1} \textcircled{2} + \textcircled{1} \otimes \textcircled{1} + \textcircled{1} \textcircled{2} \otimes | \\ \Delta(\textcircled{1} \textcircled{2} \textcircled{3}) &= | \otimes \textcircled{1} \textcircled{2} \textcircled{3} + \textcircled{1} \otimes \textcircled{1} \textcircled{2} + \textcircled{1} \otimes \textcircled{2} \textcircled{3} + \textcircled{1} \textcircled{2} \otimes \textcircled{1} + \textcircled{1} \textcircled{2} \otimes \textcircled{1} + \textcircled{1} \textcircled{2} \textcircled{3} \otimes | \end{aligned}$$

2.11. Charge Hopf algebra. There is another coproduct (“charge Hopf algebra” of Brouder and Frabetti) in [17], also mentioned in [26, sec. 2.6] which is more asymmetric than the Loday-Ronco Hopf algebra. First we define an associative multiplication s/t by putting $|/s = s/| = s$ and otherwise grafting s onto the leftmost leaf of t . Recursively, $s/(t_1 \vee t_2) = (s/t_1) \vee t_2$. This makes sense for labeled trees as well (just keeping the labels) and for LR-ordered trees we define s/t by shifting the labels of t such that all labels of s are less than the labels of t before carrying out the product. Then the algebra is generated by all elements of the form $| \vee_k t$ which we denote by $V_k(t)$. The Brouder-Frabetti coproduct is defined recursively as

$$\begin{aligned} \Delta^\gamma(|) &= | \otimes | \\ \Delta^\gamma(\textcircled{1}) &= \textcircled{1} \otimes | + | \otimes \textcircled{1} \\ \Delta^\gamma(V_k(s \vee_l t)) &= V_k(s \vee_l t) \otimes | + id \otimes V_k(\Delta^\gamma(s)/(\Delta^\gamma(V_l(t)) - V_l(t) \otimes |) \end{aligned}$$

Again the monotone trees form a subalgebra. Examples:

$$\begin{aligned} \Delta^\gamma \left(\begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \right) &= \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \otimes | + | \otimes \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \\ \Delta^\gamma \left(\begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \right) &= \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \otimes | + 2 \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{1} \end{array} \otimes \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{1} \end{array} + | \otimes \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \\ \Delta^\gamma \left(\begin{array}{c} \textcircled{3} \\ | \\ \textcircled{2} \\ | \\ \textcircled{1} \end{array} \right) &= \begin{array}{c} \textcircled{3} \\ | \\ \textcircled{2} \\ | \\ \textcircled{1} \end{array} \otimes | + \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{1} \end{array} \otimes \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} + | \otimes \begin{array}{c} \textcircled{3} \\ | \\ \textcircled{2} \\ | \\ \textcircled{1} \end{array} \end{aligned}$$

2.12. Conclusion. These interpretations of the sequence s_n invite to Fock space like constructions with corresponding creation and annihilation operators, with the hope that the sum of creation and annihilation operator would have the s_n as its moments. However, we were not able to implement this idea successfully. The involved inner products usually lacked positivity.

In the next section we will give an analytic proof of the positive definiteness of the sequence s_n . Essentially, it will consist in showing that the generating power series

$$\phi(z) := \sum_{n=0}^{\infty} s_n \frac{1}{z^{n+1}}$$

is actually the Cauchy-transform of a probability measure. We are not able to determine this measure directly, but we will show its existence. Note that the recursion (2.6) implies, at least formally, for ϕ the equation

$$-\phi(z)\phi'(z) = \phi(z) - \frac{1}{z}.$$

This equation will be the starting point of our investigations in the next section and we will show that it allows an extension of ϕ as an analytic map from the upper to the lower complex half-plane, which is a characterizing property for Cauchy transforms.

Since the s_n grow of the same order as the moments of the Gauss, the above formal series has no non-trivial radius of convergence. However, it does determine uniquely an analytic map on some truncated cone at infinity; our proof will show that this map extends to the upper half-plane.

3. ANALYTIC PROOF OF THE THEOREM

In this section we shall give an analytic proof of the free infinite divisibility of the classical normal distribution. We shall obtain the free infinite divisibility of the classical Gaussian as a limiting case of a more general family of freely infinitely divisible distributions with noncompact support, namely the *Askey-Wimp-Kerov distributions*, defined in the introduction. A certain sub-family of the Askey-Wimp-Kerov distributions appears in [11]. Recall that the distributions $\{\mu_c: c \in (-1, +\infty)\}$ are determined by the continuous

fraction expansion of their Cauchy-Stieltjes transforms:

$$(3.1) \quad G_{\mu_c}(z) = \frac{1}{z - \frac{c+1}{z - \frac{c+2}{z - \dots}}}$$

