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On the Asymptotic Spectrum of Products of Independent Random Matrices.

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Abstract

We consider products of independent random matrices with independent entries. The limit distribution of the expected empirical distribution of eigenvalues of such products is computed. Let $X_{jk}^{(\nu)}, 1 \le j,r \le n$, $\nu = 1,\dots,m$ be mutually independent complex random variables with $\mathbf{E}\, X_{jk}^{(\nu)} = 0$ and $\mathbf{E}\,|X_{jk}^{(\nu)}|^2 = 1$. Let $\mathbf{X}^{(\nu)}$ denote an $n\times n$ matrix with entries $[\mathbf{X}^{(\nu)}]_{jk} = \frac{1}{\sqrt{n}} X_{jk}^{(\nu)}$, for $1 \le j,k \le n$. Denote by $\lambda_1,\dots,\lambda_n$ the eigenvalues of the random matrix $\mathbf{W} := \prod_{\nu=1}^{m} \mathbf{X}^{(\nu)}$ and define its empirical spectral distribution by

$$\mathcal{F}_n(x,y) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}\{\mathrm{Re}\,\lambda_k \le x, \mathrm{Im}\,\lambda_k \le y\},$$

where $\mathbb{I}\{B\}$ denotes the indicator of an event B. We prove that the expected spectral distribution $F_n^{(m)}(x,y) = \mathbf{E}\,\mathcal{F}_n^{(m)}(x,y)$ converges to the distribution function $G(x,y)$ corresponding to the m-th power of the uniform distribution on the unit disc in the plane $\mathbb{R}^2$.

1 Introduction

Let $m \ge 1$ be a fixed integer. For any $n \ge 1$ consider mutually independent identically distributed (i.i.d.) complex random variables $X_{jk}^{(\nu)}$, $1 \le j,k \le n$, $\nu = 1,\dots,m$ with $\mathbf{E}\, X_{jk}^{(\nu)} = 0$ and $\mathbf{E}\,|X_{jk}^{(\nu)}|^2 = 1$ defined on a common probability space $(\Omega_n, \mathbb{F}_n, \Pr)$. Let

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$\mathbf{X}^{(\nu)}$ denote an $n \times n$ matrix with entries $[\mathbf{X}^{(\nu)}]_{jk} = \frac{1}{\sqrt{n}} X_{jk}^{(\nu)}$, for $1 \le j,k \le n$. Denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of the random matrix $\mathbf{W} := \prod_{\nu=1}^{m} \mathbf{X}^{(\nu)}$ and define its empirical spectral distribution function by

$$\mathcal{F}_n(x,y) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{I}\{\operatorname{Re}\lambda_k \le x, \operatorname{Im}\lambda_k \le y\},$$

where $\mathbb{I}\{B\}$ denotes the indicator of an event B. We shall investigate the convergence of the expected spectral distribution $F_n(x,y) = \mathbf{E}\,\mathcal{F}_n(x,y)$ to the distribution function $G(x,y)$ corresponding to the m-th power of uniform distribution on the unit disc in the plane $\mathbb{R}^2$ with Lebesgue-density

$$g(x,y) = \frac{1}{\pi m (x^2+y^2)^{\frac{m-1}{m}}} I\{x^2+y^2 \le 1\}.$$

We consider the Kolmogorov distance between the distributions $F_n(x,y)$ and $G(x,y)$

$$\Delta_n := \sup_{x,y} |F_n(x,y) - G(x,y)|.$$

The main result of this paper is the following

Theorem 1.1. *Let* $\mathbf{E}\, X_{jk}^{(\nu)} = 0$, $\mathbf{E}\,|X_{jk}^{(\nu)}|^2 = 1$. *Then, for any fixed* $m \ge 1$,

$$\lim_{n\to\infty} \sup_{x,y} |F_n(x,y) - G(x,y)| = 0.$$

The result holds in the non-i.i.d. case too.

Theorem 1.2. *Let* $\mathbf{E}\, X_{jk}^{(\nu)} = 0$, $\mathbf{E}\,|X_{jk}^{(\nu)}|^2 = 1$ *and assume that the random variables* $X_{jk}^{(\nu)}$ *have uniformly integrable second moments, i. e.*

$$\max_{\nu,j,k} \mathbf{E}\,|X_{jk}^{(\nu)}|^2 I\{|X_{jk}^{(\nu)}| > M\} \to 0 \quad as \quad M \to \infty. \tag{1.1}$$

Then for any fixed $m \ge 1$,

$$\lim_{n\to\infty} \sup_{x,y} |F_n(x,y) - G(x,y)| = 0.$$

Definition 1.3. Let $\mu_n(\cdot)$ denote the empirical spectral measure of an $n \times n$ random matrix $\mathbf{X}$ and let $\mu(\cdot)$ denote the uniform distribution on the unit disc in the complex plane $\mathbb{C}$. We say that the circular law holds for random matrices $\mathbf{X}$ if $\mathbf{E}\,\mu_n(\cdot)$ converges weakly to the measure $\mu(\cdot)$ in the complex plane $\mathbb{C}$.

Remark 1.4. For $m = 1$ we recover the well-known circular law for random matrices [9], [15].

Theorems 1.1 and 1.2 describe the asymptotics of the spectral distribution of a product of m independent random matrices. This generalizes the result of [9] and [15].

1.1 Discussion of results

The proof of these results are based on the author's investigations on asymptotics of the singular spectrum of product and powers of random matrices with independent entries (see [1], [2], [3]). Our results give a full description of the complex spectral distribution of products of large random matrices. The results mentioned on the asymptotic distribution of the singular spectrum of products of independent random matrices where already obtained some time ago by the authors, see [2],[1]. Related previous results concerned bounds for the expectation of the operator norm of two independent matrices, see Bai (1986). [5]. In Bai (2007), [4], the asymptotic distribution of the product of a sample covariance matrix and an independent Wigner matrix is investigated. Some questions about the asymptotic distribution of products and powers of random matrices were studied in Free Probability. For example, in Capitaine (2008), [7], the asymptotic distribution of the singular value distribution of the product of squares of independent Gaussian random matrices is determined. In Speicher (2008), [14], the same asymptotic distribution has been obtained for the singular value distribution of products and powers of random matrices.

A related result for norms has been obtained by Haagerup and Torbjønson [10], who proved that if $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}$ is a system of independent Gaussian random matrices and $x_1, \ldots, x_r$ is a corresponding semi-circular system in a C^* probability space, then for every polynomial p in r non commuting variables we have an asymptotic norm equality

$$\lim_{n\to\infty} \|p(\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)})\| = \|p(x_1, \ldots, x_m)\| \tag{1.2}$$

which holds almost surely.

Our result on the asymptotic distribution of the complex eigenvalues of products of large (non-Hermitian and non Gaussian) random matrices seemed to be new. After finishing this paper we learned though that the case of products of *Gaussian* had been studied by Burda et al. (2010), [6], with our main result stated as conjecture, supported by simulations.

We expect that results of this type will be useful for the analysis of some models of wireless communication. See for instance, [11].

The results of both Theorems 1.1 and 1.2 may be considered as generalizations of the circular law, see e.g. [9] for some history on the circular law and its proof.

To prove the claim of both Theorems 1.1 and 1.2 we use the logarithmic potential approach as in [9]. We may divide this approach into two parts. The first part deals with the investigation of the asymptotic distribution of the singular values of shifted matrices $\mathbf{W}(z) := \mathbf{W} - z\mathbf{I}$. To study these distributions we use the method developed in [3] for the case $z = 0$. The other part will be the investigation of small singular values of matrices $\mathbf{W}(z)$ for any $z \in \mathbb{C}$. This problem may be divided again in two parts. The first part consists of the investigation of small singular values. Here we may use our results in [9] or the results in [15]. The second part deals with the investigation of the singular values between the smallest one to the jth smallest one, where $j \geq n - n^{\gamma}$ for some $0 < \gamma < 1$. Here we use a modification of techniques of Tao and Vu in [15].

In the remaining parts of the paper we give the proof of Theorem 1.2. Theorem 1.1 follows immediately from 1.2. We shall use the logarithmic potential method which is outlined in detail in [9].

In Section 3 we derive the approximation of the singular measure of the shifted matrix $\mathbf{W}(z)$ for any $z \in \mathbb{C}$. This allows us to prove the convergence of the empirical spectral measure of the matrix $\mathbf{W}(z)$ to the corresponding limit measure in $\mathbb{R}^2$. The convergence is proved in Section 6.

In the what follows we shall denote by C and c or δ, ρ, η (without indices) some general absolute constant which may be change from one line to next one. To specify a constant we shall use subindices. By $I\{A\}$ we shall denote the indicator of an event A. For any matrix $\mathbf{G}$ we denote the Frobenius norm by $\|\mathbf{G}\|_2$ and we denote by $\|\mathbf{G}\|$ its operator norm.

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2 Auxiliary Results

In this Section we describe a symmetrization of one-sided distribution and a special representation of symmetrized distributions of squares singular values of random matrices and prove some lemmas about a truncation of entries of random matrices.

2.1 Symmetrization

We shall use the following "symmetrization" of one-sided distributions. Let ξ^2 be a positive random variable with distribution function $F(x)$. Define $\widetilde{\xi} := \varepsilon\xi$ where ε is a Rademacher random variable with $\Pr\{\varepsilon = \pm 1\} = 1/2$ which is independent of ξ. Let $\widetilde{F}(x)$ denote the distribution function of $\widetilde{\xi}$. It satisfies the equation

$$\widetilde{F}(x) = 1/2(1 + \operatorname{sign}\{x\}\, F(x^2)), \tag{2.1}$$

We apply this symmetrization to the distribution of the squared singular values of the matrix $\mathbf{W}(z)$. Introduce the following matrices

$$\mathbf{V} := \begin{pmatrix} \mathbf{W} & \mathbf{O} \\ \mathbf{O} & \mathbf{W}^* \end{pmatrix}, \quad \mathbf{J}(z) := \begin{pmatrix} \mathbf{O} & z\mathbf{I} \\ \overline{z}\mathbf{I} & \end{pmatrix}, \quad \mathbf{J} := \mathbf{J}(1).$$

Here and in the what follows $\mathbf{A}^*$ denotes the adjoined (transposed and complex conjugate) matrix $\mathbf{A}$ and $\mathbf{O}$ denotes the matrix with zero-entries. Consider matrix

$$\mathbf{V}(z) := \mathbf{V}\mathbf{J} - \mathbf{J}(z). \tag{2.2}$$

Note that $\mathbf{V}(z)$ is a Hermitian matrix. The eigenvalues of the matrix $\mathbf{V}(z)$ are $-s_1, \ldots, -s_n, s_n, \ldots, s_1$. Note that the symmetrization of the distribution function $\mathcal{F}_n(x, z)$ is a function $\widetilde{\mathcal{F}}_n(x, z)$

is the empirical distribution function of the non-zero eigenvalues of the matrix $\mathbf{V}(z)$. By (2.1), we have

$$\Delta_n = \sup_x |\widetilde{F}_n(x,z) - \widetilde{G}(x,z)|,$$

where $\widetilde{F}_n(x,z) = \mathbf{E}\,\widetilde{\mathcal{F}}_n(x,z)$ and $\widetilde{G}(x,z)$ denotes the symmetrization of the distribution function $G^{(m)}(x,z)$.

2.2 Truncation

We shall now modify the random matrix $\mathbf{X}^{(\nu)}$ by truncation of its entries. In this section we shall assume that the random variables $X_{jk}^{(\nu)}$ satisfy the following Lindeberg condition: for any $\tau > 0$

$$L_n(\tau) = \max_{1\le\nu\le m} \frac{1}{n^2} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}^{(\nu)}|^2 I\{|X_{jk}^{(\nu)}| \ge \tau\sqrt{n}\} \to 0, \quad \text{as} \quad n \to \infty. \tag{2.3}$$

It is straightforward to check that this Lindeberg condition follows from uniform integrability. We introduce the random variables $X_{jk}^{(\nu,c)} = X_{jk}^{(\nu)} I_{\{|X_{jk}^{(\nu)}| \le c\tau_n\sqrt{n}\}}$ with $\tau_n \to 0$ and the matrices $\mathbf{X}^{(\nu,c)} = \frac{1}{\sqrt{n}}(X_{jk}^{(\nu,c)})$ and $\mathbf{W}^{(c)} := \prod_{\nu=1}^{m} \mathbf{X}^{(\nu,c)}$. Denote by $s_1^{(c)} \ge \ldots \ge s_n^{(c)}$ the singular values of the random matrix $\mathbf{W}^{(c)} - z\mathbf{I}$. Let $\mathbf{V}^{(c)} := \begin{pmatrix} \mathbf{W}^{(c)} & \mathbf{O} \\ \mathbf{O} & \mathbf{W}^{(c)*} \end{pmatrix}$. We define the empirical distribution of the matrix $\mathbf{V}^{(c)}(z) = \mathbf{V}^{(c)}\mathbf{J} - \mathbf{J}(z)$ by $\widetilde{\mathcal{F}}_n^{(c)}(x) = \frac{1}{2n}\sum_{k=1}^{n} I\{s_k^{(c)} \le x\} + \frac{1}{2n}\sum_{k=1}^{n} I\{-s_k^{(c)} \le x\}$. Let $s_n(\alpha,z)$ and $s_n^{(c)}(\alpha,z)$ denote the Stieltjes transforms of the distribution functions $\widetilde{F}_n(x)$ and $\widetilde{F}_n^{(c)}(x) = \mathbf{E}\,\widetilde{\mathcal{F}}_n^{(c)}(x)$ respectively. Define the resolvent matrices $\mathbf{R} = (\mathbf{V}(z) - \alpha\mathbf{I})^{-1}$ and $\mathbf{R}^{(c)} = (\mathbf{V}^{(c)}(z) - \alpha\mathbf{I})^{-1}$, where $\mathbf{I}$ denotes the identity matrix of corresponding dimension. Note that

$$s_n(\alpha,z) = \frac{1}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}, \qquad \text{and} \qquad s_n^{(c)}(\alpha,z) = \frac{1}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}^{(c)}.$$

Applying the resolvent equality

$$(\mathbf{A}+\mathbf{B}-\alpha\mathbf{I})^{-1} = (\mathbf{A}-\alpha\mathbf{I})^{-1} - (\mathbf{A}-\alpha\mathbf{I})^{-1}\mathbf{B}(\mathbf{A}+\mathbf{B}-\alpha\mathbf{I})^{-1}, \tag{2.4}$$

we get

$$|s_n(\alpha,z) - s_n^{(c)}(\alpha,z)| \le \frac{1}{2n}\mathbf{E}\,|\mathrm{Tr}\,\mathbf{R}^{(c)}(\mathbf{V}(z) - \mathbf{V}^{(c)}(z))\mathbf{J}\mathbf{R}|. \tag{2.5}$$

Let

$$\mathbf{H}^{(\nu)} = \begin{pmatrix} \mathbf{X}^{(\nu)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-\nu+1)*} \end{pmatrix} \quad \text{and} \quad \mathbf{H}^{(\nu,c)} = \begin{pmatrix} \mathbf{X}^{(\nu,c)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-\nu+1,c)*} \end{pmatrix}$$

Introduce the matrices

$$\mathbf{V}_{a,b} = \prod_{q=a}^{b} \mathbf{H}^{(q)}, \quad \mathbf{V}_{a,b}^{(c)} = \prod_{q=a}^{b} \mathbf{H}^{(q,c)}.$$

We have

$$\mathbf{V}(z) - \mathbf{V}^{(c)}(z) = [\mathbf{V} - \mathbf{V}^{(c)}]\mathbf{J} = \left[\sum_{q=1}^{m-1} \mathbf{V}_{1,q-1}^{(c)}(\mathbf{H}^{(q)} - \mathbf{H}^{(q,c)})\mathbf{V}_{q+1,m}\right]\mathbf{J}. \tag{2.6}$$

Applying $\max\{\|\mathbf{R}\|, \|\mathbf{R}^{(c)}\|\} \le v^{-1}$, inequality (2.5), and the representations (2.6) together, we get

$$|s_n^{(m)}(z) - s_n^{(c)}(z)| \le \frac{C}{\sqrt{n}} \sum_{q=1}^{m} \mathbf{E}^{\frac{1}{2}} \|(\mathbf{X}^{(q+1)} - \mathbf{X}^{(q+1,c)})\|_2^2 \frac{1}{\sqrt{n}} \mathbf{E}^{\frac{1}{2}} \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{R}\mathbf{R}^{(c)} \mathbf{V}_{q+1,m}\|_2^2. \tag{2.7}$$

