

Module categories for finite group algebras

David J. Benson, Srikanth B. Iyengar and Henning Krause

Abstract. This survey article is intended as an introduction to the recent categorical classification theorems of the three authors, restricting to the special case of the category of modules for a finite group.

2010 Mathematics Subject Classification. Primary 20J06; Secondary 20C20, 13D45, 16E45, 18E30

Keywords. Modular representation theory, local cohomology, stable module category, idempotent modules, derived category, triangulated categories, local-global principle, stratification, costratification.

Introduction

The purpose of this paper is to explain the recent work of the three authors on the classification of localising and colocalising subcategories of triangulated categories [8, 10, 9, 11], and to put it into context. In order to be as concrete as possible, we restrict our attention to categories associated with the modular representation theory of finite groups. A more leisurely discussion, filling in requisite background from commutative algebra and triangulated categories, is given in [12].

Let G be a finite group and let k be an algebraically closed field of characteristic $p > 0$. With a few exceptions in characteristic two, if the Sylow p -subgroups of G are non-cyclic then the group algebra kG has wild representation type. We therefore do not hope to classify the indecomposable finitely generated kG -modules, and so we are left with several options. We can restrict the kinds of modules that we're interested in; we can look for properties of modules that we can prove without the need for a classification; or we look for coarser classification theorems. Examples abound of all three types of approach; we shall be examining the last option.

The first indication that it might be possible to make some sort of categorical classification theorem came with Mike Hopkins' 1987 classification [21] of the thick subcategories of the derived category $D^b(\text{proj}(R))$ of perfect complexes over a commutative Noetherian ring R . His interest was in the nilpotence theorem for the stable homotopy category, for which he regarded the derived category of perfect complexes as a toy model. The parametrisation of the thick subcategories was by specialisation closed sets of prime ideals in R .

Neeman's 1992 paper [25] took up Hopkins' work, clarified it and went on to classify the localising subcategories of the unbounded derived category of R -modules $D(\text{Mod}(R))$. This is the appropriate big category whose compact objects are the perfect complexes $D^b(\text{proj}(R))$. The parametrisation for the localising sub-

categories was by *all* subsets of the set of prime ideals of R , not just the specialisation closed ones.

In 1997, the first author together with Carlson and Rickard [6] proved the analogue of Hopkins' theorem for the stable category $\mathbf{stmod}(kG)$ of finitely generated kG -modules, over an algebraically closed field k of characteristic p . The parametrisation for the thick subcategories was by specialisation closed subsets of homogeneous primes in the cohomology ring $H^*(G, k)$, ignoring the maximal ideal of all positive degree elements.

The corresponding big category is the stable category $\mathbf{StMod}(kG)$ of all kG -modules; the full subcategory of compact objects in this is the finitely generated stable module category $\mathbf{stmod}(kG)$. It was expected by analogy with $\mathbf{D}(\mathbf{Mod}(R))$ that the classification of localising subcategories of $\mathbf{StMod}(kG)$ would be by all subsets of the set of non-maximal homogeneous primes in $H^*(G, k)$. But this turned out to be very hard to prove, and it was not until a couple of years ago that we managed to achieve this classification [9].

More recently, in 2009 Neeman [26] classified the colocalising subcategories of $\mathbf{D}(\mathbf{Mod}(R))$; the corresponding classification for $\mathbf{StMod}(kG)$ is given in [11].

Other classifications follow similar models. The work of Hovey, Palmieri and Strickland [22] and the series of papers [8, 10, 9, 11] lay the general foundations. For background on the theory of support varieties we refer to Solberg's survey [32].

CONTENTS.

1. The stable module category
2. Thick subcategories of $\mathbf{stmod}(kG)$
3. The derived category
4. Rickard idempotent modules and functors
5. Classification of tensor ideal thick subcategories
6. Localising subcategories of $\mathbf{StMod}(kG)$
7. The category $\mathbf{Klnj}(kG)$
8. Support for triangulated categories
9. Tensor triangulated categories
10. The local-global principle
11. Stratifying triangulated categories
12. Graded polynomial algebras
13. The BGG correspondence
14. The Koszul construction
15. Quillen stratification
16. Applications
17. Costratification

1. The stable module category

We write $\text{Mod}(kG)$ for the module category of kG . The objects are the left kG -modules and the morphisms are the module homomorphisms. This is an abelian category. We write $\text{mod}(kG)$ for the full subcategory of finitely generated kG -modules. We refer the reader to [2] for basic constructions and facts concerning kG -modules. The following statement is well known for finitely generated modules, but is true more generally.

Lemma 1.1. *Projective and injective kG -modules coincide.*

Proof. It is an easy exercise using the group basis to show that kG is self-dual as a kG -module. This implies that every projective module is injective. For the converse, we note that every kG -module M embeds in a free kG -module, via the map $M \rightarrow kG \otimes_k M$ sending m to $\sum_{g \in G} g \otimes g^{-1}m$. Here, we make $kG \otimes_k M$ into a kG -module via $g(g' \otimes m) = gg' \otimes m$; thus a k -basis of M gives a free kG -basis for $kG \otimes_k M$. If M is injective then this embedding splits and M is a summand of a free module, hence projective. \square

Projective kG -modules are well understood. Every projective is a direct sum of finitely generated projective indecomposables, and every projective indecomposable is the projective cover of a simple module.

To work “modulo projectives”, we use the *stable module category* $\text{StMod}(kG)$. This has the same objects as the module category, but the morphisms are given by

$$\underline{\text{Hom}}_{kG}(M, N) = \text{Hom}_{kG}(M, N) / \text{PHom}_{kG}(M, N)$$

where $\text{PHom}_{kG}(M, N)$ is the linear subspace of maps that factor through some projective module.

We write $\text{stmod}(kG)$ for the full subcategory of $\text{StMod}(kG)$ whose objects are the modules stably isomorphic to finitely generated modules. Notice that if a map between finitely generated modules factors through some projective module then it factors through a finitely generated projective module.

The subcategory $\text{stmod}(kG)$ is distinguished categorically as the *compact objects* in $\text{StMod}(kG)$.

Definition 1.2. An object M in $\text{StMod}(kG)$ is said to be *compact* if given any small coproduct $\bigoplus_{\alpha} M_{\alpha}$, the natural map

$$\bigoplus_{\alpha} \underline{\text{Hom}}_{kG}(M, M_{\alpha}) \rightarrow \underline{\text{Hom}}_{kG}(M, \bigoplus_{\alpha} M_{\alpha})$$

is an isomorphism.

Although $\text{StMod}(kG)$ and $\text{stmod}(kG)$ are not abelian categories, they come with a natural structure of triangulated category. The “shift” is given by Ω^{-1} , the functor assigning to each module M the cokernel of an embedding of M into an injective module. At the level of $\text{Mod}(kG)$ this is not functorial, but for $\text{StMod}(kG)$

it is a functorial self-equivalence whose inverse is the functor Ω taking a module M to the kernel of a surjection from a projective module onto M .

The distinguished triangles

$$A \rightarrow B \rightarrow C \rightarrow \Omega^{-1}(A)$$

are by definition those isomorphic to diagrams coming from a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

of modules as follows. Given such a short exact sequence, we embed B into an injective module I and obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & I & \longrightarrow & \Omega^{-1}(A) & \longrightarrow & 0. \end{array}$$

This gives us a map $C \rightarrow \Omega^{-1}A$ which is well defined in $\text{StMod}(kG)$.

The right hand square in this diagram gives us a short exact sequence

$$0 \rightarrow B \rightarrow C \oplus I \rightarrow \Omega^{-1}(A) \rightarrow 0.$$

Since C is isomorphic to $C \oplus I$ in $\text{StMod}(kG)$, applying the construction again gives rise to the rotated triangle

$$B \rightarrow C \rightarrow \Omega^{-1}(A) \rightarrow \Omega^{-1}(B).$$

Most of the axioms for a triangulated category are easy to verify. The only one that needs any comment is the octahedral axiom, which is the expression in $\text{StMod}(kG)$ of the third isomorphism theorem in $\text{Mod}(kG)$. More details can be found in Buchweitz [18], and in Happel [20].