In this section, we shall prove the following theorem:

Theorem 3.1. *With the notations from above, the probability measures μ_c are freely infinitely divisible for all $c \in [-1, 0]$.*

To prove this result, we shall use the well-known characterization of free infinite divisibility provided by Bercovici and Voiculescu. Recall [8] that

Theorem 3.2. *A Borel probability measure μ on the real line is \boxplus -infinitely divisible if and only if its Voiculescu transform $\phi_\mu(z)$ extends to an analytic function $\phi_\mu: \mathbb{C}^+ \rightarrow \mathbb{C}^-$.*

We remind the reader that the Voiculescu transform of a probability measure μ is defined by the equality $\phi_\mu(1/z) = G_\mu^{-1}(z) - 1/z$, for z in some Stolz angle in the lower half-plane, with vertex at zero.¹ For more details and important properties of this transform, we refer to [8]. In particular, this theorem guarantees that taking weak limits preserves free infinite divisibility. Our main source for the analysis of the function G_{μ_c} will be Kerov's work [27] and the paper [4] of Askey and Wimp. It is shown there that μ_c is absolutely continuous with respect to the Lebesgue measure on the real line and

$$(3.2) \quad \frac{d\mu_c(u)}{du} = \frac{1}{\sqrt{2\pi}\Gamma(c+1)} \frac{1}{|D_{-c}(iu)|^2}, \quad u \in \mathbb{R},$$

where for $c > 0$,

$$D_{-c}(z) = \frac{e^{-z^2/4}}{\Gamma(c)} \int_0^\infty e^{-zx - \frac{x^2}{2}} x^{c-1} dx.$$

This integral representation does not hold for $c \in (-1, 0]$, but the function $D_{-c}(iu)$ is still well defined for $c \in (-1, 0)$, according to a theorem of Askey and Wimp [27, Theorem 8.2.2]. Moreover, as remarked in [4, Section 4], the function $D_{-c}(z)$ is an entire function of both c and z .

Remark 3.3. (1) We observe that, for fixed $c > -1$, equation (3.2) together with the analyticity of D in c and z , guarantees that $D_{-c}(iu) \neq 0$ for all $u \in \mathbb{R}$. Indeed, a zero of $\mathbb{R} \ni u \mapsto D_{-c}(iu) \in \mathbb{C}$ would have to be of order at least one, hence the function $\frac{1}{|D_{-c}(iu)|^2}$ would not be integrable around that particular zero.

(2) On the other hand, analyticity of D_{-c} alone guarantees that the density $\frac{d\mu_c(u)}{du}$ is strictly positive everywhere on the real line.

¹The name ‘‘Voiculescu transform’’ for ϕ_μ has been introduced in a later paper [7] of Bercovici and Pata, not in [8].

In the same paper of Kerov [27] it is shown that $G_{\mu_c}(z)$ satisfies the Riccati equation

$$(3.3) \quad G'_{\mu_c}(z) = cG_{\mu_c}(z)^2 - zG_{\mu_c}(z) + 1, \quad z \in \mathbb{C}^+.$$

It is also shown that this expression is equivalent, via the substitution

$$G_{\mu_c}(z) = -\frac{1}{c} \frac{\varphi'(z)}{\varphi(z)},$$

which holds for any $c \neq 0$, to the second order linear differential equation

$$(3.4) \quad \varphi''(z) + z\varphi'(z) + c\varphi(z) = 0, \quad z \in \mathbb{C}^+.$$

(The function φ does depend on c .) The density $\frac{d\mu_c}{du}$ is analytic around zero, so, according to [6, Lemma 2.11], it follows that the function G_{μ_c} has an analytic extension to a small enough neighbourhood of zero. Clearly $G_{\mu_c}(0) \neq 0$, so φ has an analytic extension around zero. Using (3.4), we obtain a convergent power series expansion for φ , namely $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$, where $c_{n+2} = -\frac{c+n}{(n+1)(n+2)} c_n$. We conclude that

$$c_{2k} = (-1)^k \frac{c(c+2)(c+4) \cdots (c+2k)}{(2k)!} c_0,$$

and

$$c_{2k+1} = (-1)^k \frac{(c+1)(c+3) \cdots (c+2k-1)}{(2k+1)!} c_1.$$

Trivially, these coefficients provide a power series with an infinite radius of convergence. Thus, φ is an entire function. We immediately conclude that G_{μ_c} extends to a meromorphic function defined on all of \mathbb{C} , whose poles coincide with the zeros of φ . We shall denote this extension also by G_{μ_c} . (It may be worth noting that φ is entire for any $c \in \mathbb{C}$, and that when $c \rightarrow -1$, we have $\varphi(z) \rightarrow z$, hence μ_c tends in the weak topology to the Dirac point mass at zero, δ_0 .) We shall denote

$$(3.5) \quad F_{\mu_c}(z) = \frac{1}{G_{\mu_c}(z)}, \quad z \in \mathbb{C}.$$

Again, from the above it is clear that F_{μ_c} is meromorphic, its poles coinciding with the critical points of φ . It satisfies the differential equation

$$(3.6) \quad F'_{\mu_c}(z) = -F_{\mu_c}(z)^2 + zF_{\mu_c}(z) - c, \quad z \in \mathbb{C}.$$

