

# The compensation approach for walks with small steps in the quarter plane

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## Abstract

This paper is the first application of the compensation approach to counting problems. We discuss how this method can be applied to a general class of walks in the quarter plane  $\mathbb{Z}_+^2$  with a step set that is a subset of  $\{(-1, 1), (-1, 0), (-1, -1), (0, -1), (1, -1)\}$  in the interior of  $\mathbb{Z}_+^2$ . We derive an explicit expression for the counting generating function, which turns out to be meromorphic and nonholonomic, can be easily inverted, and can be used to obtain asymptotic expressions for the counting coefficients.

*Keywords:* lattice walks in the quarter plane, compensation approach, holonomic functions

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## 1 Introduction

In the field of enumerative combinatorics, counting walks on the lattice is among the most classical topics. While counting problems have been largely resolved for unrestricted walks on  $\mathbb{Z}^2$ , walks that are confined to the quarter plane  $\mathbb{Z}_+^2$  still pose considerable challenges. In recent years, much progress has been made, in particular for walks in the quarter plane with small steps, which means that the step set  $\mathcal{S}$  is a subset of  $\{(i, j) : |i|, |j| \leq 1\} \setminus \{(0, 0)\}$ . Bousquet-Mélou and Mishna [6] constructed a thorough classification of these walks. By definition, there are  $2^8$  such walks, but after eliminating trivial cases and exploiting equivalences, it is shown in [6] that there are 79 inherently different walks that need to be studied. Let  $q_{i,j,k}$  denote the number of paths in  $\mathbb{Z}_+^2$  starting from  $(0, 0)$  and ending in  $(i, j)$  at time  $k$ , and define the counting generating function (CGF) as

$$Q(x, y; z) = \sum_{i,j,k=0}^{\infty} q_{i,j,k} x^i y^j z^k. \quad (1)$$

There are then two key challenges:

- (i) Finding an explicit expression for  $Q(x, y; z)$ .

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- (ii) Determining the nature of  $Q(x, y; z)$ : Is it rational, algebraic, holonomic (solution of a linear differential equation with polynomial coefficients) or nonholonomic?

The common approach to address these challenges is to start from a functional equation for the CGF, which for the walks with small steps takes the form (see [6])

$$\tilde{K}(x, y; z)Q(x, y; z) = A(x)Q(x, 0; z) + B(y)Q(0, y; z) + \delta Q(0, 0; z) - xy/z, \quad (2)$$

where the kernel  $\tilde{K}$  and the functions  $A, B, C$  are polynomials of degree two in  $x$  and/or  $y$ , while  $\delta$  is a constant. For  $z = 1/|\mathcal{S}|$ , (2) belongs to the generic class of functional equations (arising in the probabilistic context of random walks) studied and solved in the book [7]. For general values of  $z$ , the analysis of (2) for the 79 above-mentioned walks has been carried out in [10, 13], which settled Challenge (i).

In order to describe the known results regarding Challenge (ii), it is worth to first define the *group of the walk*, a notion introduced by Malyshev [11]. This is the group of birational transformations  $W = \langle \xi, \eta \rangle$ , with

$$\xi(x, y) = \left( x, \frac{1}{y} \frac{\sum_{(i,-1) \in \mathcal{S}} x^i}{\sum_{(i,+1) \in \mathcal{S}} x^i} \right), \quad \eta(x, y) = \left( \frac{1}{x} \frac{\sum_{(-1,j) \in \mathcal{S}} y^j}{\sum_{(+1,j) \in \mathcal{S}} y^j}, y \right), \quad (3)$$

which leaves invariant the function  $\sum_{(i,j) \in \mathcal{S}} x^i y^j$ . Clearly,  $\xi \circ \xi = \eta \circ \eta = \text{id}$ , and  $W$  is a dihedral group of even order larger than or equal to four.

Challenge (ii) is resolved for the 23 walks that have a finite group. The nature of the CGF was determined in [6] for 22 of these 23 walks: 19 walks turn out to have a CGF that is holonomic but nonalgebraic, while 3 walks have a CGF that is algebraic. For a 23rd walk, defined by  $\mathcal{S} = \{(-1, 0), (-1, -1), (1, 0), (1, 1)\}$  and known as *Gessel's walk*, it was proven in [5] that  $Q(x, y; z)$  is algebraic. Alternative proofs for the nature of the CGF for these 23 walks were given in [8].

For the remaining 56 walks, which all have an infinite group, not much is known. In [12] it was shown that 2 of them have a nonholonomic CGF. Bousquet-Mélou and Mishna [6] have conjectured that the 54 other walks also have a nonholonomic CGF. Partial evidence was provided in [13], in which it is shown that a certain conformal gluing function, which is closely related to the CGF, is nonholonomic for those 56 cases.

In this paper we consider walks on  $\mathbb{Z}_+^2$  with small steps that do not fall into the class considered in [6]. The classification in [6] builds on the assumption that the steps on the boundaries are the same steps (if possible) as those in the interior. However, when the behavior on the boundaries is allowed to be different, we have many more models to consider. In this paper we first consider the walk in Figure 1, with steps taken from  $\mathcal{S} = \{(-1, 1), (-1, -1), (1, -1)\}$  in the interior,  $\mathcal{S}_H = \{(-1, 1), (-1, 0), (1, 0)\}$  on the horizontal boundary,  $\mathcal{S}_V = \{(0, 1), (0, -1), (1, -1)\}$  on the vertical boundary, and  $\mathcal{S}_0 = \{(0, 1), (1, 0)\}$  in state  $(0, 0)$ .

Notice that the step set  $\mathcal{S} = \{(-1, 1), (-1, -1), (1, -1)\}$  in the framework of [6] would give a trivial walk, since the walk could never depart from state  $(0, 0)$ . However, by choosing the steps on the boundaries as in  $\mathcal{S}_H$ ,  $\mathcal{S}_V$  and  $\mathcal{S}_0$ , it becomes possible to start walking, and we have a rather intricate counting problem on our hands.

It turns out that our walk has an infinite group. To see this, first observe that the interior step set in Figure 1 and the one represented in Figure 2 have isomorphic groups ([6, Lemma 2] says that two step sets differing from one of the eight symmetries of the square necessary have isomorphic

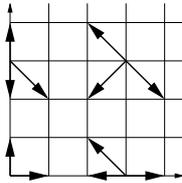


Figure 1: The jumps of the walk

groups), then note that the group associated with the step set in Figure 2 is infinite ([6, Section 3.1]).

