Submitted to the Annals of Probability

CONVERGENCE TO THE EQUILIBRIA FOR SELF-STABILIZING PROCESSES IN DOUBLE-WELL LANDSCAPE*

By Julian Tugaut

Universität Bielefeld

We investigate the convergence of McKean-Vlasov diffusions in a non-convex landscape. These processes are linked to nonlinear partial differential equation. According to our previous results, there are at least three stationary measures under simple assumptions. Hence, the convergence problem is not classical like in the convex case. By using the method in [Benedetto, Caglioti, Carrillo, Pulvirenti|1998] about the monotonicity of the free-energy and combining this with a complete description of the set of the stationary measures, we prove the global convergence of the self-stabilizing processes.

Introduction. We investigate the weak convergence in long-time of the following so-called self-stabilizing process:

$$\text{(I)} \qquad \begin{cases} X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t V'(X_s)\,ds - \int_0^t F' * u_s(X_s)\,ds \\ u_s = \mathcal{L}(X_s) \end{cases} .$$

Here, $*$ denotes the convolution. Since the own law of the process intervenes in the drift, this equation is nonlinear - in the sense of McKean. We note that X_t and u_t depend on ϵ. We do not write ϵ for simplifying the reading.

The motion of the process is generated by three concurrent forces. The first one is the derivative of a potential V - the confining potential. The second influence is a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. It allows the particle to move upwards the potential V. The third term - the so-called self-stabilizing term - represents the attraction between all the others trajectories. Indeed, we remark: $F' * u_s(X_s(\omega_0)) = \int_{\omega\in\Omega} F'(X_s(\omega_0) - X_s(\omega))\,d\mathbb{P}(\omega)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying measurable space.

This kind of processes were introduced by McKean, see [McK67] or [McK66].

*Supported by the DFG-funded CRC 701, Spectral Structures and Topological Methods in Mathematics, at the University of Bielefeld.

AMS 2000 subject classifications: Primary 60H10, 35B40; secondary 35K55, 60J60, 60G10

Keywords and phrases: McKean-Vlasov stochastic differential equations, Stationary measures, Double-well potential, Granular media equation, Self-interacting diffusion, Free-energy

Here, we will make some smoothness assumptions on the interaction potential F. Let just note that it is possible to consider non-smooth F. If F is the Heaviside step function and $V := 0$, (I) is the Burgers equation (see [SV79]). If $F := \delta_0$ and without confining potential, it is the Oelschläger equation, studied in [Oel85].

The particle X_t which verifies (I) can be seen as one particle in a continuous mean-field system of an infinite number of particles. The mean-field system that we will consider is a random dynamical system like

$$\text{(II)} \qquad \begin{cases} dX_t^1 = \sqrt{\epsilon} dB_t^1 - V'\left(X_t^1\right) dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^1 - X_t^j\right) dt \\ \vdots \\ dX_t^i = \sqrt{\epsilon} dB_t^i - V'\left(X_t^i\right) dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^i - X_t^j\right) dt \\ \vdots \\ dX_t^N = \sqrt{\epsilon} dB_t^N - V'\left(X_t^N\right) dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^N - X_t^j\right) dt \end{cases}$$

where the N brownian motions $\left(B_t^i\right)_{t\in\mathbb{R}_+}$ are independents. Mean-field systems are the subject of a rich literature: [DG87] about the large deviations for $N \to +\infty$, [Mél96] under weak assumptions on V and F. For applications, see [CDPS10] about social interactions or [CX10] about the stochastic partial differential equations.

The link between the self-stabilizing processes and the mean-field system when N tends to $+\infty$ is called the propagation of chaos, see [Szn91] under Lipschitz properties ; [BRTV98] if V is a constant ; [Mal01] or [Mal03] when both potentials are convex ; [BAZ99] for a more precise result ; [BGV07], [DPdH96] or [DG87] for a sharp estimate ; [CGM08] for a uniform result in time in the non-uniformly convex case or [Tug10] for a half-uniform propagation of chaos.

Equation (II) can be rewritten:

$$\text{(II)} \qquad d\mathcal{X}_t = \sqrt{\epsilon}\mathcal{B}_t - N\nabla\Upsilon^N\left(\mathcal{X}_t\right) dt$$

where the i-th coordinate of $\mathcal{X}_t$ (resp. $\mathcal{B}_t$) is X_t^i (resp. B_t^i) and

$$\Upsilon^N\left(\mathcal{X}\right) := \frac{1}{N}\sum_{j=1}^N V\left(\mathcal{X}_j\right) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F\left(\mathcal{X}_i - \mathcal{X}_j\right)$$

for all $\mathcal{X} \in \mathbb{R}^N$. As noted in [Tug10], the potential Υ^N converges towards a meta-potential Υ acting on the measures. Some perturbation (proportional to ϵ) of this meta-potential will play the central role in the article.

As observed in [DG87], the empirical law of the mean-field system can be seen as a perturbation of the law of the diffusion (I). Consequently, the long-time behavior of $\mathcal{L}(X_t)$ that we study in this paper provides some consequences on the exit time for the particle system (II).

Also, the convergence plays an important role in the exit problem for the self-stabilizing process since the exit time is strongly linked to the drift (according to the Kramers' law, see [DZ10] or [HIP08]) which converges towards a homogeneous function if the law of the process converges towards a stationary measure.

Let us recall briefly some of the previous results on diffusions like (I). The existence problem has been investigated by two different methods. The first one consists in the application of a fixed point theorem, see [McK67], [BRTV98], [CGM08] or [HIP08] in the non-convex case. The other consists in a propagation of chaos, see for example [Mél96].

In [McK67], the author proved - by using Weyl lemma - that the law of the (strong) solution du_t admits a $\mathcal{C}^\infty$-continuous density u_t with respect to the Lebesgue measure for all $t > 0$. Furthermore, this density satisfies a nonlinear partial differential equation of the following type:

$$\text{(III)} \qquad \frac{\partial}{\partial t}u_t(x) = \frac{\partial}{\partial x}\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_t(x) + u_t(x)\Big(V'(x) + F' * u_t(x)\Big)\right\} .$$

It is then possible to study equations like (III) by probabilistic methods which involve the diffusions (I) or (II), see [CGM08], [Fun84], [Mal03]. Reciprocally, equation (I) is a useful tool for characterizing the stationary measure(s) and the long-time behavior, see [BRTV98], [BRV98], [Tam84], [Tam87] or [Ver06]. In [HT10a], in the non-convex case, by using (III), it has been proved that the diffusion (I) admits at least three stationary measures under assumptions easy to verify. One is symmetric and the two others are not. Moreover, Theorem 3.2 in [HT10a] states the thirdness of the stationary measures if V'' is convex and F' is linear. This non-uniqueness prevents the long-time behavior to be as intuitive as in the case of unique stationary measure.

The work in [HT10b] and [HT09] provides some estimates of the small-noisse asymptotic of these three stationary measures. In particular, the convergence towards Dirac measures and its rate of convergence are investigated. This will be one of the two main tools for obtaining the convergence.

Convergence for (I) is not a new subject. In [BRV98], if V is identically equal to 0, the authors proved the convergence towards the stationary measure by using an ultracontractivity property, a Poincaré inequality and a comparison lemma for stochastic processes. The ultracontractivity property still holds if V is not convex by using the results in [KKR93]. It is possible to conserve the

Poincaré inequality by using the theorem of Muckenhoupt (see [ABC+00]) instead of the Bakry-Emery theorem. But, the comparison lemma needs some convexity properties. However, it is possible to apply these results if the initial law is symmetric in the synchronized case ($V''(0) + F''(0) \geq 0$), see Theorem 7.10 in [Tug10].
Another method consists in using the propagation of chaos in order to derive the convergence of the self-stabilizing process from the one of the mean-field system. However, we shall use it independently of the time and the classical result which is on a finite interval of time is not sufficiently strong. Cattiaux, Guillin and Malrieu proceeded a uniform propagation of chaos in [CGM08] and obtained the convergence in the convex case, including the non-uniformly convex case. See also [Mal03]. Nevertheless, according to Proposition 5.17 and Remark 5.18 in [Tug10], it is impossible to find a general result of uniform propagation of chaos. In the synchronized case, if the initial law is symmetric, it is possible to find such a uniform propagation of chaos (see Theorem 7.11 and 7.12 in [Tug10]).
The method that we will use in this paper is based on the one of [BCCP98]. See also [Mal03], [Tam84], [Mal01], [HS87], [AMTU01] for the convex case. In the non-convex case, Carrillo, McCann and Villani provided the convergence in [CMV03] under two restrictions: the center of mass is fixed and $V''(0) + F''(0) > 0$ (that means it is the synchronized case).

However, by combining the results in [HT10a], [HT10b] and [HT09] with the work of [BCCP98] (and the more rigorous proofs in [CMV03] about the free-energy), we will be able to prove the convergence in a more general setting. The principal tool of the paper is the monotonicity of the free-energy along the trajectories of (III).
Firstly, we introduce the following functional that we call the meta-potential:

$$\text{(IV)} \qquad \Upsilon(u) := \int_{\mathbb{R}} V(x)du(x) + \frac{1}{2}\iint_{\mathbb{R}^2} F(x-y)du(x)du(y)\,.$$

This meta-potential appears intuitively as the limit of the potential in (II) for $N \to +\infty$. We consider now the free-energy of the self-stabilizing process (I):

$$\Upsilon_\epsilon(u) := \frac{\epsilon}{2}\int_{\mathbb{R}} u(x)\log(u(x))dx + \Upsilon(u)$$

for all measures du which are absolutely continuous with respect to the Lebesgue measure. We can note that du_t satisfies this hypothesis.

The paper is organized as follows. After presenting the assumptions, we will state the first results, in particular the convergence of a subsequence

$(u_{t_k})_k$. This subconvergence will be used for improving the results about the thirdness of the stationary measures. Then, we will give the main statement which is the convergence towards a stationary measure, briefly discuss the assumptions of the theorem and give the proof. Subsequently, we will study the basins of attraction by two different methods and prove that these basins are not reduced to a single point. Finally, we postpone four results in the appendix including Proposition A.2 which extends the classical higher-bound for the moments of the self-stabilizing processes.

Assumptions. We assume the following properties of the confining potential V:

(V-1) V is an even polynomial function with $\deg(V) =: 2m \geq 4$.

(V-2) The equation $V'(x) = 0$ admits exactly three solutions: a, $-a$ and 0 with $a > 0$. Furthermore, $V''(a) > 0$ and $V''(0) < 0$. Then, the bottoms of the wells are located in $x = a$ and $x = -a$.

(V-3) $V(x) \geq C_4x^4 - C_2x^2$ for all $x \in \mathbb{R}$ with $C_2, C_4 > 0$.

(V-4) $\lim_{x\to\pm\infty} V''(x) = +\infty$ and $V''(x) > 0$ for all $x \geq a$.

(V-5) V'' is convex.

(V-6) Initialization: $V(0) = 0$.

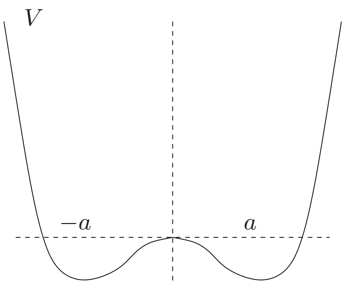

FIG 1. *Potential V*

The simplest and most studied example is $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$. Also, we would like to stress that weaker assumptions could be considered but all the mathematical difficulties are present in the polynomial case and it permits to avoid some technical and tedious computations. Let us present now the assumptions on the interaction potential F:

(F-1) F is an even polynomial function with $\deg(F) =: 2n \geq 2$.

(F-2) F and F'' are convex.

(F-3) Initialization: $F(0) = 0$.

Under these assumptions, we know by [HT10a] that (I) admits at least one symmetric stationary measure. And, if $\sum_{p=0}^{2n-2} \frac{\left|F^{(p+2)}(a)\right|}{p!} a^p < F''(0) + V''(a)$, there are at least three stationary measures: u_ϵ^0 is symmetric and u_ϵ^+ and u_ϵ^- are asymmetric. Furthermore, we know by [HT10b] that there is a unique nonnegative real x_0 such that $V'(x_0) + \frac{1}{2}F'(2x_0) = 0$ and $V''(x_0) + \frac{F''(0)+F''(2x_0)}{2} > 0$. The same paper provides that u_ϵ^0 converges weakly towards $\frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$ and $u_\epsilon^\pm$ converges weakly towards $\delta_{\pm a}$; in the small noise limit.

We present now the assumptions on the initial law du_0:

(ES) The $8q^2$-th moment of the measure du_0 is finite with $q := \max\{m, n\}$.
(FE) The probability measure du_0 admits a $\mathcal{C}^\infty$-continuous density u_0 with respect to the Lebesgue measure. And, the entropy $\int_\mathbb{R} u_0 \log(u_0)$ is finite.

Under (ES), we know by Theorem 2.12 in [HIP08] that (I) admits a strong solution. Moreover, we have the following inequality:

$$\max_{1\leq j\leq 8q^2} \sup_{t\in\mathbb{R}_+} \mathbb{E}\left[|X_t|^j\right] \leq M_0 \,. \tag{V}$$

We deduce immediately that the family $(u_t)_{t\in\mathbb{R}_+}$ is tight. The assumption (FE) ensures that the free-energy is finite. In the following, we shall use occasionnaly one of the following three additional properties concerning the two potentials V and F and the initial law du_0:

(LIN) F' is linear.
(SYN) $V''(0) + F''(0) > 0$.
(FM) For all $N \in \mathbb{N}$, we have $\int_\mathbb{R} |x|^N du_0(x) < +\infty$.

In the following, three important properties linked to the enumeration of the stationary measures for the self-stabilizing process (I) will be helpful for proving the convergence:

(M3) The process (I) admits exactly three stationary measures. One is symmetric: u_ϵ^0 and the other ones are asymmetric: u_ϵ^+ and u_ϵ^-. Furthermore, $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon(u_\epsilon^-) < \Upsilon_\epsilon(u_\epsilon^0)$.
(M3)' There exists $M > 0$ such that the diffusion (I) admits exactly three stationary measures with free-energy less than M. Furthermore, we have $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon(u_\epsilon^-) < \Upsilon_\epsilon(u_\epsilon^0)$; u_ϵ^0 is symmetric and u_ϵ^+ and u_ϵ^- are asymmetric.
(0M1) The process (I) admits only one symmetric stationary measure u_ϵ^0.

In the following, we will give some simple conditions such that (M3), (M3)' or (0M1) are true.

