

CONVERGENCE OF INVARIANT MEASURES FOR SINGULAR STOCHASTIC DIFFUSION EQUATIONS

IOANA CIOTIR

*Department of Mathematics, Faculty of Economics and Business Administration,
"Al. I. Cuza" University, Bd. Carol no. 9–11, Iași, Romania*

JONAS M. TÖLLE

*Institut für Mathematik, Technische Universität Berlin (MA 7-5), Straße des 17.
Juni 136, 10623 Berlin, Germany*

ABSTRACT. It is proved that the solutions to the singular stochastic p -Laplace equation, $p \in (1, 2)$ and the solutions to the stochastic fast diffusion equation with nonlinearity parameter $r \in (0, 1)$ on a bounded open domain $\Lambda \subset \mathbb{R}^d$ with Dirichlet boundary conditions are continuous in mean, uniformly in time, with respect to the parameters p and r respectively (in the Hilbert spaces $L^2(\Lambda)$, $H^{-1}(\Lambda)$ respectively). The highly singular limit case $p = 1$ is treated with the help of stochastic evolution variational inequalities, where \mathbb{P} -a.s. convergence, uniformly in time, is established.

It is shown that the associated unique invariant measures of the ergodic semigroups converge in the weak sense (of probability measures).

1. INTRODUCTION

Let $\Lambda \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary $\partial\Lambda$. Let $\{W(t)\}_{t \geq 0}$ be a U -valued cylindrical Wiener process on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$, where U is a separable Hilbert space.

We are interested in the following two (families of) stochastic diffusion equations, the stochastic p -Laplacian equation, $p \in (1, \infty)$, $B \in L_2(U, L^2(\Lambda))$,

$$(PL_p) \begin{cases} dX_p(t) = \operatorname{div} \left[|\nabla X_p(t)|^{p-2} \nabla X_p(t) \right] dt + B dW(t) & \text{in } (0, T) \times \Lambda, \\ X_p(t) = 0 & \text{on } (0, T) \times \partial\Lambda, \\ X_p(0) = x \in L^2(\Lambda) & \text{in } \Lambda. \end{cases}$$

E-mail addresses: ioana.ciotir@feaa.uaic.ro, toelle@math.tu-berlin.de.

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The deterministic p -Laplace equation arises from geometry, quasi-regular mappings, fluid dynamics and plasma physics, see [19, 20]. In [27], (PL_p) with $B \equiv 0$ is suggested as a model of motion of non-Newtonian fluids. See [28] for the stochastic equation.

We are also interested in the stochastic fast diffusion equation $r \in (0, \infty)$, $B \in L_2(U, H^{-1}(\Lambda))$,

$$(\text{FD}_r) \begin{cases} dY_r(t) = \Delta (|Y_r(t)|^{r-1} Y_r(t)) dt + B dW(t), & \text{in } (0, T) \times \Lambda, \\ Y_r(t) = 0, & \text{on } (0, T) \times \partial\Lambda, \\ Y_r(0) = y \in H^{-1}(\Lambda), & \text{in } \Lambda, \end{cases}$$

which models diffusion in plasma physics, curvature flows and self-organized criticality in sandpile models, see e.g. [12, 14, 36, 41] and the references therein.

The above equations considered are called *singular* for $p \in (1, 2)$, $r \in (0, 1)$ and *degenerate* for $p \in (2, \infty)$, $r \in (1, \infty)$ (porous medium equation). In this paper, we shall investigate the former case.

For $p = 1$, equation (PL_1) can be heuristically written as a stochastic evolution inclusion, $B \in L_2(U, H^{-1}(\Lambda))$,

$$(\text{PL}_1) \begin{cases} dX_1(t) \in \text{div} [\text{Sgn}(\nabla X_1(t))] dt + B dW(t) & \text{in } (0, T) \times \Lambda, \\ X_1(t) = 0 & \text{on } (0, T) \times \partial\Lambda, \\ X_1(0) = x \in L^2(\Lambda) & \text{in } \Lambda, \end{cases}$$

where $\text{Sgn} : \mathbb{R}^d \rightarrow 2^{\mathbb{R}^d}$ is defined by

$$\text{Sgn}(u) := \begin{cases} \frac{u}{|u|}, & \text{if } u \in \mathbb{R}^d \setminus \{0\}, \\ \{v \in \mathbb{R}^d \mid |v| \leq 1\}, & \text{if } u = 0. \end{cases}$$

A precise characterization of the 1-Laplace operator can be found in [2, 3, 37]. A typical 2-dimensional example for the so-called total variation flow can be found in image restoration, see [1, 3, 6] and the references therein.

We shall, however, take use of the stochastic evolution variational inequality-formulation as in [11].

We are particularly interested in continuity of the solutions in the parameters p and r , especially for the case $p \rightarrow 1$. Stochastic Trotter-type results in this direction have been obtained by the first named author in [15, 16, 17]. However, for the case $p \rightarrow 1$, we shall need the theory of Mosco convergence of convex functionals as in [4], since no strong characterization of the limit is available (which could be treated by Yosida-approximation methods). For $B = 0$ (i.e., the deterministic equation), the convergence of solutions to the evolution problem (PL_p) was proved in [23, 40]. See also [39, Ch. 8.3].

With the help of a uniqueness result for invariant measures of the equations considered, obtained by Liu and the second named author [29], we prove tightness and the weak convergence (weak continuity) of invariant measures associated to the ergodic semigroups of the equations (PL_p) and (FD_r) . See [9, 10, 18, 22] for other result in this direction.

Organization of the paper. In Section 2, we prove that the solutions to the basic examples are continuous in the parameters p and r resp.

In Section 3, The result of Section 2 is combined with the uniqueness of invariant measures proved in [29] in order to obtain the weak continuity of invariant measures in the parameters p and r resp.

In Section 4, we prove a convergence result for the stochastic p -Laplace equation as $p \rightarrow 1$, using another notion of a solution. For the limit $p = 1$, however,

uniqueness of the invariant measure is an open question. The matter is further investigated in [22].

The Appendix collects some well-known results on Mosco (variational) convergence and Mosco convergence in L^p -spaces, needed for the proof in Section 4.

2. CONVERGENCE OF SOLUTIONS

Compare with [16, Theorem 2].

Theorem 2.1. *Let $\{p_n\} \subset \left(1 \vee \frac{2d}{2+d}, 2\right]$, $n \in \mathbb{N}$, $p_0 \in \left(1 \vee \frac{2d}{2+d}, 2\right]$ such that $p_n \rightarrow p_0$. Let $X_n := X_{p_n}$, $n \in \mathbb{N}$, $X_0 := X_{p_0}$ be the solutions to (PL_{p_n}) , $n \in \mathbb{N}$, (PL_{p_0}) resp. Then for $x \in L^2(\Lambda)$.*

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} \|X_n(t) - X_0(t)\|_{L^2(\Lambda)}^2 \right] = 0.$$

Proof. For $p \in (1, \infty)$, define $a_p : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $a_p(x) := |x|^{p-2}x$. Furthermore, let $A_p : W_0^{1,p}(\Lambda) \rightarrow (W_0^{1,p})^*(\Lambda)$ be defined by $A_p(y) := -\text{div}[a_p(\nabla y)]$, where $y \in W_0^{1,p}(\Lambda)$. To be more specific,

$${}_{(W^{1,p})^*} \langle A_p(y), z \rangle_{W^{1,p}} = \int_{\Lambda} \langle a_p(\nabla y), \nabla z \rangle d\xi, \quad \forall z \in W_0^{1,p}(\Lambda).$$

We first consider the following approximating equations for (PL_p)

$$(2.1) \quad \begin{cases} dX_p^\varepsilon(t) + A_p^\varepsilon(X_p^\varepsilon(t)) dt = B dW(t) \\ X_p^\varepsilon(0) = x \end{cases}$$

where for any $u \in L^2(\Lambda)$,

$$A_p^\varepsilon(u) = -(1 - \varepsilon\Delta)^{-1} \text{div} \left[a_p^\varepsilon \left(\nabla (1 - \varepsilon\Delta)^{-1} u \right) \right]$$

and a_p^ε is the Yosida approximation of a_p i.e., for any $r \in \mathbb{R}^d$,

$$a_p^\varepsilon(r) = \frac{1}{\varepsilon} \left(1 - (1 + \varepsilon a_p)^{-1}(r) \right).$$

In particular, for $u, v \in L^2(\Lambda)$,

$$\langle A_p^\varepsilon(u), v \rangle_{L^2(\Lambda)} = \int_{\Lambda} \langle a_p^\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v) \rangle d\xi,$$

where $R_\varepsilon := (1 - \varepsilon\Delta)^{-1}$ is the resolvent of the Dirichlet Laplacian.