By multiplicative inequalities for the matrix norm, we get

$$\mathbf{E}\, \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{R}\mathbf{R}^{(c)} \mathbf{V}_{q+1,m}\|_2^2 \le \frac{C}{v^4} \mathbf{E}\, \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{V}_{q+1,m}\|_2^2$$

Applying the result of Lemma 7.2, we obtain

$$\mathbf{E}\, \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{R}\mathbf{R}^{(c)} \mathbf{V}_{q+1,m}\|_2^2 \le \frac{Cn}{v^4}. \tag{2.8}$$

Direct calculations show that

$$\frac{1}{n} \mathbf{E}\, \|\mathbf{X}^{(q)} - \mathbf{X}^{(q,c)}\|_2^2 \le \frac{C}{n^2} \sum_{j,k=1}^{n} \mathbf{E}\, |X_{jk}^{(q)}|^2 I_{\{|X_{jk}^{(q)}| \ge c\tau_n\sqrt{n}\}} \le C L_n(\tau_n). \tag{2.9}$$

Inequalities (2.7), (2.8) and 2.9) together imply

$$|s_n(\alpha, z) - s_n^{(c)}(\alpha, z)| \le \frac{C\sqrt{L_n(\tau_n)}}{v^2}. \tag{2.10}$$

Furthermore, by definition of $X_{jk}^{(c)}$, we have

$$|\mathbf{E}\, X_{jk}^{(q,c)}| \le \frac{1}{c\tau_n\sqrt{n}} \mathbf{E}\, |X_{jk}^{(q)}|^2 I_{\{|X_{jk}| \ge c\tau_n\sqrt{n}\}}.$$

This implies that

$$\|\mathbf{E}\, \mathbf{X}^{(q,c)}\|_2^2 \le \frac{C}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathbf{E}\, X_{jk}^{(q,c)}|^2 \le \frac{C L_n(\tau_n)}{c\tau_n^2}. \tag{2.11}$$

Corresponding to $\mathbf{H}^{(\nu,c)}$ introduce $\widetilde{\mathbf{H}}^{(\nu,c)} := \begin{pmatrix} \mathbf{X}^{(\nu,c)} - \mathbf{E}\, \mathbf{X}^{(\nu,c)}) & \mathbf{O} \\ \mathbf{O} & (\mathbf{X}^{(\nu,c)} - \mathbf{E}\, \mathbf{X}^{(\nu,c)})^* \end{pmatrix}$ and for the matrices $\mathbf{W}^{(c)}, \mathbf{V}^{(c)}, \mathbf{V}_{a,b}^{(c)}$ define matrices $\widetilde{\mathbf{W}}^{(c)}$, $\widetilde{\mathbf{V}}^{(c)}$, $\widetilde{\mathbf{V}}_{a,b}^{(c)}$ respectively. Denote by $\widetilde{\mathcal{F}}_n^{(c)}(x)$ the empirical distribution of the squared singular values of the matrix $\widetilde{\mathbf{V}}^{(c)}(z) :=$

$\widetilde{\mathbf{V}}^{(c)}\mathbf{J} - \mathbf{J}(z)$. Let $\widetilde{s}_n^{(c)}(\alpha, z)$ denote the Stieltjes transform of the distribution function $\widetilde{F}_n^{(c)} = \mathbf{E}\,\widetilde{\mathcal{F}}_n^{(c)}$,

$$\widetilde{s}_n^{(c)}(\alpha, z) = \int_{-\infty}^{\infty} \frac{1}{x-\alpha} d\widetilde{F}_n^{(c)}(x).$$

Similar to inequality (2.7) we get

$$|s_n^{(c)}(\alpha, z) - \widetilde{s}_n^{(c)}(\alpha, z)| \le \sum_{q=0}^{m-1} \frac{1}{\sqrt{n}} \|\mathbf{E}\,\mathbf{X}^{(q,c)}\|_2 \frac{1}{\sqrt{n}} \mathbf{E}^{\frac{1}{2}} \|\widetilde{\mathbf{V}}_{0,q}^{(c)} \mathbf{R}^{(c)} \widetilde{\mathbf{R}}^{(c)} \widetilde{\mathbf{V}}_{q+1,m}^{(c)}\|_2^2.$$

Similar to inequality (2.8), we get

$$\frac{1}{n} \mathbf{E}\, \|\widetilde{\mathbf{V}}_{0,q}^{(c)} \mathbf{R}^{(c)} \widetilde{\mathbf{R}}^{(c)} \widetilde{\mathbf{V}}_{q+1,m}^{(c)}\|_2^2 \le \frac{C}{v^4}.$$

By inequality (2.11),

$$\|\mathbf{E}\,\mathbf{X}^{(q,c)}\|_2 \le \frac{C\sqrt{L_n(\tau_n)}}{c\tau_n}.$$

The last two inequalities together imply that

$$|s_n^{(c)}(\alpha, z) - \widetilde{s}_n^{(c)}(\alpha, z)| \le \frac{C\sqrt{L_n(\tau_n)}}{\sqrt{n}\tau_n v^2} \le \frac{\tau_n}{\sqrt{n} v^2} \tag{2.12}$$

Inequalities (2.10) and (2.12) together imply that the matrices $\mathbf{W}$ and $\widetilde{\mathbf{W}}^{(c)}$ have the same limit distribution. In the what follows we may assume without loss of generality for any $\nu = 1, \dots, m$ and $j = 1, \dots n$, $k = 1, \dots, n$ and any $l = 1, \dots, m$, that

$$\mathbf{E}\, X_{jk}^{(\nu)} = 0, \quad \mathbf{E}\, {X_{jk}^{(\nu)}}^2 = 1, \quad \text{and} \quad |X_{jk}^{(\nu)}| \le c\tau_n\sqrt{n} \tag{2.13}$$

with

$$L_n(\tau_n)/\tau_n^2 \le \tau_n.$$

3 The Limit Distribution of Singular Values of the Matrices $\mathbf{V}(z)$

Recall that $\mathbf{H}^{(\nu)} = \begin{pmatrix} \mathbf{X}^{(\nu)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(\mathbf{m}-\nu+\mathbf{1}^*} \end{pmatrix}$ and $\mathbf{J}(z) := \begin{pmatrix} \mathbf{O} & z\,\mathbf{I} \\ \overline{z}\,\mathbf{I} & \mathbf{O} \end{pmatrix}$, $\mathbf{J} := \mathbf{J}(1)$. For any $1 \le \nu \le \mu \le m$, put

$$\mathbf{V}_{[\nu,\mu]} = \prod_{k=\nu}^{\mu} \mathbf{H}^{(k)}, \qquad \mathbf{V} = \mathbf{V}_{[1,m]}.$$

and

$$\mathbf{V}(z) := \mathbf{V}\mathbf{J} - \mathbf{J}(z).$$

We introduce the following functions

$$s_n(\alpha,z)=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,[\mathbf{R}(\alpha,z)]_{jj}=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,[\mathbf{R}(\alpha,z)]_{j+nj+n}=\frac{1}{2n}\sum_{j=1}^{2n}\mathbf{E}\,[\mathbf{R}(\alpha,z)]_{jj}$$
$$t_n(\alpha,z)=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,[\mathbf{R}(\alpha,z)]_{j+nj},\quad u_n(\alpha,z)=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,[\mathbf{R}(\alpha,z)]_{jj+n}. \tag{3.1}$$

Theorem 3.1. *If the random variables $X_{jk}^{(\nu)}$ satisfy the Lindeberg condition (2.3), the following limits exist*

$$y=y(z,\alpha)=\lim_{n\to\infty}s_n(\alpha,z),\quad t=t(z,\alpha)=\lim_{n\to\infty}t_n(\alpha,z),$$

and satisfy the equations

$$\begin{aligned}&1+wy+(-1)^{m+1}w^{m-1}y^{m+1}=0,\\&y(w-\alpha)^2+(w-\alpha)-y|z|^2=0,\\&w=\alpha+\frac{zt}{y}.\end{aligned} \tag{3.2}$$

Remark 3.2. Since the Lindeberg condition holds for i.i.d. random variables and for uniformly integrable random variables the conclusion of Theorem 3.1 holds by Theorem 1.1 and Theorem 1.2.

Proof. In the what follows we shall denote by $\varepsilon_n(\alpha,z)$ a generic error function such that $|\varepsilon_n(\alpha,z)|\le\frac{C\tau_n^q}{v^r}$ for some positive constants C,p,r. By the resolvent equality, we may write

$$1+\alpha s_n(\alpha,z)=\frac{1}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{V}(z)\mathbf{R}(\alpha,z)=\frac{1}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{VJR}(\alpha,z)-zt_n(\alpha,z)-\overline{z}u_n(\alpha,z). \tag{3.3}$$

In the following we shall write $\mathbf{R}$ instead of $\mathbf{R}(\alpha,z)$. Introduce the notation

$$\mathcal{A}:=\frac{1}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{VJR} \tag{3.4}$$

and represent $\mathcal{A}$ as follows

$$\mathcal{A}=\frac{1}{2}\mathcal{A}_1+\frac{1}{2}\mathcal{A}_2, \tag{3.5}$$

where

$$\mathcal{A}_1=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,[\mathbf{VJR}]_{jj},\quad \mathcal{A}_2=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,[\mathbf{VJR}]_{j+n,j+n}.$$

By definition of the matrix $\mathbf{V}$, we have

$$\mathcal{A}_1=\frac{1}{n}\sum_{j,k=1}^{n}\mathbf{E}\,X_{jk}^{(1)}[\mathbf{V}_{[2,m]}\mathbf{JR}]_{kj}. \tag{3.6}$$

Note that

$$\frac{\partial \mathbf{V}_{[2,m]}\mathbf{JR}}{\partial X_{jk}^{(1)}} = \mathbf{V}_{[2,m-1]}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{JR} - \mathbf{V}_{[2,m]}\mathbf{JRe}_j\mathbf{e}_k^T\mathbf{V}_{[2,m]}\mathbf{JR} - \mathbf{V}_{[2,m]}\mathbf{JRV}_{[1,m-1]}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{JR}. \tag{3.7}$$

Applying now the Lemmas 7.8, we obtain

$$\mathcal{A}_1 = -\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\left[\mathbf{V}_{[2,m]}\mathbf{JRV}_{[1,m-1]}\right]_{kk+n}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{JR}\right]_{j+n,j} + \varepsilon_n(z,\alpha). \tag{3.8}$$

Introduce the notation, for $\nu = 2, \ldots, m$

$$f_\nu = \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}_{[\nu,m]}\mathbf{JRV}_{[1,m-\nu+1]}\right]_{jj+n} \tag{3.9}$$

We rewrite the equality (3.8) using these notations

$$\mathcal{A}_1 = -f_2 s_n(\alpha, z) + \varepsilon_n(z,\alpha). \tag{3.10}$$

We shall investigate the asymptotics of f_ν for $\nu = 2, \ldots, m$. By definition of the matrix $\mathbf{V}_{[\nu,m]}$, we have

$$f_\nu = \frac{1}{n}\sum_{k,j=1}^{n}\mathbf{E}\, X_{jk}^{(\nu)}[\mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,m-\nu+1]}]_{kj+n} \tag{3.11}$$

For simplicity assume that $\nu \le m - \nu$. Then

$$\begin{aligned}\frac{\partial \mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,m-\nu+1]}}{\partial X^{(\nu)}} &= \mathbf{V}_{[\nu+1,m-\nu]}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{[m-\nu+2,m]}\mathbf{JRV}_{[1,m-\nu+1]} \\ &+ \mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,m-\nu]}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T \\ &+ \mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,\nu-1]}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{[\nu+1,m-\nu+1]} \\ &- \mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,\nu-1]}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,m-\nu+1]} \\ &- \mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,m-\nu]}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{[m-\nu+2,m]}\mathbf{JRV}_{[1,m-\nu+1]}\end{aligned} \tag{3.12}$$

Applying the Lemmas 7.8 again, we get

$$\begin{aligned}f_\nu &= \frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\left[\mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,m-\nu]}\right]_{kk+n} \\ &- \frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\left[\mathbf{V}_{[\nu+1,m]}\mathbf{JRV}_{[1,m-\nu]}\right]_{kk+n}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}_{[m-\nu+2,m]}\mathbf{JRV}_{[1,m-\nu+1]}\right]_{j+nj+n} \\ &= f_{\nu+1}(1 - \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}_{[m-\nu+2,m]}\mathbf{JRV}_{[1,m-\nu+1]}\right]_{j+nj+n})\end{aligned} \tag{3.13}$$

Note that

$$\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}_{[m-\nu+2,m]}\mathbf{J}\mathbf{R}\mathbf{V}_{[1,m-\nu+1]}\right]_{j+nj+n}=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}_{[1,m]}\mathbf{J}\mathbf{R}\right]_{j+nj+n} \tag{3.14}$$

Furthermore,

$$\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}_{[1,m]}\mathbf{J}\mathbf{R}\right]_{j+nj+n}=1+\alpha s_n(\alpha,z)+\overline{z}u_n(\alpha,z). \tag{3.15}$$

Relations (3.12)–(3.15) together imply

$$f_\nu=f_{\nu+1}(-\alpha s_n(\alpha,z)-\overline{z}u_n(\alpha,z))+\varepsilon_n(z,\alpha). \tag{3.16}$$

By induction we get

$$f_2=(-1)^{m-1}(\alpha s_n(\alpha,z)+\overline{z}u_n(\alpha,z))^{m-1}s_n(\alpha,z)+\varepsilon_n(z,\alpha). \tag{3.17}$$

Relations (3.10) and (3.17) together imply

$$\mathcal{A}_1=(-1)^m(\alpha s_n(\alpha,z)+\overline{z}u_n(\alpha,z))^{m-1}s_n^2(z,\alpha)+\varepsilon_n(z,\alpha). \tag{3.18}$$

Similar we get that

$$g_2=(-1)^{m-1}(\alpha s_n(\alpha,z)+zt_n(\alpha,z))^{m-1}s_n(\alpha,z)+\varepsilon_n(z,\alpha). \tag{3.19}$$

and

$$\mathcal{A}_2=(-1)^m(\alpha s_n(\alpha,z)+zt_n(\alpha,z))^{m-1}s_n^2(z,\alpha)+\varepsilon_n(z,\alpha). \tag{3.20}$$

Consider now the function $t_n(\alpha,z)$ which we may represent as follows

$$\alpha t_n(\alpha,z)=\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}(z)\mathbf{R}\right]_{j+nj}. \tag{3.21}$$

By definition of the matrix $\mathbf{H}^{(1)}$, we may write

$$\alpha t_n(\alpha,z)=\frac{1}{n}\sum_{j,k=1}^{n}\mathbf{E}\,X_{jk}^{(m)}[\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{R}]_{j+nk}-\overline{z}\;s_n(\alpha,z). \tag{3.22}$$

For the derivatives of the matrix $\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{R}$ by $X_{jk}^{(m)}$, we get

$$\begin{aligned}\frac{\partial\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{R}}{\partial X_{jk}^{(m)}}&=\mathbf{V}_{[2,m-1]}\mathbf{e}_j\mathbf{e}_k^T\mathbf{J}\mathbf{R}\\&\quad-\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{R}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{R}-\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{R}\mathbf{V}_{[1,m-1]}\mathbf{e}_j\mathbf{e}_k^T\mathbf{J}\mathbf{R}.\end{aligned} \tag{3.23}$$

Relation (3.23) and Lemmas (7.8) together imply

$$\alpha t_n(\alpha,z) = -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left[\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{R}\mathbf{V}_{[1,m-1]}\right]_{j+nj}\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\left[\mathbf{R}\right]_{k+nk} - \overline{z}s_n(\alpha,z) + \varepsilon_n(z,\alpha)$$
$$= g_2\ t_n(\alpha,z) - \overline{z}\ s_n(\alpha,z) + \varepsilon_n(z,\alpha). \tag{3.24}$$

Applying equality (3.19), we obtain

$$\alpha t_n(\alpha,z) = (-1)^m(\alpha s_n(\alpha,z)+\overline{z}u_n(\alpha,z))^{m-1}s_n(\alpha,z)t_n(\alpha,z) - \overline{z}\ s_n(\alpha,z)+\varepsilon_n(z,\alpha). \tag{3.25}$$