Finally, we recall the assumption (H) introduced in [HT10b]:

(H) A family of measures $(v_\epsilon)_\epsilon$ verifies the assumption (H) if the family of positive reals $\left(\int_\mathbb{R} x^{2n} u_\epsilon(x)dx\right)_\epsilon$ is bounded.

The aim of the weaker assumption (M3)' is to get the convergence even if there exists a family of stationary measures which does not verify the

assumption (H).
For concluding the introduction, we write the statement of the main theorem:
Theorem: *Set a probability measure du_0 which verifies (FE) and (FM). Under (M3), u_t converges weakly towards a stationary measure.*

1. First results. This section is devoted to present the tools that we will use for getting the main result of the paper. Furthermore, we provide some new results about the thirdness of the stationary measures for the self-stabilizing processes.
We introduce the following functionnal:

$$\Upsilon_\epsilon^-(u) := \frac{\epsilon}{2}\int_{\mathbb{R}} u(x)\log\left(u(x)\right)\mathbb{1}_{\{u(x)<1\}}dx + \int_{\mathbb{R}} V(x)u(x)dx\,.$$

This new functionnal does neither contain nor the interaction potential term nor the positive contribution of u in the entropy term. Due to the positivity of the interaction potential F, we get directly the inequality $\Upsilon_\epsilon(u)\geq\Upsilon_\epsilon^-(u)$ for all the measures u which verify the previous assumptions.
In the following, we will need two particular functions (the free-energy of the system and a function η_t such that $\frac{d}{dt}u_t(x)=\frac{d}{dx}\eta_t(x)$).

Definition 1.1. *For all $t\in\mathbb{R}_+$, we introduce the functions:*

$$\xi(t) := \Upsilon_\epsilon\left(u_t\right) \quad \textit{and} \quad \eta_t(x) := \frac{\epsilon}{2}\frac{\partial}{\partial x}u_t(x) + u_t(x)\left(V'(x)+F'*u_t(x)\right)\,.$$

According to (III), we remark that if η_t is identically equal to 0 then u_t is a stationary measure for (I).
We recall the following well-known entropy dissipation:

Proposition 1.2. *Let a probability measure du_0 which verifies (FE) and (ES). Then, for all $t,s\geq 0$, we have*

$$\xi(t+s)\leq\xi(t)\leq\xi(0)<+\infty\,.$$

Furthermore, we have:

$$\xi'(t)\leq-\int_{\mathbb{R}}\frac{1}{u_t(x)}\left(\eta_t(x)\right)^2dx\,.$$

See [CMV03] for a proof.

1.1. *Preliminaries.* Let's introduce the functionnal space

$$\mathcal{M}_{8q^2} := \left\{ f \in \mathcal{C}_0^2\left(\mathbb{R}, \mathbb{R}_+\right) \;\Big|\; \int_{\mathbb{R}} f(x)dx = 1 \right\}.$$

We can remark that $u_t \in \mathcal{M}_{8q^2}$ for all $t > 0$, see [McK67]. The first tool is the Proposition 1.2 (that is to say the fact that the free-energy is decreasing along the potential lines). The second one is its lower-bound.

LEMMA 1.3. *There exists $\Xi_\epsilon \in \mathbb{R}$ such that* $\inf_{u \in \mathcal{M}_{8q^2}} \Upsilon_\epsilon(u) \geq \Xi_\epsilon$.

PROOF. Let us recall $\Upsilon_\epsilon(u) \geq \Upsilon_\epsilon^-(u)$. It suffices then to prove the inequality $\inf_{u \in \mathcal{M}_{8q^2}} \Upsilon_\epsilon^-(u) \geq \Xi_\epsilon$. We proceed as in the first part of the proof of Theorem 2.1 in [BCCP98]. We show that we can minorate the negative part of the entropy by a function of the second moment. Then a growth condition of V will provide the result.
We split the negative part of the entropy into two integrals:

$$\begin{aligned} -\int_{\mathbb{R}} u(x)\log\left(u(x)\right)\mathbb{1}_{\{u(x)<1\}}dx &= -I_+ - I_- \\ \text{with} \quad I_+ := \int_{\mathbb{R}} u(x)\log\left(u(x)\right)\mathbb{1}_{\{e^{-|x|}<u(x)<1\}}dx \\ \text{and} \quad I_- := \int_{\mathbb{R}} u(x)\log\left(u(x)\right)\mathbb{1}_{\{u(x)\leq e^{-|x|}\}}dx\,. \end{aligned}$$

By definition of I_+, we have the following estimate:

$$\begin{aligned} I_+ &\geq \int_{\mathbb{R}} u(x)\log\left(e^{-|x|}\right)\mathbb{1}_{\{e^{-|x|}<u(x)<1\}}dx \\ &\geq -\int_{\mathbb{R}} |x|u(x)\mathbb{1}_{\{e^{-|x|}<u(x)<1\}}dx \\ &\geq -\int_{\mathbb{R}} |x|u(x)dx \geq -\frac{1}{2} - \frac{1}{2}\int_{\mathbb{R}} x^2u(x)dx\,. \end{aligned}$$

By putting $\gamma(x) := \sqrt{x}\log(x)\mathbb{1}_{\{x<1\}}$, a simple computation provides $\gamma(x) \geq -2e^{-1}$ for all $x < 1$. We deduce:

$$I_- = \int_{\mathbb{R}} \sqrt{u(x)}\gamma(u(x))\mathbb{1}_{\{u(x)\leq e^{-|x|}\}}dx \geq -2e^{-1}\int_{\mathbb{R}} e^{-\frac{|x|}{2}}dx = -8e^{-1}\,.$$

Consequently, it yields:

$$-\int_{\mathbb{R}} u(x)\log\left(u(x)\right)\mathbb{1}_{\{u(x)<1\}}dx \leq \frac{1}{2}\int_{\mathbb{R}} x^2u(x)dx + \frac{1}{2} + 8e^{-1}\,.$$

This implies:

$$\Upsilon_\epsilon^-(u) \geq -\frac{\epsilon}{4} - 4\epsilon e^{-1} + \int_{\mathbb{R}} \left(V(x) - \frac{\epsilon}{4}x^2\right) u(x)dx\,. \tag{1.1}$$

By hypothesis, there exist $C_2, C_4 > 0$ such that $V(x) \geq C_4 x^4 - C_2 x^2$ so the function $x \mapsto V(x) - \frac{\epsilon}{4}x^2$ is lower-bounded by some negative constant that achieves the proof. □

Let's note that the unique assumption we used is $\lim_{x\to\pm\infty} V''(x) = +\infty$.

LEMMA 1.4. *Let a probability measure du_0 which satisfies the assumptions (FE) and (ES). Then, there exists $L_0 \in \mathbb{R}$ such that $\Upsilon_\epsilon(u_t^\epsilon)$ converges towards L_0 as time elapses to infinity.*

PROOF. The assumption (FE) implies $\xi(0) = \Upsilon_\epsilon(u_0) < \infty$. As ξ is non-increasing by Lemma 1.2 and lower-bounded by a constant Ξ_ϵ according to Lemma 1.3, we deduce that the function ξ converges towards a real L_0. □

LEMMA 1.5. *If and only if $\xi'(t) = 0$, the following is true: u_t is a stationary measure u_ϵ.*

PROOF. If u_t is a stationary measure u_ϵ, then $\xi(t) = \Upsilon_\epsilon(u_t) = \Upsilon_\epsilon(u_\epsilon)$ is a constant that provides $\xi'(t) = 0$.
Reciprocally, if $\xi'(t) = 0$, Proposition 1.2 implies

$$\int_{\mathbb{R}} \frac{1}{u_t(x)} \left(\eta_t(x)\right)^2 dx = 0\,.$$

We deduce $\eta_t(x) = 0$ for all $x \in \mathbb{R}$; that means u_t is a stationary measure. □

1.2. *Subconvergence.*

THEOREM 1.6. *Let a probability measure du_0 which satisfies the assumptions (FE) and (ES). Then, there exists a stationary measure u_ϵ and a sequence $(t_k)_k$ which converges to infinity such that u_{t_k} converges weakly towards u_ϵ.*

PROOF. **Plan:** First, we use the convergence of $\int_t^\infty \xi'(s)ds$ towards 0 when t tends to infinity and we deduce the existence of a sequence $(t_k)_k$ such that $\xi'(t-k)$ tends to 0 when k tends to infinity. Then, we extract a subsequence of $(t_k)_k$ for obtaining an adherence value. By using a test function, we prove that this adherence value is a stationary measure.

Step 1: Lemma 1.4 implies that $\int_t^\infty \xi'(s)ds$ collapses at infinity. According to Proposition 1.2, the sign of ξ' is a constant so we deduce the existence of an increasing sequence $(t_k)_{k\in\mathbb{N}}$ which converges to infinity such that $\xi'(t_k) \longrightarrow 0$.

Step 2: The uniform boundedness of the first $8q^2$ moments with respect to the time allows us to use Prohorov's theorem: we can extract a subsequence (we continue to write it $(t_k)_k$ for simplicity) such that u_{t_k} converges weakly towards a probability measure u_ϵ.

Step 3: We consider now a compact function $\varphi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \bigcap \mathcal{L}_2(u_\epsilon)$ and we estimate the following quantity:

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \varphi(x)\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_{t_k}(x) + u_{t_k}(x)\left[V'(x) + (F' * u_{t_k})(x)\right]\right\}dx\right| \\
&= \left|\int_{\mathbb{R}} \varphi(x)\eta_{t_k}(x)dx\right| = \left|\int_{\mathbb{R}} \varphi(x)\sqrt{u_{t_k}(x)}\frac{|\eta_{t_k}(x)|}{\sqrt{u_{t_k}(x)}}dx\right| \\
&\leq \left(\int_{\mathbb{R}} \varphi(x)^2 u_{t_k}(x)dx\right)^{\frac{1}{2}} \times \left(\int_{\mathbb{R}} \frac{1}{u_{t_k}(x)}\left(\eta_{t_k}(x)\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \sqrt{-\xi'(t_k)}\sqrt{\int_{\mathbb{R}} \varphi(x)^2 u_{t_k}(x)} \longrightarrow 0
\end{aligned}
$$

when k tends to infinity ; by using the Cauchy-Schwarz inequality, the hypothesis about the sequence $(t_k)_k$ and the weak convergence of u_{t_k} towards u_ϵ. Thanks to the compactness of φ, we can apply this integration by parts and we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} \varphi(x)\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_{t_k}(x) + u_{t_k}(x)\left[V'(x) + F' * u_{t_k}(x)\right]\right\}dx \\
&= \int_{\mathbb{R}} \varphi(x)\left[V'(x) + F' * u_{t_k}(x)\right]u_{t_k}(x)dx - \int_{\mathbb{R}} \frac{\epsilon}{2}\varphi'(x)u_{t_k}(x)dx\,.
\end{aligned}
$$

The weak convergence of u_{t_k} towards u_ϵ implies that the previous term tends to $\int_{\mathbb{R}} \varphi(x)\left[V'(x) + (F' * u_\epsilon)(x)\right]u_\epsilon(x)dx - \int_{\mathbb{R}} \frac{\epsilon}{2}\varphi'(x)u_\epsilon(x)dx$. It has already been proved that $\int_{\mathbb{R}} \varphi(x)\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_{t_k}(x) + u_{t_k}(x)\left(V'(x) + F' * u_{t_k}(x)\right)\right\}dx$ is collapsing when k tends to ∞. We get the following statement:

$$
\int_{\mathbb{R}} \varphi(x)\left[V'(x) + F' * u_\epsilon(x)\right]u_\epsilon(x)dx - \int_{\mathbb{R}} \frac{\epsilon}{2}\varphi'(x)u_\epsilon(x)dx = 0\,. \tag{1.2}
$$

Step 4: This means that u_ϵ is a weak solution of the equation

$$
\frac{\epsilon}{2}\frac{\partial}{\partial x}u(x) + \left[V'(x) + F' * u(x)\right]u(x) = 0\,.
$$

Now, we consider a smooth function $\widetilde{\varphi}$ with compact support $[a,b]$. We put:

$$\varphi(x):=\exp\left\{\frac{2}{\epsilon}\left[V(x)+F*u_\epsilon(x)\right]\right\}\widetilde{\varphi}'(x)\,.$$

φ is also a smooth function with compact support. Indeed, the application $x\mapsto F*u_\epsilon(x)$ is a polynomial function parametrized by the moments of u_ϵ and these moments are bounded. Equality (1.2) becomes

$$\int_{\mathbb{R}}\widetilde{\varphi}''(x)\exp\left\{\frac{2}{\epsilon}\left[V(x)+F*u_\epsilon(x)\right]\right\}u_\epsilon(x)dx=0\,.$$

By applying Weyl lemma, we deduce $x\mapsto\exp\left[\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon(x)\right)\right]u_\epsilon(x)$ is smooth. Moreover, its second derivative is equal to 0. Then, there exists $A,B\in\mathbb{R}$ such that

$$u_\epsilon(x)=(Ax+B)\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon(x)\right)\right]$$

for all $x\in\mathbb{R}$. If $A\neq 0$, we get $u_\epsilon(-Ax)<0$ for x big enough. This is impossible. Consequently, $u_\epsilon(x)=Z^{-1}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon(x)\right)\right]$. This means u_ϵ is a stationary measure. □

Definition 1.7. *From now, we call $\mathcal{A}$ the set of the adherence values of the family $(u_t)_{t\in\mathbb{R}_+}$.*

Proposition 1.8. *With the assumptions and the notations of Theorem 1.6, we have the following limit:*

$$L_0:=\lim_{t\longrightarrow+\infty}\Upsilon_\epsilon(u_t)=\Upsilon_\epsilon(u_\epsilon)\,.$$

Proof. First of all, we aim to prove that $(u_{t_k})_k$ is uniformly bounded in the space $W^{1,1}$. For doing this, we will bound the integral on $\mathbb{R}$ of $\frac{\partial}{\partial x}u_{t_k}(x)$. The triangular inequality provides:

$$\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}u_{t_k}(x)\right|dx\leq\frac{2}{\epsilon}\int_{\mathbb{R}}\left|\eta_t(x)\right|dx+\frac{2}{\epsilon}\int_{\mathbb{R}}\left|V'(x)+F'*u_{t_k}(x)\right|u_{t_k}(x)dx$$

where η_t is defined in Definition 1.1. By using (V) and the growth property of V' and F', it yields

$$\int_{\mathbb{R}}\left|V'(x)+F'*u_{t_k}(x)\right|u_{t_k}(x)dx\leq C_1\int_{\mathbb{R}}\left(1+|x^{2q}|\right)u_{t_k}(x)dx\leq C_2$$

where C_2 is a constant. By using Cauchy-Schwarz inequality like in the proof of Theorem 1.6, we obtain:

$$\int_{\mathbb{R}} |\eta_t(x)|\, dx \leq \sqrt{-\xi'(t_k)}\,.$$

The quantity $\sqrt{-\xi'(t_k)}$ tends to 0 so it is bounded. We get finally

$$\int_{\mathbb{R}} \left|\frac{\partial}{\partial x} u_{t_k}(x)\right| dx \leq C_3$$

where C_3 is a constant. Consequently, $u_{t_k} \leq u_{t_k}(0) + C$ for all $x \in \mathbb{R}$. And, since the sequence $\left(u_{t_k}(0)\right)_k$ converges, it is bounded so there exists a constant C_4 such that $u_{t_k}(x) \leq C_4$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. It is then easy to prove the convergence of $\int_{\mathbb{R}} u_{t_k}(x) \log\left(u_{t_k}(x)\right) dx$ towards $\int_{\mathbb{R}} u_\epsilon(x) \log\left(u_\epsilon(x)\right) dx$.