2. Thick subcategories of $\text{stmod}(kG)$

Definition 2.1. For any triangulated category \mathbb{T} , a *thick subcategory* \mathcal{S} is a full triangulated subcategory that is closed under taking finite direct sums and summands.

It is worth spelling out what this means for the stable module category. Being a full triangulated subcategory means that the subcategory is closed under Ω and Ω^{-1} , and it has the *two in three property*: given a triangle in the stable module category for which two of the objects are in \mathcal{S} , so is the third.

Pulling back to $\text{mod}(kG)$, thick subcategories of $\text{stmod}(kG)$ are in one to one correspondence with the subcategories \mathfrak{c} of $\text{mod}(kG)$ with the following properties:

- (i) All projective modules are in \mathfrak{c} .

- (ii) \mathfrak{c} is closed under finite sums and summands.
- (iii) If M is in \mathfrak{c} then so are $\Omega(M)$ and $\Omega^{-1}(M)$.
- (iv) If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of modules with two of A , B and C in \mathfrak{c} then so is the third.

The clue to finding thick subcategories of $\mathbf{stmod}(kG)$ comes from the theory of varieties for modules, which we now briefly describe. A fuller treatment can be found in [3]. The cohomology ring $H^*(G, k)$ is defined to be $\text{Ext}_{kG}^*(k, k)$ where k denotes the trivial representation. This is a finitely generated graded commutative ring, and we write V_G for its maximal ideal spectrum.

If M is a finitely generated kG -module, we have a ring homomorphism

$$H^*(G, k) = \text{Ext}_{kG}^*(k, k) \xrightarrow{M \otimes_k -} \text{Ext}_{kG}^*(M, M)$$

given by tensoring Yoneda extensions with M . As usual, we are tensoring over k with diagonal group action, which is exact. We write I_M for the kernel of this homomorphism, and $V_G(M)$ for the subvariety of V_G determined by this ideal. Since I_M is a homogeneous ideal, $V_G(M)$ is a closed homogeneous subvariety of the homogeneous variety V_G . The following theorem summarises some of the main properties of these varieties; not all of them are easy to prove.

Theorem 2.2. *Let M, M_1, M_2, M_3 be finitely generated kG -modules.*

- (i) $V_G(M) = \{0\}$ if and only if M is a projective module.
- (ii) More generally, the dimension of the variety $V_G(M)$ determines the polynomial rate of growth of a minimal resolution of M .
- (iii) $V_G(\text{Hom}_k(M, k)) = V_G(M)$.
- (iv) $V_G(M_1 \oplus M_2) = V_G(M_1) \cup V_G(M_2)$.
- (v) $V_G(M_1 \otimes_k M_2) = V_G(M_1) \cap V_G(M_2)$.
- (vi) If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence then for $i = 1, 2, 3$, $V_G(M_i)$ is contained in the union of the varieties of the other two modules.
- (vii) If $0 \neq \zeta \in H^n(G, k)$ is represented by a cocycle $\hat{\zeta}: \Omega^n(k) \rightarrow k$, we write L_ζ for the kernel of $\hat{\zeta}$. Then we have $V_G(L_\zeta) = V_G(\langle \zeta \rangle)$, the variety determined by the principal ideal $\langle \zeta \rangle$.
- (viii) Given a closed homogeneous subvariety $V \subseteq V_G$, there exists a finitely generated module M with $V_G(M) = V$. Namely, if $V = V_G(\langle \zeta_1, \dots, \zeta_n \rangle)$ then we may take $M = L_{\zeta_1} \otimes_k \dots \otimes_k L_{\zeta_n}$ and use properties (v) and (vii).

Definition 2.3. We write \mathcal{V}_G for the collection of closed homogeneous irreducible subvarieties of V_G (including $\{0\}$), or equivalently the spectrum of homogeneous

prime ideals $\mathfrak{p} \subseteq H^*(G, k)$ (including the maximal ideal \mathfrak{m} of positive degree elements). We say that a subset $\mathcal{V} \subseteq \mathcal{V}_G$ is *specialisation closed* if $\mathfrak{p} \in \mathcal{V}$, $\mathfrak{q} \in \mathcal{V}_G$ and $\mathfrak{q} \supseteq \mathfrak{p}$ imply $\mathfrak{q} \in \mathcal{V}$.

If \mathcal{V} is a specialisation closed subset of \mathcal{V}_G then we write $\mathfrak{c}_{\mathcal{V}}$ for the full subcategory of $\mathbf{stmod}(kG)$ consisting of modules M such that $V_G(M)$ is a finite union of elements of \mathcal{V} (i.e., each irreducible component of $V_G(M)$ is in \mathcal{V}).

Lemma 2.4. *If \mathcal{V} is a specialisation closed subset of \mathcal{V}_G then $\mathfrak{c}_{\mathcal{V}}$ is a thick subcategory of $\mathbf{stmod}(kG)$.*

Proof. This follows directly from Theorem 2.2. □

We now come to an issue which will reappear in later contexts. Namely, the thick subcategories of $\mathbf{stmod}(kG)$ appearing in Lemma 2.4 are closed under tensor products with finitely generated kG -modules, by part (iii) of Theorem 2.2.

Lemma 2.5. *If \mathfrak{c} is a thick subcategory of $\mathbf{stmod}(kG)$ then the following are equivalent:*

- (i) \mathfrak{c} is closed under tensor product with finitely generated kG -modules.
- (ii) \mathfrak{c} is closed under tensor product with simple kG -modules.

These conditions are automatically satisfied if G is a finite p -group.

Proof. The equivalence of (i) and (ii) follows from the fact that every finitely generated module has a finite filtration in which the filtered quotients are simple modules. If G is a finite p -group then the trivial module is the only simple kG -module. □

Definition 2.6. We say that a thick subcategory of $\mathbf{stmod}(kG)$ is *tensor ideal*, or *tensor closed*, if the equivalent conditions of the lemma are satisfied.

We can now state the classification theorem. This was proved in Theorem 3.4 of Benson, Carlson and Rickard [6] in case k is an algebraically closed field, and in Theorem 11.4 of Benson, Iyengar and Krause [9] for general fields k .

Theorem 2.7. *There is a one to one correspondence between tensor ideal thick subcategories of $\mathbf{stmod}(kG)$ and non-empty specialisation closed subsets of \mathcal{V}_G . Under this correspondence, \mathcal{V} corresponds to $\mathfrak{c}_{\mathcal{V}}$.*

Note that the condition that \mathcal{V} is non-empty is equivalent to the condition that \mathcal{V} contains $\{0\}$. This subset plays no role for $\mathbf{stmod}(kG)$ but will make an appearance later.

If we remove the tensor ideal condition, then the classification becomes much harder. One example of a thick subcategory which is usually not tensor ideal is the full subcategory of modules in the principle block. The *principal block* $B_0(kG)$ is the block containing the trivial module. So one might ask whether the thick subcategories of $\mathbf{stmod}(B_0(kG))$ are all of the form $\mathfrak{c}_{\mathcal{V}} \cap \mathbf{stmod}(B_0(kG))$. The obstruction to this being true is given in terms of the *nucleus*, defined as follows.

Definition 2.8. The *nucleus* Y_G of a finite group G is the union of the images of $\text{res}_{G,H}^*: V_H \rightarrow V_G$, as H runs over the subgroups of G such that $C_G(H)$ is not p -nilpotent. Recall that a finite group is said to be *p -nilpotent* if it has a normal subgroup whose order is prime to p and whose index is a power of p .

Theorem 2.9 (Benson, Carlson and Robinson [7]; Benson [4]). *The subvariety Y_G of V_G is equal to the union of the $V_G(M)$ as M runs over the finitely generated modules M in $B_0(kG)$ satisfying $H^*(G, M) = 0$.*

In particular, every non-projective module in the principal block has non-trivial cohomology if and only if the centraliser of every element of order p in G is p -nilpotent.

Theorem 2.10 (Benson, Carlson and Rickard [6]). *If Y_G is empty or equal to $\{0\}$ then there is a one to one correspondence between the thick subcategories of $\text{stmod}(B_0(kG))$ and the non-empty subsets of V_G . Under this correspondence, \mathcal{V} corresponds to $\mathfrak{c}_{\mathcal{V}} \cap \text{stmod}(B_0(kG))$.*

If Y_G is bigger than $\{0\}$ then it appears to be easy to manufacture infinite collections of thick subcategories supported on each line through the origin in Y_G . It appears hopeless to classify these thick subcategories. However, if we work modulo the modules supported inside the nucleus, we again obtain a classification of thick subcategories.