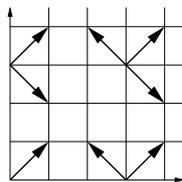


Figure 2: Walk considered in [12]

Since our walk has an infinite group, the approach in [6] cannot be applied. The methods developed in [12] also fail to work. Indeed, the main tool used there is an expression of  $Q(x, 0; z)$  and  $Q(0, y; z)$  as series involving the iterates of the roots of the kernel. While these series are convergent when the transitions  $(-1, 0)$ ,  $(-1, -1)$ ,  $(0, -1)$  are absent, they become strongly divergent in our case, see Chapter 6 of [7]. The approach via boundary value problems of [7] seems to apply, but this is more cumbersome than in [10, 13]. Indeed, contrary to the 79 walks studied there, for which in (2),  $A, B$  depend on one variable and  $\delta$  is constant, these quantities are now polynomials in two variables, since the functional equation (2) for our walk becomes

$$K(x, y; z)Q(x, y; z) = [1 + x^2 - x^2y - y]Q(x, 0; z) + [1 + y^2 - xy^2 - x]Q(0, y; z) \quad (4)$$

$$+ [x + y - 1]Q(0, 0; z) - xy/z \quad (5)$$

with the kernel

$$K(x, y; z) = 1 + x^2 + y^2 - xy/z. \quad (6)$$

We shall derive an explicit expression for  $Q(x, y; z)$ , which satisfies (4), and turns out to be a meromorphic function of  $x$  and  $y$  with infinitely many poles. This implies that  $Q(x, y; z)$  is a nonholonomic function (see e.g. [9]), which again confirms the intimate relation between infinite groups and nonholonomy.

The technique we are using is the so-called *compensation approach*. This technique has been developed in a series of papers [1, 3, 4] in the probabilistic context of *random walks*; see [2] for an overview. It does not aim directly for a solution of the CGF, but rather tries to find a solution for its coefficients

$$q_{i,j}(z) = \sum_{k=0}^{\infty} q_{i,j,k} z^k. \quad (7)$$

These coefficients satisfy certain recursion relations, which differ depending on whether the state  $(i, j)$  lies on the boundary or not. The idea is then to construct a linear combination of product forms  $\alpha^i \beta^j$ , for pairs  $(\alpha, \beta)$  such that

$$K(1/\alpha, 1/\beta; z) = 0. \quad (8)$$

By choosing only pairs  $(\alpha, \beta)$  for which (8) is satisfied, the recursion relations for  $q_{i,j}(z)$  in the interior of the quarter plane are satisfied by any linear combination of product forms  $\alpha^i \beta^j$  by virtue of the linearity of the recursion relations. The product forms have to be chosen such that the recursion relations on the boundaries are satisfied as well. As it turns out, this can be done by alternately compensating for the errors on the two boundaries, which eventually leads to an infinite series of product forms.

This paper is organized as follows. In Section 2 we obtain an explicit expression for the generating function  $Q(x, y; z)$  by applying the compensation approach. In Section 3 we present an efficient procedure for deducing the numbers of walks  $q_{i,j,k}$ . In Section 4 we derive an asymptotic expression for the coefficients  $q_{i,j,k}$  for large values of  $k$ , using the technique of singularity analysis. Because this paper is the first application of the compensation approach to counting problems, we also shortly discuss in Section 5 for which class of walks this compensation approach might work.

## 2 The compensation approach

We start from the following recursion relations:

$$q_{i,j,k+1} = q_{i-1,j+1,k} + q_{i+1,j-1,k} + q_{i+1,j+1,k}, \quad i, j > 0, \quad k \geq 0, \quad (9)$$

$$q_{i,0,k+1} = q_{i-1,1,k} + q_{i+1,1,k} + q_{i-1,0,k} + q_{i+1,0,k}, \quad i > 0, \quad k \geq 0, \quad (10)$$

$$q_{0,j,k+1} = q_{1,j-1,k} + q_{1,j+1,k} + q_{0,j-1,k} + q_{0,j+1,k}, \quad j > 0, \quad k \geq 0, \quad (11)$$

$$q_{0,0,k+1} = q_{0,1,k} + q_{1,1,k} + q_{1,0,k}, \quad k \geq 0. \quad (12)$$

Since  $q_{i,j,0} = 0$  if  $i + j > 0$  and  $q_{0,0,0} = 1$ , these relations uniquely determine all the counting numbers  $q_{i,j,k}$ . Multiplying the relations (9)–(12) by  $z^k$  and summing w.r.t.  $k \geq 0$  leads to (with the generating functions  $q_{i,j} \equiv q_{i,j}(z)$  defined in (7))

$$q_{i,j}/z = q_{i-1,j+1} + q_{i+1,j-1} + q_{i+1,j+1}, \quad i, j > 0, \quad (13)$$

$$q_{i,0}/z = q_{i-1,1} + q_{i+1,1} + q_{i-1,0} + q_{i+1,0}, \quad i > 0, \quad (14)$$

$$q_{0,j}/z = q_{1,j-1} + q_{1,j+1} + q_{0,j-1} + q_{0,j+1}, \quad j > 0, \quad (15)$$

$$q_{0,0}/z = 1/z + q_{0,1} + q_{1,1} + q_{1,0}. \quad (16)$$

Notice that

$$\sum_{i,j=0}^{\infty} q_{i,j,k} \leq 3^k, \quad (17)$$

because there are at most  $3^k$  paths of length  $k$ . This clearly implies that  $q_{i,j,k} \leq 3^k$ , so that  $q_{i,j}(z)$  is analytic at least for  $|z| < 1/3$ .

**Lemma 1.** *Equations (13)–(16) have a unique solution that is analytic in  $|z| < 1/3$ .*

*Proof.* The function  $q_{i,j}(z)$  is analytic in  $|z| < 1/3$ , and can therefore be written as a Taylor series about  $z = 0$ . Substituting these Taylor series into Equations (13)–(16), and equating coefficients of  $z^k$ , yields Equations (9)–(12). The latter obviously have a unique solution for the coefficients  $q_{i,j,k}$ , because these counting numbers can be determined recursively using  $q_{0,0,0} = 1$ .  $\square$

In order to find the unique solution for  $q_{i,j}$ , we shall employ the compensation approach, which consists of three consecutive steps:

- Characterize all product forms  $\alpha^i \beta^j$  for which the inner equations (13) are satisfied, and construct linear combinations of these product forms, which in addition to being formal solutions to (13), also satisfy (14) and (15).
- Prove the convergence of these formal solutions.
- Determine the complete unique solution to  $q_{i,j}(z)$  by taking into account the boundary condition (16).

## 2.1 Linear combinations of product forms

Substituting the product form  $\alpha^i \beta^j$  into the inner equations (13), and dividing by common powers, yields

$$\alpha\beta/z = \alpha^2 + \beta^2 + \alpha^2\beta^2. \quad (18)$$

Hence, a function  $\alpha^i \beta^j$  is a solution of (13) if and only if (18) is satisfied. Figure 3 depicts the curve (18) in  $\mathbb{R}_+^2$  for  $z = 1/4$ .