Indeed, the Lebesgue's theorem implies that $\int_{\mathbb{R}} u_{t_k}(x) \log\left(u_{t_k}(x)\right) \mathbb{1}_{\{|x|\leq R\}} dx$ converges towards $\int_{\mathbb{R}} u_\epsilon(x) \log\left(u_\epsilon(x)\right) \mathbb{1}_{\{|x|\leq R\}} dx$ for all $R \geq 0$ because the applications $x \mapsto u_{t_k}(x) \log\left(u_{t_k}(x)\right)$ are lower-bounded, uniformly with respect to k. The other integral is split into two terms. The first one is:

$$\begin{aligned}\int_{\mathbb{R}} u_{t_k}(x) \log\left(u_{t_k}(x)\right) \mathbb{1}_{\{|x|>R\,;\, u_{t_k}(x)\geq 1\}} dx &\leq \log(C) u_{t_k}\left([-R;R]^c\right)\\ &\leq \frac{\log(C)\, M_0}{R^2}\,.\end{aligned}$$

The second term is bounded as in the proof of Lemma 1.3:

$$\begin{aligned}&-\int_{\mathbb{R}} u_{t_k}(x) \log\left(u_{t_k}(x)\right) \mathbb{1}_{\{|x|>R\,;\, u_{t_k}(x)< 1\}} dx\\ &\leq \int_{[-R;R]^c} \left\{|x| u_{t_k}(x) - \gamma(u_{t_k}(x)) e^{-\frac{1}{2}|x|}\right\} dx \leq \frac{M_0}{R} + 4e^{-\frac{R}{2}}\,.\end{aligned}$$

Consequently, $\Upsilon_\epsilon(u_{t_k}^\epsilon)$ converges to $\Upsilon_\epsilon(u_\epsilon)$ then $\Upsilon_\epsilon(u_t^\epsilon)$ converges to $\Upsilon_\epsilon(u_\epsilon)$ since the free-energy is monotonous.

By taking R big enough then k big enough, we can make the following quantity arbitrarily small: $\left|\int u_{t_k} \log\left(u_{t_k}\right) - \int u_\epsilon \log\left(u_\epsilon\right)\right|$. □

1.3. *Consequences.* When V is symmetric, Proposition 3.1 (resp. Theorem 4.6) in [HT10a] states the existence of at least three stationary measures for ϵ small enough if F' is linear (resp. if $\sum_{p=0}^{\infty} \frac{\left|F^{(p+2)}(a)\right|}{p!} a^p < F''(0) + V''(a)$). Theorem 1.6 permits to extend these results.

COROLLARY 1.9. *For ϵ small enough, the process* (I) *admits at least three stationary measures: one is symmetric* (u_ϵ^0) *and two are asymmetric* (u_ϵ^+ *and* u_ϵ^-)*. Furthermore, under some critical value of* ϵ, $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon(u_\epsilon^-) < \Upsilon_\epsilon(u_\epsilon^0)$.

PROOF. We know by Theorem 4.5 in [HT10a] there exists a symmetric stationary measure u_ϵ^0. Theorem 5.4 in [HT10b] implies the weak convergence of u_ϵ^0 towards $\frac{1}{2}\left(\delta_{x_0}+\delta_{-x_0}\right)$ in the small noise limit where $x_0 \in [0;a[$ is the unique solution of

$$\begin{cases} V'(x_0)+\frac{1}{2}F'(2x_0)=0 \\ V''(x_0)+\frac{F''(0)}{2}+\frac{F''(2x_0)}{2}\geq 0 \end{cases} .$$

Lemma A.3 provides

$$\lim_{\epsilon\to 0}\Upsilon_\epsilon\left(u_\epsilon^0\right)=V(x_0)+\frac{1}{4}F(2x_0) \quad \text{and} \quad \lim_{\epsilon\to 0}\Upsilon_\epsilon\left(v_\epsilon^+\right)=V(a)$$
$$\text{with} \quad v_\epsilon^+(x):=Z^{-1}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F(x-a)\right)\right] .$$

We note that $V(x_0)+\frac{1}{4}F(2x_0) > V(a)$. Consequently, for ϵ small enough, we have $\Upsilon_\epsilon\left(v_\epsilon^+\right) < \Upsilon_\epsilon\left(u_\epsilon^0\right)$.
We consider now the process (I) starting by $u_0 := v_\epsilon^+$. This is possible because the $8q^2$-th moment of v_ϵ^+ is finite. Theorem 1.6 implies the existence of a sequence $(t_k)_k$ which satisfies $t_k \longrightarrow +\infty$ and u_{t_k} converges weakly towards a stationary measure u_ϵ satisfying $\Upsilon_\epsilon\left(u_\epsilon\right) \leq \Upsilon_\epsilon\left(u_0\right) = \Upsilon_\epsilon\left(v_\epsilon^+\right) < \Upsilon_\epsilon\left(u_\epsilon^0\right)$. So $u_\epsilon^0 \neq u_\epsilon$. We immediatly deduce there is at least two stationary measures.
If $V''(0)+F''(0)\neq 0$, we know by Theorem 7.3 and Theorem 7.4 in [HT09] that there exists a unique symmetric stationary measure for ϵ small enough. Hence u_ϵ is not symmetric
If $V''(0)+F''(0)=0$, by (1.1), we have:

$$\Upsilon_\epsilon(u)\geq -\frac{\epsilon}{4}-4\epsilon e^{-1}+\int_{\mathbb{R}}\left\{V(x)+\frac{\alpha}{2}x^2-\frac{\epsilon x^2}{4}\right\}u(x)dx$$

for all the probability measures satisfying $\int_{\mathbb{R}} xu(x)dx = 0$; in particular for the symmetric measures because $F(x)-\frac{F''(0)}{2}x^2\geq 0$ (due to the convexity of F''). Then, for ϵ small enough, $\Upsilon_\epsilon(u) > \frac{V(a)}{2}$ for all the symmetric measures. However, $\Upsilon_\epsilon(v_\epsilon^+) < \frac{V(a)}{2}$ (then $\Upsilon_\epsilon(u_\epsilon) < \frac{V(a)}{2}$) for ϵ small enough.

Consequently, the process admits at least one asymmetric stationary measure that we call u_ϵ^+. The measure $u_\epsilon^-(x) := u_\epsilon^+(-x)$ is invariant too. By construction of u_ϵ^+ and u_ϵ^-, $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon^-(u_\epsilon^-) < \Upsilon_\epsilon(u_\epsilon^0)$. □

REMARK 1.10. *By a similar method, we could also prove the existence of at least one stationary measure in the asymmetric-landscape case.*

We know by Theorem 3.2 in [HT10a] that if V'' is convex, if F' is linear then there are exactly three stationary measures for ϵ small enough. We present a more general setting. In view of the convergence, we will prove that the number of "useful" stationary measures is exactly three even if it is *a priori* possible to imagine some others stationary measures.

THEOREM 1.11. *We assume $F''(0) + V''(0) \geq 0$. Then, for all $M > 0$, there exists $\epsilon(M) > 0$ such that for all $\epsilon \leq \epsilon(M)$, the number of measures u satisfying the two following conditions is exactly three:*

1. *u is a stationary measure for the diffusion* (I)*.*
2. *$\Upsilon_\epsilon(u) \leq M$.*

Moreover, if $\deg(V) = 2m > 2n = \deg(F)$*, the diffusion* (I) *admits exactly three stationary measures for ϵ small enough.*

PROOF. **Plan:** We will begin to prove the second statement (when $m > n$). For doing this, we will Corollary (1.9) and the results in [HT10b] and [HT09]. Then, we will prove the first statement by using the second statement and a minoration of the free-energy for a sequence of stationary measures which does not verify (H).

Step 1: Corollary 1.9 implies the existence of $\epsilon_0 > 0$ such that the process (I) admits at least three stationary measures (one is symmetric and two are asymmetric) if $\epsilon < \epsilon_0$: u_ϵ^+, u_ϵ^- and u_ϵ^0.

Step 2: First, we assume that $\deg(V) > \deg(F)$.

Step 2.1: Proposition 3.1 in [HT10b] implies that each family of stationary measures for the self-stabilizing process (I) verifies Condition (H). It has also been shown that under (H), we can extract a subsequence which converges weakly from any family of stationary measures $(u_\epsilon)_{\epsilon>0}$ for the diffusion (I).

Step 2.2: Since $F''(0) + V''(0) > 0$, there are three possible limiting values: δ_0, δ_a and δ_{-a} according to Proposition 3.7 and Remark 3.8 in [HT10b].

Step 2.3: As $F''(0) + V''(0) > 0$ and V'' and F'' are convex, there is a unique stationary symmetric measure for ϵ small enough by Theorem 7.3 in [HT09]. Also, Theorem 7.2 in [HT09] implies there are exactly two asymmetric stationary measures for ϵ small enough. That achieves the proof of the statement.

Step 3: Now, we will prove the first statement. Firstly, if $m > n$, by applying the second statement, the result is obvious. We assume now $m \leq n$.

Set $M>0$. All the previous results still hold if we restrict the study to the families of stationary measures which verify Condition (H). Consequently, it is sufficient to get the following results in order to achieve the proof of the theorem:

1. $\sup\left\{\Upsilon_\epsilon\left(u_\epsilon^0\right)\,;\,\Upsilon_\epsilon\left(u_\epsilon^+\right)\,;\,\Upsilon_\epsilon\left(u_\epsilon^-\right)\right\}<M$ for ϵ small enough.
2. If $\left(u_{\epsilon_k}\right)_k$ is a sequence of stationary measures, $\int_{\mathbb{R}}x^{2n}u_{\epsilon_k}(x)dx\to\infty$ implies $\Upsilon_{\epsilon_k}\left(u_{\epsilon_k}\right)\to\infty$.

Step 3.1: Lemma A.3 tells us that $\Upsilon_\epsilon\left(u_\epsilon^0\right)$ (resp. $\Upsilon_\epsilon\left(u_\epsilon^+\right)$) tends towards 0 (resp. $V(a)<0$) when ϵ tends to 0. Hence, the first point is obvious.

Step 3.2: We will prove the second point. We recall the lower-bound (1.1):

$$\Upsilon_\epsilon^-(u)\geq-\frac{\epsilon}{4}-4\epsilon e^{-1}+\int_{\mathbb{R}}\left(V(x)-\frac{\epsilon}{4}x^2\right)u(x)dx\,.$$

As $V(x)\geq C_4x^4-C_2x^2$ and $\Upsilon_\epsilon^-(u)\leq\Upsilon_\epsilon(u)$ for all smooth u, we get

$$\Upsilon_\epsilon(u)\geq\int_{\mathbb{R}}x^2u(x)dx-C$$

where C is a constant. It is now suffficient to prove that $\int_{\mathbb{R}}x^{2n}u_{\epsilon_k}(x)dx\to\infty$ implies $\int_{\mathbb{R}}x^2u_{\epsilon_k}(x)dx\to\infty$. We will not write the index k for simplifying the reading. We proceed a *reductio ad absurdum* by assuming the existence of a sequence $\left(u_\epsilon\right)_\epsilon$ which verifies $\int_{\mathbb{R}}x^{2n}u_\epsilon(x)dx\to\infty$ and $\int_{\mathbb{R}}x^2u_\epsilon(x)dx\to C_+\in\mathbb{R}_+$.

Step 3.2.1: By taking the notations of [HT10b], we have the equality $u_\epsilon(x)=Z^{-1}\exp\left[-\frac{2}{\epsilon}\left(W_\epsilon(x)\right)\right]$ with

$$W_\epsilon(x):=V(x)+F*u_\epsilon(x)=\sum_{k=1}^{2n}\omega_k(\epsilon)\,x^k\ ,$$

$$\omega_k(\epsilon):=\frac{1}{k!}\left\{V^{(k)}(0)+(-1)^k\sum_{j\geq\frac{k}{2}}^{2n}\frac{F^{(2j)}(0)}{(2j-k)!}m_{2j-k}(\epsilon)\right\}$$

$$\text{and}\quad m_l(\epsilon):=\int_{\mathbb{R}}x^lu_\epsilon(x)dx\quad\forall l\in\mathbb{N}\,.$$

We introduce $\omega(\epsilon):=\sup\left\{\left|\omega_k(\epsilon)\right|^{\frac{1}{2n-k}}\ ;\ 1\leq k\leq2n\right\}$.

Step 3.2.2: We note that $\omega_{2n}(\epsilon)=\frac{V^{(2n)}(0)+F^{(2n)}(0)}{(2n)!}>0$. Then, $\omega(\epsilon)$ is uniformly lower-bounded. Consequently, we can divide by $\omega(\epsilon)$.

Step 3.2.3: The change of variable $x := \omega(\epsilon)y$ provides

$$\frac{m_{2l}(\epsilon)}{\omega(\epsilon)^{2l}} = \frac{\int_{\mathbb{R}} y^{2l}\exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right]dy}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right]dy} \quad \text{with} \quad \widehat{W}_\epsilon(x) := \sum_{k=1}^{2n}\frac{\omega_k(\epsilon)}{\omega(\epsilon)^{2n-k}}x^k$$

for all $l\in\mathbb{N}$, with $\widehat{\epsilon} := \frac{\epsilon}{\omega(\epsilon)^{2n}}$.