We shall use the following strategy (\mathbb{P} -a.s.)

$$\begin{aligned} & \|X_n(t) - X_0(t)\|_{L^2(\Lambda)}^2 \\ & \leq 3 \|X_n(t) - X_n^\varepsilon(t)\|_{L^2(\Lambda)}^2 + 3 \|X_n^\varepsilon(t) - X_0^\varepsilon(t)\|_{L^2(\Lambda)}^2 \\ & \quad + 3 \|X_0^\varepsilon(t) - X_0(t)\|_{L^2(\Lambda)}^2 \\ & =: I_1(n, \varepsilon) + I_2(n, \varepsilon) + I_3(\varepsilon). \end{aligned}$$

uniformly in $t \in [0, T]$.

At this point we need to prove the following lemma. We introduce the notation $r_\varepsilon^p(r) := (1 + \varepsilon a_p)^{-1}(r)$.

Lemma 2.2. *Under our assumptions, if we let X_p^ε be the solution to (2.1) and $\tilde{X}_p^\varepsilon := (1 - \varepsilon\Delta)^{-1} X_p^\varepsilon$, we have that*

$$(2.2) \quad \mathbb{E} \int_0^t \int_{\Lambda} \left| r_\varepsilon^p \left(\nabla \tilde{X}_p^\varepsilon(s) \right) \right|^p d\xi ds \leq C_t \left(\|x\|_{L^2(\Lambda)}^2 + \|B\|_{HS}^2 \right),$$

for all $t \in [0, T]$.

Proof. We know by the definition of a_p that

$$\langle a_p(r), r \rangle \geq |r|^p.$$

On the other hand we have by Itô's formula, applied to the function $u \mapsto \|u\|_{L^2(\Lambda)}^2$, that

$$(2.3) \quad \mathbb{E} \|X_p^\varepsilon(t)\|_{L^2(\Lambda)}^2 + 2\mathbb{E} \int_0^t \int_\Lambda \langle a_p^\varepsilon(\nabla \tilde{X}_p^\varepsilon(s)), \nabla \tilde{X}_p^\varepsilon(s) \rangle d\xi ds \\ \leq C_t \left(\|x\|_{L^2(\Lambda)}^2 + \|B\|_{HS}^2 \right).$$

By the definition of the Yosida approximation we have that

$$a_p^\varepsilon(r) = a_p(r_\varepsilon^p(r))$$

and

$$\langle a_p^\varepsilon(r), r \rangle = \langle a_p^\varepsilon(r_\varepsilon^p(r)), r_\varepsilon^p(r) \rangle + \frac{1}{\varepsilon} |r - r_\varepsilon^p(r)|^2.$$

We rewrite as follows

$$\mathbb{E} \int_0^t \int_\Lambda \langle a_p^\varepsilon(\nabla \tilde{X}_p^\varepsilon(s)), \nabla \tilde{X}_p^\varepsilon(s) \rangle d\xi ds \\ \geq \mathbb{E} \int_0^t \int_\Lambda \langle a_p(r_\varepsilon^p(\nabla \tilde{X}_p^\varepsilon(s))), r_\varepsilon^p(\nabla \tilde{X}_p^\varepsilon(s)) \rangle d\xi ds \\ \geq \mathbb{E} \int_0^t \int_\Lambda |r_\varepsilon^p(\nabla \tilde{X}_p^\varepsilon(s))|^p d\xi ds.$$

Plugging into (2.3) proves (2.2). \square

We shall prove now that \mathbb{P} -a.s.

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|X_p(t) - X_p^\varepsilon(t)\|_{L^2(\Lambda)}^2 = 0, \quad \text{uniformly in } p \in \left(\frac{2d}{d+2}, 2 \right).$$

We set $\tilde{X}_p^\varepsilon = (1 - \varepsilon\Delta)^{-1} X_p^\varepsilon$ and $\tilde{X}_p^\lambda = (1 - \lambda\Delta)^{-1} X_p^\lambda$. Then by (2.1), we have that

$$\frac{1}{2} \|X_p^\varepsilon(t) - X_p^\lambda(t)\|_{L^2(\Lambda)}^2 \\ + \int_0^t \int_\Lambda \langle a_p^\varepsilon(\nabla \tilde{X}_p^\varepsilon(s)) - a_p^\lambda(\nabla \tilde{X}_p^\lambda(s)), \nabla \tilde{X}_p^\varepsilon(s) - \nabla \tilde{X}_p^\lambda(s) \rangle d\xi ds = 0 \quad \mathbb{P}\text{-a.s.}$$

Setting $\nabla \tilde{X}_p^\varepsilon(s) = u^\varepsilon$ and $\nabla \tilde{X}_p^\lambda(s) = u^\lambda$ and using

$$a_p^\varepsilon(u) \in a_p \left((1 + \varepsilon a_p)^{-1}(u) \right),$$

we get by the monotonicity of a_p that

$$\langle a_p^\varepsilon(u^\varepsilon) - a_p^\lambda(u^\lambda), u^\varepsilon - u^\lambda \rangle \\ \geq \langle a_p^\varepsilon(u^\varepsilon) - a_p^\lambda(u^\lambda), \varepsilon a_p^\varepsilon(u^\varepsilon) - \lambda a_p^\lambda(u^\lambda) \rangle.$$

This leads to

$$(2.4) \quad \frac{1}{2} \|X_p^\varepsilon(t) - X_p^\lambda(t)\|_{L^2(\Lambda)}^2 \\ \leq \int_0^t \int_\Lambda \left(\varepsilon |a_p^\varepsilon(\nabla \tilde{X}_p^\varepsilon(s))|^2 + \lambda |a_p^\lambda(\nabla \tilde{X}_p^\lambda(s))|^2 \right) d\xi ds \quad \mathbb{P}\text{-a.s.}$$

We can now prove that \mathbb{P} -a.s.

$$(2.5) \quad \int_0^t \int_{\Lambda} \left| a_p^\varepsilon \left(\nabla \tilde{X}_p^\varepsilon(s) \right) \right|^2 d\xi ds \leq C_t$$

for some C_t independent of p and ε .

Using Jensen's inequality (for $t \mapsto t^{p/(2p-2)}$) and taking into account that $|a_p(r)| \leq |r|^{p-1}$, we obtain

$$(2.6) \quad \begin{aligned} & \int_0^t \int_{\Lambda} \left| a_p^\varepsilon \left(\nabla \tilde{X}_p^\varepsilon(s) \right) \right|^2 d\xi ds \\ & \leq (t|\Lambda|)^{1-((2p-2)/p)} \left(\int_0^t \int_{\Lambda} \left| a_p \left(r_\varepsilon^p \left(\nabla \tilde{X}_p^\varepsilon(s) \right) \right) \right|^{p/(p-1)} d\xi ds \right)^{(2p-2)/p} \\ & \leq (1+t|\Lambda|) \left(\int_0^t \int_{\Lambda} \left| r_\varepsilon^p \left(\nabla \tilde{X}_p^\varepsilon(s) \right) \right|^p d\xi ds \right)^{(2p-2)/p} \\ & \leq C_t + C_t \left(\int_0^t \int_{\Lambda} \left| r_\varepsilon^p \left(\nabla \tilde{X}_p^\varepsilon(s) \right) \right|^p d\xi ds \right), \end{aligned}$$

where $|\Lambda| = \int_{\Lambda} d\xi$.

Now by Lemma 2.2 we have (2.5) for a constant C_t independent of p and ε , and passing to the limit for $\varepsilon, \lambda \rightarrow 0$ in (2.4) we get that \mathbb{P} -a.s.