3. The derived category

The anomalous role of the origin in the classification of thick subcategories of $\text{stmod}(kG)$ can be repaired by moving to a slightly bigger category, namely the *derived category* $D^b(\text{mod}(kG))$. Recall that the objects in this category are the finite chain complexes of finitely generated kG -modules, and the arrows are homotopy classes of maps of complexes, with the quasi-isomorphisms inverted. A *quasi-isomorphism* is a map of complexes that induces an isomorphism in homology. The category $D^b(\text{mod}(kG))$ is a triangulated category, in which the triangles are formed using mapping cones.

We write $K^b(\text{proj}(kG))$ for the thick subcategory of $D^b(\text{mod}(kG))$ whose objects are the *perfect complexes*, i.e., the complexes quasi-isomorphic to a bounded complex of finitely generated projective kG -modules. Buchweitz [18] (see also Rickard [31]) has defined a functor from $D^b(\text{mod}(kG))$ to $\text{stmod}(kG)$ which is essentially surjective and has kernel $K^b(\text{proj}(kG))$:

$$K^b(\text{proj}(kG)) \rightarrow D^b(\text{mod}(kG)) \rightarrow \text{stmod}(kG).$$

The thick subcategory $K^b(\text{proj}(kG))$ is the unique minimal tensor ideal thick subcategory of $D^b(\text{mod}(kG))$. This allows us to extend the theory of support varieties to $D^b(\text{mod}(kG))$ in such a way that an object has the origin in its support if and only if it is non-zero. As before, if \mathcal{V} is a specialisation closed subset of V_G , we write $\mathfrak{c}_{\mathcal{V}}$ for the thick subcategory of $D^b(\text{mod}(kG))$ consisting of objects whose

support is a finite union of elements of \mathcal{V} . So the following theorem is an easy consequence of Theorem 2.7.

Theorem 3.1. *There is a one to one correspondence between tensor ideal thick subcategories of $\mathbf{D}^b(\mathbf{mod}(kG))$ and specialisation closed subsets of \mathcal{V}_G . Under this correspondence, \mathcal{V} corresponds to $\mathfrak{c}_{\mathcal{V}}$.*

Next, we observe that G is p -nilpotent if and only if k is the only simple module in the principal block. This allows us to extend Theorem 2.10 as follows.

Theorem 3.2. *If Y_G is empty (i.e., if G is p -nilpotent) then there is a one to one correspondence between the thick subcategories of $\mathbf{D}^b(\mathbf{mod}(B_0(kG)))$ and subsets of \mathcal{V}_G . Under this correspondence, \mathcal{V} corresponds to $\mathfrak{c}_{\mathcal{V}} \cap \mathbf{D}^b(\mathbf{mod}(B_0(kG)))$.*

Again, if Y_G is non-empty, we can work modulo objects supported inside the nucleus and obtain a classification of thick subcategories, but this returns us to quotients of $\mathbf{stmod}(B_0(kG))$ so nothing new is gained.

4. Rickard idempotent modules and functors

The proof of Theorem 2.7 depends in an essential way on the construction of certain infinitely generated modules, and the development of a theory of support varieties in this context.

The modules in question are *Rickard idempotent modules*. Corresponding to any specialisation closed subset \mathcal{V} of \mathcal{V}_G , Rickard constructs two idempotent functors on $\mathbf{StMod}(kG)$. He writes these as $E_{\mathcal{V}}$ and $F_{\mathcal{V}}$, but for consistency with the notation of our series of papers we shall use the notation $\Gamma_{\mathcal{V}}$ and $L_{\mathcal{V}}$. More generally, given any thick subcategory \mathfrak{c} of $\mathbf{stmod}(kG)$, we have functors $\Gamma_{\mathfrak{c}}$ and $L_{\mathfrak{c}}$, and functorial triangles

$$\Gamma_{\mathfrak{c}}(M) \rightarrow M \rightarrow L_{\mathfrak{c}}(M). \quad (4.1)$$

The defining properties of these triangles are given in terms of *localising subcategories* of $\mathbf{StMod}(kG)$.

Definition 4.1. Let \mathbf{T} be a triangulated category with small products and coproducts. A *localising subcategory* of \mathbf{T} is a thick subcategory that is closed under coproducts, while a *colocalising subcategory* of \mathbf{T} is a thick subcategory that is closed under products.

The triangle of functors (4.1) is characterised by the following properties:

- (i) $\Gamma_{\mathfrak{c}}(M)$ is in the localising subcategory $\mathbf{Loc}(\mathfrak{c})$ generated by \mathfrak{c} .
- (ii) If N is in \mathfrak{c} then $\underline{\mathbf{Hom}}_{kG}(N, L_{\mathfrak{c}}(M)) = 0$.

The functors $\Gamma_{\mathfrak{c}}$ and $L_{\mathfrak{c}}$ preserve small coproducts, see for example [8, Corollary 6.5].

If \mathfrak{c} is a tensor ideal thick subcategory then we have

$$\Gamma_{\mathfrak{c}}(k) \otimes_k M \cong \Gamma_{\mathfrak{c}}(M), \quad L_{\mathfrak{c}}(k) \otimes_k M \cong L_{\mathfrak{c}}(M).$$

If $\mathfrak{c} = \mathfrak{c}_{\mathcal{V}}$ then we abbreviate $\Gamma_{\mathfrak{c}_{\mathcal{V}}}$ to $\Gamma_{\mathcal{V}}$ and $L_{\mathfrak{c}_{\mathcal{V}}}$ to $L_{\mathcal{V}}$, so that the Rickard triangle takes the form

$$\Gamma_{\mathcal{V}}(M) \rightarrow M \rightarrow L_{\mathcal{V}}(M).$$

Rickard idempotent modules allow us to develop a theory of support for modules in $\text{StMod}(kG)$ as follows. Let $\mathfrak{p} \subseteq H^*(G, k)$ be homogeneous prime ideal. We choose specialisation closed subsets \mathcal{V} and \mathcal{W} of \mathcal{V}_G with the property that $\mathcal{V} \not\subseteq \mathcal{W}$ but $\mathcal{V} \subseteq \mathcal{W} \cup \{\mathfrak{p}\}$ (i.e., $\mathcal{V} \setminus \mathcal{W} = \{\mathfrak{p}\}$). Then we write $\Gamma_{\mathfrak{p}}$ for the functor $L_{\mathcal{W}}\Gamma_{\mathcal{V}} \cong \Gamma_{\mathcal{V}}L_{\mathcal{W}}$. The functor $\Gamma_{\mathfrak{p}}$ defined in this way is independent of the choice of \mathcal{V} and \mathcal{W} with the given properties, see [8, Theorem 6.2]. Note that if \mathfrak{p} is the maximal prime \mathfrak{m} , we have $\Gamma_{\mathfrak{m}} = 0$.

Definition 4.2. If M is an object in $\text{StMod}(kG)$, then the support $\mathcal{V}_G(M)$ is the subset of \mathcal{V}_G consisting of

$$\{\mathfrak{p} \in \mathcal{V}_G \mid \Gamma_{\mathfrak{p}}(M) \neq 0\}.$$

The following properties of $\mathcal{V}_G(M)$ are proved in [6] for k algebraically closed, and in [9] for a general field k . This theorem should be compared with Theorem 2.2.

