Cluster tilting for tilted algebras *†

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Abstract

We build a connection between iterated tilted algebras with trivial cluster tilting and tilted algebras of finite type. As a result, we can classify all tilted algebras with cluster tilting in terms of quivers, that is, all tilted algebras with cluster tilting are of finite type. Moreover, we draw the quivers of Auslander's 1-Gorenstein algebras with global dimension 2 admitting trivial cluster tilting subcategories, which implies that such algebras are tilted of finite type but not Nakayama.

1. Introduction

The notion of cluster tilting subcategories was introduced by Keller and Reiten in [KR]. This notion coincides with that of Iyama's maximal 1-orthogonal subcategories, which was introduced to develop the Auslander-Reiten theory to the higher ones. We notice that Iyama's higher dimensional Auslander-Reiten theory depends on the existence of cluster tilting subcategories. So it is interesting to study when the cluster tilting subcategories exist and the properties of algebras admitting such subcategories. Several authors worked on this topic(See [EH],[GlS],[Iy1-6],[HuZ1-3],[IO1-2],[R1-2] and so on). In [HZ2] we proved that an Auslander algebra with global dimension 2 admitting a trivial cluster tilting subcategory is a tilted algebra of finite type and a Nakayama algebra. In [HZ3] we generalized the result and proved that an Auslander's 1-Gorenstein algebra with global dimension 2 admitting a trivial cluster tilting subcategory is also tilted. So it is natural to study the connection between algebras with global dimension 2 admitting trivial cluster tilting and tilted algebras.

On the other hand, Iyama and Oppermann in [IO2] gave a complete classification of iterated tilted algebras with cluster tilting objects in terms of quivers. Notice that iterated tilted algebras are far from tilted algebras. So studying the existence of cluster tilting for tilted algebras is also important and interesting.

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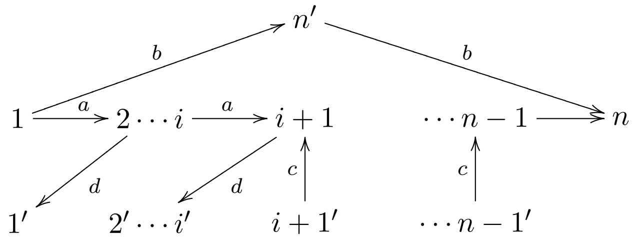
In this paper, based on Iyama and Oppermann's result, we are going to find a connection between iterated tilted algebras with trivial cluster tilting and tilted algebras. We will determine all the tilted algebras which admit cluster tilting subcategories. As a result, we can get all Auslander's 1-Gorenstein algebras with global dimension 2 admitting trivial cluster tilting subcategories are tilted of finite type but not Nakayama. The paper is organized as follows:

In Section 2, we will collect some notions and some known results. In Section 3, We will determine the cluster tilting for tilted algebras. Based on results in Section 2 especially Iyama-Oppermann's Theorem, we get the following conclusion: Let Λ be a basic and connected finite dimensional algebra over an algebraic closed field K. Λ is a tilted algebra with global dimension 2 admitting a cluster tilting subcategory of $\operatorname{mod}\Lambda$ if and only if Λ is of the following: (1) $1 \xrightarrow{a} 2 \xrightarrow{a} 3 \cdots \xrightarrow{a} \quad \xrightarrow{a} n$ with the relation $a^{n-1} = 0$

(2)



with relations $a^{n-1} = b^2$, $ac = 0$ and $da = 0$, where c is an arrow $i' \longrightarrow i$, d an arrow $i+1 \longrightarrow i'$. For some $0 \leq i \leq n-1$ the chosen arrow at vertex j' is d if $1 \leq j \leq i$ and c if $i+1 \leq j \leq n-1$.

As an application, one can determine all the Auslander's 1-Gorenstein algebras with global dimension 2. Compared to the result in [HZ2] mentioned above, such algebras are also tilted of finite type but not Nakayama. At the end of this section we give a conjecture on the existence of trivial cluster tilting for algebras of global dimension 2.