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|X_p(t) - X_p^\varepsilon(t)\|_{L^2(\Lambda)}^2 = 0, \quad \text{uniformly in } p \in \left(1 \vee \frac{2d}{d+2}, 2\right).$$

As a consequence, $I_1(n, \varepsilon)$ and $I_3(\varepsilon)$ tend to zero as $\varepsilon \downarrow 0$, uniformly in n .

For $I_2(n, \varepsilon)$, using the monotonicity of $a_{p_n}^\varepsilon$ we have

$$\begin{aligned} & \frac{1}{2} \|X_{p_n}^\varepsilon(t) - X_{p_0}^\varepsilon(t)\|_{L^2(\Lambda)}^2 \\ & + \int_0^t \int_{\Lambda} \left\langle a_{p_n}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right) - a_{p_0}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right), \nabla \tilde{X}_{p_n}^\varepsilon(s) - \nabla \tilde{X}_{p_0}^\varepsilon(s) \right\rangle_d d\xi ds \leq 0. \end{aligned}$$

Since

$$\begin{aligned} & \frac{1}{2} \|X_{p_n}^\varepsilon(t) - X_{p_0}^\varepsilon(t)\|_{L^2(\Lambda)}^2 \\ & \leq \int_0^t \int_{\Lambda} \left[(1 - \varepsilon \Delta)^{-1} \operatorname{div} \left(a_{p_n}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right) - a_{p_0}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right) \right) \right] [X_{p_n}^\varepsilon(s) - X_{p_0}^\varepsilon(s)] d\xi ds \\ & \leq \left(\int_0^t \int_{\Lambda} \left((1 - \varepsilon \Delta)^{-1} \operatorname{div} a_{p_n}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right) - (1 - \varepsilon \Delta)^{-1} \operatorname{div} a_{p_0}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right) \right)^2 d\xi ds \right)^{1/2} \\ & \quad \times \left(\int_0^t \int_{\Lambda} (X_{p_n}^\varepsilon(s) - X_{p_0}^\varepsilon(s))^2 d\xi ds \right)^{1/2} \end{aligned}$$

We only need to prove that

$$\left(\int_0^t \int_{\Lambda} \left((1 - \varepsilon \Delta)^{-1} \operatorname{div} a_{p_n}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right) - (1 - \varepsilon \Delta)^{-1} \operatorname{div} a_{p_0}^\varepsilon \left(\nabla \tilde{X}_{p_0}^\varepsilon(s) \right) \right)^2 d\xi ds \right)^{1/2} \rightarrow 0$$

and that follows from

$$(2.7) \quad A_{p_n}^\varepsilon(u) \rightarrow A_{p_0}^\varepsilon(u), \quad \text{strongly in } L^2((0, T) \times \Lambda),$$

where $A_{p_n}^\varepsilon(u) = (1 - \varepsilon \Delta)^{-1} \operatorname{div} a_{p_n}^\varepsilon(u)$ (as in (2.1)).

Indeed, we obtain (2.7) by the following arguments:

Since $a_{p_n}^\varepsilon(u) \rightarrow a_{p_0}^\varepsilon(u)$ pointwise, which follows from Lemma A.5 and [4, Proposition 3.29], and since $\{a_{p_n}^\varepsilon(u)\}_n$ is bounded *a.e.* on $(0, T) \times \Lambda$ we get by Lebesgue's dominated convergence theorem

$$\langle \operatorname{div} a_{p_n}^\varepsilon(u) - \operatorname{div} a_{p_0}^\varepsilon(u), v \rangle_{L^2((0, T) \times \Lambda)} = \langle a_{p_n}^\varepsilon(u) - a_{p_0}^\varepsilon(u), \nabla v \rangle_{L^2((0, T) \times \Lambda)} \xrightarrow{n} 0,$$

for all $v \in L^2((0, T) \times \Lambda)$.

That means

$$\operatorname{div} a_{p_n}^\varepsilon(u) \rightarrow \operatorname{div} a_{p_0}^\varepsilon(u), \text{ weakly in } L^2((0, T) \times \Lambda)$$

and this leads to

$$(1 - \varepsilon \Delta)^{-1} \operatorname{div} a_{p_n}^\varepsilon(u) \rightarrow (1 - \varepsilon \Delta)^{-1} \operatorname{div} a_{p_0}^\varepsilon(u), \text{ strongly in } L^2((0, T) \times \Lambda),$$

which is (2.7).

We have proved that

$$\lim_n \sup_{t \in [0, T]} \|X_n(t) - X_0(t)\|_{L^2(\Lambda)} = 0 \quad \mathbb{P}\text{-a.s.}$$

The convergence

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} \|X_n(t) - X_0(t)\|_{L^2(\Lambda)}^2 \right] = 0$$

is established by Lebesgue's dominated convergence theorem and [28, Eq. (1.3)], where the constant can be controlled uniformly in p by Itô's formula, Poincaré inequality and Grönwall's lemma. We refer to [38] for the p -dependence of Poincaré constants. \square

Theorem 2.3. *Let $\{r_n\} \subset \left(0 \vee \frac{d-2}{d+2}, 1\right]$, $n \in \mathbb{N}$, $r_0 \in \left(0 \vee \frac{d-2}{d+2}, 1\right]$ such that $r_n \rightarrow r_0$. Let $Y_n := Y_{r_n}$, $n \in \mathbb{N}$, $Y_0 := Y_{r_0}$ be the solutions to (FD_{r_n}) , $n \in \mathbb{N}$, (FD_{r_0}) resp. Then for $y \in H^{-1}(\Lambda)$,*

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} \|Y_n(t) - Y_0(t)\|_{H^{-1}(\Lambda)}^2 \right] = 0.$$

Proof. We need to show that

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} \|Y_n(t) - Y_0(t)\|_{H^{-1}(\Lambda)}^2 \right] = 0.$$

Using the same approximation as in [10] consider

$$\begin{aligned} & \|Y_n(t) - Y_0(t)\|_{H^{-1}(\Lambda)} \\ & \leq \|Y_n(t) - Y_n^\varepsilon(t)\|_{H^{-1}(\Lambda)} + \|Y_n^\varepsilon(t) - Y_0^\varepsilon(t)\|_{H^{-1}(\Lambda)} + \|Y_0^\varepsilon(t) - Y_0(t)\|_{H^{-1}(\Lambda)} \\ & = I_1 + I_2 + I_3. \end{aligned}$$

For I_1 and I_3 we have the convergence uniformly in r_n for $r_n > 1/2$, arguing as in [10], Proposition 2.6 and using at the end Jensen's inequality for $L^2(\Lambda) \subset L^{2r_n}(\Lambda)$.

For I_2 note that the pointwise convergence of $\Psi_{r_n}(x) = |x|^{r_n-1}x$ to $\Psi_{r_0}(x) = |x|^{r_0-1}x$ imply the convergence of the resolvent in \mathbb{R} and then we get the result arguing as in [15]. \square

3. CONVERGENCE OF INVARIANT MEASURES

In this section, we shall present a result on convergence of invariant measures associated to equations (PL_p), (FD_r) respectively.

Let $\{X_p^x(t)\}_{t \geq 0}$ be the variational solution associated to equation (PL_p) starting at $x \in L^2(\Lambda)$. Similarly, let $\{Y_r^y(t)\}_{t \geq 0}$ be the variational solution associated to equation (FD_r) starting at $y \in H^{-1}(\Lambda)$.

Let

$$P_t^p F(x) := \mathbb{E} [F(X_p^x(t))], \quad F \in C_b(L^2(\Lambda)), \quad t \geq 0,$$

be the semigroup associated to equation (PL_p).

Let

$$Q_t^r G(y) := \mathbb{E} [G(Y_r^y(t))], \quad G \in C_b(H^{-1}(\Lambda)), \quad t \geq 0,$$

be the semigroup associated to equation (FD_r).