Theorem 4.3. *Let M, M_1, M_2, M_3 be kG -modules.*

- (i) $\mathcal{V}_G(M) = \emptyset$ if and only if M is a projective module.
- (ii) If M is finitely generated then $\mathcal{V}_G(M)$ is equal to the set of closed homogeneous irreducible subvarieties of $\mathcal{V}_G(M)$.
- (iii) For a small family M_{α} of kG -modules we have

$$\mathcal{V}_G\left(\bigoplus_{\alpha} M_{\alpha}\right) = \bigcup_{\alpha} \mathcal{V}_G(M_{\alpha}).$$

- (iv) $\mathcal{V}_G(M_1 \otimes_k M_2) = \mathcal{V}_G(M_1) \cap \mathcal{V}_G(M_2)$.
- (v) If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence then for $i = 1, 2, 3$, $\mathcal{V}_G(M_i)$ is contained in the union of the supports of the other two modules.
- (vi) $\mathcal{V}_G(\Gamma_{\mathcal{V}}(k)) = \mathcal{V} \setminus \{\mathfrak{m}\}$, $\mathcal{V}_G(L_{\mathcal{V}}(k)) = \mathcal{V}_G \setminus \mathcal{V}$ and $\mathcal{V}_G(\Gamma_{\mathfrak{p}}(k)) = \{\mathfrak{p}\}$ for $\mathfrak{p} \neq \mathfrak{m}$.
- (vii) Given a subset $\mathcal{V} \subseteq \mathcal{V}_G$, there exists a module M with $\mathcal{V}_G(M) = \mathcal{V}$. For example we could take $M = \bigoplus_{\mathfrak{p} \in \mathcal{V}} \Gamma_{\mathfrak{p}}(k)$.

The proof of the tensor product theorem given in [6] depends on comparison with rank varieties and the following version of Dade's lemma for modules in $\text{StMod}(kG)$, which appeared as Theorem 5.2 of [5].

Theorem 4.4. *Let $E = \langle g_1, \dots, g_r \rangle \cong (\mathbf{Z}/p)^r$, let k be an algebraically closed field of characteristic p , and let K be an algebraically closed extension field of k of transcendence degree at least $r - 1$. Then a kE -module M is projective if and only if for every choice of $(\lambda_1, \dots, \lambda_r) \in K^r$ with not all the λ_i equal to zero, the restriction of $K \otimes_k M$ to the cyclic subgroup of KE of order p generated by*

$$1 + \lambda_1(g_1 - 1) + \dots + \lambda_r(g_r - 1)$$

is projective.

The role of this version of Dade's lemma in the development given in [5, 6] is what necessitates the requirement that k is algebraically closed. Later we shall describe another proof avoiding rank varieties and avoiding any version of Dade's lemma, and which works for a general field k .

Remark 4.5 (Maximal versus prime ideals). The reader will have observed that when we were dealing with the stable category $\mathbf{stmod}(kG)$ of finitely generated modules, we used V_G , the spectrum of maximal ideals in $H^*(G, k)$, whereas for the stable category $\mathbf{StMod}(kG)$ of all modules we used \mathcal{V}_G , the spectrum of homogeneous prime ideals in $H^*(G, k)$. In the case of a finitely generated module, $\mathcal{V}_G(M)$ is determined by $V_G(M)$, see Theorem 4.3 (ii), whereas for an infinitely generated module it is not, see part (vii) of the same theorem.

It would have been possible to use \mathcal{V}_G consistently throughout, but we chose not to, partly for historical reasons. The origin of the use of V_G is Quillen's work [27, 28], and much of the literature on finitely generated modules has been written in this context.

A theorem of Hilbert states that for the maximal ideal spectrum and the homogeneous prime ideal spectrum of a finitely generated commutative algebras over a field, each determines the other. A graded commutative ring is commutative modulo its nil radical, so this applies here.

The homogeneous prime spectrum of a nonstandardly graded ring can be quite confusing. For example a polynomial ring can give a spectrum with singularities even though it is regular in the commutative algebra sense. An explicit example of this is the ring $k[x, y, z]$ with $|x| = 2$, $|y| = 4$, $|z| = 6$. The affine open patch corresponding to $y \neq 0$ has coordinate ring generated by $\alpha = x^2/y$, $\beta = xz/y^2$, $\gamma = z^2/y^3$, with the single relation $\alpha\gamma = \beta^2$. This has a singularity at the origin. This example arises as $H^*(G, k)$ modulo its nil radical with $G = (\mathbf{Z}/p)^3 \rtimes \Sigma_3$ for $p \geq 5$, and k a field of characteristic p .

Finally, in the Quillen stratification theorem as described in [27, 28] (see also Section 15 of this survey), part of the statement is that $N_G(E)/C_G(E)$ acts freely on the stratum corresponding to E . This is true for inhomogeneous maximal, but not for homogeneous prime ideals. Fortunately, our use of Quillen stratification does not involve this feature.

5. Classification of tensor ideal thick subcategories

The key step in the classification of tensor ideal thick subcategories of $\text{stmod}(kG)$ (Theorem 2.7) is the following theorem.

Theorem 5.1. *Let M be a finitely generated kG -module with k algebraically closed, and let \mathcal{V} be the collection of closed homogeneous irreducible subvarieties of $V_G(M)$. Then the tensor ideal thick subcategory generated by M is equal to $\mathfrak{c}_{\mathcal{V}}$.*

Proof. Write \mathfrak{c} for the tensor ideal thick subcategory generated by M . It is clear that \mathfrak{c} is contained in $\mathfrak{c}_{\mathcal{V}}$. So the natural transformation $\Gamma_{\mathfrak{c}}\Gamma_{\mathcal{V}} \rightarrow \Gamma_{\mathfrak{c}}$ is an isomorphism. So if N is any kG -module then we obtain a triangle

$$\Gamma_{\mathfrak{c}}(N) \rightarrow \Gamma_{\mathcal{V}}(N) \rightarrow L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N).$$

The first two terms in this triangle are in $\text{Loc}(\mathfrak{c}_{\mathcal{V}})$, and hence so is the third. So

$$\mathcal{V}_G(L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)) \subseteq \mathcal{V}.$$

If S is any simple kG -module then

$$\underline{\text{Hom}}_{kG}(S, M^* \otimes_k L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)) \cong \underline{\text{Hom}}_{kG}(S \otimes_k M, L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)) = 0.$$

Thus $M^* \otimes_k L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)$ is projective, and hence

$$\emptyset = \mathcal{V}_G(M^* \otimes_k L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)) = \mathcal{V}_G(M) \cap \mathcal{V}_G(L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)) = \mathcal{V}_G(L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)).$$

It follows that $L_{\mathfrak{c}}\Gamma_{\mathcal{V}}(N)$ is projective, and so $\Gamma_{\mathfrak{c}}(N) \rightarrow \Gamma_{\mathcal{V}}(N)$ is a stable isomorphism. Thus $\mathfrak{c} = \mathfrak{c}_{\mathcal{V}}$. \square

6. Localising subcategories of $\text{StMod}(kG)$

First we describe the goal. As with the thick subcategories of $\text{stmod}(kG)$, we only hope to classify the tensor ideal localising subcategories of $\text{StMod}(kG)$; this is all localising subcategories only in the case where G is a p -group.

There are a lot more tensor ideal localising subcategories of $\text{StMod}(kG)$ than tensor ideal thick subcategories of $\text{stmod}(kG)$, as we can see by comparing Theorems 2.2 and 4.3. For *any* subset \mathcal{V} , not just for a specialisation closed one, let $\mathfrak{C}_{\mathcal{V}}$ be the full subcategory of $\text{StMod}(kG)$ consisting of those modules M with $\mathcal{V}_G(M) \subseteq \mathcal{V}$. The following lemma is the analogue of Lemma 2.4.

Lemma 6.1. *If \mathcal{V} is a subset of \mathcal{V}_G then $\mathfrak{C}_{\mathcal{V}}$ is a tensor ideal localising subcategory of $\text{StMod}(kG)$.*

The statement of the classification theorem is as follows.

Theorem 6.2 (Benson, Iyengar and Krause [9]). *There is a one to one correspondence between tensor ideal localising subcategories of $\text{StMod}(kG)$ and subsets of $\mathcal{V}_G \setminus \{0\}$. Under this correspondence, \mathcal{V} corresponds to $\mathfrak{C}_{\mathcal{V}}$.*

The proof of this classification theorem is much harder than the corresponding proof for tensor ideal thick subcategories of $\mathbf{stmod}(kG)$. If one tries to mimic the arguments of Neeman [25], there is a basic obstruction, which is the lack of appropriate field objects. Specifically, given a prime $\mathfrak{p} \subseteq H^*(G, k)$, a field object for \mathfrak{p} would be a module M such that $\hat{H}^*(G, M) = \underline{\mathbf{Hom}}_{kG}^*(k, M)$ is isomorphic to the graded field of fractions of $H^*(G, k)/\mathfrak{p}$. Such an object does not always exist, as can be seen using the obstruction theory of Benson, Krause and Schwede [14, 15].