2. Preliminaries

Throughout the paper Λ is a basic and connected finite dimensional algebra over an algebraic closed field K. All modules are finitely generated right modules over Λ and $\mathrm{D} = \mathrm{Hom}_K(-, K)$ is the ordinary duality.

Let us recall a classical concept due to Auslander-Smalo [ASm]. A subcategory $\mathscr{C}$ of an additive category $\mathscr{X}$ is called contravariantly finite if for any $X \in \mathscr{X}$, there exists a morphism $f \in \mathrm{Hom}_{\mathscr{X}}(C, X)$ with $C \in \mathscr{C}$ such that $\mathrm{Hom}_{\mathscr{X}}(-, C) \longrightarrow \mathrm{Hom}_{\mathscr{X}}(-, X) \longrightarrow 0$ is exact on C. Dually, a covariantly finite subcategory is defined. A contravariantly and covariantly finite subcategory is called functorially finite.

Definition 2.1 Let $\mathscr{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$, $\mathscr{C}$ is called *cluster tilting* if $\mathscr{C} = \{X \in \operatorname{mod}\Lambda | \operatorname{Ext}^1_\Lambda(X, \mathscr{C}) = 0\} = \{X \in \operatorname{mod}\Lambda | \operatorname{Ext}^1_\Lambda(\mathscr{C}, X) = 0\}$, moreover, $\mathscr{C}$ is called a cluster tilting object if $\mathscr{C} = addM$ for some $M \in \mathscr{C}$

It is not difficult to show that all projective and all injective modules are in a cluster tilting subcategory. But in general all projectives and injectives can not form a cluster tilting subcategory of $\operatorname{mod}\Lambda$. We call $\mathscr{C}$ is *trivial cluster tilting* if it is cluster tilting and M is either projective or injective for any indecomposable $M \in \mathscr{C}$.

For a module $M \in \operatorname{mod}\Lambda$, we use $\operatorname{pd}_\Lambda M$ and $\operatorname{id}_\Lambda M$ to denote the projective dimension and the injective dimension of M, respectively.

Definition 2.2 (1) ([HRS]) Λ is called *almost hereditary* if the following conditions are satisfied: (a) $\operatorname{gl.dim}\Lambda \leq 2$; and (b) If $X \in \operatorname{mod}\Lambda$ is indecomposable, then either $\operatorname{pd}_\Lambda X \leq 1$ or $\operatorname{id}_\Lambda X \leq 1$.

(2) ([HRS]) Λ is called *quasi-tilted* if $\Lambda = \operatorname{End}(T)^{op}$, where $\mathcal{H}$ is a locally finite hereditary abelian R-category and T is a tilting object in $\mathcal{H}$. It should be noted that an algebra is quasi-tilted if and only if it is almost hereditary.

(3) ([HRi]) Λ is called *tilted* if Λ is of the form $\Lambda = \operatorname{End}(T_\Gamma)$, where T_Γ is a tilting module and Γ is a hereditary Artinian algebra. It is trivial that a tilted algebra is quasi-tilted.

Now we have the following conclusion on cluster tilting subcategories for quasi-tilted algebras.

Proposition 2.3 *Let Λ be a quasi-tilted algebra with global dimension 2. If Λ admits a cluster tilting subcategory $\mathscr{C}$, then $\mathscr{C}$ is trivial.*

Proof. For any indecomposable M in $\mathscr{C}$, by Definition 2.2 we get $\operatorname{pd}_\Lambda M \leq 1$ or $\operatorname{id}_\Lambda M \leq 1$. Taking a minimal projective or injective resolution of M, one gets the assertion. □