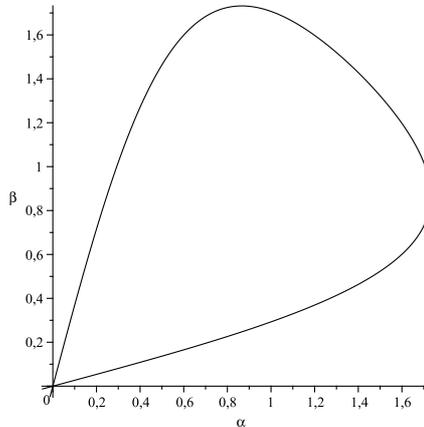


Figure 3: For  $z = 1/4$ , the curve (18) in  $\mathbb{R}_+^2$

Now we construct a linear combination of the product forms introduced above, which will give a formal solution to the balance equations (13)–(15). The first term of this combination, say  $\alpha_0^i \beta_0^j$ , has to satisfy both (13) and (14). In other words, the pair  $(\alpha_0, \beta_0)$  has to satisfy (18) as well as

$$\alpha\beta/z = \beta^2 + \alpha^2\beta^2 + \beta + \alpha^2\beta. \quad (19)$$

The motivation to start with a term satisfying both (18) and (19) will be explained at the end of this subsection (see Remark 3).

**Lemma 2.** For each  $|z| < 1/3$ , there exists exactly one product form  $\alpha^i \beta^j$ , with  $0 < |\alpha| < 1$  and  $0 < |\beta| < 1$ , that satisfies both (13) and (14). The factors  $\alpha$  and  $\beta$  of this product form are given by

$$\alpha_0 = \frac{1 - \sqrt{1 - 8z^2}}{4z}, \quad \beta_0 = \frac{\alpha_0^2}{1 + \alpha_0^2}. \quad (20)$$

*Proof.* If  $\beta = 0$ , then (18) gives  $\alpha = 0$ . If  $\beta \neq 0$ , then (19) yields

$$\alpha/z = \beta + \alpha^2\beta + 1 + \alpha^2 = (1 + \beta)(1 + \alpha^2). \quad (21)$$

On the other hand, subtracting (19) from (18) gives  $\beta(1 + \alpha^2) = \alpha^2$ . From the latter equality we get  $(1 + \beta)(1 + \alpha^2) = 1 + 2\alpha^2$ , and together with (21) this gives  $\alpha/z = 1 + 2\alpha^2$ . The last equation has the two solutions  $(1 \pm \sqrt{1 - 8z^2})/(4z)$ . For each  $|z| < 1/3$ , it follows from Rouché's theorem, applied to the unit circle, that one of them,  $\alpha_0$ , lies inside the unit circle, and the other one lies outside the unit circle.  $\square$

The function  $\alpha_0^i \beta_0^j$  thus satisfies (13) and (14), but it fails to satisfy (15). That is, if  $\alpha^i \beta^j$  is a solution to (15), then  $(\alpha, \beta)$  should satisfy

$$\alpha\beta/z = \alpha^2 + \alpha^2\beta^2 + \alpha + \alpha\beta^2, \quad (22)$$

and  $(\alpha_0, \beta_0)$  is certainly not a solution to (22).

Now we start adding compensation terms. We consider  $c_0\alpha_0^i\beta_0^j + d_1\alpha^i\beta^j$ , where  $(\alpha, \beta)$  satisfies (18) and is such that  $c_0\alpha_0^i\beta_0^j + d_1\alpha^i\beta^j$  satisfies (15) on the vertical boundary. Notice that with (15), we are forced to take  $\beta = \beta_0$ . Then, thanks to (18), we get that  $\alpha$  is the companion root of  $\alpha_0$  in Equation (18). Denote this new root by  $\alpha_1$ .

Hence,  $\alpha_0$  and  $\alpha_1$  are the two roots of Equation (18), where  $\beta$  is replaced by  $\beta_0$ . In other words,  $\alpha_0$  and  $\alpha_1$  satisfy

$$(1 + \beta_0^2)\alpha^2 - (\beta_0/z)\alpha + \beta_0^2 = 0. \quad (23)$$

Due to the root-coefficient relationships, we obtain

$$\frac{\beta_0/z}{1 + \beta_0^2} = \alpha_0 + \alpha_1. \quad (24)$$

We now determine  $d_1$  in terms of  $c_0$ . Note that  $c_0\alpha_0^i\beta_0^j + d_1\alpha_1^i\beta_0^j$  is a solution to (15) if and only if

$$(\beta_0/z)(c_0 + d_1) = (\alpha_0c_0 + \alpha_1d_1)(1 + \beta_0^2) + (c_0 + d_1)(1 + \beta_0^2). \quad (25)$$

The latter identity can be rewritten as

$$\frac{\beta_0/z}{1 + \beta_0^2}(c_0 + d_1) = (\alpha_0c_0 + \alpha_1d_1) + (c_0 + d_1), \quad (26)$$

and thus, using (24),

$$d_1 = -\frac{1 - \alpha_1}{1 - \alpha_0}c_0. \quad (27)$$

The solution  $c_0\alpha_0^i\beta_0^j + d_1\alpha_1^i\beta_0^j$  after one compensation step satisfies (13) and (15) for the interior

and the vertical boundary. However, the compensation term  $d_1\alpha_1^i\beta_0^j$  has generated a new error at the horizontal boundary. To compensate for this, we must add another compensation term, and so on. In this way, the compensation approach can be continued, which eventually leads to

$$x_{i,j} = c_0\alpha_0^i\beta_0^j + d_1\alpha_1^i\beta_0^j + c_1\alpha_1^i\beta_1^j + d_2\alpha_2^i\beta_1^j + \dots, \quad (28)$$

where, by construction and (18), for all  $k \geq 0$  we have

$$\beta_k = f(\alpha_k), \quad \alpha_{k+1} = f(\beta_k), \quad f(t) = \frac{1 - \sqrt{1 - 4z^2(1+t^2)}}{2z(1+t^2)}t, \quad (29)$$

see Figure 4, and

$$d_{k+1} = -\frac{1 - \alpha_{k+1}}{1 - \alpha_k}c_k, \quad c_{k+1} = \frac{1 - \beta_{k+1}}{1 - \beta_k}d_{k+1}, \quad \forall k \geq 0. \quad (30)$$

Notice that (28) is the formal solution to (13)–(16), except in the states  $(0, 1)$ ,  $(0, 0)$  and  $(1, 0)$ .

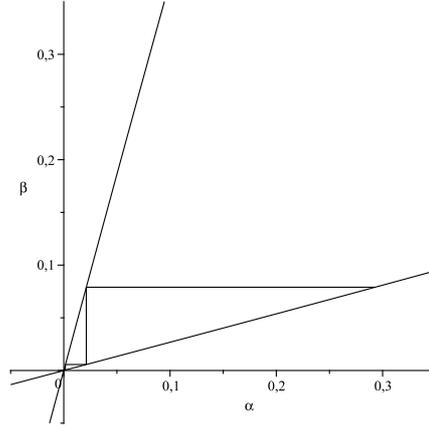


Figure 4: For  $z = 1/4$ , the curve (18) near 0 and the construction of the product forms

An easy calculation starting from (30) yields

$$d_{k+1} = -\frac{(1 - \alpha_{k+1})(1 - \beta_k)}{(1 - \alpha_0)(1 - \beta_0)}c_0, \quad c_{k+1} = -\frac{(1 - \beta_{k+1})(1 - \alpha_{k+1})}{(1 - \alpha_0)(1 - \beta_0)}c_0, \quad \forall k \geq 0, \quad (31)$$

so that choosing (arbitrarily)  $c_0 = (1 - \alpha_0)(1 - \beta_0)$  finally gives

$$x_{i,j} = \sum_{k=0}^{\infty} (1 - \beta_k)\beta_k^j [(1 - \alpha_k)\alpha_k^i - (1 - \alpha_{k+1})\alpha_{k+1}^i]. \quad (32)$$

By symmetry, we also obtain that  $x_{j,i}$  is a formal solution to (13)–(16), except in  $(0, 1)$ ,  $(0, 0)$  and  $(1, 0)$ .