There is one case where there are field objects, and that is the case of an elementary abelian 2-group E . The point here is that the group algebra of an elementary abelian 2-group is the same as an exterior algebra in characteristic two. The basic property of the cohomology in this case is that it is “formal”, in the sense that the cochains and the cohomology are equivalent—there is no higher order information. This is made precise by the Bernstein–Gelfand–Gelfand (BGG) correspondence [16], which gives a correspondence between appropriate categories of modules for the exterior algebra and for a polynomial algebra, where the polynomial algebra is regarded as the cohomology of the exterior algebra and *vice versa*. In this case, one can perform the classification for the graded polynomial ring $H^*(E, k)$ and then use the BGG correspondence to prove the classification for $\mathbf{StMod}(kE)$; see [12] for details.

For a general finite group in characteristic two, the classification can be achieved using the Quillen stratification theorem and Chouinard’s theorem to reduce to elementary abelian subgroups. We shall describe these later.

If E is an elementary abelian p -group for p odd, the problem is that the cochains are not formal. However, there is a device coming from commutative algebra that allows us to reduce to the formal case. Namely, we regard the group algebra kE as a *complete intersection*, and then use a suitable Koszul complex. We shall describe this construction later.

Before addressing these points, we wish to set up a slightly cleaner version of the module category, where the origin $\{0\} \in \mathcal{V}_G$ is not excluded, as it is in Theorems 2.7 and 6.2. This is the category $\mathbf{Klnj}(kG)$ described in the next section. It bears the same relation to $\mathbf{StMod}(kG)$ as the bounded derived category $\mathbf{D}^b(\mathbf{mod}(kG))$ does to $\mathbf{stmod}(kG)$. It might be thought that $\mathbf{D}(\mathbf{Mod}(kG))$ would play this role, but the compact objects in $\mathbf{D}(\mathbf{Mod}(kG))$ are just the *perfect complexes*, namely the complexes quasiisomorphic to finite complexes of finitely generated *projective* modules. So we would get $\mathbf{K}^b(\mathbf{proj}(kG))$ rather than the desired $\mathbf{D}^b(\mathbf{mod}(kG))$.

7. The category $\mathbf{Klnj}(kG)$

The objects of $\mathbf{Klnj}(kG)$ are the complexes of injective (or equivalently projective, see Lemma 1.1) kG -modules. We should emphasise that this means *unbounded* complexes of not necessarily finitely generated injective modules. The arrows are homotopy classes of degree preserving maps of complexes. This is a triangulated category in which the triangles come from the mapping cone construction. This

category is investigated in detail in Krause [24], Benson and Krause [13].

Let $\mathbf{K}_{\text{ac}}\text{Inj}(kG)$ be the full subcategory of $\mathbf{K}\text{Inj}(kG)$ whose objects are the acyclic complexes. These objects can be described as *Tate resolutions* of modules.

Definition 7.1. If M is a kG -module then a *Tate resolution* of M is formed by splicing together an injective resolution and a projective resolution of M :

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_1 & \rightarrow & P_0 & \longrightarrow & I_0 & \rightarrow & I_1 & \rightarrow & \cdots \\ & & & & \searrow & & \nearrow & & & & \\ & & & & & & M & & & & \\ & & & & \nearrow & & \searrow & & & & \\ & & & & 0 & & & & & & 0 \end{array}$$

Every acyclic complex of injectives

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots$$

can be regarded as a Tate resolution of the image of $P_0 \rightarrow P_{-1}$. Furthermore, given a module homomorphism $M \rightarrow M'$, it can be extended to a map of Tate resolutions. Such extensions are homotopic if and only if the homomorphisms differ by an element of $\text{PHom}_{kG}(M, M')$. It follows that Tate resolutions give an equivalence of categories between $\text{StMod}(kG)$ and $\mathbf{K}_{\text{ac}}\text{Inj}(kG)$.

We write tk for a Tate resolution of the trivial module k , as an object in $\mathbf{K}\text{Inj}(kG)$, ik for an injective resolution, and pk for a projective resolution. So there is a triangle in $\mathbf{K}\text{Inj}(kG)$ of the form

$$pk \rightarrow ik \rightarrow tk$$

expressing tk as the mapping cone of the map $pk \rightarrow ik$. We can make projective, injective and Tate resolutions of any module by tensoring:

$$M \otimes_k pk \rightarrow M \otimes_k ik \rightarrow M \otimes_k tk.$$

Tensor products in $\mathbf{K}\text{Inj}(kG)$ are taken to be tensor products over k of complexes of modules. Note that ik is the tensor identity of $\mathbf{K}\text{Inj}(kG)$.

Now $\mathbf{K}_{\text{ac}}\text{Inj}(kG)$ is a localising subcategory of $\mathbf{K}\text{Inj}(kG)$, and the quotient category $\mathbf{K}\text{Inj}(kG)/\mathbf{K}_{\text{ac}}\text{Inj}(kG)$ is the unbounded derived category $\text{D}(\text{Mod}(kG))$. The inclusion

$$\mathbf{K}_{\text{ac}}\text{Inj}(kG) \rightarrow \mathbf{K}\text{Inj}(kG)$$

and the quotient functor

$$\mathbf{K}\text{Inj}(kG) \rightarrow \text{D}(\text{Mod}(kG))$$

each have both a left and a right adjoint, so that we get a diagram of categories and functors

$$\text{StMod}(kG) \simeq \mathbf{K}_{\text{ac}}\text{Inj}(kG) \begin{array}{c} \xleftarrow{\text{Hom}_k(tk, -)} \\ \xrightarrow{\text{Hom}_k(pk, -)} \\ \xleftarrow{-\otimes_k tk} \end{array} \mathbf{K}\text{Inj}(kG) \begin{array}{c} \xleftarrow{\text{Hom}_k(pk, -)} \\ \xrightarrow{\text{Hom}_k(ik, -)} \\ \xleftarrow{-\otimes_k pk} \end{array} \text{D}(\text{Mod}(kG)).$$

The compact objects in these categories are only preserved by the left adjoints, and give us back Rickard's sequence

$$\text{stmod}(kG) \leftarrow \mathbf{D}^b(\text{mod}(kG)) \leftarrow \mathbf{K}^b(\text{proj}(kG)).$$

Lemma 7.2. *The only tensor ideal localising subcategories of the unbounded derived category $\mathbf{D}(\text{Mod}(kG))$ are zero and the whole category.*

Proof. Let \mathbf{C} be a non-zero tensor ideal localising subcategory. If X is a non-zero object in \mathbf{C} then X has non-zero homology. The homology of $kG \otimes_k X$ is thus non-zero and free. Any summand isomorphic to kG of the homology splits off as a summand of $kG \otimes_k X$, and so kG is in \mathbf{C} . But kG generates the whole of $\mathbf{D}(\text{Mod}(kG))$. \square

It follows from this lemma that corresponding to any tensor ideal localising subcategory of $\text{StMod}(kG)$, there are two tensor ideal localising subcategories of $\text{KInj}(kG)$, one of which is contained in $\mathbf{K}_{\text{ac}}\text{Inj}(kG)$ and the other of which is generated by this together with the image of $\mathbf{D}(\text{Mod}(kG))$ under $-\otimes_k pk$.

Correspondingly, we have a notion of support varieties for objects in $\text{KInj}(kG)$. The definitions involve exactly the same definitions of functors $\Gamma_{\mathbf{C}}$ and $L_{\mathbf{C}}$ as for $\text{StMod}(kG)$, and the only difference is that if X is an object in $\text{KInj}(kG)$ then the origin $\{0\}$ is either in $\mathcal{V}_{\mathbf{C}}(X)$ or not according as the complex X has homology or not.

This allows us to formulate the following version of the classification theorem for $\text{KInj}(kG)$, where the origin has lost its special role.

Theorem 7.3. *There is a one to one correspondence between tensor ideal localising subcategories of $\text{KInj}(kG)$ and subsets of $\mathcal{V}_{\mathbf{C}}$. Under this correspondence, \mathcal{V} corresponds to $\mathbf{C}_{\mathcal{V}}$.*

This is the theorem whose proof we shall outline in the sections to follow.