Let M be in $\operatorname{mod}\Lambda$. Denote by $\Omega^i M$ and $\Omega^{-i} M$ the ith syzygy and co-syzygy of M, respectively. We use $\operatorname{Tr}M$ to denote the Auslander-Bridger Transpose of M. For a subcategory $\mathscr{C}$ of $\operatorname{mod}\Lambda$. Denote by $\underline{\mathscr{C}}$ and $\overline{\mathscr{C}}$ the stable category of $\mathscr{C}$ modulo projectives and injectives respectively. Denote by $\nu = \operatorname{DHom}(-, \Lambda)$ the Nakayama functor. Following Iyama we have

Proposition 2.4 *Let Λ be an algebra and $\mathscr{C}$ a cluster tilting subcategory of* $\operatorname{mod}\Lambda$. *Then*
(1) *The functor* $\tau_2 = \operatorname{DTr}\Omega : \underline{\mathscr{C}} \to \overline{\mathscr{C}}$ is an *equivalence with the inverse* $\tau_2 = \operatorname{TrD}\Omega^-$.
(2)*If* $\operatorname{gl.dim}\Lambda = 2$, *then* $\tau_2 = \operatorname{DExt}^2_\Lambda(-, \Lambda)$ *and* $\tau_2^- = \operatorname{Ext}^2_\Lambda(\operatorname{D}-, \Lambda)$.
(3)*If* $\operatorname{gl.dim}\Lambda = 2$ *and* $0 \to P_2(M) \xrightarrow{f} P_1(M) \to P_0(M) \to M \to 0$ *a minimal projective*

resolution for $M \in \mathscr{C}$. *Then* $\tau_2 M = \ker(\nu P_2(M) \stackrel{\nu f}{\to} \nu P_1(M))$.

Remark Proposition 2.4 implies that the functor τ^2 gives a bijection from indecomposable projectives to indecomposable injectives. The following definition of τ_2-closure of $\mathrm{D}\Lambda$ due to Iyama is essential in this paper.

Definition 2.5 For an algebra Λ, a subcategory $\mathscr{M}$ of $\mathrm{mod}\,\Lambda$ is called the τ_2-*closure* of $\mathrm{D}\Lambda$ if $\mathscr{M} = add\{\tau_2^i(\mathrm{D}\Lambda) | i \geq 0\}$.

Using the properties of τ_2-closure, Iyama gives the following on the existence of cluster tilting objects in [Iy5].

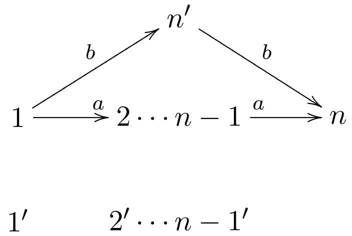
Proposition 2.6 *Let* Λ *be an algebra with global dimension* 2. Λ *has a cluster tilting object* M *if and only if* $\Lambda \in \mathscr{M}$, *where* $\mathscr{M}$ *is the* τ^2-*closure of* $\mathrm{D}\Lambda$. *Moreover,* $\mathscr{M} = addM$ *if* M *exists.*

Recall that an algebra B is called an iterated tilted algebra of Q if there is a class of algebras $A_1 = KQ, A_2, \cdots, A_n = B$ and for $1 \leq i \leq n-1$, we have $A_{i+1} \simeq \mathrm{End}_{A_i}(T_i)$ for a tilting module T_i over A_i or $A_i \simeq \mathrm{End}_{A_{i+1}}(N_i)$ for some tilting A_{i+1}-module. Now we can state the classification of Iyama and Oppermann on iterated tilted algebras with cluster tilting objects.

Theorem 2.7 *Let* Λ *be an iterated tilted algebras with global dimension* 2 *and* $n \geq 3$. Λ *admits a cluster tilting object if and only if it is one of the following:*

(1) $1 \xrightarrow{a} 2 \xrightarrow{a} 3 \cdots \xrightarrow{a} \quad \xrightarrow{a} n$ *with the relation* $a^{n-1} = 0$

(2)



with relations $a^2 = b^2$ *and* $da = ac = 0$, *where there is exactly* 1 *arrow* $c: i' \longrightarrow i$ *or* $d: i+1 \longrightarrow i'$ *at the vertex* i' *for any* $1 \leq i \leq n-1$.