(This follows easily when we divide by $-G_{\mu_c}(z)^2$ in equation (3.3).) Dividing equation (3.6) by $F_{\mu_c}(z)$ gives

$$\frac{F'_{\mu_c}(z)}{F_{\mu_c}(z)} = (z - F_{\mu_c}(z)) - \frac{c}{F_{\mu_c}(z)}.$$

Proof. (of Theorem 3.1.) We will prove our theorem by arguing that, essentially, the meromorphic function F_{μ_c} maps some simply connected domain containing the upper half-plane bijectively onto \mathbb{C}^+ . This, together with Theorem 3.2, will allow us to conclude. For further reference, we shall split the proof in a succession of remarks and lemmas. For the rest of the proof, we shall fix a $c \in (-1, 0)$ (open interval!) Let us start by studying the critical points of F_{μ_c} . It is known [1, Chapter III] that $\Im F_{\mu_c}(z) > \Im z$ for all $z \in \mathbb{C}^+$.

Moreover, if $F_{\mu_c}(z) \in \mathbb{C}^+$ while $z \notin \mathbb{C}^+$, then it is clear that $\Im(z - F_{\mu_c}(z)) < 0$. In addition, as $c < 0$, $\Im\left(-\frac{c}{F_{\mu_c}(z)}\right) < 0$ whenever $F_{\mu_c}(z) \in \mathbb{C}^+$. This, together with Remark 3.3, implies

Remark 3.4. If $\Im F_{\mu_c}(z) \geq 0$, then $F'_{\mu_c}(z) \neq 0$. In particular, F_{μ_c} has no critical points anywhere in the closure of the upper half-plane.

It will be of use to also study the behaviour of $F_{\mu_c}(iy)$, $y \in \mathbb{R}$.

Remark 3.5. The continued fraction expansion of G_{μ_c} indicates that the probability measure μ_c is symmetric with respect to the origin, so in particular $G_{\mu_c}(i[0, +\infty)) \subseteq i(-\infty, 0]$. This, of course, together with the meromorphicity of F_{μ_c} and G_{μ_c} , requires that $G_{\mu_c}(i\mathbb{R}), F_{\mu_c}(i\mathbb{R}) \subseteq i\mathbb{R} \cup \{\infty\}$.

Part of the following lemma will not be used directly in the proof of the current theorem, but we find that it is nevertheless worth mentioning these results.

- Lemma 3.6.**
- (1) *There exists $-\infty < q_0 < 0$ so that F_{μ_c} maps $i[q_0, +\infty)$ bijectively onto $i[0, +\infty)$.*
 - (2) *$F_{\mu_c}(i(-\infty, q_0))$ is a bounded subset of $i(-\infty, 0)$. In particular, F_{μ_c} has no poles on the imaginary line.*
 - (3) *We have $\Im F_{\mu_c}(iy) > y$ for all $y \in \mathbb{R}$.*
 - (4) *The function F_{μ_c} has a unique (simple) critical point is in $i(-\infty, -2\sqrt{-c})$. In addition, $\lim_{y \rightarrow -\infty} F'_{\mu_c}(iy) = \lim_{y \rightarrow -\infty} i^{-1} F_{\mu_c}(iy) = 0$.*
 - (5) *$F'_{\mu_c}(iy) < 1$ for all $y \in \mathbb{R}$.*

Proof. For convenience, we denote $f(r) = i^{-1}F_{\mu_c}(ir)$. Clearly, from the above remark, $f: \mathbb{R} \rightarrow [-\infty, +\infty]$ is real analytic, with the exceptions of a possible number of points which are poles of F_{μ_c} . It is an easy exercise to observe that all poles of F_{μ_c} must be simple: indeed, otherwise it would follow from equation (3.3) that G_{μ_c} is identically equal to zero. Equation (3.6) is re-written for f as