Step 3.2.4: The $2n$ sequences $\left(\frac{\omega_k(\epsilon)}{\omega(\epsilon)^{2n-k}}\right)_\epsilon$ are bounded so we can extract a subsequence of ϵ (that we continue to write ϵ for simplicity) such that $\frac{\omega_k(\epsilon)}{\omega(\epsilon)^{2n-k}}$ converges towards $\widehat{\omega}_k$ when $\epsilon\to 0$. We put $\widehat{W}(x) := \sum_{k=1}^{2n}\widehat{\omega}_k x^k$. We call $A_1,\cdots,A_r$ the $r\geq 1$ location(s) of the global minimum of $\widehat{W}$.

Step 3.2.5: By applying the result of Lemma A.4, we can extract a subsequence (and we continue to denote it by ϵ) such that $\frac{\int_{\mathbb{R}} y^{2l}\exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right]dy}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\widehat{\epsilon}}\widehat{W}_\epsilon(y)\right]dy}$ converges towards $\sum_{j=1}^r p_jA_j^{2l}$ where $p_1+\cdots+p_r=1$ and $p_j\geq 0$.

Step 3.2.6: If $\omega(\epsilon)_\epsilon$ is bounded, since the quantity $\sum_{j=1}^r p_jA_j^{2n}$ is finite, we deduce that $(m_{2n}(\epsilon))_\epsilon$ is bounded too. Since $m_{2n}(\epsilon)$ tends towards infinity when ϵ converges towards 0, we deduce $(\omega(\epsilon))_\epsilon$ converges towards infinity. As $m_2(\epsilon)$ is bounded, the quantity $\frac{m_2(\epsilon)}{\omega(\epsilon)^2}$ vabishes when ϵ tends to 0. This means $\sum_{j=1}^r p_jA_j^2 = 0$ then $\sum_{j=1}^r p_jA_j^{2n} = 0$. Consequently, $m_{2n}(\epsilon) = o\left\{\omega(\epsilon)^{2n}\right\}$. The Jensen's inequality provides $m_k(\epsilon) = o\left\{\omega(\epsilon)^k\right\}$.

Step 3.2.7: We recall the definition of $\omega_k(\epsilon)$:

$$\omega_k(\epsilon) = \frac{1}{k!}\left\{V^{(k)}(0) + (-1)^k\sum_{j\geq\frac{k}{2}}^{2n}\frac{F^{(2j)}(0)}{(2j-k)!}m_{2j-k}(\epsilon)\right\}.$$

We deduce $\omega_k(\epsilon) = O\left\{m_{2n-k}(\epsilon)\right\} = o\left\{\omega(\epsilon)^{2n-k}\right\}$. So

$$\omega(\epsilon) = \sup\left\{|\omega_k(\epsilon)|^{\frac{1}{2n-k}}\ ;\ 1\leq k\leq 2n\right\} = o\left\{\omega(\epsilon)\right\}.$$

This is a contradiction. This achieves the proof. □

This theorem means that - even if the diffusion (I) admits more than three stationary measures - there are only three stationary measures which play a role in the convergence. Indeed, if we take a measure u_0 with a finite free-energy, we know that for ϵ small enough, there are only three (maybe less) stationary measures which can be adherence value of the family $(u_t^\epsilon)_{t\in\mathbb{R}_+}$.

The assumption (LIN) implies (M3) (and (M3)' because it is weaker) and

(0M1) for ϵ small enough. The condition (SYN) implies (M3)' and (0M1) for ϵ small enough. Furthermore, if $\deg(V) > \deg(F)$, (SYN) implies (M3) when ϵ is less than some threshold.
This description of the stationary measures permits to obtain the principal result that is to say the long-time convergence of the process.

2. Global convergence.

2.1. *Statement of the theorem.* We write the main result of the paper:

THEOREM 2.1. *Set a probability measure du_0 which verifies (FE) and (FM). Under (M3), u_t converges weakly towards a stationary measure.*

The proof is postponed in Subsection 2.3. Before, we will discuss briefly about the assumptions.

2.2. *Remarks on the assumptions.*

du_0 is absolutely continuous with respect to the Lebesgue measure. We shall use Theorem 1.6 and prove that the family $(u_t)_{t\in\mathbb{R}_+}$ admits a unique adherence value. This theorem needs that the initial law is absolutely continuous with respect to the Lebesgue measure. However, it is possible to relax this hypothesis by using the following result (see Lemma 2.1 in [HT10a] for a proof):
Set a probability measure du_0 which verifies $\int_{\mathbb{R}} x^{8q^2} du_0(x) < +\infty$. Then, for all $t > 0$, the probability du_t is absolutely continuous with respect to the Lebesgue measure.
Consequently, it is sufficient to apply Theorem 2.1 to the probability measure u_1 since there is a unique solution to the non-linear equation (I).

The entropy of du_0 is finite. An essential point of the proof is the convergence of the free-energy. For being sure of this, we assume that it is finite at time 0. The assumption about the moments implies $\Upsilon_\epsilon(u_t) < +\infty$ if and only if $\int_{\mathbb{R}} u_t(x)\log(u_t(x))\,dx < +\infty$.
If V was convex, a little adaptation of the theorem in [OV01] (taking into account the fact that the drift is not homogeneous here) would provide the non-optimal following inequality:

$$\Upsilon_\epsilon(u_t) \leq \frac{1}{2t}\inf\left\{\sqrt{\mathbb{E}\left|X-Y\right|^2}\ ;\ \mathcal{L}(X)=u_t\ ;\ \mathcal{L}(Y)=v_t\right\}$$

$$\text{with}\quad v_t(x) := Z^{-1}\exp\left[-\frac{2}{\epsilon}\Big(V(x)+F*u_t(x)\Big)\right]$$

for all $t > 0$. The second moment of u_t is upper-bounded uniformly with respect to t. By using the convexity of V and F, we can prove the same thing for v_t. Consequently, since $t > 0$, the free-energy is finite so the entropy is finite. However, in this paper, we deal with non-convex landscape so we will not relax this hypothesis.

All the moments are finite. Theorem 1.6 tells us we can extract a sequence from the family $(u_t)_{t\in\mathbb{R}_+}$ such that it converges towards a stationary measure. The last step in order to obtain the convergence is the uniqueness of the limiting value. The most difficult part will be to prove this uniqueness when the symmetric stationary measure u_ϵ^0 is an adherence value and the only one of these adherence values to be stationary. For doing this, we will consider a function like this one:

$$\Phi(u) := \int_{\mathbb{R}} \varphi(x)u(x)dx$$

where φ is an odd and smooth function with compact support such that $\varphi(x) = x^{2l+1}$ for all x in a compact subset of $\mathbb{R}$. Then, we will prove - by proceeding a *reductio ad absurdum* - there exists an integer l such that $\Phi\left(u_\epsilon^0\right) \neq \Phi\left(u_\infty\right)$, where u_∞ would be an other limiting value. This inequality will permit to construct a stationary measure u_ϵ such that $\Phi(u_\epsilon) \notin \left\{\Phi\left(u_\epsilon^0\right) ; \Phi\left(u_\epsilon^+\right) ; \Phi\left(u_\epsilon^-\right)\right\}$. This implies the existence of a stationary measure which does not belong to $\left\{u_\epsilon^0 ; u_\epsilon^+ ; u_\epsilon^-\right\}$. Under (M3), it is impossible. We make the integration with an "almost-polynomial" function because we need the square of the derivative of such function to be uniformly bounded with respect to the time.

However, it is possible to relax the condition (FM). Indeed, according to Proposition A.2, if we assume that $\int_{\mathbb{R}} x^{8q^2}du_0(x) < +\infty$ (the condition used for the existence of a strong solution), we have

$$\int_{\mathbb{R}} x^{2l}u_t^\epsilon(x)dx < +\infty \quad \forall t > 0,\, l \in \mathbb{N}\,.$$

Hypothesis (M3). As written before, the key for getting the uniqueness of the adherence value is to proceed a *reductio ad absurdum* and then to construct a stationary measure u_ϵ such that $\Phi(u_\epsilon)$ takes a forbidden value (a value different from $\Phi\left(u_\epsilon^0\right)$, $\Phi\left(u_\epsilon^+\right)$ and $\Phi\left(u_\epsilon^-\right)$).

But, it is possible to deal with a less strong hypothesis. Indeed, by considering an initial law with finite free-energy and since the free-energy is decreasing, it is impossible for u_t to converge towards a stationary measure with a higher energy. Consequently, we can consider (M3)' instead of (M3).

All of these remarks permit to obtain the following result:

THEOREM 2.2. *Set a probability measure* du_0 *with finite entropy. If* V *and* F *are polynomial functions such that* $F''(0) + V''(0) > 0$, u_t^ϵ *converges weakly towards a stationary measure for* ϵ *small enough.*

2.3. *Proof of the theorem.* In order to get the statement of Theorem 2.1, we will provide two lemmas and one proposition about the free-energy. The lemmas state that a probability measure which verifies simple properties and with some level of energy is necessary a stationary measure for the self-stabilizing process (I). The third one permits to confine all the adherences values under some level of energy.

LEMMA 2.3. *Under (M3), if* u *is a probability measure which satisfies (FE) and (ES),* $\Upsilon_\epsilon(u) \leq \Upsilon_\epsilon(u_\epsilon^\pm)$ *implies* $u \in \{u_\epsilon^+ \,;\, u_\epsilon^-\}$.

PROOF. Set u such a measure. We consider the process (I) starting by the initial law $u_0 := u$. Theorem 1.6 implies that there exists a stationary measure u_ϵ such that $\Upsilon_\epsilon(u_t)$ converges towards $\Upsilon_\epsilon(u_\epsilon)$.
However, according to Proposition 1.2 and Proposition 1.8,

$$\Upsilon_\epsilon(u_\epsilon) = \lim_{t\to+\infty} \Upsilon_\epsilon\left(u_t\right) \leq \Upsilon_\epsilon(u_t) \leq \Upsilon_\epsilon(u) \leq \Upsilon_\epsilon(u_\epsilon^\pm)\,.$$

Condition (M3) provides $u_\epsilon \in \left\{u_\epsilon^+;\, u_\epsilon^- \,;\, u_\epsilon^0\right\}$. But, $\Upsilon_\epsilon(u_\epsilon) \leq \Upsilon_\epsilon(u_\epsilon^\pm) < \Upsilon_\epsilon(u_\epsilon^0)$ so $u_\epsilon \in \{u_\epsilon^+ \,;\, u_\epsilon^-\}$. Without loss of generality, we will assume $u_\epsilon = u_\epsilon^+$. Consequently, the function ξ (see Definition 1.1) is constant. We deduce that $\xi'(t) = 0$ for all $t \geq 0$. Lemma 1.5 implies that u_t is a stationary measure in other words $u = u_0 = u_\epsilon = u_\epsilon^+$. □

We have a similar result with the symmetric measures:

LEMMA 2.4. *Under (0M1), if* u *is a symmetric probability measure satisfying (FE) and (ES),* $\Upsilon_\epsilon(u) \leq \Upsilon_\epsilon(u_\epsilon^0)$ *implies* $u = u_\epsilon^0$.

The key-argument is the following: if the initial law is symmetric then the law at time t is still symmetric. The proof is similar to the previous so it is left to the reader's attention.

Before making the convergence, we need a last result on the adherence values: *the free-energy of a limiting value is less than the limit value of the free-energy.*

Proposition 2.5. *We assume that u_∞ is an adherence value of the family $(u_t)_{t\in\mathbb{R}_+}$. We call $L_0 := \lim_{t\to+\infty}\Upsilon_\epsilon(u_t)$. Then $\Upsilon_\epsilon(u_\infty)\leq L_0$.*

Proof. As u_∞ is an adherence value of the family $(u_t)_{t\in\mathbb{R}_+}$, there exists an increasing sequence $(t_k)_k$ which tends to infinity such that u_{t_k} converges weakly towards u_∞. We remark:

$$\begin{aligned}\Upsilon\left(u_{t_k}\right) =&V(a)+\int_{\mathbb{R}}\left(V(x)-V(a)\right)u_{t_k}(x)dx\\&+\frac{1}{2}\iint_{\mathbb{R}^2}F(x-y)u_{t_k}(x)u_{t_k}(y)dxdy\end{aligned}$$

where the meta-potential Υ is defined in (IV). As $V(x)-V(a)\geq 0$ for all $x\in\mathbb{R}$, the Fatou lemma implies $\Upsilon\left(u_\infty\right)\leq\liminf_{k\to\infty}\Upsilon\left(u_{t_k}\right)$. In the same way:

$$\begin{aligned}&\int_{\mathbb{R}}u_\infty(x)\log\left(u_\infty(x)\right)\mathbb{1}_{\{u_\infty(x)\geq 1\}}dx\\&\leq\liminf_{k\to\infty}\int_{\mathbb{R}}u_{t_k}(x)\log\left(u_{t_k}(x)\right)\mathbb{1}_{\{u_{t_k}(x)\geq 1\}}dx\,.\end{aligned}$$

Set $R>0$. By putting $\gamma_k^-(x):=u_{t_k}(x)\log\left(u_{t_k}(x)\right)\mathbb{1}_{\{u_{t_k}(x)<1\}}\mathbb{1}_{\{|x|\leq R\}}$, we not that $\left|\gamma_k^-(x)\right|\leq e^{-1}\mathbb{1}_{\{|x|\leq R\}}$ for all $x\in\mathbb{R}$ and $k\in\mathbb{N}$. We can apply the Lebesgue theorem:

$$\int_{\mathbb{R}}u_\infty(x)\log\left(u_\infty(x)\right)\mathbb{1}_{\{u_\infty(x)\geq 1\}}\mathbb{1}_{\{|x|\leq R\}}=\lim_{k\to\infty}\int_{\mathbb{R}}\gamma_k^-(x)dx\,.$$

We put $\gamma_k^+(x):=u_{t_k}(x)\log\left(u_{t_k}(x)\right)\mathbb{1}_{\{u_{t_k}(x)<1\}}\mathbb{1}_{\{|x|>R\}}$. By proceeding as in the proof of Lemma 1.3, we have:

$$\begin{aligned}-\gamma_k^+(x)=&-u_{t_k}(x)\log\left(u_{t_k}(x)\right)\mathbb{1}_{\{e^{-|x|}\leq u_{t_k}(x)<1\}}\mathbb{1}_{\{|x|>R\}}\\&-u_{t_k}(x)\log\left(u_{t_k}(x)\right)\mathbb{1}_{\{u_{t_k}(x)<e^{-|x|}\}}\mathbb{1}_{\{|x|>R\}}\\\leq&|x|u_{t_k}(x)\mathbb{1}_{\{|x|>R\}}+2e^{-1}e^{-\frac{|x|}{2}}\mathbb{1}_{\{|x|>R\}}\,.\end{aligned}$$