Recall that an algebra Λ is called Auslander's 1-Gorenstein if $I^0(\Lambda)$ is projective. In [HZ3], We considered the trivial cluster tilting for Auslander's 1-Gorenstein algebras with global dimension 2 and proved the following

Theorem 2.8 *Let* Λ *be an Auslander's 1-Gorenstein algebra with global dimension* 2. *If*

Λ *admits a trivial cluster tilting subcategory of* $\operatorname{mod}\Lambda$, *then* Λ *is tilted.*

To classify all the tilted algebras with cluster tilting, in the following we will recall some notions and properties of special biserial algebras for later use.

Let (Q, I) be a bound quiver. A path p of Q is called a *zero paths* if $p \in I$. A zero path is called a *zero relation* if none of it proper subpaths is a zero path. A pair (p, q) of non-zero paths from a vertex a to a vertex b is called a *binomial relation* of (Q, I) if $mp + nq \in I$ for some non-zero $m, n \in KQ$ and here p, q are maximal subpath of (p, q). We call a, b the start-point and the end-point of (p, q), respectively.

Definition 2.9 An algebra Λ is called *special biserial* if $\Lambda = kQ/I$ with (Q, I) a bound quiver satisfying the following:

(1) Each vertex of Q is start-point or end-point of at most two arrows.

(2) For any arrow α, there is at most one arrow β such that $\alpha\beta \notin I$ and at most one arrow γ such that $\gamma\alpha \notin I$.

Moreover Λ is called a *string algebra* if, in addition, I is generated by a set of paths of Q.

Let $\Lambda = kQ/I$ be a special biserial algebra and w a reduced walk in Q. w is called a *string* if each path contained in w is neither a zero-relation nor a maximal subpath of a binomial relation. w is a sequential pair of a zero-relation if (1) $w = p_1p_2p_3$ where the p_i are non-trivial paths such that p_1p_2 and p_2p_3 are the only zero-relation contained in w or (2)$w = pw_1q$, where w_1 is a string and p, q are paths which are the only zero-relations contained in w.

The following connection between special biserial (non-string) algebras and tilted algebras are due to Huard and Liu.

Proposition 2.10 *Let* $\Lambda = kQ/I$ *be a special biserial algebra which is not string. Then* Λ *is tilted if and only if* (Q, I) *satisfies the following:*

(1) There is no sequential pair of zero-relations.

(2) The start-point of a binomial relation does not lie in another different binomial relation.

(3) *Let* $(\alpha p\beta, \gamma q\delta)$ *be a binomial relation, where* $\alpha, \beta, \gamma, \delta$ *are arrows and* p, q *are paths. If* u *is a non-zero path with* $e(u) = s(\alpha)$, *then either* $u\alpha p$ *or* $u\gamma q$ *is non-zero. Dually, if* v *is a non-zero path with* $e(\beta) = s(v)$, *then either* $p\beta v$ *or* $q\delta v$ *is non-zero.*

Let $\Lambda = kQ/I$ be a special biserial algebra and $w = c_1c_2\cdots c_n$ a non-trivial reduced walk of Q. w is a reduced cycle if $s(w) = e(w)$ and $c_n \neq c_1^-$. Moreover, a reduced cycle w is called a *band* if each of its powers is a string and it is not a power of a string of less length. We recall the following classification of special biserial algebras of finite type from [E, II Lemma 8.1] for later use.

Lemma 2.11 *Let* Λ *be a special biserial algebra. Then the following are equivalent:*
(1) Λ *is of finite type.*
(2) Λ *has no band.*

We remark that for a special biserial algebra Λ, an indecomposable module in $\operatorname{mod}\Lambda$ is a string, a band or a projective-injective corresponding to a binomial relation (See [WW]).