$$(3.7) \quad f'(r) = f(r)^2 - rf(r) - c, \quad r \in \mathbb{R}.$$

In particular, $F'_{\mu_c}(ir) = f'(r) \in \mathbb{R}$. It is known [1] that $\lim_{y \rightarrow +\infty} F'_{\mu_c}(iy) = 1$, and $f(r) = \Im F_{\mu_c}(ir) > r$ for all $r \in (0, +\infty)$. We claim that in fact this must hold for all $r \in \mathbb{R}$, fact which excludes the existence of poles on the imaginary axis for F_{μ_c} . Indeed, continuity of f requires that for this inequality to be reversed, there must be a point $y \in \mathbb{R}$ so that $f(y) = y$. Then, from (3.7), it follows that $f'(y) = -c \in (0, 1)$. But for the real analytic $f(r)$ to cross below the first bisector as r moves towards y from the right, we clearly must have that $f'(y) \geq 1$. Contradiction. This implies also that there is no real point at which the limit of f from the right is $-\infty$. That the limit from the right cannot be $+\infty$ at any real point is trivial: that would make, according to (3.7), f' tend to $+\infty$, instead of $-\infty$, at the same point when r approaches the point from the right. This proves half of (2) and all of (3). Next, the critical points of f : First, $f'(s) = 0$ is equivalent to

$$f(s) = \frac{s \pm \sqrt{s^2 + 4c}}{2}.$$

Since $f(s)$ must be real, this excludes points $s \in (-2\sqrt{-c}, 2\sqrt{-c})$ (recall that $c \in (-1, 0)$). Assuming such a point $s > 0$ exists, we must have, from part (3), $f(s) > s$. However, that is clearly impossible, since $c < 0$. Thus, only negative s are possible, and for such an $s \leq -2\sqrt{-c}$, we have

$$f(s) = \frac{s \pm \sqrt{s^2 + 4c}}{2} < 0,$$

or, differently stated, both any critical point and any critical value of f must be negative. We shall establish that indeed there exists a unique such s (depending of course on c), but in order to do that, we will first prove part (1) of the lemma. We claim that there exists $-\infty \leq q_0 < 0$ so that f maps $[q_0, +\infty)$ onto $[0, +\infty)$. Indeed, otherwise $f(\mathbb{R}) \subseteq (0, +\infty)$ and, as seen above, then $f'(r) > 0$ for all $r \in \mathbb{R}$. Lagrange's theorem would require that there exists a sequence $\{r_n\}_n$ which tends to minus infinity so that $f'(r_n) \rightarrow 0$. But this is impossible, since it would require

$$\lim_{n \rightarrow \infty} f(r_n)(f(r_n) - r_n) = c < 0,$$

while both terms of the product are nonnegative. So $0 \in f(\mathbb{R})$. Choose the largest point in $f^{-1}(\{0\})$ to be q_0 . Now it is clear that in addition f maps $[q_0, +\infty)$ bijectively onto $[0, +\infty)$; since $f'(q_0) = -c > 0$, it follows that, first, $f^{-1}(\{0\}) = \{q_0\}$, and second, that the bijective correspondence extends to a strictly larger interval. We show next that this larger interval cannot be \mathbb{R} . To do this, first let us assume towards contradiction that $f'(r) > 0$ for all $r \in \mathbb{R}$. Then, of course, $\lim_{r \rightarrow -\infty} f(r) = d$ exists and belongs to $[-\infty, 0)$. We first assume that $d \in (-\infty, 0)$. Then, by Lagrange's theorem again, we must be able to find a sequence $\{r_n\}_n$ which tends to minus infinity so that $f'(r_n) \rightarrow 0$. As seen above, then

$$\lim_{n \rightarrow \infty} f(r_n)(f(r_n) - r_n) = c < 0,$$

or, differently said, either

$$\lim_{n \rightarrow \infty} f(r_n) = 0, \quad \text{or} \quad \lim_{n \rightarrow \infty} (f(r_n) - r_n) = 0.$$

The first case cannot happen, since it would require that there exists a critical point of f in $(-\infty, q_0)$, which we assumed not to happen. The second case would require that

$$\lim_{n \rightarrow \infty} f(r_n) = -\infty,$$

which would contradict $d \in (-\infty, 0)$ again. We consider then the situation in which $d = -\infty$. To fulfill this condition, and in addition to avoid that $f'(s) = 0$ for some $s \in (-\infty, q_0)$, it is necessary that

$$r < f(r) < \frac{r - \sqrt{r^2 + 4c}}{2},$$

for all $r < -2\sqrt{-c}$. (The necessity of the first inequality was proved before.) However, recall that $f(q_0) = 0 \implies f'(q_0) = -c \in (0, 1)$, so there must be then some point