**Remark 3.** The approach outlined above is initialized with a term satisfying both (18) and (19). Alternatively, we could also start with an arbitrarily chosen term with  $\alpha_0$  and  $\beta_0$  satisfying (18)

only. This term would violate (19) as well as (22), and therefore generate two sequences of terms, one starting with compensation of (19) and the other with (22). It is readily seen that in one of the two sequences, at least one of the parameters  $\alpha$  or  $\beta$  will exceed one (before converging to zero). Hence, the resulting formal solution would fail to be convergent, cf. Proposition 4.

## 2.2 Convergence of the formal solutions

**Proposition 4.** *For each  $|z| < 1/3$ , the  $\alpha_k$  and  $\beta_k$  appearing in (28) satisfy the following properties:*

- (i)  $1/2 > |\alpha_0| > |\beta_0| > |\alpha_1| > |\beta_1| > \dots$ .
- (ii)  $0 \leq |\alpha_k| \leq 1/2^{2k+1}$  and  $0 \leq |\beta_k| \leq 1/2^{2k+2}$ .

*Proof.* Let us show that for  $f$  defined in (29),

$$|f(t)| \leq \frac{|t|}{2}, \quad \forall |t| \leq 1, \quad \forall |z| < 1/3. \quad (33)$$

But before proving (33), notice that Proposition 4 is an immediate consequence of (29) and (33). In order to show (33), we first note that  $f$  satisfies the algebraic relationship  $(1+t^2)f(t)^2 - tf(t)/z + t^2 = 0$ , see (18) and (29). As a consequence, the function  $f(t)/t$  is such that (with  $r = f(t)/t$ )

$$(1+t^2)r^2 - r/z + 1 = 0. \quad (34)$$

For  $|t| \leq 1$ ,  $|z| < 1/3$ , and for  $r > 0$ ,

$$|(1+t^2)r^2| \leq (1+|t|^2)r^2 \leq 2r^2 \quad (35)$$

and

$$|-r/z + 1| \geq r/|z| - 1 > 3r - 1. \quad (36)$$

Since  $2r^2 \leq 3r - 1$  if and only if  $r \in [1/2, 1]$ , it follows from Rouché's theorem, applied to the circle with radius  $1/2$ , that (34) has one unique solution  $r = f(t)/t$  with  $|r| < 1/2$ .  $\square$

**Proposition 5.** *For all  $i, j \geq 0$ , the power series  $x_{i,j}$  and  $x_{j,i}$  are convergent for  $|z| < 1/3$ . Also, for each  $|z| < 1/3$ ,*

$$\sum_{i,j=0}^{\infty} |x_{i,j}(z)| = \sum_{i,j=0}^{\infty} |x_{j,i}(z)| < \infty. \quad (37)$$

*Proof.* Let us show that for all  $i, j \geq 0$ , the following upper bound holds:

$$|x_{i,j}| \leq 3 \sum_{k=0}^{\infty} 1/2^{(2k+1)i + (2k+2)j}, \quad (38)$$

Proposition 5 will then follow immediately.

Consider the case  $i \geq 1$  (the case  $i = 0$  follows in a similar fashion). With (32) we get

$$|x_{i,j}| \leq \sum_{k=0}^{\infty} [1 - \beta_k] |\beta_k|^j [1 - \alpha_k] |\alpha_k|^i + [1 - \alpha_{k+1}] |\alpha_{k+1}|^i. \quad (39)$$

Using that  $|1 - \beta_k|$ ,  $|1 - \alpha_k|$  and  $|1 - \alpha_{k+1}|$  are less than  $3/2$ , see (i) of Proposition 4, and finally applying (ii) of Proposition 4 gives (38).  $\square$

### 2.3 Determining the unique solution

We now determine the generating functions  $q_{i,j}$  in terms of the linear combinations  $cx_{i,j} + \tilde{c}x_{j,i}$  for all states  $(i, j) \in \mathbb{Z}_+^2$ , except state  $(0, 0)$ . First, by symmetry, we have  $c = \tilde{c}$ , and we are thus left with the task of determining  $c$  and  $q_{0,0}$ . Define  $\hat{x}_{i,j} = x_{i,j} + x_{j,i}$ .

**Lemma 6.**

$$c = \frac{1}{1 - 2z + z\hat{x}_{0,0}} \quad (40)$$

and

$$q_{0,0} = \frac{1 + \hat{x}_{0,0}}{1 - 2z + z\hat{x}_{0,0}}. \quad (41)$$

*Proof.* By using not only (16), but also (14) for  $i = 1$  (or equivalently (15) for  $j = 1$ ), we obtain that

$$c = \frac{z}{\hat{x}_{1,0} - z[\hat{x}_{0,1} + \hat{x}_{2,1} + \hat{x}_{2,0}] - z^2[\hat{x}_{1,0} + \hat{x}_{0,1} + \hat{x}_{1,1}]} \quad (42)$$

and

$$q_{0,0} = 1 + zc[\hat{x}_{0,1} + \hat{x}_{1,1} + \hat{x}_{1,0}]. \quad (43)$$

In addition, tedious calculations starting from (32) give (recall that  $\hat{x}_{i,j}$  is not a solution to (13)–(16) at  $(0, 1)$ ,  $(0, 0)$  and  $(1, 0)$ )

$$\hat{x}_{1,0}/z - [\hat{x}_{0,1} + \hat{x}_{2,1} + \hat{x}_{2,0} + \hat{x}_{0,0}] = 1, \quad (1/z - 1)\hat{x}_{0,0} + 2 - [\hat{x}_{0,1} + \hat{x}_{1,1} + \hat{x}_{1,0}] = 0. \quad (44)$$

With the equations in (44) the denominator of (42) can be written as  $z[1 - 2z + z\hat{x}_{0,0}]$ , which completes the proof.  $\square$

We now summarize our results in the next three propositions.