3. Quivers for tilted algebras with cluster tilting

In this section, based on the results in section 2 we will build a connection between iterated tilting algebras with a trivial cluster tilting and tilted algebras. As a result, we can give a classification of tilted algebras with cluster tilting subcategories in terms of quivers. Moreover, we can classify all Auslander's 1-Gorenstein algebras with global dimension 2 admitting trivial cluster tilting subcategories.

Throughout this section, we assume that (Q, I) a bound quiver. We use $P(a), I(a), S(a)$ to denote the indecomposable projective,injective and simple module according to the vertex $a \in Q$, respectively.

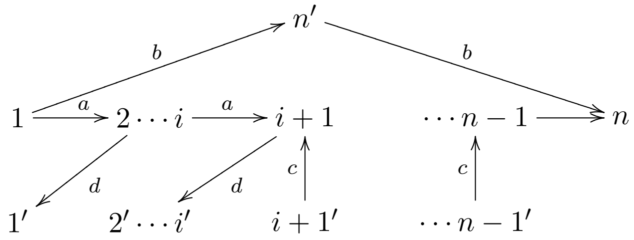
Proposition 3.1 *Let* Λ *be an algebra with global dimension* 2. Λ *admits a trivial cluster tilting subcategory* $\mathscr{C}$ *if and only if* $\tau_2 I$ *is indecomposable projective and* $\tau_2 I_1 \not\cong \tau_2 I_2$ *if* $I_1 \not\cong I_2$, *where* I, I_1,I_2 *are indecomposable injective modules.*

Proof.$\Rightarrow$ Since $\mathscr{C}$ is trivial, by Proposition 2.4 (1) one gets the assertion.

$\Leftarrow$ It is enough to show that Λ is in $\mathscr{M}$ the τ_2-closure of $\mathrm{D}\Lambda$. For any P projective-injective then $P \in \mathscr{M}$. For the case $\operatorname{id}_\Lambda P = 2$, we prove that there is an indecomposable injective I such that $\tau_2 I = P$. Denote by $\{I_1, I_2, \cdots, I_t\}$ the set of indecomposable non-projective injective modules. By the assumption, we get a set of indecomposable projective modules $\{\tau_2 I_1, \tau_2 I_2, \cdots, \tau_2 I_t\}$. Since the number of indecomposable projective and injective modules (up to isomorphism) is same, we get the assertion. □

It is not difficult to show that the algebras in the class (1) of Theorem 2.7 is Nakayama and Auslander's 1-Gorenstein. By Theorem 2.8, it is tilted. To classify all the tilted algebras from the iterated tilted algebras in Theorem 2.7, we need the following.

Proposition 3.2 *Let* Λ *be an algebra in the class* (2) *of Theorem* 2.7. Λ *admits a trivial* *cluster tilting subcategory if and only if* Λ *is given by*



with relations $a^{n-1} = b^2$, $ac = 0$ *and* $da = 0$, *where* c *is an arrow* $i' \longrightarrow i$, d *an arrow* $i+1 \longrightarrow i'$. *For some* $0 \leq i \leq n-1$ *the chosen arrow at vertex* j' *is* d *if* $1 \leq j \leq i$ *and* c *if* $i+1 \leq j \leq n-1$.

Proof. $\Rightarrow$: It is enough to prove the non-existence of cd. Suppose the first appearance of cd is at the vertexes i' and $i+1'$. We claim that the cluster tilting object is not trivial.

Taking a minimal projective resolution of $I(i')$

$$0 \longrightarrow P(i+1) \longrightarrow P(i) \longrightarrow P(i') \longrightarrow I(i') \longrightarrow 0 \,,$$

Applying the Nakayama functor ν, By Prop 2.4 we get the following minimal injective resolution of $\tau_2 I(i')$

$$0 \longrightarrow \tau_2 I(i') \longrightarrow I(i+1) \longrightarrow I(i) \longrightarrow I(i') \longrightarrow 0$$

since ν is an equivalence.