$t \in (-\infty, q_0)$ so that $f'(t) = 1$. This implies

$$f(t) = \frac{t \pm \sqrt{t^2 + 4(c+1)}}{2} \begin{cases} > 0 & \text{if choosing } + \\ < t & \text{if choosing } - \end{cases}$$

which is an obvious contradiction. This, in addition, forbids the case of $f'(t) = 1$ for any $t \in (-\infty, 0]$. Thus, there must be a critical point $s \in (-\infty, -2\sqrt{-c})$ of f . Since by equation (3.7) $f'(s) = 0 \implies f''(s) = -f(s) > 0$, any critical value of f is a local minimum, hence there exists only one such s , and $f(s)$ is the global minimum of f . So $f(r) \in (f(s), 0)$ for all $r \in (-\infty, s)$. The previous arguments about the behaviour of f near $-\infty$ can be easily reapplied to show that $\lim_{r \rightarrow -\infty} f(r) = \lim_{r \rightarrow -\infty} f'(r) = 0$. Finally, from part (4) above, it follows that $f'(r) < 1$ for all $r \in (-\infty, 0]$. Thus, $f'(t) > 1$ for some $t \in \mathbb{R}$ implies first that $t > 0$ and second, that there exists a $t_0 > 0$ so that $f'(t_0) = 1$. Differentiating in (3.7) and using part (3) gives

$$f''(t_0) = f(t_0) - t_0 > 0.$$

Thus, f' increases at t_0 whenever $f'(t_0) = 1$. In particular, this can happen for only one $t \in \mathbb{R}$, and thus $f'(r)$ must tend to one with values strictly greater than one when $r \rightarrow +\infty$. But this contradicts part (3) (namely that $f(r) > r$ for all $r \in \mathbb{R}$.) Thus it is impossible to have $f'(t_0) = 1$. This proves (5) and concludes our proof. \square

The following lemma is trivial:

Lemma 3.7. *For a fixed $c \in (-1, +\infty)$ there exists a $t > 0$ depending on c so that $F_{\mu_c}(\mathbb{C}^+) \supset \mathbb{R} + it$.*

Proof. Since F_{μ_c} is analytic, hence open, on the upper half-plane, and it increases the imaginary part, it is clearly enough to show that (i) $\lim_{x \rightarrow \pm\infty} \Re F_{\mu_c}(x+i) = \pm\infty$ and (ii) there exist $M > N \in [0, +\infty)$ so that $N \leq \Im F_{\mu_c}(x+i) \leq M$ for all $x \in \mathbb{R}$. To prove (i), just observe that, since μ_c is symmetric and has all moments, there exists a positive measure λ_c , also having all moments, so that $\lambda_c(\mathbb{R}) = \int_{\mathbb{R}} u^2 d\mu_c(u)$ and

$$(3.8) \quad F_{\mu_c}(z) = z - G_{\lambda_c}(z), \quad z \in \mathbb{C}^+.$$

Since λ_c is a finite positive measure, $\lim_{x \rightarrow \pm\infty} G_{\lambda_c}(x+i) = 0$, so (i) follows trivially. Part (ii) is equally simple. We have that

$$1 < \Im F_{\mu_c}(x+i) = 1 + \int_{\mathbb{R}} \frac{1}{1+(x-u)^2} d\lambda_c(u) \leq 1 + \lambda_c(\mathbb{R}) = 1 + \int_{\mathbb{R}} u^2 d\mu_c(u).$$

So the lemma is true for any $t \geq 1 + \int_{\mathbb{R}} u^2 d\mu_c(u)$. \square

It follows now easily that F_{μ_c} is injective on the upper half-plane and $F_{\mu_c}(\mathbb{C}^+)$ contains $\mathbb{C}^+ + it$ for some $t > 0$ depending on $c \in (-1, 0]$. However, in order to prove our theorem, we need to find a larger set $C \supset \mathbb{C}^+$ which is mapped by F_{μ_c} bijectively onto the upper half-plane. To do this, we will show that for *any* $t > 0$ there exists a $C_t \supset \mathbb{C}^+ + it$ so that $F_{\mu_c}(C_t) = \mathbb{C}^+ + it$ and F_{μ_c} is injective on C_t . This will clearly guarantee that ϕ_{μ_c} has an analytic extension to $\mathbb{C}^+ + it$ for any $t > 0$, and hence (based on the previous lemma) a unique extension to \mathbb{C}^+ , concluding the proof of Theorem 3.1 by an application of Theorem

3.2. Our strategy will be as follows: for a fixed $t > 0$ there exists, by Lemma 3.6, a unique $s > q_0$ so that $t = \frac{1}{i}F_{\mu_c}(is)$. On the other hand, equation (3.8) guarantees that there must be a number $N = N(t, c) > 0$ so that $\Im F_{\mu_c}(x + it/2) < t$ for all $x \in \mathbb{R}$, $|x| > N$. Since, by Remark 3.4, F_{μ_c} is locally injective around all these points, we conclude that there are three simple paths, one around s and two in $\{z \in \mathbb{C}^+ : \Re z > N\}$ and $\{z \in \mathbb{C}^+ : \Re z < -N\}$, respectively, which are mapped by F_{μ_c} in $\mathbb{R} + it$. We shall argue that these paths can be extended to a simple path p_t containing all of them, with the property that $F_{\mu_c}(p_t) = \mathbb{R} + it$. The correspondence, if existing, must be bijective, by Remark 3.4, and we will define C_t to be the simply connected component of $\mathbb{C} \setminus p_t$ which contains numbers with arbitrarily large imaginary part. It will then be easy to prove that C_t has the desired properties for all $t > 0$.