**Proposition 7.** *The expressions for  $q_{0,0}$  in (41) and*

$$q_{i,j} = c\hat{x}_{i,j}, \quad i + j \geq 1 \quad (45)$$

*are the unique solutions to Equations (13)–(16) that are analytic in  $|z| < 1/3$ .*

*Proof.* The functions defined in (41) and (45) are analytic in  $|z| < 1/3$  and are solutions to Equations (13)–(16) by construction, so that we conclude by Lemma 1.  $\square$

**Proposition 8.** *For all  $|z| < 1/3$  and  $|x|, |y| \leq 1$ , we have*

$$Q(x, y; z) = \sum_{i,j=0}^{\infty} q_{i,j} x^i y^j \quad (46)$$

$$= c + c \sum_{k=0}^{\infty} \left( \frac{1 - \beta_k}{1 - \beta_k y} \left[ \frac{1 - \alpha_k}{1 - \alpha_k x} - \frac{1 - \alpha_{k+1}}{1 - \alpha_{k+1} x} \right] + \frac{1 - \beta_k}{1 - \beta_k x} \left[ \frac{1 - \alpha_k}{1 - \alpha_k y} - \frac{1 - \alpha_{k+1}}{1 - \alpha_{k+1} y} \right] \right), \quad (47)$$

with  $c$  as in (40) and  $q_{0,0}$  as in (41).

*Proof.* This is immediate from (32), (40), (41) and (45).  $\square$

**Proposition 9.** For all  $|z| < 1/3$ , the three functions  $\sum_{i,j=0}^{\infty} q_{i,j}x^i y^j$ ,  $\sum_{i=0}^{\infty} q_{i,0}x^i$  and  $\sum_{j=0}^{\infty} q_{0,j}y^j$  have infinitely many poles and are therefore nonholonomic.

*Proof.* This is a direct consequence of Propositions 4 and 8.  $\square$

### 3 Retrieving coefficients

Now that we have an explicit expression for  $q_{i,j}$ , we briefly present an efficient procedure for calculating its coefficients, i.e. the numbers of walks  $q_{i,j,k}$ .

To compute  $q_{i,j}$  we need to calculate in principle an infinite series. However, we shall prove that if we are only interested in a finite number of coefficients  $q_{i,j,k}$ , then it is enough to take into account a finite number of  $\alpha_k$  and  $\beta_k$ .

**Lemma 10.** For all  $k \geq 0$ , we have  $\alpha_k = z^{2k+1}\widehat{\alpha}_k(z)$  and  $\beta_k = z^{2k+2}\widehat{\beta}_k(z)$ , where  $\widehat{\alpha}_k$  and  $\widehat{\beta}_k$  are holomorphic at  $z = 0$  and such that  $\widehat{\alpha}_k(0) = \widehat{\beta}_k(0) = 1$ .

*Proof.* First note that for  $f$  defined in (29), all  $p \geq 0$  and all real numbers  $s_1, s_2, \dots$ ,

$$f(z^p[1 + s_1z + s_2z^2 + \dots]) = z^{p+1}[1 + s_1z + (1 + s_2)z^2 + \dots]. \quad (48)$$

Because (20) yields  $\alpha_0 = z + 2z^3 + \dots$ , the proof is completed via (29).  $\square$

For  $k \geq 0$ , denote by  $k \vee 1$  the maximum of  $k$  and 1. Define

$$N_p^{i,j} = 1 + \lfloor \frac{1}{4} \max\{p - (i \vee 1 + 2(j \vee 1)), p - (2(i \vee 1) + j \vee 1)\} \rfloor. \quad (49)$$

**Proposition 11.** For any  $i, j \geq 0$ , the first  $p$  coefficients of  $q_{i,j}$  only require the series expansions of order  $p$  of  $\alpha_0, \beta_0, \dots, \alpha_{N_p^{i,j}}, \beta_{N_p^{i,j}}$ .

*Proof.* In order to obtain the series expansion of  $q_{0,0}$  of order  $p$ , it is enough to know the series expansion of  $\widehat{x}_{0,0}$  of order  $p$ , see (41). From (32) we obtain

$$\widehat{x}_{0,0} = 2 \sum_{k=0}^{\infty} (1 - \beta_k)(\alpha_{k+1} - \alpha_k) = -2\alpha_0 + 2 \sum_{k=0}^{\infty} \beta_k(\alpha_k - \alpha_{k+1}). \quad (50)$$

With Lemma 10,  $\beta_k\alpha_k = O(z^{4k+3})$  and  $\beta_k\alpha_{k+1} = O(z^{4k+5})$ , so that in order to obtain the series expansion of  $\widehat{x}_{0,0}$  of order  $p$ , it is enough to consider in (50) the values of  $k$  such that  $4k+3 \leq p$ . In other words, we have to deal with  $\alpha_0, \beta_0, \dots, \alpha_k, \beta_k, \alpha_{k+1}$  for  $4k+3 \leq p$ . Since  $N_p^{0,0} = 1 + \lfloor (p-3)/4 \rfloor$ , see (49), Proposition 11 is shown for  $i = j = 0$ . The proof for other values of  $i$  and  $j$  is similar and we omit it.  $\square$

As an application of Proposition 11, let us find the numbers  $q_{0,0,k}$  for  $k \in \{0, \dots, 10\}$ . Taking  $i = j = 0$ , we have  $N_{10}^{0,0} = 2$ ; we then calculate the series expansions of order 10 of  $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ . After using (41) and (50), we obtain the numbers:

$$1, 0, 2, 2, 10, 16, 64, 126, 454, 1004, 3404. \quad (51)$$

## 4 Asymptotic analysis

In this section we derive an asymptotic expression for the coefficients  $q_{i,j,k}$  for large values of  $k$ , using the technique of *singularity analysis* (see Flajolet and Sedgewick [9] for an elaborate exposition). This requires the investigation of the function  $q_{i,j} = q_{i,j}(z)$  near its dominant singularity (closest to the origin) in the  $z$ -plane.

The singularities of  $q_{0,0}$  and  $q_{i,j}$  in (41) and (45) are given by the singularities of  $\widehat{x}_{i,j}$  and the zeros of the denominator of  $c$  in (40). Since  $\widehat{x}_{i,j}$  is constructed from the functions  $\alpha_0, \beta_0, \alpha_1, \beta_1, \dots$ , and since all these functions follow from the iterative scheme (29), i.e. from  $\alpha_0$ , it is clear that the singularities of  $\widehat{x}_{i,j}$  are in fact the algebraic singularities of  $\alpha_0$ , namely (see (20))

$$z = \pm \frac{1}{\sqrt{8}}. \quad (52)$$

Denote the denominator of  $c$  in (40) by

$$h(z) = 1 - 2z + z\widehat{x}_{0,0}. \quad (53)$$

Let  $\rho$  denote the dominant singularity of  $c$ .