Notice that there is a minimal injective resolution of $S(i+1)$

$$0 \longrightarrow S(i+1) \longrightarrow I(i+1) \longrightarrow I(i) \longrightarrow I(i') \longrightarrow 0$$

So $\tau_2 I(1') \simeq S(i+1)$ is not projective, which contradicts Proposition 3.1.

$\Leftarrow$ It suffice to show that $\tau_2 I(j)$ and $\tau_2 I(j')$ for $1 \leq j \leq n$ are indecomposable projective by Theorem 2.7 and Proposition 2.4.

It is not difficult to show there are two projective-injective modules for Λ.

For the case $i = 0$, then all chosen arrows are c and the indecomposable projective-injective modules are $P(1) = I(n)$ and $P(1') = I(n')$.

Taking a minimal projective resolution of $I(1)$

$$0 \longrightarrow P(n) \longrightarrow P(n') \longrightarrow P(1') \longrightarrow I(1) \longrightarrow 0$$

and applying the Nakayama functor, then one can get the following exact sequence

$$0 \longrightarrow \tau_2 I(1) \longrightarrow I(n) \longrightarrow I(n') \longrightarrow I(1') \longrightarrow 0$$

by Proposition 2.4. Comparing with the minimal injective resolution of $P(2)$, we have $\tau_2 I(1) = P(2)$. For any $2 \leq j \leq n-1$, Applying the Nakayama functor on the following minimal projective resolution of $I(j)$

$$0 \longrightarrow P(n) \longrightarrow P(j) \bigoplus P(n') \longrightarrow P(j') \bigoplus P(1) \longrightarrow I(j) \longrightarrow 0\ ,$$

one gets the following exact sequence again by Proposition 2.4 and Theorem 2.7

$$0 \longrightarrow \tau_2 I(j) \longrightarrow I(n) \longrightarrow I(n') \bigoplus I(j) \longrightarrow I(1) \bigoplus I(j') \longrightarrow 0$$

which implies that $\tau_2 I(j) \simeq P(j+1)$. Similarly one can show $\tau_2 I(k') = P(k+1')$.

For the case $i = n-1$ all the chosen arrows are d, it is the opposite of the case $i = 1$, one gets the assertion easily.

For the case $1 \leq i \leq n-2$ we have the indecomposable projective-injective modules $P(1) = I(n)$ and $P(i+1') = I(i')$. In the following we will give a discussion on j for $1 \leq j \leq n-1$.

(1) if $1 \leq j \leq i$, then the minimal projective resolution of $I(j)$ is

$$0 \longrightarrow P(n) \bigoplus P(j') \longrightarrow P(j) \bigoplus P(n') \longrightarrow P(1) \longrightarrow I(j) \longrightarrow 0\ ,$$

Following the proof above, one can show $\tau_2 I(j) = P(j+1)$.

(2) if $i+2 \leq j \leq n-1$, taking the following minimal projective resolution of $I(j)$

$$0 \longrightarrow P(n) \longrightarrow P(j) \bigoplus P(n') \longrightarrow P(1) \bigoplus P(j') \longrightarrow I(j) \longrightarrow 0\ ,$$

Applying the functor ν, one gets $\tau_2 I(j) = P(j+1)$ by Proposition 2.4.

In the following we will determine the $\tau I(j')$ for $1 \leq j \leq n$ and $j \neq i$.

Applying the Nakayama functor ν on the minimal projective resolution of I(j'):

$$0 \longrightarrow P(j+1') \longrightarrow P(j+2) \longrightarrow P(j+1) \longrightarrow I(j') \longrightarrow 0\ , 1 \leq j \leq i-1$$

$$0 \longrightarrow P(j+1) \longrightarrow P(j) \longrightarrow P(j') \longrightarrow I(j') \longrightarrow 0\ , i+1 \leq j \leq n-1$$

$$0 \longrightarrow P(j+1) \longrightarrow P(j) \longrightarrow P(j') \longrightarrow I(j') \longrightarrow 0\ , j = n$$

we get $\tau_2 I(n') = P(1')$ and $\tau_2 I(j') = P(j+1')$ for $1 \leq j \leq n-1$ and $j \neq i$. $\square$

Now we can give a connection between iterated tilted algebras with trivial cluster tilting and the tilted algebras.