Lemma 3.8. *With the above notations, there exists exactly one simple curve $p = p_t$, symmetric with respect to the imaginary axis, passing through the point $is \in i\mathbb{R}$ and so that $F_{\mu_c}(p_t) = \mathbb{R} + it$. Moreover, F_{μ_c} maps p_t bijectively onto $\mathbb{R} + it$.*

Proof. For an arbitrary $t > 0$, it is a consequence of Remark 3.4 and Lemma 3.6 that F_{μ_c} is conformal on a small enough ball centered at is . Thus, there exists a simple path p_t^ε , symmetric with respect to the imaginary axis, which is mapped bijectively by F_{μ_c} onto some interval $(-\varepsilon, \varepsilon) + it$ for $\varepsilon > 0$ small enough. We shall show that $p = p_t^\varepsilon$ extends analytically to an infinite path, denoted by p_t , sent by F_{μ_c} in $\mathbb{R} + it$, on which F_{μ_c} has no critical points and so that the limit at infinity of F_{μ_c} along either half of p_t is infinite. Since F_{μ_c} is meromorphic, hence open, this will suffice to prove our lemma. Indeed, let us consider the connected component p_t^+ of $F_{\mu_c}^{-1}([0, +\infty) + it)$ which contains is . It is clear that, as F_{μ_c} is meromorphic on \mathbb{C} , the path p_t^+ must end either at infinity or at a pole of F_{μ_c} , call it ζ . If it ends at a pole, it follows easily that $F_{\mu_c}(p_t^+) = [0, +\infty) + it$ and the correspondence (by Remark 3.4) is bijective.² Let us consider the second case, namely when p_t^+ ends at infinity. In this case, the possibility of having $F_{\mu_c}(p_t^+) = [0, d) + it$ for some $0 < d < +\infty$ must be discarded first: this would correspond to when F_{μ_c} has $d + it$ as an asymptotic value at infinity along p_t^+ . Thus, let us show that

$$\lim_{z \rightarrow \infty, z \in p_t^+} F_{\mu_c}(z) = \infty.$$

Assume towards contradiction that this limit is finite, and call it x (the case when the limit does not exist is easily discarded). Of course, $\Im x = t > 0$, $\Re x = d > 0$. We shall use Equation (3.6) to obtain a contradiction: it follows from it that the differential equation satisfied by the inverse $F_{\mu_c}^{-1}$ (defined on a neighbourhood of $[0, d) + it$ and with values in a neighbourhood of p_t^+) is

$$(3.9) \quad (F_{\mu_c}^{-1})'(w) = \frac{1}{-w^2 + wF_{\mu_c}^{-1}(w) - c}.$$

²This situation will turn out later not to occur, but at this moment this is not important for our proof.

Define $r(v) = \Re F_{\mu_c}^{-1}(v + it)$, $\iota(v) = \Im F_{\mu_c}^{-1}(v + it)$, $0 \leq v < d$. Equation (3.9) translates into

$$(3.10) \quad r'(v) = \frac{v(r(v) - v) + t(t - \iota(v)) - c}{[v(r(v) - v) + t(t - \iota(v)) - c]^2 + [t(r(v) - 2v) + v\iota(v)]^2}$$

$$(3.11) \quad \iota'(v) = -\frac{t(r(v) - 2v) + v\iota(v)}{[v(r(v) - v) + t(t - \iota(v)) - c]^2 + [t(r(v) - 2v) + v\iota(v)]^2}$$

As noted before, $\lim_{v \uparrow d} F_{\mu_c}^{-1}(v + it) = \infty$, so at least one of $r(v), \iota(v)$ must be unbounded. Thus, at least one of $r'(v), \iota'(v)$ must be unbounded. From equations (3.10) and (3.11) it follows that in order for any of $r'(v), \iota'(v)$ to be unbounded, it is necessary that there exists a sequence $\{v_n\}_{n \in \mathbb{N}} \subset [0, d)$ tending to d so that

$$\lim_{n \rightarrow \infty} [v_n(r(v_n) - v_n) + t(t - \iota(v_n)) - c]^2 + [t(r(v_n) - 2v_n) + v_n\iota(v_n)]^2 = 0.$$