Consequently, we get the lower-bound

$$\int_{\mathbb{R}}u_{t_k}^\epsilon(x)\log\left(u_{t_k}^\epsilon(x)\right)\mathbb{1}_{\{u_{t_k}^\epsilon(x)<1\}}\mathbb{1}_{\{|x|>R\}}dx\geq-\frac{M_0}{R}-8e^{-1}e^{-\frac{R}{2}}$$

where M_0 is defined in (V).
By introducing $\widehat{\Upsilon_\epsilon}(u):=\Upsilon_\epsilon(u)-\frac{\epsilon}{2}\int_{\mathbb{R}}u(x)\log(u(x))\mathbb{1}_{\{u(x)<1\}}\mathbb{1}_{\{|x|>R\}}dx$, we

obtain:

$$\begin{aligned}\Upsilon_\epsilon\left(u_\infty\right)\leq\widehat{\Upsilon_\epsilon}\left(u_\infty\right) &\leq \liminf_{k\to\infty}\widehat{\Upsilon_\epsilon}\left(u_{t_k}\right)\\ &\leq \liminf_{k\to\infty}\Upsilon_\epsilon\left(u_{t_k}\right)+\frac{M_0\epsilon}{2R}+4e^{-1}\exp\left(-\frac{R}{2}\right)\epsilon\\ &\leq L_0+\frac{M_0\epsilon}{2R}+4e^{-1}\epsilon\exp\left(-\frac{R}{2}\right)\end{aligned}$$

for all $R>0$. Consequently, $\Upsilon_\epsilon(u_\infty)\leq L_0$. □

Proof of the theorem. **Plan:** The first step of the proof consists in the application of the Prohorov's theorem since the family of measure is tight. We shall prove the uniqueness of the adherence value. We will proceed a *reductio ad absurdum*. The previous results provide $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}\neq\emptyset$ where $\mathcal{A}$ is introduced in Definition 1.7. We will then study all the possible cases and we will prove that all of these case imply contradictions. The cases $\left\{u_\epsilon^0\,;\,u_\epsilon^+\right\}\subset\mathcal{A}$, $\left\{u_\epsilon^0\,;\,u_\epsilon^-\right\}\subset\mathcal{A}$ and $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\{u_\epsilon^+\}$ would imply contradiction by using the fact that u_ϵ^+ and u_ϵ^- are the unique minimizers of the free-energy. The cases $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\left\{u_\epsilon^0\right\}$ and $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\{u_\epsilon^+\,;\,u_\epsilon^-\}$ imply the existence of an other stationary measure which is an adherence value.

Step 1: The inequality (V) implies that the family of probability measures $\{u_t\,;\,t\in\mathbb{R}_+\}$ is tight. Prohorov Theorem permits to conclude that each extracted sequence of this family is relatively compact with respect to the weak convergence. So, in order to prove the statement of the theorem, it is sufficient to prove that this family admits exactly one adherence value. We proceed a *reductio ad absurdum*. We assume in the following that the family admits at least two adherence values.

Step 2: As the condition (M3) is true, there are exactly three stationary measures : u_ϵ^0, u_ϵ^+ and u_ϵ^-. By Theorem 1.6, we know that $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}\neq\emptyset$. We split this step into four cases:

- $\left\{u_\epsilon^0\,;\,u_\epsilon^+\right\}\subset\mathcal{A}$.
- $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\left\{u_\epsilon^0\right\}$.
- $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\{u_\epsilon^+\,;\,u_\epsilon^-\}$.
- $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\{u_\epsilon^+\}$.

By symmetry, we will not deal with the two following cases: $\left\{u_\epsilon^0\,;\,u_\epsilon^-\right\}\subset\mathcal{A}$ and $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\{u_\epsilon^-\}$.

Step 2.1: First case: $\left\{u_\epsilon^0\,;\,u_\epsilon^+\right\}\subset\mathcal{A}$. According to Proposition 1.8, we have the following equality:

$$\Upsilon_\epsilon\left(u_\epsilon^0\right)=L_0=\Upsilon_\epsilon\left(u_\epsilon^+\right)\quad\text{with}\quad L_0:=\lim_{t\to+\infty}\Upsilon_\epsilon\left(u_t\right)\,.$$

The hypothesis (M3) implies $\Upsilon_\epsilon\left(u_\epsilon^0\right)>\Upsilon_\epsilon\left(u_\epsilon^+\right)$. This is impossible.

Step 2.2: We will now prove that the second case: $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\}=\left\{u_\epsilon^0\right\}$ is impossible. It will be the core of the proof.

Step 2.2.1: Set u_∞ an other adherence value of the family $(u_t)_{t\in\mathbb{R}_+}$. Proposition 2.5 tells us $\Upsilon_\epsilon\left(u_\infty\right)\leq\Upsilon_\epsilon\left(u_\epsilon^0\right)$. Since $u_\infty\neq u_\epsilon^0$, Lemma 2.4 implies u_∞ is not symmetric. We deduce there exists $l\in\mathbb{N}$ such that $\int_{\mathbb{R}}x^{2l+1}u_\infty(x)dx\neq0$. Set $R>0$. We introduce the following function:

$$\begin{aligned}\varphi(x):=\;&x^{2l+1}\mathbb{1}_{[-R;R]}(x)\\&+x^{2l+1}\,\mathbb{1}_{[R;R+1]}(x)\,Z^{-1}\int_x^{R+1}\exp\left[-\frac{1}{(y-R)^2}-\frac{1}{(y-R-1)^2}\right]dy\\&+x^{2l+1}\,\mathbb{1}_{[-R-1;-R]}(x)\,Z^{-1}\int_{-R-1}^{x}\exp\left[-\frac{1}{(y+R)^2}-\frac{1}{(y+R+1)^2}\right]dy\\&\text{with}\quad Z:=\int_0^1\exp\left[-\frac{1}{z^2}-\frac{1}{(z-1)^2}\right]dz\,.\end{aligned}$$

By construction, φ is an odd function so $\int_{\mathbb{R}}\varphi(x)u_\epsilon^0(x)dx=0$. Furthermore, $|\varphi(x)|\leq|x|^{2l+1}$. By using the triangular inequality and (FM), we have:

$$\begin{aligned}\left|\int_{\mathbb{R}}\varphi(x)u_\infty(x)dx\right|&\geq\left|\int_{\mathbb{R}}x^{2l+1}u_\infty(x)dx\right|-\int_{[-R;R]^c}|x|^{2l+1}\,u_\infty(x)dx\\&\geq\left|\int_{\mathbb{R}}x^{2l+1}u_\infty(x)dx\right|-\frac{1}{R^3}C_0\end{aligned}$$

where $C_0:=\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}}|x|^{2l+4}u_t(x)dx<+\infty$. Since $\int_{\mathbb{R}}x^{2l+1}u_\infty(x)dx\neq0$, we deduce that $\int_{\mathbb{R}}\varphi(x)u_\infty(x)dx\neq0$ for R big enough. Consequently, we obtain the existence of a smooth function φ with compact support such that

$$0=\int_{\mathbb{R}}\varphi(x)u_\epsilon^0(x)dx<\int_{\mathbb{R}}\varphi(x)u_\infty(x)dx=:3\kappa$$

with $\kappa>0$. Moreover, we can verify that $\varphi'(x)^2\leq C(R)x^{4l+2}$ for all $x\in\mathbb{R}$. This implies: $\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}}\varphi'(x)^2u_t(x)dx<+\infty$.

Step 2.2.2: By definition of $\mathcal{A}$, there exist two increasing sequences $(t_k^{(1)})_k$ (resp. $(t_k^{(2)})_k$) such that $u_{t_k^{(1)}}$ (resp. $u_{t_k^{(2)}}$) converges weakly towards u_ϵ^0 (resp. u_∞). We deduce there exist two increasing sequences $(r_k)_k$ and $(s_k)_k$ such that $\int_{\mathbb{R}} \varphi(x)u_{r_k}(x)dx = \kappa$ and $\int_{\mathbb{R}} \varphi(x)u_{s_k}(x)dx = 2\kappa$. Then, for all $k \in \mathbb{N}$, we put $\widehat{r_k} := \sup\left\{t \in [0; s_k] \mid \int_{\mathbb{R}} \varphi(x)u_t(x)dx = \kappa\right\}$ then we define $\widehat{s_k} := \inf\left\{s \in [\widehat{r_k}; s_k] \mid \int_{\mathbb{R}} \varphi(x)u_s(x)dx = 2\kappa\right\}$. For simplicity, we write r_k (resp. s_k) instead of $\widehat{r_k}$ (resp. $\widehat{s_k}$). And, we have:

$$\kappa = \int_{\mathbb{R}} \varphi(x)u_{r_k}(x)dx \le \int_{\mathbb{R}} \varphi(x)u_t(x)dx \le \int_{\mathbb{R}} \varphi(x)u_{s_k}(x)dx = 2\kappa$$

for all $t \in [r_k; s_k]$.

Step 2.2.3: By applying Proposition A.1, we deduce there exists an increasing sequence $(q_k)_k$ converging to $+\infty$ such that $(u_{q_k})_k$ converges weakly towards a stationary measure u_ϵ verifying $\int_{\mathbb{R}} \varphi(x)u_\epsilon(x)dx \in [\kappa; 2\kappa]$. As the set $\mathcal{A}\bigcap\{u_\epsilon^+\,;\,u_\epsilon^-\}$ is empty, we deduce $u_\epsilon = u_\epsilon^0$. This is impossible since $\int_{\mathbb{R}} \varphi(x)u_\epsilon^0(x)dx = 0 \notin [\kappa; 2\kappa]$.

Step 2.3: We deal now with the third case: $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\} = \{u_\epsilon^+\,;\,u_\epsilon^-\}$.

Step 2.3.1: By definition of u_ϵ^+ and u_ϵ^-, we know that these measures are not symmetric. Consequently, there exists $l \in \mathbb{N}$ such that $\int_{\mathbb{R}} x^{2l+1}u_\epsilon^+(x)dx \neq 0$. As $u_\epsilon^-(x) = u_\epsilon^+(-x)$, by proceeding as in **Step 2.2.1** and **Step 2.2.2** and after the application of Proposition A.1, we deduce there exists an increasing sequence $(q_k)_{k\in\mathbb{N}}$ which converges to ∞ such that u_{q_k} converges weakly towards a stationary measure u_ϵ which verifies $\int_{\mathbb{R}} \varphi(x)u_\epsilon(x)dx \in [\kappa; 2\kappa]$ where φ is a smooth function with compact support such that $\int_{\mathbb{R}} \varphi(x)u_\epsilon^\pm(x)dx \notin [\kappa; 2\kappa]$. We deduce that $u_\epsilon = u_\epsilon^0$ which contradicts $u_\epsilon^0 \notin \mathcal{A}$.

Step 2.4: We consider now the last case: $\mathcal{A}\bigcap\left\{u_\epsilon^0\,;\,u_\epsilon^+\,;\,u_\epsilon^-\right\} = \{u_\epsilon^+\}$. Proposition 1.8 implies that $\Upsilon_\epsilon(u_t)$ converges towards $\Upsilon_\epsilon(u_\epsilon^+)$. Set u_∞ a limit value of the family $(u_t)_{t\in\mathbb{R}_+}$ which is not u_ϵ^+. By Proposition 2.5, we know that $\Upsilon_\epsilon(u_\infty) \le \Upsilon_\epsilon(u_\epsilon^+) = \lim_{t\longrightarrow+\infty} \Upsilon_\epsilon(u_t)$. Then, Lemma 2.3 implies $u_\infty = u_\epsilon^- \notin \mathcal{A}$.

Conclusion The family $(u_t)_{t\in\mathbb{R}_+}$ admits only one adherence value with respect to the weak convergence. So u_t converges weakly towards a stationary measure that achieves the proof. □

3. Bassins of attraction. Now, we shall provide some condition in order to precise the limit.

3.1. *Domain of* u_ϵ^0.

THEOREM 3.1. *Set a symmetric probability measure* du_0 *which verifies (FE) and (ES). We assume that* $V''(0)+F''(0) \neq 0$. *Then, for* ϵ *small enough* u_t *converges weakly towards* u_ϵ^0.

PROOF. $V''(0)+F''(0)\neq 0$ and both functions V'' and F'' are convex. Theorem 7.3 and 7.4 in [HT09] imply the existence and the uniqueness of a symmetric stationary measure u_ϵ^0 for ϵ small enough.

Theorem 1.6 provides the existence of a stationary measure u_ϵ and an increasing sequence $(t_k)_k$ which converges to ∞ such that u_{t_k} converges weakly towards u_ϵ and $\Upsilon_\epsilon(u_t)$ converges towards $\Upsilon_\epsilon(u_\epsilon)$. As u_t is symmetric for all $t\geq 0$, we deduce $u_\epsilon=u_\epsilon^0$, the unique symmetric stationary measure.

We proceed a *reductio ad absurdum* by assuming there exists an other sequence $(s_k)_k$ which converges to ∞ such that u_{s_k} does not converge weakly towards u_ϵ^0. The uniform boundedness of the second moment with respect to the time permits to extract a subsequence (that we continue to write $(s_k)_k$ for simplicity) such that u_{s_k} converges weakly towards $u_\infty\neq u_\epsilon^0$. Proposition 2.5 implies $\Upsilon_\epsilon(u_\infty)\leq\Upsilon_\epsilon(u_\epsilon^0)$. Lemma 2.4 implies $u_\infty=u_\epsilon^0$. This is absurd. □

REMARK 3.2. *We assume $V''(0)+F''(0)\neq 0$ in order to have a unique symmetric stationary measure for ϵ small enough. We can extend to the case $V''(0)+F''(0)=0$ by using the half-uniform propagation of chaos, see Theorem 6.5 in [Tug10]. We can also assume that $n=2$ that means $\deg(F)=4$ by Subsection 4.2 in [HT10a].*

REMARK 3.3. *In the previous theorem, if we have assumed (FM) instead of (ES), we could have applies directly Theorem 2.1.*

3.2. *Domain of $u_\epsilon^\pm$.* The principal tool of the previous theorem is the stability of the subset of all the symmetric stationary measures with a finite $8q^2$-moment. If we could find an invariant subset which contains u_ϵ^+ but neither u_ϵ^0 nor u_ϵ^-, we could apply the same method for obtaining the convergence towards u_ϵ^+.