Analogously we obtain

$$\alpha u_n(\alpha,z) = (-1)^m(\alpha s_n(\alpha,z)+zt_n(\alpha,z))^{m-1}s_n(\alpha,z)u_n(\alpha,z) - z\ s_n(\alpha,z)+\varepsilon_n(z,\alpha). \tag{3.26}$$

Multiplying equation (3.25) by z and equation (3.26) by $\overline{z}$ and subtracting the second one from the first equation, we may conclude

$$zt_n(\alpha,z) = \overline{z}u_n(\alpha,z) + \varepsilon_n(z,\alpha). \tag{3.27}$$

The last relation implies that

$$\mathcal{A}_1 = \mathcal{A}_2 + \varepsilon_n(z,\alpha). \tag{3.28}$$

Relations (3.3), 3.18), (3.20), (3.27, and (3.28) together imply

$$1+\alpha\ s_n(\alpha,z) = (-1)^m(\alpha s_n(\alpha,z) + z\ t_n(\alpha,z))^{m-1}s_n^2(z,\alpha) - z\ t_n(\alpha,z) + \varepsilon_n(z,\alpha). \tag{3.29}$$

Introduce the notations

$$y_n := s_n(\alpha,z), \quad w_n := \alpha + \frac{z\ t_n(\alpha,z)}{y_n}. \tag{3.30}$$

Using these notations we may rewrite the equations (3.29) and (3.27) as follows

$$1 + w_n y_n = (-1)^m y_n^{m+1}w_n^{m-1} + \varepsilon_n(z,\alpha)$$
$$(w_n-\alpha) + (w_n-\alpha)^2 y_n - y_n|z|^2 = \varepsilon_n(z,\alpha). \tag{3.31}$$

Let $n, n' \to \infty$. Consider the difference $y_n - y_{n'}$. From the first inequality it follows that

$$|y_n - y_{n'}| \le \frac{|\varepsilon_{n,n'}(z,\alpha)| + |w_n - w_{n'}||y_n + (-1)^{m+1}y_{n'}^{m+1}(w_n^{m-2}+\cdots+w_{n'}^{m-2})|}{|w_n + (-1)^{m+1}y_{n'}^{m+1}(w_n + (-1)^{m+1}w_n^{m-1}(y_n^m+\cdots+y_{n'}^m)|} \tag{3.32}$$

Note that $\max\{|y_n|,\ |y_{n'}|\} \le \frac{1}{v}$ and $\max\{|w_n|,\ |w_{n'}|\} \le C+v$ for some positive constant $C = C(m)$ depending of m. We may choose a sufficiently large v_0 such that for any $v \ge v_0$ we obtain

$$|y_n - y_{n'}| \le \frac{|\varepsilon_{n,n'}(z,\alpha)|}{v} + \frac{C}{v}|w_n - w_{n'}|. \tag{3.33}$$

Furthermore, the second equation implies that

$$(w_n - w_{n'})(1 + y_n(w_n + w_{n'} - 2\alpha)) = (y_n - y_{n'})((w_n - \alpha)^2 - |z|^2) + \varepsilon_{n,n'}(z, \alpha). \quad (3.34)$$

It is straightforward to check that $\max\{|w_n - \alpha|, |w_{n'} - \alpha|\} \le (1 + |\varepsilon_n(z, \alpha)|)|z|$. This implies that there exists v_1 such that for any $v \ge v_1$

$$|w_n - w_{n'}| \le |\varepsilon_{n,n'}(z, \alpha)| + 4|z|^2|y_n - y_n'|. \quad (3.35)$$

Inequalities (3.33) and (3.35) together imply that there exists a constant V_0 such that for any $v \ge V_0$

$$|y_n - y_n'| \le |\varepsilon_{n,n'}(\alpha, z)|, \quad (3.36)$$

where $\varepsilon_{n,n'}(\alpha, z) \to 0$ as $n \to \infty$ uniformly with respect to $v \ge V_0$ and $|u| \le C$ ($\alpha = u + iv$). Since $y_n, y_{n'}$ are locally bounded analytic functions in the upper half-plane we may conclude by Montel's Theorem (see, for instance, [8], p. 153, Theorem 2.9) that there exists an analytic function y_0 in the upper half-plane such that $\lim y_n = y_0$. Since y_n are Nevanlinna functions, (that is analytic functions mapping the upper half-plane into itself) y_0 will be a Nevanlinna function too and there exists some distribution function $F(x, z)$ such that $y_0 = \int_{-\infty}^{\infty} \frac{1}{x-\alpha} dF(x, z)$ and

$$\Delta_n(z) := \sup_x |F_n(x, z) - F(x, z)| \to 0 \quad \text{as} \quad n \to \infty. \quad (3.37)$$

The function y_0 satisfies the equations (3.2).

Thus Theorem 3.1 is proved.

□

4 Properties of Limit Measures

In this section we study the measure $F(x, z)$ with Stieltjes transform $s(\alpha, z) = \int_{-\infty}^{\infty} \frac{1}{x-\alpha} d\, F(x, z)$ satisfying the equations

$$\begin{aligned} &1 + wy + (-1)^{m+1} w^{m-1} y^{m+1} = 0, \\ &y(w - \alpha)^2 + (w - \alpha) - y|z|^2 = 0. \end{aligned} \quad (4.1)$$

Consider the first equation in (4.1) with $w = u = \sqrt{-1}\, v$. Assume that there are two solutions of these equation, say y_1 and y_2, which are Stieltjes transform of some measures. Then we have

$$(y_1 - y_2)w + (-1)^{m+1} w^{m-1}(y_1 - y_2)(y^m + \cdots + y_2^m) = 0. \quad (4.2)$$

Note that

$$\operatorname{Im}\{(-1)^{m+1} w^{m-1} y_j^m\} \ge 0, \quad j = 1, 2. \quad (4.3)$$

Indeed, by equation (4.1)

$$\operatorname{Im}\{(-1)^{m+1}w^{m-1}y_j^m\} = \frac{\operatorname{Im} y_j}{|y_j|^2} - v \geq v\ \frac{\mathbf{E}\,|\xi - w|^{-2}}{|\mathbf{E}\,(\xi - w)^{-1}|} \geq 0. \tag{4.4}$$

Note that if $\operatorname{Im}\xi_j^m \geq 0$ for $j = 1, 2$ then $\operatorname{Im}\{\xi_1^k \xi_2^{m-k}\} \geq 0$ for every $k = 0, \ldots, m$. This implies that

$$\operatorname{Im}\{(-1)^{m+1}w^{m-1}y_1^k y_2^{m-k}\} \geq 0. \tag{4.5}$$

From here it follows that

$$|w + (-1)^{m+1}w^{m-1}(y^m + \cdots + y_2^m)| \geq v > 0 \tag{4.6}$$

and

$$y_1 = y_2. \tag{4.7}$$

It is well-known that the Stieltjes transform of a distribution function $F(x)$ with moments given by the Fuss–Catalan numbers $FC(m,p) = \frac{1}{mp+p}\binom{mp+p}{p}$ satisfies the equation (4.1) (see,for instance, [1]). This distribution has bounded support given by $|w| \leq C_m := \sqrt{\frac{(m+1)^{m+1}}{m^m}}$.

The second equation has a solution

$$w - \alpha = \frac{-1 + \sqrt{1 + 4y^2|z|^2}}{2y}, \tag{4.8}$$

with $\operatorname{Im}\{w - \alpha\} \geq 0$ and $|w - \alpha| \leq |z|^2$.

Corollary 4.1. *Let $p(x,z)$ denote the density of the measure $\nu(x,z)$ with Stieltjes transform $s(\alpha,z)$. Then, for any $|z|$ and $|x| \geq C_m + |z|$, we have*

$$p(x,z) = 0 \tag{4.9}$$

Otherwise $p(x,z) > 0$ holds. For $z = 0$ we have

$$p(x,z) = O(|x|^{-\frac{m-1}{m+1}}) \quad as \quad x \to 0. \tag{4.10}$$

It is straightforward to check that the logarithmic potential of the measure $\mu^{(m)}$ (the m-th power of the uniform distribution on the unit circle) satisfies

$$U_{\mu^{(m)}}(z) = \begin{cases} -\log|z|, & |z| \geq 1 \\ \frac{m}{2}(1 - |z|^{\frac{2}{m}}), & |z| \leq 1 \end{cases}. \tag{4.11}$$

Corollary 4.2. *For $x = 0$ we have*

$$s(0,z) = \begin{cases} 0, & |z| > 1 \\ \sqrt{-1}\frac{\sqrt{1-|z|^{\frac{2}{m}}}}{|z|^{1-\frac{1}{m}}}, & |z| \leq 1 \end{cases}. \tag{4.12}$$

We investigate now the connection of family of measures $\nu(\cdot, z)$ with the distribution of ζ^m, where ζ is uniformly distributed on the unit disc in the complex plane. We prove the following Lemma.

Lemma 4.3. *For $z = u + iv$ we have*

$$\frac{\partial s(x,z)}{\partial u} = \frac{s(x,z)}{\sqrt{1+4|z|^2 s^2(x,z)}} \frac{\partial s(x,z)}{\partial x} \tag{4.13}$$

Proof. Let $y = s(x,z)$. Denote by $R_i(y,w,z)$, $i = 1,2$ the functions

$$R_1 := R_1(y,w,z,x) := 1 + wy + (-1)^{m+1} w^{m-1} y^{m+1},$$
$$R_2 := R_2(y,w,z,x) := (w-x)^2 y + (w-x) - |z|^2 y.$$

Differentiating both functions with respect to x and by u, we get

$$\begin{aligned} \frac{\partial y}{\partial u} &= \frac{-2yu}{\frac{\partial R_1}{\partial w}\frac{\partial R_2}{\partial y} - \frac{\partial R_2}{\partial w}\frac{\partial R_1}{\partial y}} \\ \frac{\partial y}{\partial x} &= \frac{-2(w-x)y - 1}{\frac{\partial R_1}{\partial w}\frac{\partial R_2}{\partial y} - \frac{\partial R_2}{\partial w}\frac{\partial R_1}{\partial y}}. \end{aligned} \tag{4.14}$$

It follows immediately that

$$\frac{\partial y}{\partial u} = \frac{-2uy}{-2(w-x)y - 1} \frac{\partial y}{\partial x} \tag{4.15}$$

Taking in account the equality (4.8), we get

$$\frac{\partial y}{\partial u} = 2u \frac{y}{\sqrt{1+4|z|^2 y^2}} \frac{\partial y}{\partial x}, \tag{4.16}$$

which completes the proof. □

Introduce now the function

$$V(z) = -\int_{-\infty}^{\infty} \log|x| d\nu(z,x).$$

Lemma 4.4. *The following relation holds*

$$V(z) = U_{\mu^{(m)}}(z).$$

Proof. We start from the simple equality, for $z = u + iv$,

$$\frac{\partial U_{\mu^{(m)}}(z)}{\partial u} = \begin{cases} -\frac{u}{u^2+v^2}, & |z| \geq 1 \\ -\frac{u}{(u^2+v^2)^{\frac{m-1}{m}}}, & |z| < 1 \end{cases}.$$

We prove that

$$\frac{\partial V(z)}{\partial u} = \frac{\partial U_{\mu}(z)}{\partial u}.$$

Let $\Delta(x) = -\sqrt{-1}s(z, \sqrt{-1}x)$, where $x > 0$. The symmetry of function $\nu(z, y)$ in y implies that the function $\Delta(x)$ will be real and non-negative. We have

$$\Delta(x) = \int_{-\infty}^{\infty} \frac{x}{x^2 + y^2} d\nu(z, y). \tag{4.17}$$

By Corollary 4.2, we have

$$\lim_{x \to 0} \Delta(x) = \begin{cases} 0, & |z| > 1 \\ \frac{\sqrt{1 - |z|^{\frac{2}{m}}}}{|z|^{1 - \frac{1}{m}}}, & |z| \leq 1 \end{cases} \tag{4.18}$$

Note that $\lim_{x \to \infty} \Delta(x) = 0$. We consider integral

$$B(C, z) = \int_0^C \Delta(x) dx.$$

Using the representation (4.17), we get

$$B(C, z) = -\int_{-\infty}^{\infty} \log |y| p(y, z) dy + \frac{1}{2} \int_{-\infty}^{\infty} \log(1 + \frac{y^2}{C^2}) p(y, z) dy + \log C. \tag{4.19}$$

We rewrite this equality as follows

$$V(z) = B(C, z) + \frac{1}{2} \int_{-\infty}^{\infty} \log(1 + \frac{y^2}{C^2}) p(y, z) dy + \log C, \tag{4.20}$$

which implies

$$\frac{\partial}{\partial u} V(z) = \frac{\partial}{\partial u} B(C, z) + \frac{1}{2} \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \log(1 + \frac{y^2}{C^2}) p(y, z) dy. \tag{4.21}$$

According to Lemma 4.3, we get

$$\frac{\partial \Delta(x)}{\partial u} = \frac{2u\Delta(x)}{\sqrt{1 - 4|z|^2 \Delta^2(x)}} \frac{\partial \Delta(x)}{\partial x}, \tag{4.22}$$

Note that the quantity $\Delta(x)$ satisfies $0 \leq \Delta(x) \leq \frac{1}{2|z|}$. There exists a point x_0 such that $\Delta(x_0) = \frac{1}{2|z|}$. Thus we get

$$\frac{\partial}{\partial u} \int_0^C \Delta(x) dx = \int_0^C \frac{\partial}{\partial u} \Delta(x) dx = 2u \int_0^C \frac{\Delta(x)}{\sqrt{1 - 4|z|^2 \Delta^2(x)}} \frac{\partial}{\partial x} \Delta(x) dx \tag{4.23}$$

$$= u \left(\int_{\Delta(0)}^{\frac{1}{2|z|^2}} + \int_{\Delta(C)}^{\frac{1}{2|z|^2}} \right) \frac{d(a^2)}{\sqrt{1 - 4a^2|z|^2}}$$

$$= \frac{-u}{2|z|^2} \left(\sqrt{1 - 4|z|^2 \Delta^2(C)} + \sqrt{1 - 4|z|^2 \Delta^2(0)} \right) \tag{4.24}$$

Simple calculations show that in the limit $C \to \infty$, we obtain

$$\lim_{C\to\infty} \frac{\partial}{\partial u} B(C,z) = \lim_{C\to\infty} \frac{\partial}{\partial u} \int_0^C \Delta(x)dx = \begin{cases} -\frac{u}{|z|^2}, & \text{if } |z \geq 1 \\ -\frac{u}{|z|^{2-\frac{2}{m}}}, & \text{if } |z| > 1 \end{cases} . \tag{4.25}$$

Consider now the quantity

$$A(C) = \frac{\partial}{\partial u} \int_{-\infty}^{\infty} \log\{1 + \frac{y^2}{C^2}\} p(y,z)dy.$$

By Corollary 2.7, we have

$$A(C) = \frac{\partial}{\partial u} \int_{-x_3}^{x_3} \log\left(1 + \frac{y^2}{C^2}\right) p(y,z)dy. \tag{4.26}$$

Using equality $p(y,z) = \operatorname{Im} s(z,y)$, we may rewrite equality (4.26) as follows

$$A(C) = \operatorname{Im}\left\{ \int_{-C_0}^{C_0} \log\left(1 + \frac{y^2}{C^2}\right) \frac{\partial}{\partial u} s(z,y)dy \right\}. \tag{4.27}$$

Applying Lemma 4.3, we get

$$A(C) = \operatorname{Im}\left\{ \int_{-2C_0}^{2C_0} \log\left(1 + \frac{y^2}{C^2}\right) \frac{s(y,z)}{\sqrt{1 + 4|z|^2 s^2(y,z)}} \frac{\partial s(y,z)}{\partial y} dy \right\}. \tag{4.28}$$

Integrating by parts and using the inequality $|\log(1 + \frac{y^2}{C^2})| \leq \frac{\gamma y^2}{C^2}$ with some constant $\gamma > 0$, and $|s(2C_0, z)| \leq \frac{1}{C_0}$, and $|s(0,z)| \leq \frac{1}{2|z|}$, we conclude that

$$\lim_{C\to\infty} A(C) = 0. \tag{4.29}$$

Collecting the relations (4.21), (4.25), and (4.29) concludes the proof of the Lemma. □

5 The Minimal Singular Value of the Matrix $\mathbf{W} - z\mathbf{I}$

Recall that

$$\mathbf{W} = \prod_{\nu=1}^{m} \mathbf{X}^{(\nu)},$$

where $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}$ are independent $n \times n$ matrices with independent entries. Let $\mathbf{W}(z) = \mathbf{W} - z\mathbf{I}$ and let $s_n(\mathbf{A})$ denote the minimal singular value of a matrix $\mathbf{A}$. Note that

$$s_n(\mathbf{A}) = \inf_{\mathbf{x}:\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|_2.$$

Introduce the matrix $\mathbf{W}^{(1)} = \prod_{\nu=2}^{m} \mathbf{X}^{(\nu)}$. We may write

$$s_n(\mathbf{W}(z)) = \inf_{\mathbf{x}:\|\mathbf{x}\|=1} \|\mathbf{W}(z)\mathbf{x}\|_2 \geq \inf_{\mathbf{x}:\|\mathbf{x}\|=1} \|(\mathbf{X}^{(1)} - z(\mathbf{W}^{(1)})^{-1})\mathbf{x}\|_2 \inf_{\mathbf{x}:\|\mathbf{x}\|=1} \|\mathbf{W}^{(1)}\mathbf{x}\|_2. \quad (5.1)$$

By induction, we obtain

$$s_n(\mathbf{W}(z)) \geq s_n(\mathbf{X}^{(1)} - z(\mathbf{W}^{(1)})^{-1})) \prod_{\nu=2}^{m} s_n(\mathbf{X}^{(\nu)}). \quad (5.2)$$

Lemma 5.1. *Let $X_{jk}^{(\nu)}$ be independent complex random variables with $\mathbf{E}\, X_{jk} = 0$ and $\mathbf{E}\,|X_{jk}|^2 = 1$, which are uniformly integrable , i.e.*

$$\max_{j,k,\nu} \mathbf{E}\,|X_{jk}^{(\nu)}|^2 I_{\{|X_{jk}|>M\}} \to 0 \quad as \quad M \to \infty. \quad (5.3)$$

Let $K \geq 1$. Then there exist constants $c, C, B > 0$ depending on θ and K such that for any $z \in \mathbb{C}$ and positive ε we have

$$\Pr\{s_n \leq \varepsilon/n^B;\ \max_{1\leq\nu\leq m} s_1(\mathbf{X}^{(\nu)}) \leq Kn\} \leq \exp\{-c\ n\} + \frac{C\sqrt{\ln n}}{\sqrt{n}}, \quad (5.4)$$

where $s_n = s_n(\mathbf{W}(z))$.