Theorem 3.3 *The algebras* Λ *in Theorem* 3.2 *are tilted of finite type, that is, an iterated tilted algebra admitting a trivial cluster tilting is tilted of finite type.*

Proof. It is not difficult to show the algebras in Theorem 3.2 are special biserial but not string by Definition 2.9.

(1) Λ is tilted.

We only need to show Λ satisfying the conditions in Proposition 2.10. Since the only binomial relation are $(b^2, a^{n-1}) = (b\varepsilon_{n'}b, aa^{n-3}a)$, then (2) and (3) hold true. Notice that there is no reduced walk $w = p_1p_2p_3$ such that p_i are non-trivial paths and p_1p_2, p_2p_3 are the only zero-relation in w. Moreover, there does not exist a reduced walk $w = pw_1q$ such that w_1 a string and p, q are paths which are the only zero-relation in w.

(2) Λ is of finite type.

Using the relations of Λ, we get that there is no non-trivial reduced cycle, and hence there is no band in $\operatorname{mod}\Lambda$. Then by Lemma 2.11 Λ is of finite type. $\square$

Notice that a tilted algebra with a cluster tilting is contained in the class of algebras given in Theorem 2.6. We are now in a position to the main result.

Theorem 3.4 Λ *is a tilted algebra admitting a cluster tilting subcategory of* $\operatorname{mod}\Lambda$ *if and only if it is given by*

(1) $1 \xrightarrow{a} 2 \xrightarrow{a} 3 \cdots \xrightarrow{a} \xrightarrow{a} n$ *with the relation* $a^{n-1} = 0$

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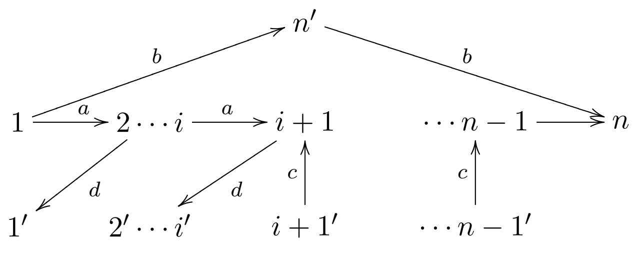


with relations $a^{n-1} = b^2$, $ac = 0$ *and* $da = 0$, *where* c *is an arrow* $i' \longrightarrow i$, d *an arrow* $i+1 \longrightarrow i'$. *For some* $0 \leq i \leq n-2$ *the chosen arrow at vertex* j' *is* d *if* $1 \leq j \leq i$ *and* c *if* $i+1 \leq j \leq n-1$.

Proof. The case (1) is well-known and the case (2) is straight result of Theorem 3.3. $\square$

In the following we give a classification of Auslander's 1-Gorenstein algebras with trivial cluster tilting following [HZ3].

Lemma 3.5 *Let* Λ *be an algebra in the class* (2) *of Theorem* 2.7. Λ *is Auslander's 1-Gorenstein if and only if* Λ *is given by the chosen arrows for the vertexes* $1', 2' \cdots, n-1'$ *are* $cdcdcd\cdots$ *or* $dcdcdc\cdots$ *and the relations are same as in Theorem* 2.7.

Proof. $\Leftarrow$ For the case $cdcdcd\cdots$.

If $n = 2m+1, m \geq 1$, then the last arrow at the vertex $n-1'$ must be d. We have the following projective and injective modules: $P(1) = I(n)$, $P(1') = I(n')$ and $P(2j+1') = I(2j')$ for $1 \leq j \leq m$.

It is not difficult to show that the injective envelope $I^0(P(k))$ of $P(k)$ is $I(n)$ if $k = 1$ or $k = 2j, 1 \leq j \leq m$, $I(n) \bigoplus I(k-1')$ if $k = 2j+1, 1 \leq j \leq m-1$ and $I(n-1')$ if $k = 2m+1$, which implies that $I^0(P(k))$ is projective.