As noted in the comments preceding Remark 3.4, $\iota(v_n) \leq t$. Let us choose a subsequence of $\{v_n\}_{n \in \mathbb{N}}$, also denoted by v_n , on which $\iota(v_n)$ converges. If it converges to $\ell \in (-\infty, t]$, then we know that $r(v_n) \rightarrow \infty$ (since $F_{\mu_c}^{-1}(v_n) \rightarrow \infty$). But then, in order for the above displayed limit to hold, it is also necessary that $\iota(v_n)$ tend to infinity, a contradiction. So we must have that both $r(v_n)$ and $\iota(v_n)$ tend to infinity (plus or minus). Then $\lim_{n \rightarrow \infty} [v_n(r(v_n) - v_n) + t(t - \iota(v_n)) - c]^2 = 0 \implies \lim_{n \rightarrow \infty} v_n \frac{r(v_n)}{\iota(v_n)} = t$, or equivalently $\lim_{n \rightarrow \infty} \frac{r(v_n)}{\iota(v_n)} = \frac{t}{d}$. Similarly, $\lim_{n \rightarrow \infty} \frac{r(v_n)}{\iota(v_n)} = \lim_{n \rightarrow \infty} \frac{r(v_n) - 2v_n}{\iota(v_n)} = -\frac{d}{t}$. So $\frac{d}{t} = -\frac{t}{d}$, which implies $d^2 = -t^2$, a contradiction. Thus, it is impossible for $F_{\mu_c}^{-1}(v_n)$ to tend to infinity when $v_n \rightarrow d$, so F_{μ_c} cannot have a finite asymptotic value along p_t^+ . Since μ_c is symmetric, this concludes the proof of our lemma. \square

By proving the previous lemma, we have also proved that the inverse $F_{\mu_c}^{-1}$ of F_{μ_c} admits an analytic extension around $i(0, +\infty)$ and around $\mathbb{R} + it$ for any $t > 0$. We shall argue that all these extensions agree with each other, and provide us with an analytic map $F_{\mu_c}^{-1} : \mathbb{C}^+ \rightarrow \mathbb{C}$ which (1) decreases the imaginary part, and (2) satisfies $\lim_{y \rightarrow +\infty} F_{\mu_c}^{-1}(iy)/iy = 1$. Let us denote

$$t_0 = \inf\{t > 0 \mid \mathbb{R} + ir \subset F_{\mu_c}(\mathbb{C}^+ \cup \mathbb{R}) \forall r \geq t\},$$

and s_0 the unique number greater than q_0 so that $F_{\mu_c}(is_0) = it_0$. As noted after the proof of Lemma 3.7, it is clear that $F_{\mu_c}^{-1} : \mathbb{C}^+ + it_0 \rightarrow \mathbb{C}^+$ satisfies both (1) and (2). Clearly, this function has, by Remark 3.4 and Lemma 3.6, a unique analytic continuation to a small enough neighbourhood of $i[0, +\infty)$ in \mathbb{C}^+ , which we will still denote by $F_{\mu_c}^{-1}$. Now, for an arbitrary $t \in (0, t_0)$, we have proved in Lemma 3.8 that $F_{\mu_c}^{-1}$ admits an analytic continuation to a small enough neighbourhood of $\mathbb{R} + it$ which coincides on a neighbourhood of it with $F_{\mu_c}^{-1}$. Since \mathbb{C}^+ is simply connected and $\mathbb{C}^+ = (\mathbb{C}^+ + it_0) \cup \bigcup_{t \in (0, t_0)} (\mathbb{R} + it)$, we conclude

that $F_{\mu_c}^{-1}$ admits a unique extension to the upper half-plane, with values in \mathbb{C} . Let us denote $C = F_{\mu_c}^{-1}(\mathbb{C}^+)$. It follows easily from the identity principle for analytic functions that $F_{\mu_c}^{-1}(F_{\mu_c}(z)) = z$ for all $z \in C$ and $F_{\mu_c}(F_{\mu_c}^{-1}(z)) = z$ for all $z \in \mathbb{C}^+$. Moreover, $\Im F_{\mu_c}(z) > \Im z$ for all $z \in \mathbb{C}^+ \cup \mathbb{R}$, and if $z \in \mathbb{C}^-$, $F_{\mu_c}(z) \in \mathbb{C}^+$, then it is obvious that

the inequality $\Im F_{\mu_c}(z) > \Im z$ still holds. Thus, we conclude that both (1) and (2) are satisfied by $F_{\mu_c}^{-1}$. Now our theorem follows: we define $\phi_{\mu_c}(z) = F_{\mu_c}^{-1}(z) - z$, $z \in \mathbb{C}^+$. This function is obviously well defined, and maps the upper half-plane into \mathbb{C}^- , thus satisfying the requirements of Theorem 3.2. \square

Since the classical Gaussian γ equals μ_0 , we are now able to conclude its free infinite divisibility from Theorem 3.1 and Theorem 3.2.