8. Support for triangulated categories

At this stage, it is appropriate to describe the general setup introduced in [8] for discussing support for objects in triangulated categories. This setup allows us to move from one category to another without too much effort. Then the game plan, which is inspired by the work of Avramov, Buchweitz, Iyengar, and Miller [1], is as follows:

- (i) Reduce from a finite group to its elementary abelian subgroups.
- (ii) Use a Koszul construction to move from complexes of modules over an elementary abelian group to differential graded modules over a graded exterior algebra.
- (iii) Use a version of the BGG correspondence to move from a graded exterior algebra to a graded polynomial algebra.

- (iv) Use a version of Neeman's classification [25] to deal directly with differential graded modules over graded polynomial algebras.

Definition 8.1. Let \mathbb{T} be a compactly generated triangulated category with small coproducts. We write Σ for the shift in \mathbb{T} and \mathbb{T}^c for the full subcategory of compact objects in \mathbb{T} .

We write $Z^*(\mathbb{T})$ for the *graded centre* of \mathbb{T} . Namely, $Z^*(\mathbb{T})$ is the graded ring whose degree n component $Z^n(\mathbb{T})$ is the set of natural transformations

$$\eta: \text{Id}_{\mathbb{T}} \rightarrow \Sigma^n$$

satisfying

$$\eta\Sigma = (-1)^n \Sigma\eta.$$

Then $Z^*(\mathbb{T})$ is a *graded commutative ring*, in the sense that for $x, y \in Z^*(\mathbb{T})$ we have

$$yx = (-1)^{|x||y|}xy.$$

Let R be a graded commutative Noetherian ring. We say that \mathbb{T} is an *R -linear triangulated category* if we are given a homomorphism of graded commutative rings $\phi: R \rightarrow Z^*(\mathbb{T})$. This amounts to giving, for each object X in \mathbb{T} , a homomorphism of graded rings

$$\phi_X: R \rightarrow \text{End}_{\mathbb{T}}^*(X),$$

such that for X and Y objects in \mathbb{T} , the two induced actions of an element $r \in R$ on $\alpha \in \text{Hom}_{\mathbb{T}}^*(X, Y)$ via ϕ_X and ϕ_Y are related by

$$\phi_Y(r)\alpha = (-1)^{|r||\alpha|}\alpha\phi_X(r).$$

Example 8.2. If $\mathbb{T} = \text{StMod}(kG)$ then $\mathbb{T}^c = \text{stmod}(kG)$ and $\Sigma = \Omega^{-1}$. In this case, we take $R = H^*(G, k)$. Although it might seem more natural to use Tate cohomology $\hat{H}^*(G, k) = \underline{\text{Hom}}_{kG}^*(k, k)$, the problem is that this ring is usually not Noetherian. This is related to the awkward role of the origin in the theory of the support varieties for $\text{StMod}(kG)$.

If $\mathbb{T} = \text{KInj}(kG)$ then $\mathbb{T}^c = \text{D}^b(\text{mod}(kG))$ and Σ is the usual shift. In this case, we also take $R = H^*(G, k)$. In this case, R is just the graded endomorphism ring of the tensor identity ik , and the origin no longer has an awkward role.

We write $\text{Spec } R$ for the set of *homogeneous* prime ideals in R . For each prime $\mathfrak{p} \in \text{Spec } R$ and each graded R -module M , we denote by $M_{\mathfrak{p}}$ the graded localisation of M at \mathfrak{p} .

If \mathcal{V} is a specialisation closed subset of $\text{Spec } R$, we set

$$\mathbb{T}_{\mathcal{V}} = \{X \in \mathbb{T} \mid \text{Hom}_{\mathbb{T}}^*(C, X)_{\mathfrak{p}} = 0 \text{ for all } C \in \mathbb{T}^c, \mathfrak{p} \in \text{Spec } R \setminus \mathcal{V}\}.$$

This is a localising subcategory of \mathbb{T} , and there is a localisation functor $L_{\mathcal{V}}: \mathbb{T} \rightarrow \mathbb{T}$ such that $L_{\mathcal{V}}(X) = 0$ if and only if X is in $\mathbb{T}_{\mathcal{V}}$. We then define $\Gamma_{\mathcal{V}}(X)$ by completing $X \rightarrow L_{\mathcal{V}}(X)$ to a triangle:

$$\Gamma_{\mathcal{V}}(X) \rightarrow X \rightarrow L_{\mathcal{V}}(X).$$

Example 8.3. In the case where \mathbb{T} is either $\text{StMod}(kG)$ or $\text{KInj}(kG)$, and $R = H^*(G, k)$, this is exactly Rickard's triangle for the subset \mathcal{V} of $\text{Spec } H^*(G, k)$.

As in Section 4, if $\mathfrak{p} \in \text{Spec } R$ we choose specialisation closed subsets \mathcal{V} and \mathcal{W} of $\text{Spec } R$ satisfying $\mathcal{V} \not\subseteq \mathcal{W}$ and $\mathcal{V} \subseteq \mathcal{W} \cup \mathfrak{p}$ and define

$$\Gamma_{\mathfrak{p}} = L_{\mathcal{W}}\Gamma_{\mathcal{V}} = \Gamma_{\mathcal{V}}L_{\mathcal{W}}.$$

Again, this is independent of choice of \mathcal{V} and \mathcal{W} with the given properties.

Definition 8.4. If X is an object in \mathbb{T} then the *support* of X is the subset

$$\text{supp}_R(X) = \{\mathfrak{p} \in \text{Spec } R \mid \Gamma_{\mathfrak{p}}(X) \neq 0\}.$$

This is a subset of

$$\text{supp}_R(\mathbb{T}) = \{\mathfrak{p} \in \text{Spec } R \mid \Gamma_{\mathfrak{p}}(\mathbb{T}) \neq 0\} \subseteq \text{Spec } R.$$

Example 8.5. Continuing Example 8.3, this notion of support agrees with $\mathcal{V}_G(X)$ as defined in Section 4. We have

$$\begin{aligned} \text{supp}_{H^*(G, k)}(\text{StMod}(kG)) &= \mathcal{V}_G \setminus \{0\}, \\ \text{supp}_{H^*(G, k)}(\text{KInj}(kG)) &= \mathcal{V}_G. \end{aligned}$$

The following theorem summarises the properties of support for R -linear triangulated categories; confer Theorems 2.2 and 4.3. Proofs can be found in [8].

Theorem 8.6. *Let X, Y and Z be objects in \mathbb{T} . Then*

- (i) $X = 0$ if and only if $\text{supp}_R(X) = \emptyset$.
- (ii) $\text{supp}_R(\Sigma X) = \text{supp}_R(X)$.
- (iii) For a small family of objects X_{α} we have

$$\text{supp}_R\left(\bigoplus_{\alpha} X_{\alpha}\right) = \bigcup_{\alpha} \text{supp}_R(X_{\alpha}).$$

- (iv) If $X \rightarrow Y \rightarrow Z$ is a triangle in \mathbb{T} then

$$\text{supp}_R(Y) \subseteq \text{supp}_R(X) \cup \text{supp}_R(Z).$$

- (v) For $\mathcal{V} \subseteq \text{Spec } R$ we have

$$\begin{aligned} \text{supp}_R(\Gamma_{\mathcal{V}}(X)) &= \mathcal{V} \cap \text{supp}_R(X) \\ \text{supp}_R(L_{\mathcal{V}}(X)) &= (\text{Spec } R \setminus \mathcal{V}) \cap \text{supp}_R(X). \end{aligned}$$

- (vi) If $\text{cl}(\text{supp}_R(X)) \cap \text{supp}_R(Y) = \emptyset$ then $\text{Hom}_{\mathbb{T}}(X, Y) = 0$.

Here, $\text{cl}(\mathcal{V})$ denotes the specialisation closure of a subset \mathcal{V} .