**Lemma 12.**  $\rho$  is the unique root of the function  $h(z)$  in the interval  $(1/3, 1/\sqrt{8})$ .

The proof of Lemma 12 is presented in Appendix A. It seems hard to find a closed-form expression for  $\rho$ , but in Corollary 18 we prove that  $\rho \in [0.34499975, 0.34499976]$ . Here is then our main result on the asymptotic behavior of large counting numbers:

**Proposition 13.** The exact asymptotic of the numbers of walks  $q_{i,j,k}$  as  $k \rightarrow \infty$  are

$$q_{i,j,k} \sim C_{i,j} \rho^{-k}, \quad (54)$$

with  $\rho$  as defined in Lemma 12,

$$C_{0,0} = \frac{3\rho - 1}{-\rho^2 h'(\rho)}, \quad (55)$$

and

$$C_{i,j} = \frac{\widehat{x}_{i,j}(\rho)}{-\rho h'(\rho)}, \quad i + j \geq 1. \quad (56)$$

*Proof.* Consider first the case  $i = j = 0$ . Thanks to (41), (53) and Lemma 12, we obtain that  $q_{0,0}$  has a pole at  $\rho$ , and is holomorphic within the domain

$$\{z \in \mathbb{C} : |z| < (1 + \epsilon)\rho\} \setminus [\rho, (1 + \epsilon)\rho] \quad (57)$$

for any  $\epsilon > 0$  small enough. Moreover, the pole of  $q_{0,0}$  at  $\rho$  is of order one, as we show now. For this it is sufficient to prove that  $1 + \widehat{x}_{0,0}(\rho) \neq 0$  and that  $h'(\rho) \neq 0$ , see again (41) and (53). In this perspective, note that with (53) and Lemma 12 we have  $h(\rho) = 1 - 2\rho + \rho\widehat{x}_{0,0}(\rho) = 0$ , so that  $1 + \widehat{x}_{0,0}(\rho) = (3\rho - 1)/\rho$ , which is positive by Lemma 12. On the other hand, it is a consequence of Lemma 19 that  $h'(\rho) \neq 0$ . In particular, the behavior of  $q_{0,0}$  near  $\rho$  is given by

$$q_{0,0} = \frac{1 + \widehat{x}_{0,0}(\rho)}{h'(\rho)(z - \rho)[1 + O(z - \rho)]} = \frac{3\rho - 1}{-\rho^2 h'(\rho)(1 - z/\rho)} [1 + O(z - \rho)]. \quad (58)$$

The holomorphy of  $q_{0,0}$  within the domain (57) and the behavior (58) of  $q_{0,0}$  near  $\rho$  immediately give the asymptotic (54) for  $i = j = 0$  (see e.g. [9]). The proof for other values of  $i$  and  $j$  is similar and we omit it.  $\square$

To conclude Section 4, let us illustrate Proposition 13 with the following table, obtained by approximating  $\rho$  by 0.34499975 and  $C_{0,0}$  by 0.0531, see Corollary 18 and Lemma 19 in Appendix A.

Value of $k$	Exact value of $q_{0,0,k}$ (see Section 3)	Approximation of $q_{0,0,k}$ (see Proposition 13)	Ratio
10	$3.404 \cdot 10^3$	$2.222 \cdot 10^3$	0.653
20	$1.106 \cdot 10^8$	$9.305 \cdot 10^7$	0.840
30	$4.254 \cdot 10^{12}$	$3.895 \cdot 10^{12}$	0.915
40	$1.714 \cdot 10^{17}$	$1.630 \cdot 10^{17}$	0.951
50	$7.037 \cdot 10^{21}$	$6.825 \cdot 10^{21}$	0.969
60	$2.913 \cdot 10^{26}$	$2.857 \cdot 10^{26}$	0.980
70	$1.211 \cdot 10^{31}$	$1.196 \cdot 10^{31}$	0.987
80	$5.051 \cdot 10^{35}$	$5.007 \cdot 10^{35}$	0.991
90	$2.109 \cdot 10^{40}$	$2.096 \cdot 10^{40}$	0.993
100	$8.814 \cdot 10^{44}$	$8.774 \cdot 10^{44}$	0.995

## 5 Discussion

### 5.1 A wider range of applicability

The compensation approach for our counting problem leads to an exact expression for the counting generation function  $Q(x, y; z)$ . A detailed exposition of the compensation approach can be found in [1, 3, 4], in which it has been shown to work for two-dimensional random walks on the lattice of the first quadrant that obey the following conditions:

- Step size: Only transitions to neighboring states.
- Forbidden steps: No transitions from interior states to the North, North-East, and East.
- Homogeneity: The same transitions occur for all interior points, and similarly for all points on the horizontal boundary, and for all points on the vertical boundary.

Although the theory has been developed for the bivariate transform of stationary distributions of recurrent two-dimensional random walks, the results in this paper suggest that similar conclusions can be obtained for the trivariate function  $Q(x, y; z)$ . The fact that  $Q(x, y; z)$  has one additional variable does not seem to matter much. We therefore expect that the compensation approach will work for walks in the quarter plane that obey the three conditions mentioned above. This will be a topic for future research. Another topic is to see whether the above conditions can be relaxed, and in fact, in the next section we present an example of a walk, not satisfying these properties, yet amenable to the compensation approach.

## 5.2 Another example

We now consider the walk with  $\mathcal{S} = \{(-1, 0), (-1, -1), (0, -1), (1, -1), (1, 0)\}$  in the interior,  $\mathcal{S}_H = \{(-1, 0), (1, 0)\}$  on the horizontal boundary,  $\mathcal{S}_V = \{(0, 1), (0, -1), (1, -1), (1, 0)\}$  on the vertical boundary, and  $\mathcal{S}_0 = \{(0, 1), (1, 0)\}$ ; see the left display in Figure 5. This walk has a rather special behavior, as it can only move upwards on the vertical boundary. By simple enumeration, we get the following recursion relations for  $q_{i,j}$ :

$$q_{i,j}/z = q_{i-1,j} + q_{i-1,j+1} + q_{i,j+1} + q_{i+1,j} + q_{i+1,j+1}, \quad i, j > 0, \quad (59)$$

$$q_{0,j}/z = q_{0,j-1} + q_{0,j+1} + q_{1,j} + q_{1,j+1}, \quad j > 0, \quad (60)$$

$$q_{0,0}/z = 1/z + q_{0,1} + q_{1,1} + q_{1,0}. \quad (61)$$

**Proposition 14.** *The unique solution to Equations (59)–(61) is given by  $q_{i,j} = c_0 \alpha_0^i \beta_0^j$ , with  $\beta_0$  defined as the smallest (in modulus) solution to the fourth-degree polynomial*

$$\beta/z = 1 + \beta^2 + (1 + \beta)(\beta^2 + \beta^3), \quad (62)$$

$\alpha_0 = \beta_0(1 + \beta_0)$  and  $c_0 = 1/[1 - z(\alpha_0 + \alpha_0\beta_0 + \beta_0)]$ . Therefore, the CGF is rational and given by

$$Q(x, y; z) = \frac{c_0}{(1 - \alpha_0 x)(1 - \beta_0 y)}. \quad (63)$$

*Proof.* Substituting  $q_{i,j} = \alpha^i \beta^j$  into (59) and (60) yields

$$\alpha\beta/z = \beta + \beta^2 + \alpha\beta^2 + \alpha^2\beta + \alpha^2\beta^2, \quad (64)$$

$$\alpha\beta/z = \alpha + \alpha\beta^2 + \alpha^2\beta + \alpha^2\beta^2. \quad (65)$$