Corollary 3.9. *The classical normal distribution $d\gamma(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$ is freely infinitely divisible.*

Next we shall discuss some properties of the Askey-Wimp-Kerov distributions μ_c with parameter $c \in (0, 1)$. As mentioned in the introduction, numerical computations show that μ_c is not freely infinitely divisible for certain values of $c > 0$. More specific, direct (computer assisted) calculation of the Hankel determinants of the free cumulants of $\mu_{9/10}$, for example, shows that the 97th determinant is negative, and thus the cumulant series of $\mu_{9/10}$ is not the moment sequence of a positive measure on \mathbb{R} ; Theorem 3.2 allows us then to conclude that $\mu_{9/10}$ is not freely infinitely divisible. (More such computations have been performed, and they seem to indicate that the size of the first Hankel matrix whose determinant is negative rather tends to decrease as $c > 0$ increases: for example, the 83th Hankel determinant corresponding to μ_1 is negative.)

However, the family $\{\mu_c : c \in (0, 1]\}$ turns out to be of some interest from the point of view of the arithmetic properties of free additive convolution. This subject is not new (implicit results on the arithmetic of free convolutions can be found in many works), but it is the rather recent preprint [20] that has first addressed the problem of the decomposability of measures in “free convolution factors” in an explicit and systematic way. However, the subject is by no means exhausted, the number of results is rather small (we would like to mention among them a remarkable idecomposability result given in [9]), so we feel it is worth mentioning the following by-product of equation (3.1) and our main free infinite divisibility result from Theorem 3.1.

Indeed, it follows from the continued fraction expansion (3.1) and analytic continuation that for any $c \in (-1, 0]$,

$$(3.12) \quad G_{\mu_{c+1}}(z) = \frac{z - F_{\mu_c}(z)}{c + 1}, \quad z \in \mathbb{C}.$$

In addition, for any fixed $c \in (-1, 0]$, the dilation transformation $\mu_c \mapsto \mu_c^1$ induced by $F_{\mu_c^1}(z) = \frac{1}{\sqrt{c+1}}F_{\mu_c}(z\sqrt{c+1})$ provides us with a probability μ_c^1 of variance one, and thus

$$(3.13) \quad F_{\mu_c^1}(z) = z - \sqrt{c+1}G_{\mu_{c+1}}(z\sqrt{c+1}) = z - G_{\tilde{\mu}_{c+1}}(z), \quad z \in \mathbb{C},$$

where of course $\tilde{\mu}_{c+1}$ is a probability measure obtained by a dilation with a factor of $\sqrt{1+c}$ of μ_{c+1} . It is noted in [5, Theorems 1.2 and 1.6] (see also references therein) that a probability measure λ with variance one and first moment zero is freely infinitely divisible if and only if there exist two probabilities ν, ρ on \mathbb{R} so that

$$(a) \quad F_{\rho}(z) = z - G_{\nu}(z) \text{ for all } z \in \mathbb{C}^+,$$

- (b) $F_\lambda(z) = z - G_{\nu \boxplus \mathcal{S}}(z)$, $z \in \mathbb{C}^+$, where $\lambda = (\rho^{\boxplus 2})^{\boxplus 1/2}$ and \mathcal{S} is the centered Wigner (semicircular) distribution of variance one. (Operation \boxplus is called Boolean convolution - see [39].)

We apply this observation to $\lambda = \mu_c^1$ to conclude from (3.13) that the Voiculescu transform $\phi_{\mu_c^1}$ of μ_c^1 is also the Cauchy-Stieltjes transform of a probability measure ν_c playing the role of ν in (a) and (b) above and moreover

$$\tilde{\mu}_{c+1} = \nu_c \boxplus \mathcal{S}.$$

This provides us with another interesting decomposition result in the arithmetic of free additive convolution, stating that

Remark 3.10. For any $c \in (-1, 0]$, the Askey-Wimp-Kerov distribution μ_{c+1} can be written as a free additive convolution of the Wigner law with another probability ν_c on \mathbb{R} .

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DEPARTMENT OF MATHEMATICS AND STATISTICS,, UNIVERSITY OF SASKATCHEWAN,, AND INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY;; 106 WIGGINS ROAD, SASKATOON, SK, S7N 5E6, CANADA

E-mail address: belinschi@math.usask.ca

INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCŁAWSKI,, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND

E-mail address: bozejko@math.uni.wroc.pl

INSTITUTE FOR MATHEMATICAL STRUCTURE THEORY,, GRAZ TECHNICAL UNIVERSITY,, STEYR-ERGASSE 30, 8010 GRAZ, AUSTRIA

E-mail address: lehner@finanz.math.tu-graz.ac.at

QUEEN'S UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTICS,, JEFFERY HALL, KINGSTON, ON K7L 3N6, CANADA

E-mail address: speicher@mast.queensu.ca