9. Tensor triangulated categories

In this section we consider compactly generated triangulated categories with some extra structure. Namely, we want to consider an internal tensor product

$$\otimes: \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T},$$

exact in each variable, preserving small coproducts, and with a unit $\mathbb{1}$. It then follows (using the Brown representability theorem) that there are *function objects* $\mathcal{H}om(X, Y)$ in \mathbb{T} with natural isomorphisms

$$\mathrm{Hom}_{\mathbb{T}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathbb{T}}(X, \mathcal{H}om(Y, Z)).$$

We define the *dual* of an object X in \mathbb{T} to be

$$X^{\vee} = \mathcal{H}om(X, \mathbb{1}).$$

Definition 9.1. We say that $(\mathbb{T}, \otimes, \mathbb{1})$ is a *tensor triangulated category* if the following hold:

- (i) \mathbb{T} is a compactly generated triangulated category with small coproducts.
- (ii) The tensor product \otimes and unit $\mathbb{1}$ make \mathbb{T} a symmetric monoidal category.
- (iii) The tensor product is exact in each variable and preserves small coproducts.
- (iv) The unit $\mathbb{1}$ is compact.
- (v) Compact objects are *strongly dualisable* in the sense that if C is compact and X is any object in \mathbb{T} then the canonical map

$$C^{\vee} \otimes X \rightarrow \mathcal{H}om(C, X)$$

is an isomorphism.

See [8, 11] for further discussion of this structure.

Example 9.2. Both $\mathrm{StMod}(kG)$ and $\mathrm{Klnj}(kG)$ are tensor triangulated categories.

The symmetric monoidal structure ensures that the endomorphism ring of the tensor identity $\mathrm{End}_{\mathbb{T}}^*(\mathbb{1})$ is a graded commutative ring. Given any object X , this ring acts on $\mathrm{End}_{\mathbb{T}}^*(X)$ via

$$\mathrm{End}_{\mathbb{T}}^*(\mathbb{1}) \xrightarrow{X \otimes -} \mathrm{End}_{\mathbb{T}}^*(X).$$

So if R is a graded commutative Noetherian ring, a homomorphism $R \rightarrow \mathrm{End}_{\mathbb{T}}^*(\mathbb{1})$ gives \mathbb{T} a structure of an R -linear category.

Definition 9.3. We say that the action of R on \mathbb{T} is *canonical* if it arises from a homomorphism $R \rightarrow \mathrm{End}_{\mathbb{T}}^*(\mathbb{1})$ as described in the previous paragraph.

Proposition 9.4. *If the action of R on \mathbb{T} is canonical then for each specialisation closed subset $\mathcal{V} \subseteq \mathrm{Spec} R$ and each prime $\mathfrak{p} \in \mathrm{Spec} R$ there are natural isomorphisms*

$$\Gamma_{\mathcal{V}}(X) \cong X \otimes \Gamma_{\mathcal{V}}(\mathbb{1}), \quad L_{\mathcal{V}}(X) \cong X \otimes L_{\mathcal{V}}(\mathbb{1}), \quad \Gamma_{\mathfrak{p}}(X) \cong X \otimes \Gamma_{\mathfrak{p}}(\mathbb{1}).$$

10. The local-global principle

If X is an object or collection of objects in a triangulated category \mathbb{T} , we write $\text{Loc}_{\mathbb{T}}(X)$ for the localising subcategory of \mathbb{T} generated by X .

Definition 10.1. Let \mathbb{T} be an R -linear triangulated category with small coproducts. We say that the *local-global principle holds* for \mathbb{T} if for each object X in \mathbb{T} we have

$$\text{Loc}_{\mathbb{T}}(X) = \text{Loc}_{\mathbb{T}}(\{I_{\mathfrak{p}}(X) \mid \mathfrak{p} \in \text{supp}_R(\mathbb{T})\}).$$

This condition is usually satisfied, because of the following theorem [10, Corollary 3.5]:

Theorem 10.2. *The local-global principle holds provided R has finite Krull dimension.*

For a tensor triangulated category \mathbb{T} , we modify the definition slightly. If X is an object or collection of objects in \mathbb{T} , we write $\text{Loc}_{\mathbb{T}}^{\otimes}(X)$ for the tensor ideal localising subcategory of \mathbb{T} generated by X . The following theorem says that for a tensor triangulated category the appropriate analogue of the local-global principle always holds [10, Theorem 7.2]:

Theorem 10.3. *Let \mathbb{T} be an R -linear tensor triangulated category with canonical R -action. Then for each object X in \mathbb{T} the following holds:*

$$\text{Loc}_{\mathbb{T}}^{\otimes}(X) = \text{Loc}_{\mathbb{T}}^{\otimes}(\{I_{\mathfrak{p}}(X) \mid \mathfrak{p} \in \text{Spec } R\}).$$

When the local-global principle holds, the classification of localising subcategories can be achieved one prime at a time. The way to express this is via the following maps:

$$\left\{ \begin{array}{c} \text{Localising} \\ \text{subcategories of } \mathbb{T} \end{array} \right\} \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} \left\{ \begin{array}{c} \text{Families } (\mathbf{S}(\mathfrak{p}))_{\mathfrak{p} \in \text{supp}_R(\mathbb{T})} \text{ with } \mathbf{S}(\mathfrak{p}) \text{ a} \\ \text{localising subcategory of } I_{\mathfrak{p}}(\mathbb{T}) \end{array} \right\}$$

where $\sigma(\mathbf{S}) = (\mathbf{S} \cap I_{\mathfrak{p}}(\mathbb{T}))$ and $\tau(\mathbf{S}(\mathfrak{p})) = \text{Loc}_{\mathbb{T}}(\mathbf{S}(\mathfrak{p}) \mid \mathfrak{p} \in \text{supp}_R(\mathbb{T}))$. The following is [10, Proposition 3.6].

Theorem 10.4. *If the local-global principle holds then the maps σ and τ are mutually inverse bijections.*

In other words, specifying a localising subcategory of \mathbb{T} is equivalent to specifying a localising subcategory of $I_{\mathfrak{p}}(\mathbb{T})$ for each prime $\mathfrak{p} \in \text{supp}_R(\mathbb{T})$.

Since the tensor ideal version of the local-global principle always holds in a tensor triangulated category, we have the following theorem.

Theorem 10.5. *Let \mathbb{T} be an R -linear tensor triangulated category with canonical R -action. Then the maps σ and τ above induce mutually inverse bijections between the tensor ideal localising subcategories of \mathbb{T} and families $(\mathbf{S}(\mathfrak{p}))_{\mathfrak{p} \in \text{supp}_R(\mathbb{T})}$ with $\mathbf{S}(\mathfrak{p})$ a tensor ideal localising subcategory of $I_{\mathfrak{p}}(\mathbb{T})$.*

Remark 10.6. If the tensor identity $\mathbb{1}$ generates \mathbb{T} then every localising subcategory of \mathbb{T} is tensor ideal. In the case of $\text{StMod}(kG)$ and $\text{Klnj}(kG)$ this holds if and only if G is a finite p -group.

11. Stratifying triangulated categories

In the last section, we showed how to classify localising subcategories of an R -linear triangulated category \mathbb{T} one prime at a time. The easiest case is where each $\Gamma_{\mathfrak{p}}(\mathbb{T})$ with $\mathfrak{p} \in \text{supp}_R(\mathbb{T})$, is a minimal subcategory.

Definition 11.1. We say that a localising subcategory of \mathbb{T} is *minimal* if it is non-zero and has no proper non-zero localising subcategories.

We say that \mathbb{T} is *stratified* by R if the local-global principle holds, and each $\Gamma_{\mathfrak{p}}(\mathbb{T})$ with $\mathfrak{p} \in \text{supp}_R \mathbb{T}$ is a minimal localising subcategory.

Suppose that \mathbb{T} is stratified by R , and consider the maps σ and τ of Section 10. To name a family $(\mathbb{S}(\mathfrak{p}))_{\mathfrak{p} \in \text{supp}_R(\mathbb{T})}$ with $\mathbb{S}(\mathfrak{p})$ a localising subcategory of $\Gamma_{\mathfrak{p}}(\mathbb{T})$, we just have to name the set of $\mathfrak{p} \in \text{supp}_R(\mathbb{T})$ for which $\mathbb{S}(\mathfrak{p}) = \Gamma_{\mathfrak{p}}(\mathbb{T})$, since the remaining primes \mathfrak{p} will satisfy $\mathbb{S}(\mathfrak{p}) = 0$. So the following is a direct consequence of Theorem 10.4.