Proof. The proof is similar to the proof of Theorem 4.1 in [9]. Applying inequality (5.2), we get

$$\begin{aligned}
&\Pr\{s_n \leq \varepsilon/n^B;\ \max_{1\leq\nu\leq m} s_1(\mathbf{X}^{(\nu)}) \leq Kn\} \\
&\quad \leq \Pr\{s_n(\mathbf{X}^{(1)} - z(\mathbf{W}^{(1)})^{-1}) \leq \varepsilon^{\frac{1}{m}} n^{-\frac{B}{m}};\ \max_{1\leq\nu\leq m} s_1(\mathbf{X}^{(\nu)}) \leq Kn\} \\
&\qquad + \sum_{\nu=2}^{m} \Pr\{s_n(\mathbf{X}^{(\nu)}) \leq \varepsilon^{\frac{1}{m}} n^{-\frac{B}{m}};\ \max_{1\leq\nu\leq m} s_1(\mathbf{X}^{(\nu)}) \leq Kn\}. \quad (5.5)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\Pr\{s_n(\mathbf{X}^{(1)} - z(\mathbf{W}^{(1)})^{-1}) \leq \varepsilon^{\frac{1}{m}} n^{-\frac{B}{m}};\ \max_{1\leq\nu\leq m} s_1(\mathbf{X}^{(\nu)}) \leq Kn\} \\
&\quad \leq \Pr\{s_n(\mathbf{X}^{(1)} - z(\mathbf{W}^{(1)})^{-1}) \leq \varepsilon^{\frac{1}{m}} n^{-\frac{B}{m}};\ s_1(\mathbf{X}^{(\nu)}) \leq Kn;\ s_1({\mathbf{W}^{(1)}}^{-1}) \leq n^B\} \\
&\qquad + \Pr\{s_1({\mathbf{W}^{(1)}}^{-1}) \geq n^B;\ \max_{1\leq\nu\leq m} s_1(\mathbf{X}^{(\nu)}) \leq Kn\}. \quad (5.6)
\end{aligned}$$

Note that

$$s_1({\mathbf{W}^{(1)}}^{-1}) \leq \prod_{\nu=2}^{m} s_1({\mathbf{X}^{(\nu)}}^{-1}) = \prod_{\nu=2}^{m} s_n^{-1}(\mathbf{X}^{(\nu)}). \quad (5.7)$$

Applying this inequality and Theorem 4.1 in [9], we obtain

$$\Pr\{s_1({\mathbf{W}^{(1)}}^{-1}) \geq n^B;\ \max_{1\leq\nu\leq m} s_1(\mathbf{X}^{(\nu)}) \leq Kn\} \leq \exp\{-c\ n\} + \frac{C\sqrt{\ln n}}{\sqrt{n}}. \tag{5.8}$$

with some positive constants $C, c > 0$. Moreover, adapting the proof of Theorem 4.1 in [9], we see that this theorem holds for all matrices $\mathbf{X}^{(1)} - z\mathbf{B}$ uniformly for all non-random matrices $\mathbf{B}$ such that $\|\mathbf{B}\|_2 \leq Cn^Q$ for some positive constant $Q > 0$, i.e.

$$\Pr\{s_n(\mathbf{X}^{(1)} - z\mathbf{B}) \leq \varepsilon n^{-B}, s_1(\mathbf{X}^{(1)}) \leq Kn\} \leq \exp\{-c\ n\} + \frac{C\sqrt{\ln n}}{\sqrt{n}}. \tag{5.9}$$

with a constant depending on C and Q and not depending on the matrix B. Since the matrices $\mathbf{X}^{(1)}$ and $\mathbf{W}^{(1)}$ are independent, we may apply this result and get

$$\Pr\{s_n(\mathbf{X}^{(1)} - z{\mathbf{W}^{(1)}}^{-1}) \leq \varepsilon n^{-B}, s_1(\mathbf{X}^{(1)}) \leq Kn; s_1({\mathbf{W}^{(1)}}^{-1}) \leq Cn^B\} \leq \exp\{-c\ n\} + \frac{C\sqrt{\ln n}}{\sqrt{n}} \tag{5.10}$$

Collecting the inequalities (5.5)–(5.10), we conclude the proof of the Lemma.

□

Following Tao and Vu [15], we may prove sharper results about the behavior of small singular values of a matrix product.

We shall use the following well-known fact. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ denote matrices and let $s_1(\mathbf{A}) \geq \cdots \geq s_n(\mathbf{A})$ resp. ($s_1(\mathbf{B}) \geq \cdots \geq s_n(\mathbf{B})$ and $s_1(\mathbf{AB}) \geq \cdots \geq s_n(\mathbf{AB})$) denote the singular value of a matrix $\mathbf{A}$ (and the matrices $\mathbf{B}$ and $\mathbf{AB}$ respectively). Then for any $1 \leq k \leq n$ we have

$$\prod_{j=k}^{n} s_j(\mathbf{AB}) \geq \prod_{j=k}^{n} s_j(\mathbf{A}) s_j(\mathbf{B}), \tag{5.11}$$

and

$$\prod_{j=1}^{n} s_j(\mathbf{AB}) = \prod_{j=1}^{n} s_j(\mathbf{A}) s_j(\mathbf{B}) \tag{5.12}$$

(see, for instance [12], p.171, Theorem 3.3.4).

We need to prove a bound similar to the bound (45) in [15], namely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=n-n\delta_n}^{n-n^\gamma} \ln s_j(\mathbf{W} - z\mathbf{I}) = 0, \tag{5.13}$$

for any sequence $\delta_n \to 0$. To prove this bound it is enough to prove that for any $\nu = 1, \ldots, m$ and any fixed sequence of matrices $\mathbf{M}_n$ with $\|\mathbf{M}_n\|_2 \leq Cn^B$ for some positive constant $B > 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=n-n\delta_n}^{n-n^\gamma} \ln s_j(\mathbf{X}^{(\nu)} + \mathbf{M}_n) = 0. \tag{5.14}$$

Indeed, it follows from (5.11), that

$$\frac{1}{n}\sum_{j=n-n\delta_n}^{n-n^\gamma}\ln s_j(\mathbf{W}-z\mathbf{I}) \geq \frac{1}{n}\sum_{\nu=1}^{m-1}\sum_{j=n-n\delta_n}^{n-n^\gamma}\ln s_j(\mathbf{X}^{(\nu)})+\frac{1}{n}\sum_{j=n-n\delta_n}^{n-n^\gamma}\ln s_j(\mathbf{X}^{(m)}+\mathbf{M}_n), \quad (5.15)$$

where $\mathbf{M}_n^{-1} = \prod_{\nu=1}^{m-1}\mathbf{X}^{(\nu)}$. Note that the matrices $X^{(m)}$ and $\mathbf{M}_n$ are independent and it follows from our results in [9], Lemma A1, that $\|\mathbf{M}_n\|_2 \leq Cn^B$ for some $B > 0$ with probability close to one. The relations

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=n-n\delta_n}^{n-n^\gamma}\ln s_j(\mathbf{X}^{(\nu)}) = 0, \quad \text{for} \quad \nu = 1,\dots,m-1,$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=n-n\delta_n}^{n-n^\gamma}\ln s_j(\mathbf{X}^{(m)}+\mathbf{M}_n) = 0 \quad (5.16)$$

follow from the bound

$$s_j(\mathbf{X}^{(\nu)}+\mathbf{M}_n) \geq c\sqrt{\frac{n-j}{n}}, \quad 1 \leq j \leq n-n^\gamma. \quad (5.17)$$

To prove this we need the following simple Lemma.

Lemma 5.2. *Let* $\lim_{n\to\infty}\delta_n = 0$ *and let* s_j, *for* $n-n\delta_n \leq j \leq n-n^\gamma$ *with* $0 < \gamma < 1$ *denote numbers satisfying the inequality*

$$s_j \geq c\sqrt{\frac{n-j}{n}}. \quad (5.18)$$

Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{n-n\delta_n\leq j\leq n-n^\gamma}\ln s_j = 0. \quad (5.19)$$

Proof. Without loss of generality we may assume that $0 < s_j \leq 1$. By the conditions of Lemma 5.2, we have

$$0 \geq \frac{1}{n}\sum_{n-n\delta_n\leq j\leq n-n^\gamma}\ln s_j \geq \frac{1}{n}\sum_{n-n\delta_n\leq j\leq n-n^\gamma}\ln\{\frac{n-j}{n}\} = A. \quad (5.20)$$

After summation and using Stirling's formula, we get

$$\begin{aligned}|A| &\leq \frac{1}{n}\ln\{\frac{[n-n\delta_n]!}{[n-n^\gamma]!n^{n\delta_n-n^\gamma}}\}\\ &\leq \delta_n|\ln\delta_n| + (1-\gamma)n^{\gamma-1}\ln n \to 0, \quad \text{as} \quad n\to\infty. \end{aligned} \quad (5.21)$$

This proves Lemma 5.2.

□

It remains to prove inequality (5.17). This result was proved by Tao and Vu in [15] (see inequality (8.4) in [15]). It represents the crucial result in their proof of the circular law assuming a second moment only. For completeness we repeat this proof here. We start from the following

Proposition 5.1. *Let $1 \le d \le n - n^\gamma$ with $\frac{8}{15} < \gamma < 1$. and $0 < c < 1$, and $\mathbb{H}$ be a (deterministic) d-dimensional subspace of $\mathbb{C}^n$. Let X be a row of $\mathbf{A}_n := \mathbf{X} + \mathbf{M}_n$. Then*

$$\Pr\{\mathrm{dist}(X, \mathbb{H}) \le c\sqrt{n-d}\} = O(\exp\{-n^{\frac{\gamma}{8}}\}), \tag{5.22}$$

where $\mathrm{dist}(X, \mathbb{H})$ *denotes the Euclidean distance between a vector X and a subspace $\mathbb{H}$ in $\mathbb{C}^n$.*

Proof. It was proved by Tao and Vu in [15] (see Proposition 5.1). Here we sketch their proof. As shown in [15] we may reduce the problem to the case that $\mathbf{E}\, X = 0$. For this it is enough to consider vectors X' and v such that $X = X' + v$ and $\mathbf{E}\, X' = 0$. Instead of the subspace $\mathbb{H}$ we may consider subspace $\mathbb{H}' = \mathrm{span}(\mathbb{H}, v)$ and note that

$$\mathrm{dist}(X, \mathbb{H}) \ge \mathrm{dist}(X', \mathbb{H}'). \tag{5.23}$$

The claim follows now from a corresponding result for random vectors with mean zero. In what follows we assume that $\mathbf{E}\, X = 0$. We reduce the problem to vectors with bounded coordinates. Let $\xi_j = I\{|X_j| \ge n^{\frac{1-\gamma}{2}}\}$, where X_j denotes the j-th coordinate of a vector X. Note that $p_n := \mathbf{E}\, \xi_j \le n^{-(1-\gamma)}$. Applying Chebyshev's inequality, we get, for any $h > 0$

$$\Pr\{\sum_{j=1}^{n} \xi_j \ge 2n^\gamma\} \le \exp\{-hn^\gamma\} \exp\{np_n(\mathrm{e}^h - 1 - h)\}. \tag{5.24}$$

Choosing $h = \frac{1}{4}$, we obtain

$$\Pr\{\sum_{j=1}^{n} \xi_j \ge 2n^\gamma\} \le \exp\{-\frac{n^\gamma}{8}\}. \tag{5.25}$$

Let $J \subset \{1, \dots, n\}$ and $E_J := \{\prod_{j \in J}(1 - \xi_j) \prod_{j \notin J} \xi_j = 1\}$. Inequality (5.25) implies

$$\Pr\{\bigcup_{J : |J| \ge n - 2n^\gamma} E_J\} \ge 1 - \exp\{-\frac{n^\gamma}{8}\}. \tag{5.26}$$

Let J with $|J| \ge n - 2n^\gamma$ be fixed. Without loss of generality we may assume that $J = 1, \dots, n'$ with some $n - 2n^\gamma \le n' \le n$. It is now suffices to prove that

$$\Pr\{\mathrm{dist}(X, \mathbb{H}) \le c\sqrt{n-d} | E_J\} = O(\exp\{-\frac{n^\gamma}{8}\}). \tag{5.27}$$

Let π denote the orthogonal projection $\pi : \mathbb{C}^n \to \mathbb{C}^{n'}$. We note that

$$\mathrm{dist}(X, \mathbb{H}) \ge \mathrm{dist}(\pi(X), \pi(\mathbb{H})). \tag{5.28}$$

Let $\widetilde{x}$ be a random variable x conditioned on the event $|x| \le n^{1-\gamma}$ and let $\widetilde{X} = (\widetilde{x}_1, \dots, \widetilde{x}_n)$. The relation (5.27) will follow now from

$$\Pr\{\mathrm{dist}(\widetilde{X}', \mathbb{H}') \le c\sqrt{n-d} \,||x_j| \le n^{1-\gamma}, j \notin J\} = O(\exp\{-\frac{n^\gamma}{8}\}), \tag{5.29}$$

where $\mathbb{H}' = \pi(\mathbb{H})$ and $\widetilde{X}' = \pi(\widetilde{X})$. We may represent the vector $\widetilde{X}$ as $\widetilde{X} = \widetilde{X}' + v$, where $v = \mathbf{E}\,\widetilde{X}$ and $\mathbf{E}\,\widetilde{X}' = 0$. We reduce the claim to the bound

$$\Pr\{\mathrm{dist}(\widetilde{X}', \mathbb{H}'') \le c\sqrt{n-d} \,||x_j| \le n^{1-\gamma}, j \notin J\} = O(\exp\{-\frac{n^\gamma}{8}\}), \tag{5.30}$$

where $\mathbb{H}'' = \mathrm{span}(v, \mathbb{H}')$. In the what follows we shall omit the symbol $'$ in the notations. To prove (5.30) we shall apply the following result of Maurey. Let $\mathbb{X}$ denote a normed space and f denote a convex function on $\mathbb{X}$. Define the functional Q as follows

$$Qf(x) := \inf_{y \in \mathbb{X}} [f(y) + \frac{\|x-y\|^2}{4}]. \tag{5.31}$$

Definition 5.2. We say that a measure μ satisfies the convex property (τ) if for any convex function f on $\mathbb{X}$

$$\int_{\mathbb{X}} \exp\{Qf\} d\mu \int_{\mathbb{X}} \exp\{-f\} d\mu \le 1. \tag{5.32}$$