Instead of this, we will consider some inequality linked to the meta-potential and we will exhibit a simple subset included in the domain of attraction of u_ϵ^+. Let us first introduce the following hyperplan:

$$\mathcal{H}:=\left\{u\in\mathcal{C}^\infty\left(\mathbb{R}\,;\,\mathbb{R}_+\right)\;\Big|\;\int_{\mathbb{R}}x^{8q^2}u(x)dx<\infty\text{ and }\int_{\mathbb{R}}xu(x)dx=0\right\}.$$

THEOREM 3.4. *Set a probability measure du_0 which verifies (FE) and (FM). We assume also*

$$\Upsilon_\epsilon\left(u_0\right)<\inf_{u\in\mathcal{H}}\Upsilon_\epsilon(u)\quad\text{and}\quad\int_{\mathbb{R}}xu_0(x)dx>0\,.$$

Under (M3), u_t converges weakly towards u_ϵ^+.

PROOF. We know by Theorem 2.1 that there exists a stationary measure u_ϵ such that $(u_t)_t$ converges weakly towards u_ϵ. And, by Proposition 1.8, $\Upsilon_\epsilon(u_t)$ converges towards $\Upsilon_\epsilon(u_\epsilon)$.

Step 1: As $\int_{\mathbb{R}} x u_\epsilon^0(x)dx = 0$ and $\int_{\mathbb{R}} x^{8q^2} u_\epsilon^0(x)dx < +\infty$, we have

$$\Upsilon_\epsilon\left(u_\epsilon^0\right) \geq \inf_{u\in\mathcal{H}} \Upsilon_\epsilon(u) > \Upsilon_\epsilon\left(u_0\right) .$$

We deduce $u_\epsilon \neq u_\epsilon^0$ since $t \mapsto \xi(t) = \Upsilon_\epsilon(u_t)$ is nonincreasing.

Step 2. We proceed now a *reductio ad absurdum* by assuming $u_\epsilon = u_\epsilon^-$. There exists $t_0 > 0$ such that $\int_{\mathbb{R}} x u_{t_0}(x)dx = 0$. Consequently:

$$\Upsilon_\epsilon\left(u_{t_0}\right) \geq \inf_{u\in\mathcal{H}} \Upsilon_\epsilon(u) > \Upsilon_\epsilon\left(u_0\right)$$

which contradicts the fact that ξ is non-increasing.

Step 3. Assumption (M3) implies the weak convergence towards u_ϵ^+. □

We use now Theorem 3.4 in some particular cases.

THEOREM 3.5. *Set a probability measure du_0 which verifies (FE) and (FM). We assume also*

$$\Upsilon\left(u_0\right) < V(x_0) + \frac{1}{4}F(2x_0) \quad \textit{and} \quad \int_{\mathbb{R}} x u_0(x)dx > 0$$

where x_0 is defined in the introduction. Under either Condition (LIN) or Condition (SYN), for ϵ small enough u_t converges weakly towards u_ϵ^+.

PROOF. **Step 1.** Theorem 3.2 in [HT10a] and Theorem 1.11 imply Condition (M3) under (LIN) or (SYN).

Step 2. Lemma A.3 provides the following limit:

$$\lim_{\epsilon\longrightarrow 0} \Upsilon_\epsilon(u_\epsilon^0) = V(x_0) + \frac{1}{4}F(2x_0)\,.$$

Then, we deduce

$$\lim_{\epsilon\longrightarrow 0} \inf_{u\in\mathcal{H}} \Upsilon_\epsilon(u) \leq V(x_0) + \frac{1}{4}F(2x_0)\,. \tag{3.1}$$

Step 3. We prove now that $V(x_0) + \frac{1}{4}F(2x_0) = \lim_{\epsilon\longrightarrow 0} \inf_{u\in\mathcal{H}} \Upsilon_\epsilon(u)$. Indeed, if u is a probability measure such that $\int_{\mathbb{R}} x u(x)dx = 0$, it verifies the following inequality:

$$\begin{aligned}\Upsilon_\epsilon(u) &\geq \Upsilon_\epsilon^-(u) + \frac{F''(0)}{4}\iint_{\mathbb{R}^2}(x-y)^2u(x)u(y)dxdy\\ &\geq \Upsilon_\epsilon^-(u) + \frac{F''(0)}{2}\int_{\mathbb{R}}x^2u(x)dx\,.\end{aligned}$$

By using (1.1), it yields

$$\Upsilon_\epsilon(u) \geq -\frac{\epsilon}{4} - \frac{4\epsilon}{\exp(1)} + \int_{\mathbb{R}}\left\{V(x) + \frac{F''(0)}{2}x^2 - \frac{\epsilon x^2}{4}\right\}u(x)dx\,. \tag{3.2}$$

We split now the study depending on whether we use Condition (LIN) or Condition (SYN):

(LIN) If F' is linear, $\frac{\alpha}{2}x^2 = \frac{1}{4}F(2x)$. So the minimum of $x \mapsto V(x)+\frac{1}{4}F(2x)$ is $V(x_0) + \frac{1}{4}F(2x_0)$. We can easily prove that

$$\min_{x\in\mathbb{R}}\left(V(x) + \frac{\alpha}{2}x^2 - \frac{\epsilon}{4}x^2\right) = V(x_0) + \frac{1}{4}F(2x_0) - \frac{\epsilon}{4}x_0^2 + o(\epsilon)\,.$$

Consequently:

$$\Upsilon_\epsilon(u) \geq -\frac{\epsilon}{4} - \frac{4\epsilon}{\exp(1)} + V(x_0) + \frac{1}{4}F(2x_0) - \frac{\epsilon}{4}x_0^2 + o(\epsilon)$$

for all $u \in \mathcal{H}$. Then, $\lim_{\epsilon\longrightarrow 0}\min_{u\in\mathcal{H}}\Upsilon_\epsilon(u) \geq V(x_0)+\frac{1}{4}F(2x_0)$. The inequality (3.1) provides $\lim_{\epsilon\longrightarrow 0}\inf_{u\in\mathcal{H}}\Upsilon_\epsilon(u) = V(x_0) + \frac{1}{4}F(2x_0)$.

(SYN) If $V''(0)+F''(0) > 0$, (3.2) implies $\Upsilon_\epsilon(u) \geq -\frac{\epsilon}{4} - \frac{4\epsilon}{\exp(1)}$ for all $u \in \mathcal{H}$ since ϵ is less than $2\left(V''(0) + F''(0)\right)$. We deduce that $\lim_{\epsilon\longrightarrow 0}\inf_{u\in\mathcal{H}}\Upsilon_\epsilon(u) \geq 0$. However, as $V''(0) + F''(0) > 0$, Theorem 5.4 in [HT10b] implies $x_0 = 0$ so $V(x_0) + \frac{1}{4}F(2x_0) = 0$. The inequality (3.1) provides the following limit: $\lim_{\epsilon\longrightarrow 0}\inf_{u\in\mathcal{H}}\Upsilon_\epsilon(u) = 0 = V(x_0) + \frac{1}{4}F(2x_0)$.

Step 4. Consequently, $\Upsilon_\epsilon(u_0) < \inf_{u\in\mathcal{H}}\Upsilon_\epsilon(u)$ for ϵ small enough. Then, we apply Theorem 3.4. □

Remark 3.6. *We can replace $\int_{\mathbb{R}}xu_0(x)dx > 0$ by $\int_{\mathbb{R}}xu_0(x)dx < 0$ in Theorem 3.4 and 3.5 then the same results holds with u_ϵ^- instead of u_ϵ^+.*

APPENDIX A: APPENDIX

In this appendix, we present some results used previously in the proofs of the main theorems which were postponed here.

Proposition A.1 permits to ensure that even if the free-energy does not reach its global minimum on the stationary measure u_ϵ^0, if the unique symmetric stationary measure is an adherence value, then it is unique.

Proposition A.2 is a general result on the self-stabilizing processes. Indeed, it is well-known that du_t is absolutely continuous with respect to the Lebesgue measure. Proposition A.2 extends this instantaneous regularization to the finiteness of all the moments since $t>0$.

Lemma A.3 consists in asymptotic computation of the free-energy in the small-noise limit for some useful measures. Lemma A.4 is a Laplace method tedious computation necessary for avoiding to assume that any family of stationary measures verify Condition (H).

We present now the essential proposition for proving Theorem 2.1.

PROPOSITION A.1. *Set a probability measure du_0 which verifies (FE) and (FM). We assume the existence of two polynomial functions $\mathcal{P}$ and $\mathcal{Q}$, a smooth function φ with compact support such that $|\varphi(x)|\leq\mathcal{P}(x)$ and $|\varphi'(x)|^2\leq\mathcal{Q}(x)$, $\kappa>0$ and two sequences $(r_k)_k$ and $(s_k)_k$ which converge to ∞ such that for all $r_k\leq t\leq s_k<r_{k+1}$:*

$$\kappa=\int_{\mathbb{R}}\varphi(x)u_{r_k}(x)dx\leq\int_{\mathbb{R}}\varphi(x)u_t(x)dx\leq\int_{\mathbb{R}}\varphi(x)u_{s_k}(x)dx=2\kappa\,.$$

Then, there exists a stationary measure u_ϵ which verifies $\int_{\mathbb{R}}\varphi(x)u_\epsilon(x)dx\in[\kappa;2\kappa]$ and an increasing sequence $(q_k)_k$ which converges to ∞ such that u_{q_k} converges weakly towards u_ϵ.

PROOF. **Step 1:** We will prove that $\limsup\limits_{k\longrightarrow+\infty}(s_k-r_k)>0$. We introduce the function :

$$\Phi(t):=\int_{\mathbb{R}}\varphi(x)u_t(x)dx\,.$$

This function is well-defined since $|\varphi|$ is bounded by a polynomial function. The derivation of Φ, the use of equation (III) and an integration by part lead to:

$$\begin{aligned}\Phi'(t) &= -\int_{\mathbb{R}}\varphi'(x)\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_t(x)+u_t(x)\left(V'(x)+F'*u_t(x)\right)\right\}dx\\ &= -\int_{\mathbb{R}}\varphi'(x)\eta_t(x)dx\,.\end{aligned}$$

The Cauchy-Schwarz inequality implies

$$\left|\Phi'(t)\right| \leq \sqrt{|\xi'(t)|}\sqrt{\int_{\mathbb{R}} \left(\varphi'(x)\right)^2 u_t(x)dx}$$

where we recall that $\xi(t) = \Upsilon_\epsilon(u_t)$. The function $(\varphi')^2$ is bounded by a polynomial function and $\int_{\mathbb{R}} x^{2N} u_t(x)$ is uniformly bounded with respect to $t \in \mathbb{R}_+$ for all $N \in \mathbb{N}$. So, there exists $C > 0$ such that $\int_{\mathbb{R}} \left(\varphi'(x)\right)^2 u_t(x)dx \leq C^2$ for all $t \in \mathbb{R}_+$. We deduce

$$\left|\Phi'(t)\right| \leq C\sqrt{|\xi'(t)|}\,. \tag{A.1}$$

By definition of the two sequences $(r_k)_k$ and $(s_k)_k$, we have

$$\Phi(s_k) - \Phi(r_k) = \kappa\,.$$

Combining this identity with (A.1), it yields

$$C\int_{r_k}^{s_k} \sqrt{|\xi'(t)|}dt \geq \kappa\,.$$

We apply the Cauchy-Schwarz inequality and obtain:

$$C\sqrt{s_k - r_k}\sqrt{\xi(r_k) - \xi(s_k)} \geq \kappa$$

since ξ is non-increasing (see Proposition 1.2). Moreover, $\xi(t)$ converges as t converges to ∞ (see Lemma 1.4). It implies the convergence of $\xi(r_k) - \xi(s_k)$ towards 0 when k tends to $+\infty$. Consequently, $s_k - r_k$ converges to $+\infty$ so $\limsup_{k\longrightarrow+\infty} s_k - r_k > 0$.

Step 2: By Lemma 1.4, $\Upsilon_\epsilon(u_t) - \Upsilon_\epsilon(u_\epsilon) = \int_t^\infty \xi'(s)ds$ converges to 0. As ξ' is nonpositive, we deduce that $\sum_{k=N}^\infty \int_{r_k}^{s_k} \xi'(s)ds$ converges also to 0 when N tends to $+\infty$. As $\limsup_{k\longrightarrow+\infty} s_k - r_k > 0$, we deduce there exists an increasing sequence $q_k \in [r_k; s_k]$ which converges to ∞ and such that $\xi'(q_k)$ converges to 0 when k tends to ∞. Furthermore, $\int_{\mathbb{R}} \varphi(x)u_{q_k}(x)dx \in [\kappa; 2\kappa]$ for all $k \in \mathbb{N}$.