Recently, Liu and the second named author obtained the following result:

Proposition 3.1. *Suppose that $p \in \left(1 \vee \frac{2d}{2+d}, 2\right]$, $r \in \left(0 \vee \frac{d-2}{d+2}, 1\right]$. Then $\{P_t^p\}$ and $\{Q_t^r\}$ are ergodic and admit unique invariant measures μ_p, ν_r respectively. It holds that μ_p is supported by $W_0^{1,p}(\Lambda)$ and ν_r is supported by $L^{r+1}(\Lambda)$. Also*

$$(3.1) \quad \int_{L^2(\Lambda)} \|x\|_{1,p}^p \mu_p(dx) < +\infty,$$

and

$$(3.2) \quad \int_{H^{-1}(\Lambda)} \|y\|_{r+1}^{r+1} \nu_r(dy) < +\infty.$$

Proof. See [29, Propositions 3.2 and 3.4]. □

Theorem 3.2. (i) *Let $\{p_n\} \subset \left(1 \vee \frac{2d}{2+d}, 2\right]$, $n \in \mathbb{N}$, $p_0 \in \left(1 \vee \frac{2d}{2+d}, 2\right]$ such that $p_n \rightarrow p_0$. Set $P_t^n := P_t^{p_n}$, $P_t^0 := P_t^{p_0}$.*

Then the unique invariant measures μ_n , $n \in \mathbb{N}$, μ_0 resp. associated to $\{P_t^n\}$, $n \in \mathbb{N}$, $\{P_t^0\}$ converge in the weak sense, i.e.

$$\lim_n \int_{L^2(\Lambda)} F(x) \mu_n(dx) = \int_{L^2(\Lambda)} F(x) \mu_0(dx) \quad \forall F \in C_b(L^2(\Lambda)).$$

(ii) *Let $\{r_n\} \subset \left(0 \vee \frac{d-2}{d+2}, 1\right]$, $n \in \mathbb{N}$, $r_0 \in \left(0 \vee \frac{d-2}{d+2}, 1\right]$ such that $r_n \rightarrow r_0$. Set $Q_t^n := Q_t^{r_n}$, $Q_t^0 := Q_t^{r_0}$.*

Then the unique invariant measures ν_n , $n \in \mathbb{N}$, ν_0 resp. associated to $\{Q_t^n\}$, $n \in \mathbb{N}$, $\{Q_t^0\}$ converge in the weak sense, i.e.

$$\lim_n \int_{L^2(\Lambda)} F(x) \nu_n(dx) = \int_{L^2(\Lambda)} F(x) \nu_0(dx) \quad \forall F \in C_b(L^2(\Lambda)).$$

Proof. Let us prove (i) first. By Proposition 3.1, we see that $\{P_t^n\}$, $n \in \mathbb{N}$, $\{P_t^0\}$ admit unique invariant measures μ_n , $n \in \mathbb{N}$, μ_0 resp. Let $p_1 := \inf_n p_n$. By the convergence $p_n \rightarrow p_0$, $p_1 \in \left(1 \vee \frac{2d}{2+d}, 2\right]$ and the embedding $W_0^{1,p_1}(\Lambda) \subset L^2(\Lambda)$ is compact.

Let $\theta > 0$. Set

$$K_\theta := \left\{ x \in L^2(\Lambda) \mid \|x\|_{1,p_1}^{p_1} \leq \theta^{-1} + |\Lambda| \right\}.$$

Clearly, K_θ is compact in $L^2(\Lambda)$. Now by (3.1),

$$\mu_n \{K_\theta^c\} = \mu_n \left\{ \|\cdot\|_{1,p_1}^{p_1} - |\Lambda| \geq \theta^{-1} \right\} \leq \theta \int_{L^2(\Lambda)} \|x\|_{1,p_n}^{p_n} \mu_n(dx) \leq \theta \|B\|_{HS}^2.$$

Hence the family of measures $\{\mu_n\}_{n \in \mathbb{N}}$ is tight and has a weak accumulation point $\tilde{\mu}$, i.e. $\mu_{n_k} \rightarrow \tilde{\mu}$ weakly. By the Krylov–Bogoliubov theorem, for $F \in C_b(L^2(\Lambda))$,

$$\begin{aligned} \int_{L^2(\Lambda)} F(x) \mu_{n_k}(dx) &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T P_t^{n_k} F(x) dt \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T (P_t^{n_k} F(x) - P_t^0 F(x)) dt \\ &\quad + \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T P_t^0 F(x) dt \\ &=: \varepsilon_k + \int_{L^2(\Lambda)} F(x) \mu_0(dx) \end{aligned}$$

By Theorem 2.1 and dominated convergence, $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$ and hence

$$\int_{L^2(\Lambda)} F(x) \tilde{\mu}(dx) = \int_{L^2(\Lambda)} F(x) \mu_0(dx).$$

As a consequence, for the whole sequence, $\mu_n \rightarrow \mu_0$ weakly.

The proof for (ii) can be carried out by similar arguments. \square

4. THE CASE $p = 1$

For $p = 1$, the situation is more complicated. We would like to find a convex functional Φ^1 such that the stochastic 1-Laplace equation

$$(PL_1) \begin{cases} dX_1(t) = \operatorname{div} \left[\frac{\nabla X_1(t)}{|\nabla X_1(t)|} \right] dt + B dW(t) & \text{in } (0, T) \times \Lambda, \\ X_1(t) = 0 & \text{on } (0, T) \times \partial\Lambda, \\ X_1(0) = x & \text{in } \Lambda, \end{cases}$$

can be written as

$$(4.1) \quad \begin{cases} dX_1(t) \in -\partial\Phi^1(X_1(t)) dt + B dW(t) & \text{in } (0, T), \\ X_1(0) = x, \end{cases}$$

where $\partial\Phi^1$ is the subdifferential of Φ^1 .

We shall need the spaces $BV(\Lambda)$ and $BV(\mathbb{R}^d)$. For $f \in L^1_{\text{loc}}(\Lambda)$, define the *total variation*

$$\|Df\|(\Lambda) = \sup \left\{ \int_{\Lambda} f \operatorname{div} \psi d\xi \mid \psi \in C_0^\infty(\Lambda; \mathbb{R}^d), |\psi| \leq 1 \right\}$$

$BV(\Lambda)$ is defined to be equal to $\{f \in L^1(\Lambda) \mid \|Df\|(\Lambda) < \infty\}$. Denote the $d-1$ -dimensional Hausdorff measure on $\partial\Lambda$ by \mathcal{H}^{d-1} . For $f \in BV(\Lambda)$ there is an element $f^\Lambda \in L^1(\partial\Lambda, d\mathcal{H}^{d-1})$ called the *trace* such that

$$\int_{\Lambda} f \operatorname{div} \psi d\xi = - \int_{\Lambda} \langle \psi, d[Df] \rangle + \int_{\partial\Lambda} \langle \psi, \nu \rangle f^\Lambda d\mathcal{H}^{d-1} \quad \forall \psi \in C^1(\bar{\Lambda}; \mathbb{R}^d),$$

where $[Df]$ denotes the distributional gradient of f on Λ (which is a \mathbb{R}^d -valued Radon measure here) and ν denotes the outer unit normal on $\partial\Lambda$. $BV(\mathbb{R}^d)$ is defined similarly by setting $\Lambda = \mathbb{R}^d$. Define also $\|Df\|(\mathbb{R}^d)$ in the above manner. Note that for $f \in BV(\Lambda)$ (extended by zero outside Λ) it holds that $f \in BV(\mathbb{R}^d)$ and that

$$(4.2) \quad \|Df\|(\mathbb{R}^d) = \|Df\|(\Lambda) + \int_{\partial\Lambda} |f^\Lambda| d\mathcal{H}^{d-1},$$

cf. [1, Theorem 3.87].

Remark 4.1. *By Ambrosio et al. [1, Corollary 3.49], if $d \in \{1, 2\}$, then*

$$W_0^{1,1}(\Lambda) \subset BV(\Lambda) \subset L^2(\Lambda)$$

continuously. If $d = 1$, then

$$BV(\Lambda) \subset\subset L^2(\Lambda)$$

compactly.

For further results in spaces of functions of bounded variation, we refer to [1, Ch. 3].

We shall return to equation (4.1). Recall that the subdifferential $\partial\Phi^1$ in $L^2(\Lambda)$ is defined by $\eta \in \partial\Phi^1(x)$ iff

$$(4.3) \quad \Phi^1(x) - \Phi^1(y) \leq \int_{\Lambda} \eta(x-y) d\xi, \quad \forall y \in \text{dom } \Phi^1.$$

One possible choice for Φ^1 is the (homogeneous) energy

$$\tilde{\Phi}(u) := \begin{cases} \int_{\Lambda} |\nabla u| d\xi, & \text{if } u \in W_0^{1,1}(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus W_0^{1,1}(\Lambda). \end{cases}$$

In this case, if $u \in W_0^{1,1}(\Lambda)$, and if $U := -\text{div}(\text{sgn}(\nabla u)) \in L^2(\Lambda)$, then we have that $u \in \text{dom } \partial\tilde{\Phi}$ and $U = \partial\tilde{\Phi}(u)$.