We reformulate the following result of Maurey (see [13], Theorem 3)

Theorem 5.3. *Let $(\mathbb{X}_i)$ be a family of normed spaces; for each i, let μ_i be a probability measure with diameter ≤ 1 on $\mathbb{X}_i$, for $x \in \mathbb{X}_i$. If μ is the product of a family (μ_i), then μ satisfies the convex property (τ).*

As corollary of Theorem 5.3 we get

Corollary 5.3. *Let μ_i be a probability measure with diameter ≤ 1 on $\mathbb{X}$, $i = 1, \dots, n$. Let g denote a convex 1-Lipshitz function on $\mathbb{X}^n$. Let $M(g)$ denote a median of g. If μ is the product of the family (μ_i), then*

$$\mu\{|g - M(g)| \ge h\} \le 4\exp\{-\frac{h^2}{4}\}. \tag{5.33}$$

Applying Corollary 5.3 to μ_i, being the distribution of $\widetilde{x}_i$, we get

$$\Pr\left\{|\mathrm{dist}(\widetilde{X}, \mathbb{H}) - M(\mathrm{dist}(\widetilde{X}, \mathbb{H}))| \ge r n^{\frac{1-\gamma}{2}}\right\} \le 4\exp\{-r^2/16\}. \tag{5.34}$$

The last inequality implies that there exists a constant $C > 0$ such that

$$|\mathbf{E}\,\mathrm{dist}(\widetilde{X}, \mathbb{H}) - M(\mathrm{dist}(\widetilde{X}, \mathbb{H}))| \le C n^{\frac{1-\gamma}{2}}, \tag{5.35}$$

and

$$\mathbf{E}\,\mathrm{dist}(\widetilde{X}, \mathbb{H}) \ge \sqrt{\mathbf{E}\,(\mathrm{dist}(\widetilde{X}, \mathbb{H}))^2} - C n^{\frac{1-\gamma}{2}}. \tag{5.36}$$

By Lemma 5.3 in [15]

$$\mathbf{E}\,(\mathrm{dist}(\widetilde{X},\mathbb{H}))^2 = (1-o(1)(n-d). \tag{5.37}$$

Since $n-d \geq n^\gamma$ the inequalities (5.35), (5.36) and (5.37) together imply (5.22). Thus Proposition 5.1 is proved. □

Now we prove (5.17). We repeat the proof of Tao and Vu [15], inequality (8.4). Fix j. Let $\mathbf{A}_n = \mathbf{X}^{(m)} - z\mathbf{M}_n$ and let $\mathbf{A}'_n$ denote a matrix formed by the first $n-k$ rows of $\mathbf{A}_n$ with $k = j/2$. Let σ'_l, $1 \leq l \leq n-k$, be singular values of $\mathbf{A}'_n$ (in decreasing order). By the interlacing property and re-normalizing we get

$$\sigma_{n-j} \geq \frac{1}{\sqrt{n}}\sigma'_{n-j}. \tag{5.38}$$

By Lemma A.4 in [15]

$$T := {\sigma'_1}^{-2} + \cdots + {\sigma'_{n-k}}^{-2} = \mathrm{dist}_1^{-2} + \cdots + \mathrm{dist}_{n-k}^{-2}. \tag{5.39}$$

Note that

$$T \geq (j-k){\sigma'}_{n-j}^{-2} = \frac{j}{2}{\sigma'}_{n-j}^{-2}. \tag{5.40}$$

Applying Proposition 5.1, we get that with probability $1-\exp\{-n^\gamma\}$

$$T \leq \frac{n}{j}. \tag{5.41}$$

Combining the last inequalities, we get (5.17).

Lemma 5.4. *Under the conditions of Theorem 1.1 there exists a constant C such that for any $k \leq n(1-C\Delta_n^{\frac{1}{m+1}}(z))$,*

$$\Pr\{s_k \leq \Delta_n(z)\} \leq C\Delta_n^{\frac{1}{m+1}}(z). \tag{5.42}$$

Proof. Recall that $F_n(x,z) = \mathbf{E}\,\mathcal{F}_n(x,z)$ denotes the mean of the spectral distribution function $\mathcal{F}_n(x,z)$ of the matrix $\mathbf{H}(z)$ and that $F(x,z) = \lim_{n\to\infty} F_n(x,z)$. According to Theorem 3.1, the Stieltjes transform of the distribution function $F_n(x,z)$ satisfies the system of algebraic equations (3.2) and

$$\Delta_n(z) = \sup_x |F_n(x,z) - F(x,z)| \to 0 \quad \text{as} \quad n\to\infty. \tag{5.43}$$

We may write, for any $k = 1,\ldots,n$,

$$\Pr\{s_k \leq \Delta_n(z)\} \leq \Pr\{\mathcal{F}_n(s_k,z) \leq \mathcal{F}_n(\Delta_n(z)\} \leq \Pr\{\frac{n-k}{n} \leq \mathcal{F}_n(\Delta_n(z)\}. \tag{5.44}$$

Applying Chebyshev's inequality, we obtain

$$\Pr\{s_k \leq \Delta_n(z)\} \leq \frac{n\mathbf{E}\,\mathcal{F}_n(\Delta_n(z))}{n-k} \leq \frac{n(F(\Delta_n(z),z)+\Delta_n(z)}{n-k}. \tag{5.45}$$

It is straightforward to check that from the system of equations (3.2) it follows

$$F(\Delta_n(z), z) \le C\Delta_n^{\frac{2}{m+1}}(z). \tag{5.46}$$

The last inequality concludes the proof of Lemma 5.4.

□

Lemma 5.5. *Let* $\Delta_n(z) := \sup_x |F_n(x,z) - F(x,z)|$. *Then there exists some absolute positive constant* R *such that*

$$\Pr\{|\lambda_{k_1}| > R\} \le C\sqrt{\Delta_n(z)}, \tag{5.47}$$

where $k_1 := \left[\Delta_n^{\frac{1}{4}}(z)n\right]$.

Proof. It is straightforward to check from (3.31) that the distribution $F(x,z)$ is compactly supported. Fix R such that $F(R,z) = 1$. Let us introduce $k_0 := \left[\Delta_n^{\frac{1}{2}}n\right]$. Using Chebyshev's inequality we obtain, for $R > 0$,

$$\Pr\{s_{k_0} > R\} \le \frac{1 - \mathbf{E}\, F_n(R)}{k_0/n} \le \Delta_n^{\frac{1}{2}}.$$

On the other hand,

$$\Pr\{|\lambda_{k_1}| > R\} \le \Pr\{\prod_{\nu=1}^{k_1} |\lambda_\nu| > R^{k_1}\} \le \Pr\{\prod_{\nu=1}^{k_1} s_\nu > R^{k_1}\} \le \Pr\{\frac{1}{k_1}\sum_{\nu=1}^{k_1} \ln s_\nu^{(m)} > \ln R\}.$$

Let $k_2 = \max\{1 \le j \le k_0 :\ \sigma_j \ge \Delta_n^{-1}(z)\}$. If $\sigma_1 \le \Delta_n^{-1}(z)$ then $k_2 = 0$. Furthermore, for any value $R_1 \ge 1$, splitting into the events $s_{k_0} > R$ and $s_{k_0} \le R$, we get

$$\begin{aligned}\Pr\{\frac{1}{k_1}\sum_{\nu=1}^{k_1} \ln s_\nu > \ln R_1\} &\le \Pr\{s_{k_0} > R\} \\ &+ \Pr\{\frac{1}{k_1}\sum_{j=k_2+1}^{k_0} \ln s_j + \ln R > \frac{1}{2}\ln R_1\} + \Pr\{\frac{1}{k_1}\sum_{j=1}^{k_2} \ln s_j > \frac{1}{2}\ln R_1\}\end{aligned} \tag{5.48}$$

Applying Chebyshev's inequality, we get

$$\begin{aligned}\Pr\{\frac{1}{k_1}\sum_{\nu=1}^{k_1} \ln s_\nu > \ln R_1\} &\le \Pr\{s_{k_0} > R\} \\ &+ \Pr\{\frac{k_0}{k_1}\ln\Delta_n^{-1}(z) > \frac{1}{2}\ln\frac{R_1}{R^2}\} + \frac{n}{k_1}\int_{\Delta_n^{-1}(z)} \ln x dF_n(x,z).\end{aligned}$$

Now choose $R_1 := 2R^2$. Thus, since $k_1/k_0 \sim \Delta_n^{\frac{1}{4}}(z)$, and $\Delta_n^{\frac{1}{4}}(z)|\ln \Delta_n(z)| \to 0$, we get for sufficiently large n

$$\Pr\{|\lambda_{k_1}| > R\} \le \Delta_n^{\frac{1}{2}} + \frac{n}{k_1}\int_{\Delta_n^{-1}(z)} \ln x d\ F_n(x,z).$$

Taking into account that the function $\frac{\ln x}{x^2}$ decreases in the interval $[\delta_n^{-1}(z), \infty)$, we get

$$\frac{n}{k_1}\int_{\Delta_n^{-1}(z)}^{\infty} \ln x d\ F_n(x,z) \le \frac{n\Delta_n^2(z)}{k_1}\ln \Delta_n^{-1}(z)\int_0^{\infty} x^2 d\ F_n(x,z) \le \Delta_n^{\frac{1}{2}}(z)\ln \Delta_n^{-1}(z). \tag{5.49}$$

Thus the Lemma is proved. □

6 Proof of the Main Theorem

In this Section we give the proof of Theorem 1.1. For any $z \in \mathbb{C}$ and an absolute constant $c > 0$ we introduce the set $\Omega_n(z) = \{\omega \in \Omega :\ c/n^B \le s_n(z),\ s_1 \le n,\ |\lambda_{k_1}| \le R\ s_{k_2} \ge \Delta_n(z)\}$. According to Lemma 7.4

$$\Pr\{s_1(\mathbf{X}) \ge n\} \le Cn^{-1}.$$

Due to Lemma 5.1 with $\varepsilon = c$, we have

$$\Pr\{c/n^B \ge s_n(z)\} \le \frac{C\sqrt{\ln n}}{\sqrt{n}} + \Pr\{s_1 \ge n\}.$$

According to Lemma 5.5, we have

$$\Pr\{|\lambda_{k_1}| \le R\} \le C\sqrt{\Delta_n}. \tag{6.1}$$

Furthermore, in view of Lemma 5.4,

$$\Pr\{s_k \le \Delta_n(z)\} \le C\Delta_n^{\frac{1}{m+1}}(z). \tag{6.2}$$

These inequalities imply

$$\Pr\{\Omega_n(z)^c\} \le C\Delta_n^{\frac{1}{m+1}}(z). \tag{6.3}$$

The remaining part of the proof of Theorem 1.1 is similar to the proof of Theorem 1.1 in the paper of Götze and Tikhomirov [9]. For completeness we shall repeat it here. Let $r = r(n)$ be such that $r(n) \to 0$ as $n \to \infty$. A more specific choice will be made later. Consider the potential $U_{\mu_n}^{(r)}$. We have

$$\begin{aligned} U_{\mu_n}^{(r)} &= -\frac{1}{n}\mathbf{E}\log|\det(\mathbf{W} - z\mathbf{I} - r\xi\mathbf{I})| \\ &= -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\log|\lambda_j^{(m)} - r\xi - z|I_{\Omega_n(z)} - \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\log|\lambda_j^{(m)} - r\xi - z|I_{\Omega_n^{(c)}(z)} \\ &= \overline{U}_{\mu_n}^{(r)} + \widehat{U}_{\mu_n}^{(r)}, \end{aligned}$$

where I_A denotes an indicator function of an event A and $\Omega_n(z)^c$ denotes the complement of $\Omega_n(z)$.

Lemma 6.1. *Assuming the conditions of Theorem 5.1, for r such that*

$$\ln(1/r)\,(\Delta_n^{\frac{1}{4}}(z)) \to \infty \quad as \quad n \to 0$$

we have

$$\widehat{U}_{\mu_n}^{(r)} \to 0, \ as\ n \to \infty. \tag{6.4}$$

Proof. By definition, we have

$$\widehat{U}_{\mu_n}^{(r)} = -\frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\,\log|\lambda_j^{(m)} - r\xi - z| I_{\Omega_n^{(c)}(z)}. \tag{6.5}$$

Applying Cauchy's inequality, we get, for any $\tau > 0$,

$$\begin{aligned}
|\widehat{U}_{\mu_n}^{(r)}| &\le \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}^{\frac{1}{1+\tau}} |\log|\lambda_j^{(m)} - r\xi - z||^{1+\tau} \left(\Pr\{\Omega_n^{(c)}\}\right)^{\frac{\tau}{1+\tau}} \\
&\le \left(\frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\,|\log|\lambda_j^{(m)} - r\xi - z||^{1+\tau}\right)^{\frac{1}{1+\tau}} \left(\Pr\{\Omega_n^{(c)}\}\right)^{\frac{\tau}{1+\tau}}.
\end{aligned} \tag{6.6}$$

Furthermore, since ξ is uniformly distributed in the unit disc and independent of λ_j, we may write

$$\mathbf{E}\left|\log|\lambda_j - r\xi - z|\right|^{1+\tau} = \frac{1}{2\pi}\mathbf{E}\int_{|\zeta|\le 1} \left|\log|\lambda_j^{(m)} - r\zeta - z|\right|^{1+\tau} d\zeta = \mathbf{E}\,J_1^{(j)} + \mathbf{E}\,J_2^{(j)} + \mathbf{E}\,J_3^{(j)},$$

where

$$\begin{aligned}
J_1^{(j)} &= \frac{1}{2\pi}\int_{|\zeta|\le 1,\ |\lambda_j^{(m)} - r\zeta - z|\le\varepsilon} |\log|\lambda_j^{(m)} - r\zeta - z||^{1+\tau} d\zeta, \\
J_2^{(j)} &= \frac{1}{2\pi}\int_{|\zeta|\le 1,\ \frac{1}{\varepsilon} > |\lambda_j^{(m)} - r\zeta - z| > \varepsilon} |\log|\lambda_j^{(m)} - r\zeta - z||^{1+\tau} d\zeta, \\
J_3^{(j)} &= \frac{1}{2\pi}\int_{|\zeta|\le 1,\ |\lambda_j^{(m)} - r\zeta - z| > \frac{1}{\varepsilon}} |\log|\lambda_j^{(m)} - r\zeta - z||^{1+\tau} d\zeta.
\end{aligned}$$

Note that

$$|J_2^{(j)}| \le \log\left(\frac{1}{\varepsilon}\right).$$

Since for any $b > 0$, the function $-u^b \log u$ is not decreasing on the interval $[0, \exp\{-\frac{1}{b}\}]$, we have for $0 < u \le \varepsilon < \exp\{-\frac{1}{b}\}$,

$$-\log u \le \varepsilon^b u^{-b} \log\left(\frac{1}{\varepsilon}\right).$$

Using this inequality, we obtain, for $b(1+\tau) < 2$,

$$|J_1^{(j)}| \le \frac{1}{2\pi}\varepsilon^{b(1+\tau)}\left(\log\left(\frac{1}{\varepsilon}\right)\right)^{1+\tau}\int_{|\zeta|\le 1,\ |\lambda_j^{(m)}-r\zeta-z|\le\varepsilon}|\lambda_j^{(m)}-r\zeta-z|^{-b(1+\tau)}d\zeta \tag{6.7}$$

$$\le \frac{1}{2\pi r^2}\varepsilon^{b(1+\tau)}r^{-2}\log\left(\frac{1}{\varepsilon}\right)\int_{|\zeta|\le\varepsilon}|\zeta|^{-b(1+\tau)}d\zeta \le C(\tau,b)\varepsilon^2 r^{-2}\left(\log\left(\frac{1}{\varepsilon}\right)\right)^{1+\tau}. \tag{6.8}$$

If we choose $\varepsilon = r$, then we get

$$|J_1^{(j)}| \le C(\tau,b)\left(\log\left(\frac{1}{r}\right)\right)^{1+\tau}. \tag{6.9}$$