Step 3: By proceeding similarly as in the proof of Theorem 1.6, we extract a subsequence of $(q_k)_k$ (we continue to write it q_k for simplifying the reading) such that u_{q_k} converges weakly towards a stationary measure u_ϵ. Moreover, u_ϵ verifies $\int_{\mathbb{R}} \varphi(x)u_\epsilon(x)dx \in [\kappa; 2\kappa]$. □

We provide now a result that allows us to get the results of the main theorem (Theorem 2.1) with less strong condition:

PROPOSITION A.2. *Let a probability measure du_0 which verifies (FE) and (ES). Then, for all $t>0$, du_t satisfies (FM).*

PROOF. **Step 1:** If du_0 verifies (FM) then du_t satisfies (FM) for all $t>0$, see Theorem 2.12 in [HIP08]. We assume now that du_0 does not satisfy (FM). Let us introduce $l_0:=\min\left\{l\geq 0\ \mid\ \mathbb{E}\left[X_0^{2l}\right]\right\}=+\infty$. We know that $\mathbb{E}\left[X_t^{2l_0-2}\right]<+\infty$ for all $t\geq 0$.
Step 2: Set $t_0>0$. We proceed a *reduction ad absurdum* by assuming that $\mathbb{E}\left[X_{t_0}^{2l_0}\right]=+\infty$. This implies directly $\mathbb{E}\left[X_t^{2l_0}\right]=+\infty$ for all $t\in[0,t_0]$. We recall that $2m$ (resp. $2n$) is the degree of the confining (resp. interaction) potential V (resp. F). Also, $q:=\max\left\{m\ ;\ n\right\}$. For all $t\in[0,t_0]$, the application $x\mapsto F'*u_t(x)$ is a polynomial function with parameters $m_1(t),\cdots,m_{2n-1}(t)$, where $m_j(t)$ is the j-th moment of the law du_t. We recall the inequality (V):

$$\sup_{1\leq j\leq 8q^2}\sup_{t\in[0,t_0]}m_j(t)\leq M_0\,.$$

Consequently, the application $x\mapsto V'(x)+F'*u_t(x)$ is a polynomial function with degree $2q-1$. Furthermore, the principal term does not depend of the moments of the law du_t so we can write:

$$V'(x)+F'*u_t(x)=\kappa_{2q-1}x^{2q-1}+\mathcal{P}_t(x)$$

where $\kappa_{2q-1}\in\mathbb{R}_+^*$ is a constant, and $\mathcal{P}_t$ is a polynomial function with degree at most $2q-2$. Moreover, $\mathcal{P}_t$ is parametrized by the $2n$ first moments only. Set $l\in\mathbb{N}$. We introduce the function $\mathcal{Q}_t(x):=2lx^{2q-1}\mathcal{P}_t(x)-l(2l-1)\epsilon x^{2l-2}$. As $\mathcal{Q}_t$ is a polynomial function of degree less than $2l+2q-3$, we have the following inequality:

$$2l\kappa_{2q-1}x^{2l+2q-2}+\mathcal{Q}_t(x)\geq C_l\left(x^{2l+2q-2}-1\right)\tag{A.2}$$

where C_l is some positive constant. The application of Ito formula provides:

$$dX_t^{2l}=2lX_t^{2l-1}\sqrt{\epsilon}dB_t-\left[2l\kappa_{2q-1}X_t^{2l+2q-2}+\mathcal{Q}_t\left(X_t\right)\right]dt\,.$$

After integration, we obtain:

$$\begin{aligned}X_{t_0}^{2l} &= X_0^{2l}+2l\sqrt{\epsilon}\int_0^{t_0}X_t^{2l-1}dB_t\\ &- \int_0^{t_0}\left[2l\kappa_{2q-1}X_t^{2l+2q-2}+\mathcal{Q}_t\left(X_t\right)\right]dt\,.\end{aligned}$$

We put $\Omega_R := \mathbb{1}_{\{\sup_{u\in[0,t_0]}|X_u|\leq R\}}$ for all $R>0$. We take now the expectation after multiplying by Ω_R:

$$\begin{aligned}\mathbb{E}\left[X_{t_0}^{2l}\Omega_R\right] &= \mathbb{E}\left[X_0^{2l}\Omega_R\right] - \int_0^{t_0}\mathbb{E}\left[\left(2l\kappa_{2q-1}X_t^{2l+2q-2}+\mathcal{Q}_t\left(X_t\right)\right)\Omega_R\right]dt\\ &\leq \mathbb{E}\left[X_0^{2l}\right]+C_l t_0 - C_l\int_0^{t_0}\mathbb{E}\left[X_t^{2l+2q-2}\Omega_R\right]dt\end{aligned}$$

after using (A.2). We take $l := l_0+1-q$:

$$0\leq \mathbb{E}\left[X_{t_0}^{2l_0+2-2q}\Omega_R\right]\leq C_1 - C_2\int_0^{t_0}\mathbb{E}\left[X_t^{2l_0}\Omega_R\right]dt$$

where C_1 and C_2 are positive constants. As $R\mapsto\Omega_R$ is increasing and converges towards 1 almost surely when R tends to $+\infty$, we deduce the pointwise convergence of $\mathbb{E}\left[X_t^{2l_0}\Omega_R\right]$ towards $\mathbb{E}\left[X_t^{2l_0}\right]=+\infty$ when R tends to $+\infty$. Also, the application $R\mapsto\mathbb{E}\left[X_t^{2l_0}\Omega_R\right]$ being increasing for all $t\in[0,t_0]$, it yields the convergence of $\int_0^{t_0}\mathbb{E}\left[X_t^{2l_0}\Omega_R\right]dt$ towards $+\infty$ when R tends towards $+\infty$ which implies $\mathbb{E}\left[X_{t_0}^{2l_0+2-2q}\Omega_R\right]<0$ for R big enough. This contradicts the fact that $\mathbb{E}\left[X_{t_0}^{2l_0+2-2q}\Omega_R\right]$ is positive for all R and increasing with respect to R. Consequently, for all $t_0>0$, $\mathbb{E}\left[X_{t_0}^{2l_0}\right]<+\infty$.

Step 3: Set $T>0$ and $l_1\in\mathbb{N}$ such that $l_1\geq l_0$ where the integer l_0 is defined as previously: $l_0 := \min\left\{l\geq 0\ \mid\ \mathbb{E}\left[X_0^{2l}\right]=+\infty\right\}$. If $l_1=l_0$, the application of Step 2 leads to $\mathbb{E}\left[X_T^{2l_1}\right]<+\infty$. If $l_1>l_0$, we put $t_i := \frac{i}{l_1+1-l_0}T$ for all $1\leq i\leq l_1+1-l_0$. We apply Step 2 to t_1 and we deduce $\mathbb{E}\left[X_{t_1}^{2l_0}\right]<+\infty$. By recurrence, we deduce $\mathbb{E}\left[X_{t_i}^{2l_0+2i}\right]<+\infty$ for all $1\leq i\leq l_1+1-l_0$, in particular $\mathbb{E}\left[X_{t_{l_0-l_1}}^{2l_0+2(l_1-l_0)}\right]<+\infty$ that means $\mathbb{E}\left[X_T^{2l_1}\right]<+\infty$. This inequality holds for all $l_1\geq l_0$ so the probability measure du_T satisfies (FM). □

In order to get the thirdness of the stationary measure (or a weaker result, see Theorem 1.11), we need to compute the small noise limit of the free-energy for the stationary measures u_ϵ^+, u_ϵ^- and u_ϵ^0.

LEMMA A.3. *Set ϵ_0 such that there exist three families of stationary measures $\left(u_\epsilon^+\right)_{\epsilon\in]0;\epsilon_0]}$, $\left(u_\epsilon^-\right)_{\epsilon\in]0;\epsilon_0]}$ and $\left(u_\epsilon^0\right)_{\epsilon\in]0;\epsilon_0]}$ which verify*

$$\lim_{\epsilon\to0}u_\epsilon^\pm=\delta_{\pm a}\quad\textit{and}\quad\lim_{\epsilon\to0}u_\epsilon^0=\frac12\delta_{x_0}+\frac12\delta_{-x_0}$$

where x_0 is defined in the introduction. Then, we have the following limits:

$$\lim_{\epsilon\to 0}\Upsilon_\epsilon\left(u_\epsilon^\pm\right)=V(a)\quad \textit{and}\quad \lim_{\epsilon\to 0}\Upsilon_\epsilon\left(u_\epsilon^0\right)=V(x_0)+\frac{1}{4}F\left(2x_0\right).$$

Plus, by considering the measure $v_\epsilon^+(x):=Z^{-1}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F(x-a)\right)\right]$, we have:

$$\lim_{\epsilon\to 0}\Upsilon_\epsilon\left(v_\epsilon^+\right)=V(a)\,.$$

PROOF. **Step 1:** We begin to prove the result for u_ϵ^0.
Step 1.1: We can write $u_\epsilon^0(x)=Z^{-1}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^0(x)\right)\right]$ since it is a stationary measure. Hence

$$\begin{aligned}\Upsilon_\epsilon\left(u_\epsilon^0\right) &= -\frac{\epsilon}{2}\log\left(\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^0(x)\right)\right]dx\right)\\ &- \frac{1}{2}\iint_{\mathbb{R}^2}F(x-y)u_\epsilon^0(x)u_\epsilon^0(y)dxdy\,.\end{aligned}$$

It has been proved in [HT09] (Theorem 3.1 if $V''(0)+F''(0)>0$, Corollary 3.6 if $V''(0)+F''(0)=0$ and Theorem 4.5 if $V''(0)+F''(0)<0$ applied with $f_{2l}(x):=x^{2l}$) that the $2l$-th moment of u_ϵ^0 tends towards x_0^{2l} for all $l\in\mathbb{N}$. Since F is a polynomial function, we get the convergence of $\iint_{\mathbb{R}^2}F(x-y)u_\epsilon^0(x)u_\epsilon^0(y)dxdy$ towards $\frac{F(2x_0)}{2}$.
Step 1.2: If $V''(0)+F''(0)\neq 0$, we can apply Lemma A.4 in [HT09] to $f(x):=1$ and $U_\epsilon(x):=V(x)+F*u_\epsilon^0(x)$. This provides

$$\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^0(x)\right)\right]dx=C_\epsilon\exp\left[-\frac{2}{\epsilon}\left(V(x_0)+\frac{F(2x_0)}{2}\right)\right]$$

where the constant C_ϵ verifies $\epsilon\log(C_\epsilon)\longrightarrow 0$ in the small noise limit. We deduce

$$-\frac{\epsilon}{2}\log\left(\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^0(x)\right)\right]dx\right)\longrightarrow V(x_0)+\frac{F(2x_0)}{2}$$

when $\epsilon\longrightarrow 0$. Consequently, we get the following limit:

$$\Upsilon_\epsilon\left(u_\epsilon^0\right)\longrightarrow V(x_0)+\frac{1}{4}F(2x_0)\,.$$

Step 1.3: We assume now $V''(0)+F''(0)=0$. Then $x_0=0$ according to Proposition 3.7 and Remark 3.8 in [HT10b]. Proposition 3.2 and 3.3 in

[HT09] imply

$$0<\liminf_{\epsilon\to0}\epsilon^{\frac{1}{2m_0}}\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^0(x)\right)\right]dx$$
$$\text{and}\quad \limsup_{\epsilon\to0}\epsilon^{\frac{1}{2m_0}}\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^0(x)\right)\right]dx<+\infty$$

where $m_0\in\mathbb{N}^*$ depends only on V and F. We deduce

$$-\frac{\epsilon}{2}\log\left(\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^0(x)\right)\right]dx\right)\longrightarrow0$$

when $\epsilon\longrightarrow0$. Consequently, we get the following limit:

$$\Upsilon_\epsilon\left(u_\epsilon^0\right)\longrightarrow0=V(x_0)+\frac{1}{4}F(2x_0)\,.$$

Step 2: We prove now the result for u_ϵ^+ (the proof is similar for u_ϵ^-).
Step 2.1: We can write $u_\epsilon^+(x)=Z^{-1}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^+(x)\right)\right]$ since it is a stationary measure. Hence

$$\begin{aligned}\Upsilon_\epsilon\left(u_\epsilon^+\right)&=-\frac{\epsilon}{2}\log\left(\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^+(x)\right)\right]dx\right)\\&-\frac{1}{2}\iint_{\mathbb{R}^2}F(x-y)u_\epsilon^+(x)u_\epsilon^+(y)dxdy\,.\end{aligned}$$

It has been proved in [HT09] (Theorem 6.3 applied with $f_l(x):=x^l$) that the l-th moment of u_ϵ^+ tends towards a^l for all $l\in\mathbb{N}$. Since F is a polynomial function, we obtain the convergence of $\iint_{\mathbb{R}^2}F(x-y)u_\epsilon^+(x)u_\epsilon^+(y)dxdy$ towards 0.
Step 2.2: Since the second derivative of the application $x\mapsto V(x)+F(x-a)$ in a is positive, we can apply Lemma A.4 in [HT09] to $f(x):=1$ and $U_\epsilon(x):=V(x)+F*u_\epsilon^+(x)$ (after noting that $U_\epsilon^{(i)}(x)$ tends to $V^{(i)}(x)+F^{(i)}(x-a)$ uniformly on each compact for all $i\in\mathbb{N}$). This provides

$$\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^+(x)\right)\right]dx=C_\epsilon\exp\left[-\frac{2}{\epsilon}V(a)\right]$$

where the constant C_ϵ verifies $\epsilon\log(C_\epsilon)\longrightarrow0$ in the small noise limit. We deduce

$$-\frac{\epsilon}{2}\log\left(\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(x)+F*u_\epsilon^+(x)\right)\right]dx\right)\longrightarrow V(a)$$

when $\epsilon\longrightarrow0$. Consequently, we get the following limit:

$$\Upsilon_\epsilon\left(u_\epsilon^0\right)\longrightarrow V(a)\,.$$

Step 3: We proceed similarly for v_ϵ^+. □

We provide here a useful asymptotic result linked to the Laplace method.

LEMMA A.4. *Let U_k and $U \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ such that for all $i \in \mathbb{N}$, $U_k^{(i)}$ converges uniformly on all compact subset when k tends to $+\infty$. Set a sequence $(\epsilon_k)_k$ which tends to 0 as k tends to $+\infty$. If U has r global minimum locations $A_1 < \cdots < A_r$ and if there exist $R > 0$ and k_c such that $U_k(x) > x^2$ for all $|x| > R$ and $k > k_c$, then, for k big enough, we get:*
1. *U_k has exactly one global minimum location $A_j^{(k)}$ on each interval I_j, where I_j represents the Voronoï cells corresponding to the central points A_j, with $1 \leq j \leq r$.*
2. *$A_j^{(k)}$ tends to A_j when k tends to $+\infty$.*
Furthermore, for all $N \in \mathbb{N}$, there exists $p_1, \cdots, p_r$ which verify $p_1+\cdots+p_r = 1$ and $p_i \geq 0$ for all $1 \leq i \leq r$ such that we can extract a subsequence $\psi(k)$ which satisfies

$$\lim_{k\to+\infty} \frac{\int_{\mathbb{R}} x^l \exp\left[-\frac{2}{\epsilon_{\psi(k)}} U_{\psi(k)}\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon_{\psi(k)}} U_{\psi(k)}\right] dx} = \sum_{j=1}^{r} p_j A_j^l$$

for all $1 \leq l \leq N$.

PROOF. **1.** The first point of the lemma is exactly the one of Lemma A.4 in [HT09].
2. Since $U_k(x) \geq x^2$ for $x \geq R$ and $k > k_c$, we can confine each $A_j^{(k)}$ in a compact subset. Then, the uniform convergence on all the compact subset implies the convergence of $A_j^{(k)}$ towards A_j when k tends to $+\infty$.
3. Set $\rho > 0$ arbitrarily small such that $[A_j - \rho, A_j + \rho] \subset I_j$. For obvious reasons, we can extract a subsequence such that

$$\frac{\int_{A_i-\rho}^{A_i+\rho} \exp\left[-\frac{2}{\epsilon_{\psi(k)}} U_{\psi(k)}(x)\right] dx}{\sum_{j=1}^{r} \int_{A_j-\rho}^{A_j+\rho} \exp\left[-\frac{2}{\epsilon_{\psi(k)}} U_{\psi(k)}(x)\right] dx} \longrightarrow \lambda_i(\rho)$$

with $\lambda_i(\rho) \geq 0$ for all $1 \leq i \leq r$ and $\sum_{j=1}^{r} \lambda_j(\rho) = 1$.
We can note that the generation of the sequence $\psi(k)$ depends on the choice of ρ. Consequently, in the following, we can take ρ arbitrarily small then $\epsilon_{\psi(k)}$ arbitrarily small.
As the r families $(\lambda_j(\rho))$ are bounded, we can extract a subsequence $(\rho_p)_p$ such that $\lambda_j(\rho_p)$ tends to λ_j when p tends to $+\infty$. Furthermore, $\lambda_j \geq 0$ for all $1 \leq j \leq r$ and $\sum_{j=1}^{r} \lambda_j = 1$. For simplicity, we will write ρ (resp. k)

instead of ρ_p (resp. $\psi(k)$).