However, $\tilde{\Phi}$ fails to be lower semi-continuous in $L^2(\Lambda)$ which is a necessary ingredient for the theory. Therefore, it is convenient to consider its relaxed functional in $L^2(\Lambda)$, which is equal to

$$\Phi^1(u) := \begin{cases} \|Du\|(\mathbb{R}^d), & \text{if } u \in BV(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus BV(\Lambda), \end{cases}$$

see equation (4.2) above. Φ^1 is proper, convex and lower semi-continuous in $L^2(\Lambda)$ and an extension of $\tilde{\Phi}$ in the sense that $\text{dom } \Phi^1 \supset \text{dom } \tilde{\Phi}$ and $\Phi^1 \leq \tilde{\Phi}$. Compare with [3, 24, 37, 40].

Following the approach of Barbu, Da Prato and Röckner [11], we shall give the definition of a solution for equations (PL_p) , $p \in [1, 2]$.

Definition 4.2. *Set $V_p := W_0^{1,p}(\Lambda)$, $p \in (1, 2]$, $V_1 := BV(\Lambda)$. Let Φ^1 be defined as above. For $p \in (1, 2]$, let*

$$\Phi^p(x) := \begin{cases} \frac{1}{p} \int_{\Lambda} |\nabla x|^p d\xi, & \text{if } x \in W_0^{1,p}(\Lambda), \\ +\infty, & \text{if } x \in L^2(\Lambda) \setminus W_0^{1,p}(\Lambda). \end{cases}$$

A stochastic process $X = X^x$ with \mathbb{P} -a.s. continuous sample paths in $H := L^2(\Lambda)$ is said to be a solution to equation (PL_p) , $p \in [1, 2]$ if

$$X \in C_W([0, T]; H) \cap L^p((0, T) \times \Omega, V_p), \quad X(0) = x \in H$$

and

$$\begin{aligned} & \frac{1}{2} \|X(t) - Y(t)\|_{L^2(\Lambda)}^2 + \int_0^t (\Phi^p(X(s)) - \Phi^p(Y(s))) ds \\ & \leq \frac{1}{2} \|x - Y(0)\|_{L^2(\Lambda)}^2 + \int_0^t (G(s), X(s) - Y(s))_{L^2(\Lambda)} ds, \quad t \in [0, T], \end{aligned}$$

for all $G \in L_W^2(0, T; H)$ and $Y \in C_W([0, T]; H) \cap L^p((0, T) \times \Omega; V_p)$ satisfying the equation

$$(4.4) \quad dY(t) + G(t) dt = B dW(t), \quad t \in [0, T].$$

Suppose for a while that $1 < p < 2$, $d = 1, 2$. Arguing as in [31, Example 4.1.9, Theorem 4.2.4], we can easily prove existence and uniqueness of the solution X_p for equation (PL_p) , in the usual (strong) variational sense, as in Pardoux, Krylov, Rozovskii [26, 30]. We shall refer to Prévôt, Röckner [31, Definition 4.2.1]. By Itô's formula, we see that X_p is also a solution in the sense of the definition above.

Here, $W(t)$ is a cylindrical Wiener process on $L^2(\Lambda)$ of the form

$$W(t) = \sum_{n=1}^{\infty} \gamma_n(t) e_n, \quad t \geq 0,$$

where $\{\gamma_n\}$ is a sequence of mutually independent real Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and $\{e_n\}$ is an orthonormal basis of $L^2(\Lambda)$. We shall make further specifications. BB^* is assumed to be a linear, continuous, non-negative, symmetric operator on $L^2(\Lambda)$ with eigenbasis $\{e_n\}$ and corresponding sequence of eigenvalues $\{\lambda_n\}$. Let $(-\Delta, \text{dom}(-\Delta))$ be the Dirichlet Laplacian in $L^2(\Lambda)$, in particular, $\text{dom}(-\Delta) = H^2(\Lambda) \cap H_0^1(\Lambda)$. Assume for simplicity that $\{e_n\}$ is an eigenbasis of $-\Delta$ with corresponding sequence of eigenvalues $\{\mu_n\}$. We shall assume that

$$(4.5) \quad \sum_{n=1}^{\infty} \lambda_n^{1+\kappa} \mu_n < \infty$$

for some $\kappa > 0$. For the situation considered in this paper, it is enough to set $Q := (-\Delta)^{-1-\delta}$ with $\delta > \frac{1}{2} + \kappa$ for $d = 1$ and $\delta > 1 + \kappa$ for $d = 2$.

Regarding equation (PL_1) , well-posedness of the problem as well as existence and uniqueness of the solution were proved by Barbu, Da Prato and Röckner in [11].

Remark 4.3. Note that in [11], the space $BV_0(\Lambda)$ is introduced, consisting of $BV(\Lambda)$ -functions with zero trace. They claim, however, that the energy

$$\Psi(u) := \begin{cases} \|Du\|(\Lambda), & \text{if } u \in BV_0(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus BV_0(\Lambda). \end{cases}$$

is lower semi-continuous which is not the case. Consider, for example, a sequence u_n of trace zero Lipschitz functions on Λ with $\|Du_n\|(\Lambda) = 1$ converging in $L^2(\Lambda)$ to $\mathbb{1}_\Lambda$. Then

$$\varliminf_n \Psi(u_n) = 1 < +\infty = \Psi(\mathbb{1}_\Lambda).$$

Fortunately, all results of [11] remain true, if one replaces Ψ (denoted by Φ in their paper) by Φ^1 . We do not repeat the steps taken in the proof of [11] here, but note that for their existence and uniqueness result relies on an approximation $\{\Psi^\varepsilon\}$ of Ψ which “does not see” the trace-term in (4.2), i.e. maps $L^2(\Lambda)$ functions on a joint subspace of $BV_0(\Lambda)$ and $\text{dom}(\Phi^1)$. In fact, $\{\Psi^\varepsilon\}$ is defined similarly to (4.6).

Other results of stochastic evolution variational inequalities can be found in [8, 13, 32, 33, 34].

We are now able to formulate the main result of this section.

Theorem 4.4. Let $d \in \{1, 2\}$. The sequence of solutions $\{X_p\}_p$ to equations (PL_p) is convergent for $p \rightarrow 1$ to the solution X_1 of equation (PL_1) , strongly in $L^2(\Lambda)$, uniformly on $[0, T]$, \mathbb{P} -a.s., i.e.,

$$\lim_{p \rightarrow 1} \sup_{t \in [0, T]} \|X_p(t) - X_1(t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P} - \text{a.s.}$$

There is some evidence that the following conjecture is true, see [21, 22, 25].

Conjecture 4.5. *Let $d \in \{1, 2\}$. Then the semigroup*

$$P_t^1 F(x) := \mathbb{E}[F(X_1(t, x))], \quad F \in C_b(L^2(\Lambda)),$$

admits a unique invariant measure μ_1 .