The following bound holds for $\frac{1}{n}\sum_{j=1}^n \mathbf{E}\, J_3^{(j)}$. Note that $|\log x|^{1+\tau} \le \varepsilon^2|\log\varepsilon|^{1+\tau}x^2$ for $x \ge \frac{1}{\varepsilon}$ and sufficiently small ε. Using this inequality, we obtain

$$\frac{1}{n}\sum_{j=1}^n \mathbf{E}\, J_3^{(j)} \le C(\tau)\varepsilon^2|\log\varepsilon|^{1+\tau}|\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\lambda_j^{(\varepsilon)}-r\zeta-z|^2 \le C(\tau)(1+|z|^2+r^2)\varepsilon^2|\log\varepsilon|^{1+\tau}$$

$$\le C(\tau)(2+|z|^2)r^2|\log r|^{1+\tau}|. \tag{6.10}$$

The inequalities (6.7)–(6.10) together imply that

$$|\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\log|\lambda_j^{(m)}-r\xi-z||^{1+\tau}| \le C\left(\log\left(\frac{1}{r}\right)\right)^{1+\tau}. \tag{6.11}$$

Furthermore, the inequalities (6.3), (6.5), (6.6), and (6.11) together imply

$$|\widehat{U}_{\mu_n}^{(r)}| \le C\log\left(\frac{1}{r}\right)((\Delta_n^{\frac{1}{2}}(z))^{\frac{\tau}{1+\tau}}.$$

We choose $\tau = 1$ and rewrite the last inequality as follows

$$|\widehat{U}_{\mu_n}^{(r)}| \le C\log\left(\frac{1}{r}\right)\Delta_n^{\frac{1}{4}}(z) \tag{6.12}$$

If we choose $r = \Delta_n(z)$ we obtain $\log(1/r)\Delta_n^{\frac{1}{4}}(z) \to 0$, then (6.4) holds and the Lemma is proved. □

We shall investigate $\overline{U}_{\mu_n}^{(r)}$ now. Let $\nu_n(\cdots,z,r) = \mathbf{E}_\zeta\nu_n(\cdot,z+r\zeta)$ and $\nu(\cdot,z,r) = \mathbf{E}\,\nu(\cdot,z+r\zeta)$. We may write

$$\overline{U}_{\mu_n}^{(r)} = -\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,\log|\lambda_j^{(\varepsilon)}-z-r\xi|I_{\Omega_n(z)} = -\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,\log(s_j(\mathbf{X}^{(\varepsilon)}(z,r))I_{\Omega_n(z)}$$

$$= -\int_{n^{-B}}^{K_n+|z|}\log x d\mathbf{E}\,\overline{F}_n(x,z,r), \tag{6.13}$$

where $\overline{F}_n(\cdot,z,r)$ ($F(x,z,r)$) is the distribution function corresponding to the restriction of the measure $\nu_n(\cdot,z,r)$ ($\nu(\cdot,z,r)$) to the set $\Omega_n(z)$. Introduce the notation

$$\overline{U}_\mu = -\int_{\Delta_n(z)}^{n+|z|} \log x dF(x,z,r). \tag{6.14}$$

Integrating by parts, we get

$$\begin{aligned}\overline{U}^{(r)}_{\mu_n} - \overline{U}_\mu = &-\int_{\Delta_n(z)}^{n+|z|} \frac{\mathbf{E}\, F_n(x,z,r) - F(x,z,r)}{x} dx \\ &+ C\sup_x |\mathbf{E}\, F_n(x,z,r) - F(x,z,r)||\log(\Delta_n(z))| + \mathbf{E}\left\{\frac{1}{n}\sum_{j=k_2}^{n} \ln s_j I\{\Omega_n(z)\}\right\}.\end{aligned} \tag{6.15}$$

This implies that

$$|\overline{U}^{(r)}_{\mu_n} - \overline{U}_\mu| \le C|\log(\Delta_n(z))| \sup_x |\mathbf{E}\, F_n(x,z,r) - F(x,z)|. \tag{6.16}$$

Note that, for any $r>0$, $|s_j(z) - s_j(z,r)| \le r$. This implies that

$$\mathbf{E}\, F_n(x-r,z) \le \mathbf{E}\, F_n(x,z,r) \le \mathbf{E}\, F_n(x+r,z). \tag{6.17}$$

Hence, we get

$$\sup_x |\mathbf{E}\, F_n(x,z,r) - F(x,z)| \le \sup_x |\mathbf{E}\, \mathcal{F}_n(x,z) - F(x,z)| + \sup_x |F(x+r,z) - F(x,z)|. \tag{6.18}$$

Since the distribution function $F(x,z)$ has a density $p(x,z)$ which is bounded for $|z|>0$ and $p(x,0) = O(x^{-\frac{m-1}{m+1}})$ (see Remark 2.7) we obtain

$$\sup_x |\mathbf{E}\, \mathcal{F}_n(x,z,r) - F(x,z)| \le \sup_x |\mathbf{E}\, \mathcal{F}^{(\varepsilon)}_n(x,z) - F(x,z)| + Cr^{\frac{2}{m+1}}. \tag{6.19}$$

Choose $r = \Delta_n(z)$. Inequalities (6.19) and (6.18) together imply

$$\sup_x |\mathbf{E}\, \overline{\mathcal{F}}_n(x,z,r) - \overline{F}(x,z)| \le C\Delta_n^{\frac{2}{m+1}}(z). \tag{6.20}$$

From inequalities (6.20) and (6.16) and lemma 5.2 it follows that

$$|\overline{U}^{(r)}_{\mu_n} - \overline{U}_\mu| \le C\Delta_n^{\frac{2}{m+1}}(z)|\ln \Delta_n(z)|.$$

Note that

$$|\overline{U}^{(r)}_{\mu_n} - U_\mu| \le |\int_0^{\Delta_n(z)} \log x dF(x,z)| \le C\Delta_n^{\frac{2}{m+1}}(z)|\ln(\Delta_n(z))|.$$

Let $\mathcal{K} = \{z \in \mathbb{C} : \ |z| \le R\}$ and let $\mathcal{K}^c$ denote $\mathbb{C} \setminus \mathcal{K}$. According to Lemma 5.5, we have, for k_1 and R from Lemma 5.5,

$$1 - q_n := \mathbf{E}\,\mu_n^{(r)}(\mathcal{K}^c) \le \frac{k_1}{n} + \Pr\{|\lambda_{k_1}| > R\} \le C\delta_n^{\frac{1}{2}}(z). \tag{6.21}$$

Furthermore, let $\overline{\mu}_n^{(r)}$ and $\widehat{\mu}_n^{(r)}$ be probability measures supported on the compact set K and $K^{(c)}$ respectively, such that

$$\mathbf{E}\,\mu_n^{(r)} = q_n\overline{\mu}_n^{(r)} + (1 - q_n)\widehat{\mu}_n^{(r)}. \tag{6.22}$$

Introduce the logarithmic potential of the measure $\overline{\mu}_n^{(r)}$,

$$U_{\overline{\mu}_n^{(r)}} = -\int \log|z - \zeta| d\overline{\mu}_n^{(r)}(\zeta).$$

Similar to the proof of Lemma 6.1 we show that

$$|U_{\mu_n}^{(r)} - U_{\overline{\mu}_n^{(r)}}| \le C\Delta_n^{\frac{1}{4}}(z)|\ln\Delta_n(z)|.$$

This implies that

$$\lim_{n\to\infty} U_{\overline{\mu}_n^{(r)}}(z) = U_\mu(z)$$

for all $z \in \mathbb{C}$. According to equality (1.1), $U_\mu(z)$ is equal to the potential of the m-th power of the uniform distribution on the unit disc. This implies that the measure μ coincides with the m-th power of uniform distribution on the unit disc. Since the measures $\overline{\mu}_n^{(r)}$ are compactly supported, Theorem 6.9 from [16] and Corollary 2.2 from [16] together imply that

$$\lim_{n\to\infty} \overline{\mu}_n^{(r)} = \mu \tag{6.23}$$

in the weak topology. Inequality (6.21) and relations (6.22) and (6.22) together imply that

$$\lim_{n\to\infty} \mathbf{E}\,\mu_n^{(r)} = \mu$$

in the weak topology. Finally, by Lemma 1.1 in [9], we get

$$\lim_{n\to\infty} \mathbf{E}\,\mu_n = \mu \tag{6.24}$$

in the weak topology. Thus Theorem 1.1 is proved.

7 Appendix

Define $\mathbf{V}_{\alpha,\beta} := \prod_{\nu=\alpha}^{\beta} \mathbf{X}^{(\nu)}$.

Lemma 7.1. *Under the conditions of Theorem 1.1 we have, for any $j = 1, \dots, n$, $k = 1, \dots, n$ and for any $1 \le \alpha \le \beta \le m$,*

$$\mathbf{E}\,[\mathbf{V}_{\alpha,\beta}]_{jk} = 0$$

Proof. For $\alpha = \beta$ the claim is easy. Let $\alpha < \beta$ and $1 \le j \le n$, $1 \le k \le n$. Direct calculations show that

$$\mathbf{E}\,[\mathbf{V}_{\alpha,\beta}]_{jk} = \frac{1}{n^{\frac{\beta-\alpha+1}{2}}} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_{\beta-\alpha}=1}^{n} \mathbf{E}\, X_{j,j_1}^{(\alpha)} X_{j_1,j_2}^{(\alpha+1)} \cdots X_{j_{\beta-\alpha-1},k}^{(\beta)} = 0$$

Thus the Lemma is proved. □

In all Lemmas below we shall assume that

$$\mathbf{E}\, X_{jk}^{(\nu)} = 0, \quad \mathbf{E}\,|X_{jk}^{(\nu)}|^2 = 1, \quad |X_{jk}^{(\nu)}| \le c\tau_n\sqrt{n} \quad \text{a. s.} \tag{7.1}$$

with $\tau_n = o(1)$ such that $\tau_n^{-2} L_n(\tau_n) \le \tau_n^2$.

Lemma 7.2. *Assuming the conditions of Theorem 1.1 as well as (7.1), we have, for any* $1 \le \alpha \le \beta \le m$,

$$\mathbf{E}\,\|\mathbf{V}_{\alpha,\beta}\|_2^2 \le Cn \tag{7.2}$$

Proof. We shall consider the case $\alpha < \beta$ only. Other case is easy. Direct calculations show that

$$\mathbf{E}\,\|\mathbf{V}_{\alpha,\beta}\|_2^2 \le \frac{C}{n^{\beta-\alpha+1}} \sum_{j=1}^{n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_{\beta-\alpha}=1}^{p_{\beta-\alpha-1}} \sum_{k=1}^{p_{\beta-\alpha}} \mathbf{E}\,[X_{j,j_1}^{(\alpha)} X_{j_1,j_2}^{(\alpha+1)} \cdots X_{j_{\beta-\alpha},k}^{(\beta)}]^2$$

By independence of random variables, we get

$$\mathbf{E}\,\|\mathbf{V}_{\alpha,\beta}\|_2^2 \le Cn$$

Thus the Lemma is proved. □

Lemma 7.3. *Assuming the conditions of Theorem 1.1 as well as (7.1) we have, for any* $j = 1, \dots n$, $k = 1, \dots, n$ *and* $r \ge 1$,

$$\mathbf{E}\,\|\mathbf{V}_{a,b}\mathbf{e}_k\|_2^{2r} \le C_r, \tag{7.3}$$

and

$$\mathbf{E}\,\|\mathbf{e}_j^T\mathbf{V}_{a,b}\|_2^{2r} \le C_r, \tag{7.4}$$

with some positive constant C_r *depending on* r.

Proof. By definition of the matrices $\mathbf{V}_{a,b}$, we may write

$$\|\mathbf{e}_j\mathbf{V}_{a,b}\|_2^2 = \frac{1}{n^{b-a+1}} \sum_{l=1}^{n} \left| \sum_{j_a=1}^{n} \cdots \sum_{j_{b-1}=1}^{n} X_{jj_a}^{(a)} \cdots X_{j_{b-1}l}^{(b)} \right|^2 \tag{7.5}$$

Using this representation, we get

$$\mathbf{E}\,\|\mathbf{V}_{a,b}\mathbf{e}_k\|_2^{2r} = \frac{1}{n^{r(b-a)}}\sum_{l_1=1}^{n}\cdots\sum_{l_r=1}^{n}\mathbf{E}\prod_{q=1}^{r}\left(\sum_{j_a=1}^{n}\cdots\sum_{j_{b-1}=1}^{n}\sum_{\widehat{j}_a=1}^{n}\cdots\sum_{\widehat{j}_{b-1}=1}^{n}A^{(l_q)}_{(j_a,\ldots,j_b,\widehat{j}_1,\ldots,\widehat{j}_b)}\right) \tag{7.6}$$

where

$$A^{(l_q)}_{(j_a,\ldots,j_b,\widehat{j}_1,\ldots,\widehat{j}_b)} = X^{(a)}_{jj_a}\overline{X}^{(a)}_{j\widehat{j}_a}X^{(a)}_{j_aj_{a+1}}\overline{X}^{(a)}_{\widehat{j}_a\widehat{j}_{a+1}}\cdots X^{(b)}_{j_{b-2}j_{b-1}}\overline{X}^{(b-1)}_{\widehat{j}_{b-2}\widehat{j}_{b-1}}X^{(b)}_{j_{b-1}l_q}\overline{X}^{(b)}_{\widehat{j}_{b-1}l_q}. \tag{7.7}$$

By $\overline{x}$ we denote the complex conjugate of the number x. Expanding the product on the r.h.s of (7.6), we get

$$\mathbf{E}\,\|\mathbf{V}_{a,b}\mathbf{e}_k\|_2^{2r} = \sum\nolimits^{**}\mathbf{E}\prod_{q=1}^{r}A^{(l_q)}_{(j_a^{(q)},\ldots,j_b^{(q)},\widehat{j}_1^{(\nu)},\ldots,\widehat{j}_b^{(q)})}, \tag{7.8}$$

where $\sum^{**}$ is taken over all set of indices $j_a^{(q)},\ldots,j_{b-1}^{(q)},l_q$ and $\widehat{j}_a^{(\nu)},\ldots,\widehat{j}_{b-1}^{(q)}$ where $j_k^{(q)},\widehat{j}_k^{(q)} = 1,\ldots,p_k$, $k=a,\ldots,b-1$, $l_q=1,\ldots,p_b$ and $q=1,\ldots,r$. Note that the summands in the right hand side of (7.7) is equal 0 if there is at least one term in the product 7.7 which appears only once. This implies that the summands in the right hand side of (7.7) are not equal zero only if the union of all sets of indices in r.h.s of (7.7) consist of at least r different terms and each term appears at least twice.