All pairs  $(\alpha, \beta)$  that satisfy both (64) and (65) are such that  $\beta(1 + \beta) = \alpha$ . Substituting the latter into (64) gives (62). If we denote by  $\beta_k$ ,  $k \in \{0, \dots, 3\}$ , the four roots of (62) and let  $\alpha_k = \beta_k(1 + \beta_k)$ , we know that there exist four constants  $c_k$ ,  $k \in \{0, \dots, 3\}$ , such that  $q_{i,j} = c_0 \alpha_0^i \beta_0^j + \dots + c_3 \alpha_3^i \beta_3^j$ . Suppose now that  $z \in (0, 1/6)$ . It is easily seen that two roots of (62), say  $\beta_2$  and  $\beta_3$ , are complex conjugate, while two others, say  $\beta_0$  and  $\beta_1$ , are real positive. Among the latter, one belongs to  $]0, 1[$ , say  $\beta_0$ , whereas the other one,  $\beta_1$ , is always larger than 1. Since by the same reasoning as in (17),  $\sum_{i,j=0}^{\infty} q_{i,j}$  is finite for all  $|z| < 1/5$ , and hence also for  $z \in (0, 1/6)$ , we have  $c_1 = c_2 = c_3 = 0$ . Finally, the value of  $c = c_0$  follows from (61).

Now we show that  $\beta_0$  can be characterized as the smallest solution to (62). It is enough to prove that for all  $|z| < 1/5$ , (62) has only one solution smaller than  $1/2$  in modulus. But the latter fact comes from applying Rouché's theorem to the polynomial  $[1 + \beta^2 + (1 + \beta)(\beta^2 + \beta^3)] - [\beta/z]$  (see (62)) on the circle of radius  $1/2$ .  $\square$

We now modify the boundary behavior of this walk and we show that this has severe consequences for the CGF. Consider again the walk with  $\mathcal{S} = \{(-1, 0), (-1, -1), (0, -1), (1, -1), (1, 0)\}$  in the interior, but this time with  $\mathcal{S}_V = \{(0, -1), (1, -1), (1, 0)\}$  on the vertical boundary,  $\mathcal{S}_0 = \{(1, 0)\}$ , and the following rather special step set on the horizontal boundary: if the walk is in state  $(i, 0)$  with  $i > 0$ , the possible steps are  $\{(-1, 0), (1, 0)\}$  and a *big* step to  $(0, i)$ ; see the right display in Figure 5. We should note that a *random* walk, with similar unusual boundary behavior, has



This gives

$$\beta_1 = \alpha_0, \quad \alpha_1 = g(\beta_0), \quad c_1 = c_0 \frac{\alpha_1}{1 + \beta_1}. \quad (74)$$

By adding the second product form, we also obtain an extra term  $c_1 \alpha_1^j$  on the left-hand side of (73). That is why we add a third product form  $c_2 \alpha_2^i \beta_2^j$ , with  $(\alpha_2, \beta_2)$  on the curve  $\alpha = g(\beta)$ , and via similar reasoning,

$$\beta_2 = \alpha_1, \quad \alpha_2 = g(\beta_1), \quad c_2 = c_1 \frac{\alpha_2}{1 + \beta_2}. \quad (75)$$

Continuing this procedure means that we keep adding product forms  $c_k \alpha_k^i \beta_k^j$ , with  $c_k$ ,  $\alpha_k$  and  $\beta_k$  as in (68), so that (69) is a formal solution to Equations (59), (61) and (66).

Let  $z \in (0, 1/5)$ . Let us now prove that  $\alpha_k$  and  $\beta_k$  converge to the unique  $\beta_* \in (0, 1)$  such that

$$\beta_*^3 + 2\beta_*^2 + \beta_*(1 - 1/z) + 1 = 0. \quad (76)$$

Note that there is only one  $\beta_* \in (0, 1)$  satisfying (76) because of Rouché's theorem applied to the unit circle and the polynomial (76) written as  $[\beta_*^3 + 2\beta_*^2 + 1] + [\beta_*(1 - 1/z)]$ .

In order to show this, it is enough to prove that with  $\beta_*$  defined as above, the function  $g$  defined in (67) is increasing on  $[0, \beta_*]$ , such that  $g(t) > t$  for  $t \in [0, \beta_*)$  and  $g(\beta_*) = \beta_*$ .

First, the equality  $g(\beta_*) = \beta_*$  follows directly from Equations (67) and (76). Moreover, noting that  $g(0) > 0$  and that  $\beta_*$  is the smallest positive value of  $t$  for which  $g(t) = t$ , we reach the conclusion that  $g(t) > t$  for  $t \in [0, \beta_*[$ . Finally, the fact that  $g$  is increasing on  $[0, \beta_*]$  follows from

$$g'(t) = \frac{1 + 1/z}{t + 1} \frac{g(t)}{\sqrt{(1/z - t)^2 - 4(1 + t)^2}} \geq 0, \quad (77)$$

that is straightforward starting from (67). The convergence of the series  $q_{i,j}$  is then immediate from (68) and (69).  $\square$

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## A Remaining proofs

Let us start by expressing  $h$ , defined in (53), as an alternating sum. The reason why we wish to formulate  $h$  differently is twofold. First, this will enable us to show that  $\rho \in [0.34499975, 0.34499976]$ , see Corollary 18—and in fact, in a similar way, we can approximate  $\rho$  up to any level of precision. Also, this is actually a key lemma for proving Lemma 12.

**Lemma 16.**

$$h(z) = 1 + 2z \left( -1 - \alpha_0 + \sum_{k=0}^{\infty} (-1)^k T_k \right), \quad T_k = \begin{cases} \beta_{k/2} \alpha_{k/2} & \text{if } k \text{ is even,} \\ \beta_{(k-1)/2} \alpha_{(k+1)/2} & \text{if } k \text{ is odd.} \end{cases} \quad (78)$$

*Proof.* Equation (78) follows immediately from (50) and (53). □

As a preliminary result, we also need the following extension of Proposition 4. The proof, similar to that of Proposition 4, is omitted.

**Lemma 17.** Assume that  $z \in (0, 1/\sqrt{8})$ . Then both sequences  $\{\alpha_k\}_{k \geq 0}$  and  $\{\beta_k\}_{k \geq 0}$  are positive and decreasing. Moreover,

$$0 \leq \alpha_k \leq 1/\sqrt{2}^{2k+1}, \quad 0 \leq \beta_k \leq 1/\sqrt{2}^{2k+2}, \quad \forall k \geq 0. \quad (79)$$

**Corollary 18.**  $\rho \in [0.34499975, 0.34499976]$ .

*Proof.* For  $z \in (0, 1/\sqrt{8})$ , the sequence  $\{T_k\}_{k \geq 0}$  defined in Lemma 16 is positive and decreasing, see Lemma 17. In particular, denoting

$$\Lambda^p = \sum_{k=0}^p (-1)^k T_k \quad (80)$$

and using (78), for all  $p \geq 0$  and all  $z \in (0, 1/\sqrt{8})$  we have

$$1 + 2z(-1 - \alpha_0 + \Lambda^{2p+1}) < h(z) < 1 + 2z(-1 - \alpha_0 + \Lambda^{2p}). \quad (81)$$

Applying the last inequalities to  $p = 4$  and noting that the right-hand side (resp. left-hand side) evaluated at 0.34499976 (resp. 0.34499975) is negative (resp. positive) concludes the proof.  $\square$

*Proof of Lemma 12.* First, using the explicit expression of  $\alpha_k$  and  $\beta_k$ , and employing calculus software, we obtain that the algebraic function  $1 + 2z(-1 - \alpha_0 + \Lambda^5)$  has only one zero within the circle of radius  $1/\sqrt{8}$ , and that

$$\inf_{|z|=1/\sqrt{8}} |1 + 2z(-1 - \alpha_0 + \Lambda^5)| > 10^{-2}. \quad (82)$$

Note that the lower bound  $10^{-2}$  in (82) is almost optimal; it is rather small because like  $\rho$ , the unique zero of  $1 + 2z(-1 - \alpha_0 + \Lambda^5)$  within the disc of radius  $1/\sqrt{8}$  is very close to  $1/\sqrt{8}$ .