Theorem 4.6. *Let $d = 1$. Suppose that Conjecture 4.5 is true. Let $X_p = X_p(t, x)$ be the solution to equation (PL_p), $p \in [1, 2]$. Let $\{p_n\} \subset (1, 2]$ such that $\lim_n p_n = 1$. Let*

$$P_t^p F(x) := \mathbb{E}[F(X_p^x(t))], \quad \varphi \in C_b(L^2(\Lambda)),$$

be the semigroup associated to equation (PL_p). Let μ_{p_n} , $n \in \mathbb{N}$, μ_1 be the associated unique invariant measures on $L^2(\Lambda)$. Then

$$\mu_{p_n} \rightarrow \mu_1 \quad \text{in the weak sense.}$$

Proof. Note that by Remark 4.1, the embedding $BV(\Lambda) \subset L^2(\Lambda)$ is compact. The proof is similar to that of Theorem 3.2, $W_0^{1,p_1}(\Lambda)$ therein replaced by $BV(\Lambda)$. \square

Proof of Theorem 4.4. For each $\varepsilon > 0$, let $R_\varepsilon := (1 - \varepsilon\Delta)^{-1}$ be the resolvent of the (Dirichlet) Laplace operator $(-\Delta, \text{dom}(-\Delta))$, where $\text{dom}(-\Delta) = H_0^1(\Lambda) \cap H^2(\Lambda)$. For $p \in [1, 2]$, $\varepsilon > 0$, let

$$(4.6) \quad \Phi_\varepsilon^p(u) := \int_\Lambda j_\varepsilon^p(\nabla R_\varepsilon u) d\xi, \quad u \in L^2(\Lambda).$$

Lemma 4.7. *Let $\{p_n\} \subset [1, 2]$ such that $\lim_n p_n = 1$. Let $\varepsilon > 0$. Then for $u \in L^2(\Lambda)$, we have that*

$$(4.7) \quad \lim_n \Phi_\varepsilon^{p_n}(u) = \Phi_\varepsilon^1(u).$$

Furthermore, if $u_n \rightharpoonup u$ converges weakly in $L^2(\Lambda)$, we have that

$$(4.8) \quad \varliminf_n \Phi_\varepsilon^{p_n}(u_n) \geq \Phi_\varepsilon^1(u).$$

Also, each Φ_ε^p , $p \in [1, 2]$, $\varepsilon > 0$, is continuous w.r.t. the weak topology of $L^2(\Lambda)$.

Proof. Since R_ε maps to $\text{dom}(-\Delta) \subset H_0^1(\Lambda)$, it is clear that $\nabla R_\varepsilon u \in L^2(\Lambda; \mathbb{R}^d)$ and hence (4.7) follows from (A.2).

Let $u_n \in L^2(\Lambda)$, $n \in \mathbb{N}$, $u \in L^2(\Lambda)$, such that $u_n \rightharpoonup u$ weakly in $L^2(\Lambda)$. If we can prove that $\nabla R_\varepsilon u_n \rightharpoonup \nabla R_\varepsilon u$ weakly in $L^2(\Lambda; \mathbb{R}^d)$, we can apply (A.3) and (4.7) follows. Indeed, we even have that $\nabla R_\varepsilon u_n \rightarrow \nabla R_\varepsilon u$ strongly in $L^2(\Lambda; \mathbb{R}^d)$.

The last part follows by repeating the compactness argument above and the strong $L^2(\Lambda; \mathbb{R}^d)$ -continuity of the Ψ_ε^p 's. \square

We first consider the following approximating equations for (PL_p)

$$(4.9) \quad \begin{cases} dX_p^\varepsilon(t) + A_p^\varepsilon(X_p^\varepsilon) dt = B dW(t) \\ X_p^\varepsilon(0) = x \end{cases}$$

where for any $u \in L^2(\Lambda)$,

$$A_p^\varepsilon(u) = -(1 - \varepsilon\Delta)^{-1} \text{div} \left[a_p^\varepsilon \left(\nabla (1 - \varepsilon\Delta)^{-1} u \right) \right]$$

and a_p^ε is the Yosida approximation of a_p i.e., for any $r \in \mathbb{R}^d$,

$$a_p^\varepsilon(r) = \frac{1}{\varepsilon} \left(1 - (1 + \varepsilon a_p)^{-1}(r) \right).$$

In particular, for $u, v \in L^2(\Lambda)$,

$$(A_p^\varepsilon(u), v)_{L^2(\Lambda)} = \int_\Lambda \langle a_p^\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v) \rangle d\xi.$$

We shall consider a similar approximation for equation (PL₁)

$$(4.10) \quad \begin{cases} dX_1^\varepsilon(t) + A^\varepsilon(X_1^\varepsilon) dt = B dW(t) \\ X_1^\varepsilon(0) = x \end{cases}$$

where for any $u \in L^2(\Lambda)$,

$$A^\varepsilon(u) = -(1 - \varepsilon\Delta)^{-1} \operatorname{div} \left[\beta^\varepsilon \left(\nabla (1 - \varepsilon\Delta)^{-1} u \right) \right].$$

with

$$\beta^\varepsilon(r) = \begin{cases} \frac{r}{\varepsilon}, & \text{if } |r| \leq \varepsilon, \\ \frac{r}{|r|}, & \text{if } |r| > \varepsilon. \end{cases}$$

In particular, for $u, v \in L^2(\Lambda)$,

$$(A^\varepsilon(u), v)_{L^2(\Lambda)} = \int_{\Lambda} \langle \beta^\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v) \rangle d\xi.$$

Note that β^ε is the Yosida approximation of the sign function, i.e., for any $r \in \mathbb{R}^d$,

$$\beta^\varepsilon(r) = \frac{1}{\varepsilon} \left(1 - (1 + \varepsilon \operatorname{sgn})^{-1}(r) \right).$$

In particular, $\beta^\varepsilon = \nabla j^\varepsilon$, where j^ε is the convex function defined by

$$j^\varepsilon(r) = \begin{cases} \frac{|r|^2}{2\varepsilon}, & \text{if } |r| \leq \varepsilon, \\ |r| - \frac{\varepsilon}{2}, & \text{if } |r| > \varepsilon. \end{cases}$$

We shall use the following strategy to prove the main result

$$\begin{aligned} & \|X_p(t) - X_1(t)\|_{L^2(\Lambda)} \\ & \leq \|X_p(t) - X_p^\varepsilon(t)\|_{L^2(\Lambda)} + \|X_p^\varepsilon(t) - X_1^\varepsilon(t)\|_{L^2(\Lambda)} + \|X_1^\varepsilon(t) - X_1(t)\|_{L^2(\Lambda)} \end{aligned}$$

\mathbb{P} -a.s. and uniformly in $t \in [0, T]$.

Step I

We note that, taking Remark 4.3 into account, the result of [11, equation (4.8)] remains valid in our case. Hence,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|X_1^\varepsilon(t) - X_1(t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s.}$$

Step II

Note that we have proved above (proof of Theorem 2.1) that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|X_p(t) - X_p^\varepsilon(t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s. uniformly in } p \in (1, 2).$$

Step III

In order to complete the proof we still need to show that for all $\varepsilon > 0$ fixed we have

$$\lim_{p \rightarrow 1} \sup_{t \in [0, T]} \|X_p^\varepsilon(t) - X_1^\varepsilon(t)\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s.}$$

To this aim, we consider the definition of the solution for equations

$$\begin{cases} dX_p^\varepsilon(t) + A_p^\varepsilon(X_p^\varepsilon) dt = B dW(t) \\ X_p^\varepsilon(0) = x \end{cases}$$

as

$$\begin{aligned} & \frac{1}{2} \|X_p^\varepsilon(t) - Y(t)\|_{L^2(\Lambda)}^2 + \int_0^t (\Phi_\varepsilon^p(X_p^\varepsilon(s)) - \Phi_\varepsilon^p(Y(s))) ds \\ & \leq \frac{1}{2} \|x - Y(0)\|_{L^2(\Lambda)}^2 + \int_0^t (G(s), X_p^\varepsilon(s) - Y(s))_{L^2(\Lambda)} ds, \\ & \text{for all } t \in [0, T], \mathbb{P}\text{-a.s.} \end{aligned}$$

We take $Y = X_1^\varepsilon$, the solution of equation

$$\begin{cases} dX_1^\varepsilon(t) + A^\varepsilon(X_1^\varepsilon) dt = B dW(t) \\ X_1^\varepsilon(0) = x. \end{cases}$$

and using the definition of the subdifferential we get that

$$\begin{aligned} (4.11) \quad & \frac{1}{2} \|X_p^\varepsilon(t) - X_1^\varepsilon(t)\|_{L^2(\Lambda)}^2 \\ & + \int_0^t (\Phi_\varepsilon^p(X_p^\varepsilon(s)) - \Phi_\varepsilon^p(X_1^\varepsilon(s)) + \Phi_\varepsilon^1(X_1^\varepsilon(s)) - \Phi_\varepsilon^1(X_p^\varepsilon(s))) ds \\ & \leq \frac{1}{2} \|x - X_1^\varepsilon(0)\|_{L^2(\Lambda)}^2 = 0, \end{aligned}$$

for $t \in [0, T]$ and \mathbb{P} -a.s. By estimate (2.3), we can extract a subsequence $\{p_n\}$ with $\lim_n p_n = 1$ such that for $X_n^\varepsilon := X_{p_n}^\varepsilon$ we have that for dt -a.a. $t \in [0, T]$, $X_n^\varepsilon(t) \rightharpoonup Z^\varepsilon(t)$ weakly in $L^2(\Lambda)$ \mathbb{P} -a.s. for some $dt \otimes \mathbb{P}$ -measurable Z^ε that satisfies

$$\sup_{t \in [0, T]} \|Z^\varepsilon(t)\|_{L^2(\Lambda)} \leq \liminf_n \sup_{t \in [0, T]} \|X_n(t)\|_{L^2(\Lambda)} \quad \mathbb{P}\text{-a.s.}$$

We shall need following lemma. Set $\Phi_\varepsilon^n := \Phi_{p_n}^\varepsilon$.