Similarly we can get the injective envelope $I^0(P(k'))$ of $P(k')$ is $I(n')$ if $k = 1$, $I(2j')$ if $k = 2j, 2j+1$ for $1 \leq j \leq m$. Together with the proof above we get that $I^0(\Lambda)$ is projective, and hence Λ is Auslander's 1-Gorenstein.

If $n = 2m+2, m \geq 1$, then the last chosen arrow at vertex $n-1'$ is c. One can get the following projective-injectives: $P(1) = I(n)$, $P(1') = I(n')$ and $P(2j+1') = I(2j')$ for $1 \leq j \leq m$.

Then the injective envelope $I^0(P(k))$ of $P(k)$ is $I(n)$ if $k = 1$ or $k = 2j, 1 \leq j \leq m+1$, $I(n) \bigoplus I(k-1')$ if $k = 2j+1, 1 \leq j \leq m$. Similarly, $I^0(P(k'))$ is isomorphic to $I(n')$ if $k = 1$, $I(2j')$ if $k = 2j, 2j+1$ for $1 \leq j \leq m$ and $I(n)$ if $k = n$, which implies that Λ is also Auslander's 1-Gorenstein. We leave the case $dcdc\cdots$ to the reader.

$\Rightarrow$ It is enough to show there is no cc or dd in the chosen arrows.

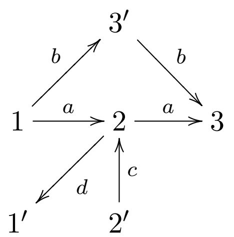
For the non-existence of cc: Suppose the first appearance of cc is at the vertex j' and $j+1'$ for some $1 \leq j \leq n-2$. Then $\mathrm{Soc}P(j+1') = S(i+1)$. So $I^0(P(i+1')) = I(i+1)$ is not projective, which contradicts that Λ is Auslander's 1-Gorenstein.

For the non-existence of dd: We suppose that the first appearance of dd is at the vertex $j-1'$ and j' for some $2 \leq j \leq n-2$. Considering the indecomposable projective $P(j)$, we get $\mathrm{Soc}P(j) = S(j-1') \bigoplus S(n)$. That is, $I(j-1')$ is a direct summand of $I^0(P(j))$ but not projective contradicting that Λ is Auslander's 1-Gorenstein. $\square$

By using Theorem 3.4 and Lemma 3.5 and Theorem 2.7, one can get the following classification of Auslander's 1-Gorenstein algebras with global dimension 2 which admit trivial cluster tilting subcategories.

Theorem 3.6 Λ *is an Auslander's 1-Gorenstein algebra with global dimension 2 admitting a trivial cluster tilting subcategory of* $\mathrm{mod}\,\Lambda$ *if and only if it is one of the following:*

(1) $1 \xrightarrow{a} 2 \xrightarrow{a} 3 \cdots \xrightarrow{a} \ \xrightarrow{a} n$ *with the relation* $a^{n-1} = 0 (n \geq 3)$.

(2) 

with relations $a^2 = b^2$ *and* $ac = 0 = da$.

Proof. It is an immediate result of Theorem 3.4 and Lemma 3.5 and Theorem 2.8.

Remark (1) It is not difficult to show that an Auslander's 1-Gorenstein algebra with global dimension 2 admitting trivial cluster tilting is tilted of finite type but not Nakayama by Theorem 3.6.

(2)By Theorem 3.4 and Theorem 3.6, one gets a class of non-Auslander's 1-Gorenstein tilted algebras with (trivial) cluster tilting subcategories. Comparing the results get in [HZ2] and [HZ3] and Theorem 3.3, we conjecture the following:

Let Λ *be an algebra with global dimension 2. If* Λ *admits a trivial cluster tilting subcategory of* $\mathrm{mod}\,\Lambda$, *then it is of finite type.*

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