By Rouché's theorem, applied to the circle with radius  $1/\sqrt{8}$ , it is now enough to prove that

$$\sup_{|z|=1/\sqrt{8}} |[1 + 2z(-1 - \alpha_0 + \Lambda^5)] - [1 + 2z(-1 - \alpha_0 + \Lambda^\infty)]| < 10^{-2}. \quad (83)$$

For this we write

$$|\Lambda^\infty - \Lambda^p| = \left| \sum_{k=p+1}^{\infty} (-1)^k T_k \right| \leq \sum_{k=p+1}^{\infty} |T_k| \leq \frac{1/\sqrt{2}^{2p+5}}{1 - (1/\sqrt{2})^2} < 1/\sqrt{2}^{2p+3}, \quad (84)$$

where the last but one upper bound is obtained from the inequality  $|T_k| \leq 1/\sqrt{2}^{2k+3}$ , see (78) and Lemma 17. By (84) we then obtain

$$|[1 + 2z(-1 - \alpha_0 + \Lambda^p)] - [1 + 2z(-1 - \alpha_0 + \Lambda^\infty)]| \leq 2|z| \frac{1}{2^{p+3/2}}. \quad (85)$$

If  $|z| < 1/\sqrt{8}$ , the last quantity is bounded from above by  $1/2^{p+2}$ . For  $p = 5$ , we obviously get  $1/2^{p+2} < 10^{-2}$ , and (83) is proven.  $\square$

**Lemma 19.** *We have*

$$C_{0,0} = 0.0531 \cdot [1 + O(10^{-3})]. \quad (86)$$

*Proof.* With exactly the same calculations as in the proof of Proposition 13, we obtain that

$$C_{0,0} = \frac{3\rho - 1}{\rho[1 - \rho^2 \widehat{x}'_{0,0}(\rho)]}, \quad (87)$$

so that the main difficulty lies in approximating  $\widehat{x}'_{0,0}(\rho)$ . For this, we use (50) and (78) to write

$$\widehat{x}'_{0,0}(z) = -2\alpha'_0(z) + \sum_{k=0}^{\infty} (-1)^k T'_k(z). \quad (88)$$

While it is easy to control the series  $\sum_{k=0}^{\infty} (-1)^k T_k(z)$ , because  $\alpha_k(z)$  and  $\beta_k(z)$ , and hence  $T_k(z)$ , decrease exponentially fast to 0 as  $k \rightarrow \infty$ , see Proposition 4 and Lemma 17, it is not obvious how we should deal with  $\sum_{k=0}^{\infty} (-1)^k T'_k(z)$ , where terms like  $\alpha'_k(z)$  and  $\beta'_k(z)$  appear. We next show that  $\alpha'_k(z)$  and  $\beta'_k(z)$  actually also decrease exponentially fast to 0 as  $k \rightarrow \infty$ , at least for  $z \in [1/4, 0.35]$ . Note that this assumption on  $z$  is not restrictive, since  $\rho$  belongs to the interval  $[1/4, 0.35]$  by Corollary 18.

Consider the sequence  $\{\gamma_k(z)\}_{k \geq 0}$  defined by  $\gamma_0(z) = a_0(z)$  and, for  $k \geq 0$ , by  $\gamma_{k+1}(z) = f(\gamma_k(z))$ , with  $a_0(z)$  and  $f$  as defined in (20) and (29), respectively. Note that (29) gives  $\gamma_{2k}(z) = \alpha_k(z)$  and  $\gamma_{2k+1}(z) = \beta_k(z)$ . The sequence  $\{\gamma'_k(z)\}_{k \geq 0}$  satisfies the recurrence relation

$$\gamma'_{k+1}(z) = \gamma'_k(z) \partial_t f(\gamma_k(z)) + \partial_z f(\gamma_k(z)). \quad (89)$$

Below we study the coefficients  $\partial_t f(\gamma_k(z))$  and  $\partial_z f(\gamma_k(z))$  of (89). By using expression (29) of  $f$ , we easily obtain that

$$|\partial_t f(\gamma_k(z))| \leq 4\sqrt{2}/9, \quad \forall z \in [0, 1/\sqrt{8}], \quad \forall t \in [0, a_0(z)]. \quad (90)$$

In addition,  $\partial_z f(t) = f(t)/[z\sqrt{1 - 4z^2(1 + t^2)}]$ , in such a way that  $|\partial_z f(t)| \leq |2/z| \cdot |f(t)|$  for all  $z \in [0, 1/\sqrt{8}]$  and all  $t \in [0, a_0(z)]$ . Using then Lemma 17, we obtain

$$|\partial_z f(\gamma_k(z))| \leq |2/z| \cdot |f(\gamma_k(z))| = |2/z| \cdot |\gamma_{k+1}(z)| \leq |2/z| \cdot 1/\sqrt{2}^{k+1} \leq 8/\sqrt{2}^{k+1}, \quad (91)$$

where the last inequality follows from the assumption  $z \in [1/4, 0.35]$ . From (89)-(91) we get

$$|\gamma'_{k+1}(z)| \leq \eta \cdot |\gamma'_{k+1}(z)| + \xi_k, \quad \eta = 4\sqrt{2}/9, \quad \xi_k = 8/\sqrt{2}^{k+1}. \quad (92)$$

We deduce that

$$|\gamma'_{k+1}(z)| \leq \eta^{k+1} \cdot |\gamma'_0(z)| + \sum_{p=0}^k \eta^p \xi_{k-p} \leq \eta^{k+1} \cdot |\gamma'_0(z)| + 72/\sqrt{2}^{k+1} \leq 100/\sqrt{2}^{k+1}, \quad (93)$$

where the last inequality follows from  $\sup_{z \in [0.25, 0.35]} |\gamma'_0(z)| \leq 13$  and  $\eta \leq 1/\sqrt{2}$ . With Lemma 17 and (93), we obtain

$$|T'_k(z)| \leq 200/\sqrt{2}^{k+2}, \quad (94)$$

and thanks to (88) we can obtain approximations of  $\widehat{x}'_{0,0}(\rho)$  up to any level of precision. Lemma 19 follows.  $\square$