Lemma 4.8.

$$\Phi_\varepsilon^n(X_1^\varepsilon(\cdot)) - \Phi_\varepsilon^n(X_n^\varepsilon(\cdot)) + \Phi_\varepsilon^1(X_n^\varepsilon(\cdot)) - \Phi_\varepsilon^1(X_1^\varepsilon(\cdot))$$

is \mathbb{P} -a.s. bounded above by a function in $L^\infty(0, T)$.

Proof. Set $u := X_n^\varepsilon(\cdot)$, $v := X_1^\varepsilon(\cdot)$. Recall that in our notation, $R_\varepsilon := (1 - \varepsilon\Delta)^{-1}$.

Let us treat the term $\Phi_\varepsilon^1(u) - \Phi_\varepsilon^1(v)$ first. By the definition of the subgradient it is bounded by $(\nabla\Phi_\varepsilon^1(u), u - v)_{L^2(\Lambda)}$. But this term is equal to

$$\int_\Lambda \langle \beta^\varepsilon(\nabla R_\varepsilon(u)), \nabla R_\varepsilon(u - v) \rangle d\xi.$$

Since $|\beta^\varepsilon| \leq 1$, we get that the latter is bounded by $\|\nabla R_\varepsilon(u - v)\|_{L^2(\Lambda; \mathbb{R}^d)}$. By the proof of Lemma 4.7, ∇R_ε is a bounded operator from $L^2(\Lambda)$ to $L^2(\Lambda; \mathbb{R}^d)$.

We get that

$$\Phi_\varepsilon^1(X_n^\varepsilon(\cdot)) - \Phi_\varepsilon^1(X_1^\varepsilon(\cdot)) \leq C \sup_n \|X_n^\varepsilon(\cdot)\|_{L^2(\Lambda)} + C \|X_1^\varepsilon(\cdot)\|_{L^2(\Lambda)}$$

which is \mathbb{P} -a.s. in $L^\infty(0, T)$ again by estimate (2.3).

We continue with the term $\Phi_\varepsilon^n(v) - \Phi_\varepsilon^n(u)$. By the definition of the subgradient it is bounded by $(\nabla\Phi_\varepsilon^n(v), v - u)_{L^2(\Lambda)}$, which is equal to

$$\int_\Lambda \langle a_p^\varepsilon(\nabla R_\varepsilon(v)), \nabla R_\varepsilon(v - u) \rangle d\xi.$$

Noticing that r_ε^p is a contraction on \mathbb{R}^d , we can use a similar estimate as in (2.6) to get that the latter is bounded by

$$C + C \|\nabla R_\varepsilon(v)\|_{L^2(\Lambda; \mathbb{R}^d)} \|\nabla R_\varepsilon(v - u)\|_{L^2(\Lambda; \mathbb{R}^d)}.$$

Arguing as above, we see that this term is bounded by

$$C + C \sup_n \|X_n^\varepsilon(\cdot)\|_{L^2(\Lambda)} \|X_1^\varepsilon(\cdot)\|_{L^2(\Lambda)} + C \|X_1^\varepsilon(\cdot)\|_{L^2(\Lambda)}^2,$$

which is \mathbb{P} -a.s. in $L^\infty(0, T)$ by estimate (2.3). \square

We take the limit superior in (4.11) and continue investigating

$$\overline{\lim}_n \int_0^t [\Phi_\varepsilon^n(X_1^\varepsilon(s)) - \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s))] ds.$$

By Lemma 4.8, we can apply (reverse) Fatou's lemma such that it is sufficient to prove that

$$\overline{\lim}_n [\Phi_\varepsilon^n(X_1^\varepsilon(s)) - \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s))] \leq 0$$

\mathbb{P} -a.s. and for ds -a.e. $s \in [0, T]$. At this point, we apply Lemma 4.7 and get that

$$\begin{aligned} & \overline{\lim}_n [\Phi_\varepsilon^n(X_1^\varepsilon(s)) - \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s))] \\ & \leq \overline{\lim}_n \Phi_\varepsilon^n(X_1^\varepsilon(s)) - \underline{\lim}_n \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \overline{\lim}_n \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s)) \\ & \leq \Phi_\varepsilon^1(X_1^\varepsilon(s)) - \Phi_\varepsilon^1(Z^\varepsilon(s)) + \Phi_\varepsilon^1(Z^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s)) \\ & = 0, \end{aligned}$$

\mathbb{P} -a.s. and for ds -a.e. $s \in [0, T]$.

Final step. Going back to

$$\begin{aligned} & \|X_p(t) - X_1(t)\|_{L^2(\Lambda)} \\ & \leq \|X_p(t) - X_p^\varepsilon(t)\|_{L^2(\Lambda)} + \|X_p^\varepsilon(t) - X_1^\varepsilon(t)\|_{L^2(\Lambda)} + \|X_1^\varepsilon(t) - X_1(t)\|_{L^2(\Lambda)} \end{aligned}$$

\mathbb{P} -a.s. and uniformly in $t \in [0, T]$, we can complete the proof using Steps I–III as follows. Let $\delta > 0$. Pick $\varepsilon_0 > 0$, independent of p , such that the first and the third term are less than $\delta/3$. Having fixed ε_0 in such a way, we can pick p such that the second term is less than $\delta/3$. \square

APPENDIX A. SOME RESULTS ON VARIATIONAL CONVERGENCE

Let H be a separable Hilbert space. For a proper, convex functional $\Phi : H \rightarrow (-\infty, +\infty]$, the *Legendre transform* Φ^* is defined by

$$\Phi^*(y) := \sup_{x \in H} [(x, y)_H - \Phi(x)], \quad y \in H.$$

For two functionals $F, G : H \rightarrow (-\infty, +\infty]$ the *infimal convolution* $F \# G$ is defined by

$$(F \# G)(y) := \inf_{x \in H} [F(x) + G(y - x)], \quad y \in H.$$

For a proper, convex, l.s.c. functional $\Phi : H \rightarrow (-\infty, +\infty]$, for each $\varepsilon > 0$, define the *Moreau-Yosida regularization*

$$\Phi_\varepsilon := \Phi \# \frac{1}{2\varepsilon} \|\cdot\|_H^2.$$

Φ_ε is a continuous convex function. Also, $\lim_{\varepsilon \searrow 0} \Phi_\varepsilon = \Phi$ pointwise.

It holds that

$$(A.1) \quad (\Phi_\varepsilon)^* = \Phi^* + \frac{\varepsilon}{2} \|\cdot\|_H^2.$$

see e.g. [7, §2.2] and [4, Ch. 3].

Recall following definition.

Definition A.1 (Mosco convergence). *Let $\Phi^n : H \rightarrow (-\infty, +\infty]$, $n \in \mathbb{N}$, $\Phi : H \rightarrow (-\infty, +\infty]$ be proper, convex, l.s.c. functionals. We say that $\Phi^n \xrightarrow{M} \Phi$ in the Mosco sense if*

$$(M1) \quad \forall x \in H \quad \forall x_n \in H, n \in \mathbb{N}, x_n \rightharpoonup x \text{ weakly in } H : \varliminf_n \Phi^n(x_n) \geq \Phi(x).$$

$$(M2) \quad \forall y \in H \quad \exists y_n \in H, n \in \mathbb{N}, y_n \rightarrow y \text{ strongly in } H : \overline{\lim}_n \Phi^n(y_n) \leq \Phi(y).$$

We shall need following theorem.