Introduce the following random variables, for $\nu=a+1,\ldots,b-1$,

$$\zeta^{(\nu)}_{j_{\nu-1}^{(1)},\ldots,j_{\nu-1}^{(r)},j_\nu^{(1)},\ldots,j_\nu^{(r)},\widehat{j}_{\nu-1}^{(1)},\ldots,\widehat{j}_{\nu-1}^{(r)},\widehat{j}_\nu^{(1)},\ldots,\widehat{j}_\nu^{(r)}} = X^{(\nu)}_{j_{\nu-1}^{(1)},j_\nu^{(1)}}\cdots X^{(\nu)}_{j_{\nu-1}^{(r)},j_\nu^{(r)}}\overline{X}^{(\nu)}_{\widehat{j}_{\nu-1}^{(1)},\widehat{j}_\nu^{(1)}},\cdots\overline{X}^{(\nu)}_{\widehat{j}_{\nu-1}^{(r)},\widehat{j}_\nu^{(r)}}, \tag{7.9}$$

and

$$\begin{aligned}\zeta^{(a)}_{j_1^{(1)},\ldots,j_1^{(r)},\widehat{j}_1^{(1)},\ldots,\widehat{j}_1^{(r)}} &= X^{(a)}_{jj_1^{(a)}}\cdots X^{(a)}_{j_a^{(r)}j_{a+1}^{(r)}}\overline{X}^{(a)}_{j\widehat{j}_a^{(1)}}\cdots\overline{X}^{(a)}_{\widehat{j}_a^{(r)},\widehat{j}_{a+1}^{(r)}}\\ \zeta^{(b)}_{j_{b-1}^{(1)},\ldots,j_{b-1}^{(r)},\widehat{j}_{b-1}^{(1)},\ldots,\widehat{j}_{b-1}^{(r)},l_q} &= X^{(b)}_{j_{b-1}^{(1)}j_b^{(1)}}\cdots X^{(b)}_{j_{b-1}^{(r)}l_q}\overline{X}^{(b)}_{\widehat{j}_{b-1}^{(1)},l_q},\cdots\overline{X}^{(b)}_{\widehat{j}_{b-1}^{(r)},l_q}.\end{aligned}$$

Let the set of indices $j_a^{(1)},\ldots,j_a^{(r)},\widehat{j}_a^{(1)},\ldots,\widehat{j}_a^{(r)}$ contain t_a different indices, say $i_1^{(a)},\ldots,i_{t_a}^{(a)}$ with multiplicities $k_1^{(a)},\ldots,k_{t_a}^{(a)}$ respectively, $k_1^{(a)}+\ldots+k_{t_a}^{(a)}=2r$. Note that $\min\{k_1^{(a)},\ldots,k_{t_a}^{(a)}\}\le 2$. Otherwise, $|\mathbf{E}\,\zeta^{(a)}_{j_a^{(1)},\ldots,j_a^{(r)},\widehat{j}_a^{(1)},\ldots,\widehat{j}_a^{(r)}}| = 0$. By assumption (7.1), we have

$$|\mathbf{E}\,\zeta^{(a)}_{j_a^{(1)},\ldots,j_a^{(r)},\widehat{j}_a^{(1)},\ldots,\widehat{j}_a^{(r)}}| \le C(\tau_n\sqrt{n})^{2r-2t_a} \tag{7.10}$$

A similar bound we get for $|\mathbf{E}\,\zeta^{(b)}_{j_{b-1}^{(1)},\ldots,j_1^{(r)},\widehat{j}_{b-1}^{(1)},\ldots,\widehat{j}_{b-1}^{(r)},l_q}|$. Assume that the set of indices $\{j_{b-1}^{(1)},\ldots,j_{b-1}^{(r)},\ \widehat{j}_{b-1}^{(1)},\ldots,\widehat{j}_{b-1}^{(r)}\}$ contains t_{b-1} different indices, say, $i_1^{(b-1)},\ldots,i_{t_{b-1}}^{(a)}$ with

multiplicities
$k_1^{(b-1)}, \ldots, k_{t_{b-1}}^{(a)}$ respectively, $k_1^{(b-1)} + \ldots + k_{t_{b-1}}^{(a)} = 2r$. Then

$$|\mathbf{E}\, \zeta^{(b)}_{j_{b-1}^{(1)},\ldots,j_1^{(r)},\widehat{j}_{b-1}^{(1)},\ldots,\widehat{j}_{b-1}^{(r)},l_q}| \le C(\tau_n\sqrt{n})^{2r-2t_{b-1}} \tag{7.11}$$

Furthermore, assume that for $a+1 \le \nu \le b-2$ there are t_ν different pairs of indices, say, $(i_a, i'_a), \ldots (i_{t_b}, i'_{t_b})$ in the set
$\{j_a^{(1)}, \ldots, j_a^{(r)}, \widehat{j}_a^{(1)}, \ldots, \widehat{j}_a^{(r)}, \ldots, j_{b-1}^{(1)}, \ldots, j_{b-1}^{(r)}, \widehat{j}_{b-1}^{(1)}, \ldots, \widehat{j}_{b-}^{(r)}, l_1, l_r\}$ with multiplicities
$k_1^{(\nu)}, \ldots, k_{t_\nu}^{(\nu)}$. Note that

$$k_1^{(\nu)} + \ldots + k_{t_\nu}^{(\nu)} = 2r \tag{7.12}$$

and

$$\mathbf{E}\, \zeta^{(\nu)}_{j_{\nu-1}^{(1)},\ldots,j_{\nu-1}^{(r)},j_\nu^{(1)},\ldots,j_\nu^{(r)},\widehat{j}_{\nu-1}^{(1)},\ldots,\widehat{j}_{\nu-1}^{(r)},\widehat{j}_\nu^{(1)},\ldots,\widehat{j}_\nu^{(r)}} \le C(\tau_n\sqrt{n})^{2r-2t_\nu}. \tag{7.13}$$

The inequalities (7.10)-(7.13) together yield

$$|\mathbf{E} \prod_{q=1}^{r} A^{(l_q)}_{(j_a^{(q)},\ldots,j_b^{(q)},\widehat{j}_1^{(\nu)},\ldots,\widehat{j}_b^{(q)})}| \le C(\tau_n\sqrt{n})^{2r(b-a)-2(t_1+\ldots+t_{b-a})}. \tag{7.14}$$

It is straightforward to check that the number $\mathcal{N}(t_a, \ldots, t_b)$ of sequences of indices $\{j_a^{(1)}, \ldots, j_a^{(r)}, \widehat{j}_a^{(1)}, \ldots, \widehat{j}_a^{(r)}, \ldots, j_{b-1}^{(1)}, \ldots, j_{b-1}^{(r)}, \widehat{j}_{b-1}^{(1)}, \ldots, \widehat{j}_{b-}^{(r)}, l_1, \ldots, l_r\}$ with $t_a, \ldots, t_b$ of different pairs satisfies the inequality

$$\mathcal{N}(t_a, \ldots, t_b) \le C n^{t_a+\ldots+t_b}, \tag{7.15}$$

with $1 \le t_i \le r, \quad i = a, \ldots, b$. Note that in the case $t_a = \cdots = t_b = r$ the inequalities (7.10)–(7.13) imply

$$\mathbf{E}\, \zeta^{(\nu)}_{j_{\nu-1}^{(1)},\ldots,j_{\nu-1}^{(r)},j_\nu^{(1)},\ldots,j_\nu^{(r)},\widehat{j}_{\nu-1}^{(1)},\ldots,\widehat{j}_{\nu-1}^{(r)},\widehat{j}_\nu^{(1)},\ldots,\widehat{j}_\nu^{(r)}} \le C \tag{7.16}$$

The inequalities (7.15), (7.14), (7.16), and representation (7.6) together conclude the proof. □

The Largest Singular Value. Recall that $|\lambda_1^{(m)}| \ge \ldots \ge |\lambda_n^{(m)}|$ denotes the eigenvalues of the matrix $\mathbf{W}$ ordered by decreasing absolute values and let $s_1^{(m)} \ge \ldots \ge s_n^{(m)}$ denote the singular values of the matrix $\mathbf{W}$.

We show the following

Lemma 7.4. *Under the conditions of Theorem 1.1 we have, for sufficiently large $K \ge 1$*

$$\Pr\{s_1^{(m)} \ge n\} \le C/n \tag{7.17}$$

for some positive constant $C > 0$.

Proof. Using Chebyshev's inequality, we get

$$\Pr\{s_1^{(m)} \geq n\} \leq \frac{1}{n^2}\mathbf{E}\,\mathrm{Tr}\left(\mathbf{W}\mathbf{W}^*\right) \leq \frac{1}{n} \tag{7.18}$$

Thus the Lemma is proved. □

Lemma 7.5. *Under conditions of Theorem 1.1 assuming (7.1), we have*

$$\mathbf{E}\,|\frac{1}{n}(\mathrm{Tr}\,\mathbf{R} - \mathbf{E}\,\mathrm{Tr}\,\mathbf{R})| \leq \frac{C}{nv^2}.$$

Proof. Consider the matrix $\mathbf{X}^{(1,j)}$ obtained from the matrix $\mathbf{X}^{(1)}$ by replacing its j-th row by a row with zero-entries. We define the following matrices

$$\mathbf{H}^{(\nu,j)} = \mathbf{H}^{(\nu)} - \mathbf{e}_j\mathbf{e}_j^T\mathbf{H}^{(\nu)},$$

and

$$\widetilde{\mathbf{H}}^{(m-\nu+1,j)} = \mathbf{H}^{(m-\nu+1)} - \mathbf{H}^{(m-\nu+1)}\mathbf{e}_{j+n}\mathbf{e}_{j+n}^T.$$

For the simplicity we shall assume that $\nu \leq m - \nu + 1$. Define

$$\mathbf{V}^{(\nu,j)} = \prod_{q=1}^{\nu-1}\mathbf{H}^{(q)}\,\mathbf{H}^{(\nu,j)}\prod_{q=\nu+1}^{m-\nu}\mathbf{H}^{(q)}\widetilde{\mathbf{H}}^{(m-\nu+1,j)}\prod_{q=m-\nu+2}^{m}\mathbf{H}^{(q)}.$$

Let $\mathbf{V}(\nu,j)(z) = \mathbf{V}(\nu,j)\mathbf{J} - \mathbf{J}(z)$. We shall use the following inequality. For any Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$ with spectral distribution function $F_A(x)$ and $F_B(x)$ respectively, we have

$$|\mathrm{Tr}\,(\mathbf{A} - \alpha\mathbf{I})^{-1} - \mathrm{Tr}\,(\mathbf{B} - \alpha\mathbf{I})^{-1}| \leq \frac{\mathrm{rank}(\mathbf{A} - \mathbf{B})}{v}, \tag{7.19}$$

where $\alpha = u + iv$. It is straightforward to show that

$$\mathrm{rank}(\mathbf{V}(z) - \mathbf{V}^{(\nu,j)}(z)) = \mathrm{rank}(\mathbf{V}\mathbf{J} - \mathbf{V}^{(\nu,j)}\mathbf{J}) \leq 4m. \tag{7.20}$$

Inequality (7.19) and (7.20) together imply

$$|\frac{1}{2n}(\mathrm{Tr}\,\mathbf{R} - \mathrm{Tr}\,\mathbf{R}^{(\nu,j)})| \leq \frac{C}{nv}.$$

After this remark we may apply a standard martingale expansion procedure. We introduce σ-algebras $\mathcal{F}_{\nu,j} = \sigma\{X_{lk}^{(\nu)},\, j < l \leq n, k = 1,\dots,n; X_{pk}^{(q)},\, q = \nu+1,\dots m,\, p = 1,\dots,n,\, k = 1,\dots,n\}$ and use the representation

$$\mathrm{Tr}\,\mathbf{R} - \mathbf{E}\,\mathrm{Tr}\,\mathbf{R} = \sum_{\nu=1}^{m}\sum_{j=1}^{n}(\mathbf{E}_{\nu,j-1}\mathrm{Tr}\,\mathbf{R} - \mathbf{E}_{\nu,j}\mathrm{Tr}\,\mathbf{R}),$$

where $\mathbf{E}_{\nu,j}$ denotes conditional expectation given the σ-algebra $\mathcal{F}_{\nu,j}$. Note that $\mathcal{F}_{\nu,n} = \mathcal{F}_{\nu+1,0}$ □

Lemma 7.6. *Under the conditions of Theorem 1.1 we have, for* $1 \le a, \le m$,

$$\mathbf{E}\,|\frac{1}{n}(\sum_{k=1}^{n}[\mathbf{V}_{a+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a}]_{k,k+n} - \mathbf{E}\sum_{j=1}^{n}[\mathbf{V}_{a+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a}]_{kk+n})|^2 \le \frac{C}{nv^4}.$$

and, for $1 \le a, \le m-1$,

$$\mathbf{E}\,|\frac{1}{n}(\sum_{k=1}^{n}[\mathbf{V}_{m-a+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{k,k} - \mathbf{E}\sum_{j=1}^{n}[\mathbf{V}_{m-a+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{kk})|^2 \le \frac{C}{nv^4}.$$

Proof. We prove the first inequality only. The proof of the other one is similar. For $\nu = 1, \dots, m$ and for $j = 1, \dots, n$, we introduce the matrices, $\mathbf{X}^{(\nu,j)} = \mathbf{X}^{(\nu)} - \mathbf{e}_j\mathbf{e}_j^T\mathbf{X}^{(\nu)}$, and $\mathbf{H}^{(\nu,j)} = \mathbf{H}^{(\nu)} - \mathbf{e}_j\mathbf{e}_j^T\mathbf{H}^{(\nu)}$ and
$\widetilde{\mathbf{H}}^{(m-\nu+1,j)} = \mathbf{H}^{(m-\nu+1,j)} - \mathbf{H}^{(m-\nu+1)}\mathbf{e}_{j+n}\mathbf{e}_{j+n}^T$. Note that the matrix $\mathbf{X}^{(\nu,j)}$ is obtained from the matrix $\mathbf{X}^{(\nu)}$ by replacing its j-th row by a row of zeros. Similar to the proof of the previous Lemma we introduce the matrices $\mathbf{V}_{c,d}^{(\nu,j)}$ by replacing in the definition of $\mathbf{V}_{c,d}$ the matrix $\mathbf{H}^{(\nu)}$ by $\mathbf{H}^{(\nu,j)}$ and the matrix $\mathbf{H}^{(m-\nu+1)}$ by $\widetilde{\mathbf{H}}^{(m-\nu+1,j)}$. For instance, if $c \le \nu \le m-\nu+1 \le d$ we get

$$\mathbf{V}_{c,d}^{(\nu,j)} = \prod_{q=a}^{\nu-1}\mathbf{H}^{(q)}\,\mathbf{H}^{(\nu,j)}\prod_{q=\nu+1}^{m-\nu}\mathbf{H}^{(q)}\widetilde{\mathbf{H}}^{(m-\nu+1,j)}\prod_{q=m-\nu+1}^{b}\mathbf{H}^{(q)}$$

.

Let $\mathbf{V}^{(\nu,j)} := \mathbf{V}_{1,m}^{(\nu,j)}$ and $\mathbf{R}^{(j)} := (\mathbf{V}^{(\nu,j)}(z) - \alpha\mathbf{I})^{-1}$. Introduce the following quantities, for $\nu = 1 \dots, m$ and $j = 1, \dots, n$,

$$\Xi_j := \sum_{k=1}^{n}[\mathbf{V}_{a+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{kk+n} - \sum_{k=1}^{n}[\mathbf{V}_{a+1,m}^{(\nu,j)}\mathbf{J}\mathbf{R}^{(\nu,j)}\mathbf{V}_{1,m-a+1}^{(\nu,j)}]_{kk+n}$$

We represent them in the following form

$$\Xi_j := \Xi_j^{(1)} + \Xi_j^{(2)} + \Xi_j^{(3)},$$

where

$$\Xi_{\nu,j}^{(1)} == \sum_{k=1}^{n}[(\mathbf{V}_{a+1,m} - \mathbf{V}_{a+1,m}^{(\nu,j)})\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{k,k+n},$$

$$\Xi_{\nu,j}^{(2)} = \sum_{k=1}^{n}[\mathbf{V}_{a+1,m}^{(\nu,j)}\mathbf{J}(\mathbf{R} - \mathbf{R}^{(\nu,j)})\mathbf{J}\mathbf{V}_{1,m-a+1}]_{kk+n},$$

$$\Xi_{\nu,j}^{(3)} = \sum_{k=1}^{n}[\mathbf{V}_{a+1,m}^{(j)}\mathbf{J}\mathbf{R}^{(\nu,j)}(\mathbf{V}_{1,m-a+1} - \mathbf{V}_{1,m-a+1}^{(\nu,j)})]_{kk+n}.$$

Note that

$$\mathbf{V}_{a+1,m} - \mathbf{V}^{(\nu,j)}_{a+1,m} = \mathbf{V}_{a+1,\nu-1}(\mathbf{H}^{(\nu)} - \mathbf{H}^{(\nu,j)})\mathbf{V}_{\nu+1,m} + \mathbf{V}_{a+1,\nu-1}\mathbf{H}^{(\nu,j)}\mathbf{V}_{\nu+1,m-\nu}(\widetilde{\mathbf{H}}_{m-\nu+1} - \widetilde{\mathbf{H}}^{\nu,j}_{m-\nu+1})\mathbf{V}_{m-\nu+2,m}.$$

By definition of the matrices $\mathbf{H}^{\nu,j}$ and $\widetilde{\mathbf{H}}^{m-\nu+1,j}$, we have

$$\sum_{k=1}^{n}[(\mathbf{V}_{a+1,m} - \mathbf{V}^{(\nu,j)}_{a+1,m})\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}]_{k,k+n} = [\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,\nu}]_{j,j} + [\mathbf{V}_{m-\nu+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,m-a+1}]_{j+n,j+n},$$

where

$$\widetilde{\mathbf{J}} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

This equality implies that

$$|\Xi^{(1)}_j| \le |[\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,\nu}]_{j,j+n}| + |[\mathbf{V}_{m-\nu+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,m-\nu+1}]_{j+n,j+n}|.$$

Using the obvious inequality $\sum_{j=1}^{n} a_{jj}^2 \le \|\mathbf{A}\|_2^2$ for any matrix $\mathbf{A} = (a_{jk})$, $j,k = 1,\dots,n$, we get

$$T_1 := \sum_{j=1}^{n}\mathbf{E}\,|\Xi^{(1)}_j|^2 \le \mathbf{E}\,\|\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,\nu}\|_2^2 + \mathbf{E}\,\|\mathbf{V}_{m-\nu+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,m-\nu+1}\|_2^2.$$