We introduce the function $\zeta_l^{(k)}(x) := x^l \exp\left[-\frac{2}{\epsilon_k}U_{\epsilon_k}(x)\right]$ for all $l\in\mathbb{N}$. By using classical analysis' inequality, we obtain:

(A.3)
$$\left|\frac{\int_{\mathbb{R}}\zeta_l^{(k)}(x)dx}{\int_{\mathbb{R}}\zeta_0^{(k)}(x)dx}-\sum_{j=1}^r\lambda_jA_j^l\right|\leq\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5$$

with $\mathcal{T}_1(\rho) := \left|\sum_{j=1}^r\left(\lambda_j-\lambda_j(\rho)\right)A_j^l\right|$, $\mathcal{T}_2\left(\rho,R\right) := \rho l R^{l-1}$,

$\mathcal{T}_3\left(\rho,k\right) := \sum_{j=1}^r\frac{\int_{I_j\bigcap[A_j-\rho,A_j+\rho]^c}\zeta_l^{(k)}(x)dx}{\int_{\mathbb{R}}\zeta_0^{(k)}(x)dx}$, $\mathcal{T}_4\left(R,k\right) := 2\frac{\int_R^{+\infty}\zeta_l^{(k)}(x)dx}{\int_{\mathbb{R}}\zeta_0^{(k)}(x)dx}$ and

$\mathcal{T}_5\left(\rho,R,k\right) := \sum_{j=1}^r|A_j|^l\left|\frac{\int_{A_j-\rho}^{A_j+\rho}\zeta_0^{(k)}(x)dx}{\int_{\mathbb{R}}\zeta_0^{(k)}(x)dx}-\lambda_j(\rho)\right|\leq\left(\sum_{j=1}^r|A_j|^l\right)\left(\mathcal{T}_3+\mathcal{T}_4\right)$.

Set $\tau>0$ arbitrarily small. We take $R\geq2$ such that

$$\max_{z\in[A_1-1;A_1+1]}U(z)+2<\frac{R^2}{2}$$

3.1: The convergence of $\lambda_j(\rho)$ towards λ_j implies the existence of $\rho_0>0$ such that for all $\rho<\rho_0$, we have:

$$\mathcal{T}_1(\rho)\leq\frac{\tau}{5} \tag{A.4}$$

for all $1\leq l\leq N$.

3.2: By taking $\rho<\min\left\{\rho_0\,;\,\min_{1\leq l\leq N}\frac{\tau}{5lR^{l-1}}\right\}$, we get:

$$\mathcal{T}_2\left(\rho,R\right)\leq\frac{\tau}{5} \tag{A.5}$$

for all $1\leq l\leq N$.

3.3: We will prove that the third term tends to 0. It is sufficient to prove the following convergence:

$$\frac{\int_{I_j\bigcap[A_j-\rho,A_j+\rho]^c}\zeta_l^{(k)}(x)dx}{\int_{[A_j-\rho,A_j+\rho]}\zeta_0^{(k)}(x)dx}\longrightarrow0$$

for all $1 \leq j \leq r$. Since $I_j \subset [-R, R]$, we have

$$\frac{\int_{I_j \bigcap [A_j-\rho,A_j+\rho]^c} \zeta_l^{(k)}(x)dx}{\int_{[A_j-\rho,A_j+\rho]} \zeta_0^{(k)}(x)dx} \leq R^l \frac{\int_{I_j \bigcap [A_j-\rho,A_j+\rho]^c} \zeta_0^{(k)}(x)dx}{\int_{[A_j-\rho,A_j+\rho]} \zeta_0^{(k)}(x)dx}.$$

Let us prove the convergence towards 0 of the right hand term:

$$\frac{\int_{I_j \bigcap [A_j-\rho,A_j+\rho]^c} \zeta_0^{(k)}(x)dx}{\int_{[A_j-\rho,A_j+\rho]} \zeta_0^{(k)}(x)dx} \leq R^{l+1} \frac{\sup\left\{\zeta_0^{(k)}(z)\ ;\ z \in I_j \bigcap [A_j - \rho, A_j + \rho]^c\right\}}{\int_{A_j-\frac{\rho}{2}}^{A_j+\frac{\rho}{2}} \zeta_0^{(k)}(x)dx}$$

$$\leq \frac{R^{l+1}}{\rho} \frac{\sup\left\{\zeta_0^{(k)}(z)\ ;\ z \in I_j \bigcap [A_j - \rho, A_j + \rho]^c\right\}}{\inf\left\{\zeta_0^{(k)}(z)\ ;\ z \in \left[A_j - \frac{\rho}{2}, A_j + \frac{\rho}{2}\right]\right\}}$$

$$\leq \frac{R^{l+1}}{\rho} \exp\left\{-\frac{2}{\epsilon_k}\left[\inf_{z \in I_j \cap [A_j-\rho,A_j+\rho]^c} U_k(z) - \sup_{z \in \left[A_j-\frac{\rho}{2},A_j+\frac{\rho}{2}\right]} U_k(z)\right]\right\}$$

Let $\rho_1 > 0$ such that for all $\rho < \rho_1$, we have:

$$\min_{1\leq j\leq r}\left\{\inf_{z \in I_j \cap [A_j-\rho,A_j+\rho]^c} U(z) - \sup_{z \in \left[A_j-\frac{\rho}{2},A_j+\frac{\rho}{2}\right]} U(z)\right\} \geq \delta > 0\,.$$

We take $\rho < \min\left\{\rho_0, \rho_1, \min_{1\leq l\leq N} \frac{\tau}{5lR^{l-1}}\right\}$. As U_k converges uniformly towards U on all the compact subset, we deduce that for $k \geq k_0$, we have:

$$\mathcal{T}_3\left(\rho, k\right) \leq \frac{\tau}{5\left(1 + \max_{1\leq l\leq N} \sum_{j=1}^r |A_j|^l\right)} \tag{A.6}$$

for all $1 \leq l \leq N$.

3.4: By using the growth property on U_k then the change of variable $x := \sqrt{\epsilon_k}y$, it yields

$$\int_R^{+\infty} \zeta_l^{(k)}(x) \leq \int_R^{+\infty} x^l \exp\left[-\frac{2}{\epsilon_k}x^2\right] dx \leq C(l) e^{-\frac{R^2}{\epsilon_k}} \epsilon_k^{\frac{l+1}{2}}$$

where $C(l)$ is a constant. We recall that we assume $\max_{z\in[A_1-1;A_1+1]} U(z) + 2 < \frac{R^2}{2}$. Since U_k converges U uniformly on all the compact subset, we have $\max_{z\in[A_1-1;A_1+1]} U_k(z)+1<\frac{R^2}{2}$ for $k\geq k_1$ (independantly of ρ). Consequently:

$$\begin{aligned}\mathcal{T}_4\left(R,k\right) &\leq \frac{2C(l)\epsilon_k^{\frac{l+1}{2}}\exp\left[-\frac{2}{\epsilon_k}\left(\max_{z\in[A_1-1;A_1+1]}U_k(z)+1\right)\right]}{\int_{A_1-1}^{A_1+1}\exp\left[-\frac{2}{\epsilon_k}U_k(z)\right]dx}\\ &\leq C(l)\epsilon_k^{\frac{l+1}{2}}\exp\left[-\frac{2}{\epsilon_k}\right].\end{aligned}$$

For $k\geq k_2$, we have the inequality

$$\epsilon_k^{\frac{l+1}{2}}\exp\left[-\frac{2}{\epsilon_k}\right]\leq\frac{\tau}{5\max_{1\leq l\leq N}C(l)\times\left(1+\max_{1\leq l\leq N}\sum_{j=1}^r|A_j|^l\right)}.$$

By taking $k\geq\max\{k_0,k_1,k_2\}$, we get:

$$\mathcal{T}_4\left(R,k\right)\leq\frac{\tau}{5\left(1+\max_{1\leq l\leq N}\sum_{j=1}^r|A_j|^l\right)} \tag{A.7}$$

for all $1\leq l\leq N$.

3.5 By taking $\rho<\min\left\{\rho_0,\rho_1,\frac{\tau}{5lR^{l-1}}\right\}$ and $k\geq\max\{k_0,k_1,k_2\}$, Inequalities (A.3), (A.4), (A.5), (A.6) and (A.7) provide

$$\left|\frac{\int_{\mathbb{R}}\zeta_l^{(k)}(x)dx}{\int_{\mathbb{R}}\zeta_0^{(k)}(x)dx}-\sum_{j=1}^r\lambda_jA_j^l\right|<\tau$$

for all $1\leq l\leq N$. This achieves the proof. □

Remark A.5. *This lemma seems weaker than Lemma A.4 in [HT09]. However, in Lemma A.4 we do not assume that the second derivative of U is positive in all the global minimum locations.*

It is a great pleasure to thank Samuel Herrmann for his remarks concerning this work. The most of the ideas of this work have been found while I was at the Institut Élie Cartan in Nancy. And so I wanted to mention that I would not have been able to write it without the hospitality I received from the beginning.
Finalement, un très grand merci à Manue et à Sandra pour tout.

References.

[ABC+00] Cécile Ané, Sébastien Blachère, Djalil Chafaï, Pierre Fougères, Ivan Gentil, Florent Malrieu, Cyril Roberto, and Grégory Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.

[AMTU01] Anton Arnold, Peter Markowich, Giuseppe Toscani, and Andreas Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100, 2001.

[BAZ99] G. Ben Arous and O. Zeitouni. Increasing propagation of chaos for mean field models. *Ann. Inst. H. Poincaré Probab. Statist.*, 35(1):85–102, 1999.

[BCCP98] D. Benedetto, E. Caglioti, J. A. Carrillo, and M. Pulvirenti. A non-Maxwellian steady distribution for one-dimensional granular media. *J. Statist. Phys.*, 91(5-6):979–990, 1998.

[BGV07] François Bolley, Arnaud Guillin, and Cédric Villani. Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields*, 137(3-4):541–593, 2007.

[BRTV98] S. Benachour, B. Roynette, D. Talay, and P. Vallois. Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.*, 75(2):173–201, 1998.

[BRV98] S. Benachour, B. Roynette, and P. Vallois. Nonlinear self-stabilizing processes. II. Convergence to invariant probability. *Stochastic Process. Appl.*, 75(2):203–224, 1998.

[CDPS10] Francesca Collet, Paolo Dai Pra, and Elena Sartori. A simple mean field model for social interactions: dynamics, fluctuations, criticality. *J. Stat. Phys.*, 139(5):820–858, 2010.

[CGM08] P. Cattiaux, A. Guillin, and F. Malrieu. Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields*, 140(1-2):19–40, 2008.

[CMV03] José A. Carrillo, Robert J. McCann, and Cédric Villani. Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoamericana*, 19(3):971–1018, 2003.

[CX10] Dan Crisan and Jie Xiong. Approximate McKean-Vlasov representations for a class of SPDEs. *Stochastics*, 82(1-3):53–68, 2010.

[DG87] Donald A. Dawson and Jürgen Gärtner. Large deviations from the McKean-Vlasov limit for weakly interacting diffusions. *Stochastics*, 20(4):247–308, 1987.

[DPdH96] Paolo Dai Pra and Frank den Hollander. McKean-Vlasov limit for interacting random processes in random media. *J. Statist. Phys.*, 84(3-4):735–772, 1996.

[DZ10] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[Fun84] Tadahisa Funaki. A certain class of diffusion processes associated with nonlinear parabolic equations. *Z. Wahrsch. Verw. Gebiete*, 67(3):331–348, 1984.

[HIP08] Samuel Herrmann, Peter Imkeller, and Dierk Peithmann. Large deviations and a Kramers' type law for self-stabilizing diffusions. *Ann. Appl. Probab.*, 18(4):1379–1423, 2008.

[HS87] Richard Holley and Daniel Stroock. Logarithmic Sobolev inequalities and stochastic Ising models. *J. Statist. Phys.*, 46(5-6):1159–1194, 1987.

[HT09] S. Herrmann and J. Tugaut. Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small noise limit. *Prépublications de l'Institut Élie Cartan*, 2009.

[HT10a] S. Herrmann and J. Tugaut. Non-uniqueness of stationary measures for self-stabilizing processes. *Stochastic Process. Appl.*, 120(7):1215–1246, 2010.

[HT10b] S. Herrmann and J. Tugaut. Stationary measures for self-stabilizing processes: asymptotic analysis in the small noise limit. *to appear in Electronic Journal of Probability*, 2010.

[KKR93] Otared Kavian, Gérard Kerkyacharian, and Bernard Roynette. Quelques remarques sur l'ultracontractivité. *J. Funct. Anal.*, 111(1):155–196, 1993.

[Mal01] F. Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.*, 95(1):109–132, 2001.

[Mal03] Florent Malrieu. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.*, 13(2):540–560, 2003.

[McK66] H. P. McKean, Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, 56:1907–1911, 1966.

[McK67] H. P. McKean, Jr. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57. Air Force Office Sci. Res., Arlington, Va., 1967.

[Mél96] Sylvie Méléard. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.

[Oel85] Karl Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 69(2):279–322, 1985.

[OV01] Felix Otto and Cédric Villani. Comment on: "Hypercontractivity of Hamilton-Jacobi equations" by S. G. Bobkov, I. Gentil and M. Ledoux. *J. Math. Pures Appl. (9)*, 80(7):697–700, 2001.

[SV79] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.

[Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[Tam84] Yozo Tamura. On asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(1):195–221, 1984.

[Tam87] Yozo Tamura. Free energy and the convergence of distributions of diffusion processes of McKean type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):443–484, 1987.

[Tug10] J. Tugaut. *Processus autostabilisants dans un paysage multi-puits*. PhD thesis, Université Henri Poincaré, Nancy, 2010.

[Ver06] A. Yu. Veretennikov. On ergodic measures for McKean-Vlasov stochastic equations. In *Monte Carlo and quasi-Monte Carlo methods 2004*, pages 471–486. Springer, Berlin, 2006.

Fakultät für Mathematik
Universität Bielefeld
D-33615 Bielefeld
Germany
E-mail: jtugaut@math.uni-bielefeld.de