Theorem A.2. *Let $\Phi^n : H \rightarrow (-\infty, +\infty]$, $n \in \mathbb{N}$, $\Phi : H \rightarrow (-\infty, +\infty]$ be proper, convex, l.s.c. functionals. Then the following conditions are equivalent.*

$$(i) \quad \Phi^n \xrightarrow{M} \Phi.$$

$$(ii) \quad (\Phi^n)^* \xrightarrow{M} \Phi^*.$$

$$(iii) \quad \forall \varepsilon > 0, \forall x \in H : \lim_n \Phi_\varepsilon^n(x) = \Phi_\varepsilon(x).$$

Proof. See [4, Theorems 3.18 and 3.26]. □

Corollary A.3. *Suppose that $\Phi^n \xrightarrow{M} \Phi$. Then for each $\varepsilon > 0$, $\Phi_\varepsilon^n \xrightarrow{M} \Phi_\varepsilon$, too.*

Proof. Suppose that $\Phi^n \xrightarrow{M} \Phi$. By Theorem A.2, $(\Phi^n)^* \xrightarrow{M} \Phi^*$, too.

If we can prove for each $\varepsilon > 0$ that $(\Phi_\varepsilon^n)^* \xrightarrow{M} (\Phi_\varepsilon)^*$, we are done by Theorem A.2. (M2) in Definition A.1 follows easily, using equation (A.1) and (M2) for $\{(\Phi_n)^*\}$ and Φ^* .

Let $x_n \in H$, $n \in \mathbb{N}$, $x \in H$ such that $x_n \rightharpoonup x$ weakly in H . By (A.1), weak lower semi-continuity of the norm and (M1) in Definition A.1 for $\{(\Phi_n)^*\}$ and Φ^* we get that

$$\begin{aligned} \varliminf_n (\Phi_\varepsilon^n)^*(x_n) &= \varliminf_n \left[(\Phi^n)^*(x_n) + \frac{\varepsilon}{2} \|x_n\|_H^2 \right] \\ &\geq \varliminf_n (\Phi^n)^*(x_n) + \varliminf_n \frac{\varepsilon}{2} \|x_n\|_H^2 \geq \Phi^*(x) + \frac{\varepsilon}{2} \|x\|_H^2 = (\Phi_\varepsilon)^*(x). \end{aligned}$$

□

A.1. The L^p -case. Let $p \in [1, 2]$. We define $j^p : \mathbb{R}^d \rightarrow \mathbb{R}$ by $j^p(x) := \frac{1}{p} |x|^p$. Obviously, if $p > 1$, each j^p is a convex C^1 -function. For $\varepsilon > 0$, let

$$j_\varepsilon^p(x) := \inf_{y \in \mathbb{R}^d} \left[j^p(y) + \frac{1}{2\varepsilon} |x - y|^2 \right]$$

be its regularization. For $u \in L^2(\Lambda; \mathbb{R}^d)$, set

$$\Psi^p(u) := \int_\Lambda j^p(u) d\xi.$$

Ψ^p is a continuous convex functional on $L^2(\Lambda; \mathbb{R}^d)$ for each $p \in [1, 2]$.

Lemma A.4. *For $\varepsilon > 0$, let Ψ_ε^p be the Moreau-Yosida regularization of Ψ^p in $L^2(\Lambda; \mathbb{R}^d)$. Then*

$$\Psi_\varepsilon^p(v) = \int_\Lambda j_\varepsilon^p(v) d\xi \quad \forall v \in L^2(\Lambda; \mathbb{R}^d).$$

Proof. Straightforward from [35, Theorem 14.60]. □

We would like to prove a convergence result, which shall be useful later. See the appendix for the terminology. Compare also with [5].

Lemma A.5. *Let $\{p_n\} \subset [1, 2]$, $p_0 \in [1, 2]$ such that $\lim_n p_n = p_0$. Then*

$$\Psi^{p_n} \xrightarrow{M} \Psi^{p_0} \quad \text{in the Mosco sense in } L^2(\Lambda; \mathbb{R}^d).$$

Proof. Let us prove (M1) in Definition A.1 first. Let $u_n \in L^2(\Lambda; \mathbb{R}^d)$, $n \in \mathbb{N}$, $u \in L^2(\Lambda; \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $L^2(\Lambda; \mathbb{R}^d)$. W.l.o.g. $\liminf_n \Psi^{p_n}(u_n) < +\infty$. Extract a subsequence (also denoted by $\{u_n\}$) such that

$$\liminf_n \Psi^{p_n}(u_n) = \lim_n \Psi^{p_n}(u_n).$$

Let $v \in L^\infty(\Lambda; \mathbb{R}^d)$. Clearly,

$$\lim_n \int_\Lambda \langle u_n, v \rangle d\xi = \int_\Lambda \langle u, v \rangle d\xi.$$

Also, by Hölder's inequality,

$$\frac{1}{p_n} \left| \int_\Lambda \langle u_n, v \rangle d\xi \right|^{p_n} \leq \Psi^{p_n}(u_n) \times \begin{cases} |\Lambda|^{p_n-1} \|v\|_{L^\infty(\Lambda; \mathbb{R}^d)}^{p_n}, & \text{if } p_0 = 1, \\ \left(\int_\Lambda |v|^{p_n/(p_n-1)} d\xi \right)^{p_n-1}, & \text{if } p_0 > 1, \end{cases}$$

(here $|\Lambda| = \int_\Lambda d\xi$). Upon taking the limit $n \rightarrow \infty$, we get that

$$\frac{1}{p_0} \left| \int_\Lambda \langle u, v \rangle d\xi \right|^{p_0} \leq \liminf_n \Psi^{p_n}(u_n) \times \begin{cases} \|v\|_{L^\infty(\Lambda; \mathbb{R}^d)}, & \text{if } p_0 = 1, \\ \left(\int_\Lambda |v|^{p_0/(p_0-1)} d\xi \right)^{p_0-1}, & \text{if } p_0 > 1. \end{cases}$$

Taking the supremum over all $v \in L^\infty(\Lambda; \mathbb{R}^d)$ with $\|v\|_{L^\infty(\Lambda; \mathbb{R}^d)}^{p_0/(p_0-1)} \leq 1$ and using the l.s.c. property of the supremum, we get that

$$\Psi^{p_0}(u) = \frac{1}{p_0} \int_\Lambda |u|^{p_0} d\xi \leq \liminf_n \Psi^{p_n}(u_n).$$

Since the same argument works for any subsequence of $\{u_n\}$, we have proved (M1).

We are left to prove (M2) in Definition A.1. Let $u \in L^2(\Lambda; \mathbb{R}^d)$. Clearly for a.e. $\xi \in \Lambda$

$$\lim_n \frac{1}{p_n} |u(\xi)|^{p_n} = \frac{1}{p_0} |u(\xi)|^{p_0}.$$

But for all $p \in [1, 2]$,

$$\frac{1}{p} |u|^p \leq 1_\Lambda + |u|^2 \in L^1(\Lambda).$$

Hence an application of Lebesgue's dominated convergence theorem yields

$$\lim_n \Psi^{p_n}(u) = \Psi^{p_0}(u).$$

(M2) is proved. □

Theorem A.2, Corollary A.3 and Lemmas A.4, A.5 together give:

Corollary A.6. *Let $\{p_n\} \subset [1, 2]$ such that $\lim_n p_n = 1$. Let $\varepsilon > 0$. Then for $u \in L^2(\Lambda; \mathbb{R}^d)$, we have that*

$$(A.2) \quad \lim_n \int_\Lambda j_\varepsilon^{p_n}(u) d\xi = \int_\Lambda j_\varepsilon^1(u) d\xi.$$

Furthermore, if $u_n \rightharpoonup u$ converges weakly in $L^2(\Lambda; \mathbb{R}^d)$, we have that

$$(A.3) \quad \liminf_n \int_\Lambda j_\varepsilon^{p_n}(u_n) d\xi \geq \int_\Lambda j_\varepsilon^1(u) d\xi.$$

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