By Lemma 7.2, we get

$$T_1 \le \frac{C}{v^2}\mathbf{E}\,\|\mathbf{V}_{a+1,m}\mathbf{V}_{1,m-a+1}\|_2^2 \le \frac{Cn}{v^2} \tag{7.21}$$

Consider now the term

$$T_2 = \sum_{j=1}^{n}\mathbf{E}\,|\Xi^{(2)}_j|^2.$$

Using that $\mathbf{R} - \mathbf{R}^{(j)} = -\mathbf{R}^{(j)}(\mathbf{V}(z) - \mathbf{V}^{(\nu,j)}(z))\mathbf{R}$, we get

$$|\Xi^{(2)}_j| \le |\sum_{k=1}^{n}[\mathbf{V}^{(\nu,j)}_{a,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\nu-1}\mathbf{e}_j\mathbf{e}_j^T\mathbf{V}_{\nu,m}\mathbf{R}\mathbf{V}_{1,b}]_{k,k+n}| \le [\mathbf{J}\mathbf{H}^{(\alpha+1)}\mathbf{V}_{\alpha+2,m-\alpha}\mathbf{H}^{(m-\alpha+1,j)}\mathbf{V}_{m-\alpha+2,m}\mathbf{R}\mathbf{V}_{1,m-\alpha}\mathbf{V}^{(j)}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\alpha}]_{jj}.$$

This implies that

$$T^{(2)} \le C\mathbf{E}\,\|[\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,b}\mathbf{V}_{a,m}\mathbf{J}\mathbf{R}V_{1,\nu}\|_2^2.$$

It is straightforward to check that

$$T^{(2)} \leq \frac{C}{v^4}\mathbf{E}\,\|\mathbf{V}_{1,\alpha}\mathbf{J}\mathbf{H}^{(\alpha+1)}\mathbf{V}_{\alpha+2,m-\alpha}\mathbf{H}^{(m-\alpha+1,j)}\mathbf{V}_{m-\alpha+2,m}\|_2^2 = \mathbf{E}\,\|\mathbf{Q}\|_2^2 \tag{7.22}$$

The matrix on the right hand side of equation (7.22) may be represented in the following form

$$Q = \prod_{\nu=1}^{m} \mathbf{H}^{(\nu)^{\varkappa_\nu}},$$

where $\varkappa_\nu = 0$ or $\varkappa_\nu = 1$ or $\varkappa_\nu = 2$. Since $X_{ss}^{(\nu)} = 0$, for $\varkappa = 1$ or $\varkappa = 2$, we have

$$\mathbf{E}\,|\mathbf{H}^{(\nu)^{\varkappa}}_{kl}|^2 \leq \frac{C}{n}.$$

This implies that

$$T_2 \leq Cn. \tag{7.23}$$

Similar we prove that

$$T_3 := \sum_{j=1}^{n} \mathbf{E}\,|\Xi_j^{(3)}|^2 \leq Cn. \tag{7.24}$$

Inequalities (7.21), (7.23) and (7.24) together imply

$$\sum_{j=1}^{n} \mathbf{E}\,|\Xi_j|^2 \leq Cn$$

Applying now a martingale expansion with respect to the σ-algebras $\mathcal{F}_j$ generated by the random variables $X_{kl}^{(\alpha+1)}$ with $1 \leq k \leq j$, $1 \leq l \leq n$ and all other random variables $X_{sl}^{(q)}$ except $q = \alpha + 1$, we get

$$\mathbf{E}\,|\frac{1}{n}(\sum_{k=1}^{n}[\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha}]_{kk+n} - \mathbf{E}\sum_{j=1}^{n}[\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha}]_{kk+n})|^2 \leq \frac{C}{nv^4}.$$

Thus the Lemma is proved.

□

Lemma 7.7. *Under the conditions of Theorem 1.1 we have, for $\alpha = 1, \ldots, m$, there exists a constant C such that*

$$\frac{1}{n^{\frac{3}{2}}}\mathbf{E}\left|\sum_{j=1}^{n}\sum_{k=1}^{n}(X_{jk}^{(\alpha)} + {X_{jk}^{(\alpha)}}^3)\left[\frac{\partial^2(\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha+1})}{\partial {X_{jk}^{(\alpha)}}^2}(\theta_{jk}^{(\alpha)}X_{jk}^{(\alpha)})\right]_{kj}\right| \leq C\tau_n v^{-4},$$

and

$$\frac{1}{n^{\frac{3}{2}}}\mathbf{E}\left|\sum_{j=1}^{p_{m-\alpha}}\sum_{k=1}^{p_{m-\alpha+1}}(X_{jk}^{(m-\alpha+1)}+{X_{jk}^{(m-\alpha+1)}}^3)\right.$$
$$\left.\times\left[\frac{\partial^2(\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha+1})}{\partial {X_{jk}^{(m-\alpha+1)}}^2}(\theta_{jk}^{(m-\alpha+1)}X_{jk}^{(m-\alpha+1)})\right]_{j+n,k}\right|\leq C\tau_n v^{-4},\ (7.25)$$

where $\theta_{jk}^{(\alpha)}$ *and* $X_{jk}^{(\alpha)}$ *are independent in aggregate for* $\alpha=1,\dots,m$ *and* $j=1,\dots,n$, $k=1,\dots,n$, *and* $\theta_{jk}^{(\alpha)}$ *are r.v. which are uniformly distributed on the unit interval.*
By $\frac{\partial^2}{\partial {X_{jk}^{(\alpha)}}^2}\mathbf{A}(\theta_{jk}^{(\alpha)}X_{jk}^{(\alpha)})$ *we denote the matrix obtained from* $\frac{\partial^2}{\partial {X_{jk}^{(\alpha)}}^2}\mathbf{A}$ *by replacing its entries* $X_{jk}^{(\alpha)}$ *by* $\theta_{jk}^{(\alpha)}X_{jk}^{(\alpha)}$.

Proof. The proof of this lemma is rather technical. But we shall include it for completeness. By the formula for the derivatives of a resolvent matrix, we have

$$\frac{\partial(\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha+1})}{\partial X_{jk}^{(\alpha)}}=\sum_{l=1}^{5}Q_l,\qquad(7.26)$$

$$\begin{aligned}
\mathbf{Q}_1=&\frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\alpha-1}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{\alpha+1,m-\alpha+1}I_{\{\alpha\leq m-\alpha+1\}})\\
\mathbf{Q}_2=&\frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha}\mathbf{e}_{k+n}\mathbf{e}_{j+n}\\
\mathbf{Q}_3=&-\frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\alpha-1}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha+1}\\
\mathbf{Q}_4=&-\frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha}\mathbf{e}_{k+p_{m-\alpha}}\mathbf{e}_{j+p_{m-\alpha+1}}^T\mathbf{V}_{m-\alpha+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha+1}\\
\mathbf{Q}_5=&\frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m-\alpha}\mathbf{e}_{k+p_{m-\alpha}}\mathbf{e}_{j+n}^T\mathbf{V}_{m-\alpha+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha+1}I_{\{\alpha\leq m-\alpha+1\}}).
\end{aligned}$$

Introduce the notations

$$\mathbf{U}_\alpha:=\mathbf{V}_{\alpha+1,m},\quad \mathbf{V}_\alpha=\mathbf{V}_{1,m-\alpha+1}.$$

From formula (7.26) it follows that

$$\frac{\partial^2(\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_\alpha)}{\partial {X_{jk}^{(\nu)}}^2}=\sum_{l=1}^{5}\frac{\partial\mathbf{Q}_l}{\partial X_{jk}^{(\alpha)}}.$$

Since all the calculations will be similar we consider the case $l=3$ only. Simple calculations of derivatives show that

$$\frac{\partial\mathbf{Q}_3}{\partial X_{jk}^{(\alpha)}}=\sum_{m=1}^{7}\mathbf{P}^{(m)},\qquad(7.27)$$

where

$$\mathbf{P}^{(1)} = -\frac{1}{n}\mathbf{V}_{\alpha+1,m-\alpha}\mathbf{e}_{k+p_{m-\alpha}}\mathbf{e}_{j+n}^T\mathbf{U}_{m-\alpha+1}\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_\alpha$$
$$\mathbf{P}^{(2)} = -\frac{1}{n}\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{\alpha+1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T$$
$$\mathbf{P}^{(3)} = -\frac{1}{n}\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{\alpha+1,m-\alpha}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{U}_{m-\alpha+1}\mathbf{J}\mathbf{R}\mathbf{V}_\alpha$$
$$\mathbf{P}^{(4)} = -\frac{1}{n}\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_\alpha$$
$$\mathbf{P}^{(5)} = \frac{1}{n}\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{\alpha+1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{U}_{m-\alpha+1}\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_\alpha$$
$$\mathbf{P}^{(6)} = \frac{1}{n}\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{\alpha+1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{U}_{m-\alpha+1}\mathbf{J}\mathbf{R}\mathbf{V}_\alpha$$
$$\mathbf{P}^{(7)} = \frac{1}{n}\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\mathbf{e}_k^T\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_\alpha.$$

Consider now the quantity, for $\mu = 1, \ldots, 5$,

$$L_\mu = \frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^n\sum_{k=1}^n \mathbf{E}\, {X_{j,k}^{(\alpha)}}^3 \left[\frac{\partial \mathbf{Q}_\mu}{\partial X_{jk}^{(\alpha)}}\right]_{kj}. \tag{7.28}$$

We bound L_3 only. The others terms are bounded in a similar way. First we note that

$$\sum_{j=1}^n\sum_{k=1}^n \mathbf{E}\, {X_{j,k}^{(\alpha)}}^3 [\mathbf{P}^{(\nu)}]_{kj} = 0, \quad \text{for} \quad \nu = 1, 2, 3. \tag{7.29}$$

Furthermore,

$$\mathbf{E}\,|{X_{j,k}^{(\alpha)}}^3||[\mathbf{P}^{(4)}]_{kj}| \le \mathbf{E}\,|X_{jk}^{(\alpha)}|^3|[\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}]_{kj}|^2|[\mathbf{U}_\alpha\mathbf{J}\mathbf{R}\mathbf{V}_\alpha]_{kj}|. \tag{7.30}$$

Let $\mathbf{U}_\alpha^{(jk)}$ ($\mathbf{V}_\alpha^{(j,k)}$) denote matrix obtained from $\mathbf{U}_\alpha$ ($\mathbf{V}_\alpha$) by replacing $X_{jk}^{(\alpha)}$ by zero. We may write

$$\mathbf{U}_\alpha = \mathbf{U}_\alpha^{(jk)} + \frac{1}{\sqrt{n}}X_{jk}^{(\alpha)}\mathbf{V}_{\alpha+1,m-\alpha+1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{m-\alpha+2,m}. \tag{7.31}$$

and

$$\mathbf{V}_\alpha = \mathbf{V}_\alpha^{(j,k)} + \frac{1}{\sqrt{n}}X_{jk}\mathbf{V}_{1,m-\alpha+1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T.$$

Using these representations and taking in account that

$$[\mathbf{V}_{\alpha+1,m-\alpha}]_{k,k+n} = [\mathbf{V}_{1,m-\alpha}]_{k,k+n} = 0, \tag{7.32}$$

we get by differentiation

$$\mathbf{E}\,|X_{j,k}^{(\alpha)}{}^3||[\mathbf{P}^{(4)}]_{kj}| \le \frac{1}{n}\mathbf{E}\,|X_{j,k}^{(\alpha)}{}^3|\,|[\mathbf{U}_\alpha \mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}]_{kj}|^2\,|[\mathbf{U}_\alpha^{(j,k)}\mathbf{J}\mathbf{R}\mathbf{V}_\alpha^{(j,k)}]_{kj}|. \tag{7.33}$$

Furthermore,

$$\begin{aligned} |[\mathbf{U}_\alpha \mathbf{J}\mathbf{R}\mathbf{V}_{m-\alpha+2}]_{k,j}| &\le \frac{1}{v}\|\mathbf{V}_{m-\alpha+2}\mathbf{e}_j\|_2\|\mathbf{e}_k^T\mathbf{U}_\alpha\|_2 \\ |[\mathbf{U}_\alpha^{(j,k)}\mathbf{J}\mathbf{R}\mathbf{V}_\alpha^{(j,k)}]_{kj}| &\le \frac{1}{v}\|\mathbf{V}_\alpha^{(j,k)}\mathbf{e}_k\|_2\|\mathbf{e}_j^T\mathbf{U}_\alpha^{(j,k)}\|_2. \end{aligned} \tag{7.34}$$

Applying inequalities (7.33) and (7.34) and taking in account the independence of entries, we get

$$\mathbf{E}\,|X_{j,k}^{(\alpha)}{}^3||[\mathbf{P}^{(4)}]_{kj}| \le \frac{1}{nv^2}\mathbf{E}\,|X_{j,k}^{(\alpha)}{}^3\mathbf{E}\,\|\mathbf{V}_{m-\alpha+2}\mathbf{e}_k\|_2^2\|\mathbf{e}_j^T\mathbf{U}_\alpha\|_2^2\|\mathbf{V}_\alpha^{(j,k)}\mathbf{e}_k\|_2\|\mathbf{e}_j^T\mathbf{U}_\alpha^{(j,k)}\|_2 \tag{7.35}$$

Applying Lemma 7.3, we get

$$\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^n\sum_{k=1}^n \mathbf{E}\,|X_{jk}^{(\alpha)}|^3|[\mathbf{P}^{(4)}]_{kj}| \le \frac{C}{n^{\frac{5}{2}}}\sum_{j=1}^n\sum_{k=1}^n \mathbf{E}\,|X_{jk}^{(\alpha)}|^3 \tag{7.36}$$

The assumption (7.1) now yields

$$\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^n\sum_{k=1}^n \mathbf{E}\,|X_{jk}^{(\alpha)}|^3|[\mathbf{P}^{(4)}]_{kj}| \le C\tau_n. \tag{7.37}$$

Similar we get corresponding bounds for $\nu = 5, 6, 7$

$$\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^n\sum_{k=1}^n \mathbf{E}\,|X_{jk}^{(\alpha)}|^3|[\mathbf{P}^{(\nu)}]_{kj}| \le C\tau_n. \tag{7.38}$$

and

$$|L_\mu| \le C\tau_n, \quad \mu = 1, \dots, 5. \tag{7.39}$$

The bound of the quantity

$$\widehat{L}_\mu = \sum_{j=1}^n\sum_{k=1}^n \mathbf{E}\,X_{j,k}^{(\alpha)}\left[\frac{\partial \mathbf{Q}_\nu}{\partial X_{jk}^{(\alpha)}}\right]_{kj}. \tag{7.40}$$

is similar. Thus, the Lemma is proved.

□

Lemma 7.8. *Under the conditions of Theorem 1.1 we have*

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,X_{jk}^{(\nu)}[\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}]_{kj}=\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,\left[\frac{\partial\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}}{\partial X_{jk}^{(\nu)}}\right]_{kj}+\varepsilon_n(z,\alpha))$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,X_{j,k}^{(m-\nu+1)}[\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}]_{j+n,k}$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,\left[\frac{\partial\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}}{\partial X_{jk}^{(\nu)}}\right]_{j+n,k}+\varepsilon_n(z,\alpha)),$$

where $|\varepsilon_n(z,\alpha))|\le\frac{C\tau_n}{v^4}$.

Proof. By Taylor expansion we have,

$$\mathbf{E}\,\xi f(\xi)=f'(0)\mathbf{E}\,\xi^2+\mathbf{E}\,\xi^3 f''(\theta\xi),$$

and

$$f'(0)=\mathbf{E}\,f'(\xi)+\mathbf{E}\,\xi f''(\theta\xi)\tag{7.41}$$

where θ denotes a r.v. which uniformly distributed on the unit interval and is independent on ξ. After simple calculations we get

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,X_{jk}^{(\nu)}[\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}]_{kj}=\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,\left[\frac{\partial\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}}{\partial X_{jk}^{(\nu)}}\right]_{kj}$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,(X_{jk}^{(\nu)}+{X_{jk}^{(\nu)}}^3)\left[\frac{\partial^2\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}}{\partial {X_{jk}^{(\nu)}}^2}(\theta_{jk}^{(\nu)}X_{jk}^{(\nu)})\right]_{kj}.$$

Using the results of Lemma 7.7, we conclude the proof.

□

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