Theorem 11.2. *Let \mathbb{T} be an R -linear triangulated category. If \mathbb{T} is stratified by R then the maps σ and τ establish a bijection between the localising subcategories of \mathbb{T} and the subsets of $\text{supp}_R(\mathbb{T})$.*

In the case of an R -linear tensor triangulated category \mathbb{T} , because the corresponding local-global principle is automatic (Theorem 10.3), we say that \mathbb{T} is *stratified* by R if for each $\mathfrak{p} \in \text{supp}_R(\mathbb{T})$, $\Gamma_{\mathfrak{p}}(\mathbb{T})$ is minimal as a tensor ideal localising subcategory.

Theorem 11.3. *Let \mathbb{T} be an R -linear tensor triangulated category. If \mathbb{T} is stratified by R then the maps σ and τ establish a bijection between the tensor ideal localising subcategories of \mathbb{T} and the subsets of $\text{supp}_R(\mathbb{T})$.*

In order to prove stratification in a given situation, we need a criterion for minimality of localising subcategories. The following can be found in Lemma 4.1 of [10] and Lemma 3.9 of [9].

Lemma 11.4. (i) *Let \mathbb{T} be an R -linear triangulated category. A non-zero localising subcategory \mathbb{S} of \mathbb{T} is minimal if and only if for all non-zero objects X and Y in \mathbb{S} we have $\text{Hom}_{\mathbb{T}}^*(X, Y) \neq 0$.*

(ii) *Let \mathbb{T} be an R -linear tensor triangulated category. A non-zero tensor ideal localising subcategory \mathbb{S} of \mathbb{T} is minimal if and only if for all non-zero objects X and Y in \mathbb{S} there exists an object Z in \mathbb{T} such that $\text{Hom}_{\mathbb{T}}^*(X \otimes Z, Y) \neq 0$. The object Z may be taken to be a compact generator for \mathbb{T} .*

We can now restate Theorems 6.2 and 7.3.

Theorem 11.5. *The tensor triangulated categories $\text{StMod}(kG)$ and $\text{Klnj}(kG)$ are stratified by $H^*(G, k)$.*

In the next few sections, we outline the proof of this theorem. A lot of details will be swept under the carpet here, but are spelt out in [9].

12. Graded polynomial algebras

The first step in proving Theorem 11.5 is to stratify the derived category of differential graded modules over a polynomial ring. Let S be a graded polynomial algebra over the field k on a finite number of indeterminates. If k does not have characteristic two, we assume that the degrees of the indeterminates are even, so that S is graded commutative.

We view S as a *differential graded* (abbreviated to dg) algebra with zero differential, and we write $D(S)$ for the derived category of dg S -modules. The objects of this category are the dg S -modules, and the morphisms are homotopy classes of degree preserving chain maps with quasi-isomorphisms inverted. See for example Keller [23] for further details.

The category $D(S)$ is a tensor triangulated category in which the tensor product is the derived tensor product over S , $X \otimes_S^{\mathbf{L}} Y$ and the tensor identity is S , which is a compact generator for $D(S)$. In particular, every localising subcategory is tensor ideal. The canonical action of S on $D(S)$ makes $D(S)$ into an S -linear tensor triangulated category. The following theorem is a dg analogue of the theorem of Neeman [25], and is proved in Theorem 5.2 of [9]. The existence of field objects plays a crucial role in the proof.

Theorem 12.1. *The category $D(S)$ is stratified by the canonical S -action. So the maps σ and τ of Section 10 are mutually inverse bijections between the localising subcategories of $D(S)$ and subsets of $\text{Spec } S$.*

13. A BGG correspondence

The second step in the proof of Theorem 11.5 is to use a version of the BGG correspondence to transfer the stratification from polynomial rings to exterior algebras.

Let k be a field and let A be an exterior algebra on indeterminates ξ_1, \dots, ξ_c of negative odd degrees. We view A as a dg algebra with zero differential. Let S be a graded polynomial algebra on variables x_1, \dots, x_c with $|x_i| = -|\xi_i| + 1$. Let J be the dg $A \otimes_k S$ -module with

$$J^{\natural} = \text{Hom}_k(A, k) \otimes_k S$$

and

$$d(f \otimes s) = \sum_i \xi_i f \otimes x_i s.$$

If M is a dg A -module then $\text{Hom}_A(J, M)$ is in a natural way a dg S -module.

In general, for a dg algebra A , we write A^{\natural} for the underlying graded algebra of A , forgetting the differential. If M is a dg A -module, we write M^{\natural} for the underlying graded A^{\natural} -module. We say that a dg A -module I is *graded-injective* if I^{\natural} is injective in the category of graded A^{\natural} -module. Finally, we write $\text{KInj}(A)$ for the homotopy category of graded-injective dg A -modules.

The following version of the BGG correspondence [16] extends the one from [1].

Theorem 13.1. *The functor*

$$\mathrm{Hom}_\Lambda(J, -): \mathrm{Klnj}(\Lambda) \rightarrow \mathrm{D}(S)$$

is an equivalence of categories.

We give Λ a comultiplication

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i.$$

This allows us to define a tensor product of dg Λ -modules, and this makes $\mathrm{Klnj}(\Lambda)$ into a tensor triangulated category. Its tensor identity is ik , which is a compact generator. We have

$$S \cong \mathrm{Ext}_\Lambda^*(k, k) \cong \mathrm{Hom}_{\mathrm{Klnj}(\Lambda)}(J, J)$$

which makes $\mathrm{Klnj}(\Lambda)$ into an S -linear tensor triangulated category with canonical S -action. The equivalence described in Theorem 13.1 allows us to stratify $\mathrm{Klnj}(\Lambda)$.

Theorem 13.2. *The category $\mathrm{Klnj}(\Lambda)$ is stratified by the canonical action of S . So the maps σ and τ of Section 10 give a bijection between localising subcategories of $\mathrm{Klnj}(\Lambda)$ and subsets of $\mathrm{Spec} S$.*

14. The Koszul construction

If E is an elementary abelian 2-group and k has characteristic two, then kE is an exterior algebra on generators of degree zero. So we can use Theorem 13.2 to stratify $\mathrm{Klnj}(kE)$. This proves Theorem 11.5 in this case; see [12] for details. Elementary abelian p -groups with p odd cannot be treated this way. In this section, we show how to use a Koszul construction to deal with this case.

So let p be a prime, k be a field of characteristic p , and

$$E = \langle g_1, \dots, g_r \rangle \cong (\mathbf{Z}/p)^r$$

be an elementary abelian p -group. Let $z_i = g_i - 1 \in kE$, so that

$$kE = k[z_1, \dots, z_r]/(z_1^p, \dots, z_r^p).$$

We regard kE as a complete intersection, and form the Koszul construction as follows. Let A be the dg algebra

$$A = kE\langle y_1, \dots, y_r \rangle$$

where kE is in degree zero, and y_1, \dots, y_r are exterior generators of degree -1 with

$$y_i^2 = 0, \quad y_i y_j = -y_j y_i, \quad d(y_i) = z_i, \quad d(z_i) = 0.$$

Let A be an exterior algebra on generators ξ_1, \dots, ξ_r of degree -1 , regarded as a dg algebra with zero differential, and S be a polynomial algebra on generators x_1, \dots, x_r of degree 2, again regarded as dg algebra with zero differential. The following is Lemma 7.1 of [9].

Lemma 14.1. *The map $\phi: \Lambda \rightarrow A$ defined by*

$$\phi(\xi_i) = z_i^{p-1} y_i$$

is a quasi-isomorphism of dg algebras. In particular,

$$\mathrm{Ext}_A^*(k, k) \cong \mathrm{Ext}_\Lambda^*(k, k) \cong S.$$

We give A a comultiplication

$$\Delta(z_i) = z_i \otimes 1 + 1 \otimes z_i, \quad \Delta(y_i) = y_i \otimes 1 + 1 \otimes y_i.$$

This gives $\mathrm{Klnj}(A)$ the structure of a tensor triangulated category with a canonical action of S . The following is a special case of Proposition 4.6 of [9].

Theorem 14.2. *The map $\phi: \Lambda \rightarrow A$ induces an equivalence of tensor triangulated categories*

$$\mathrm{Hom}_\Lambda(A, -): \mathrm{Klnj}(\Lambda) \rightarrow \mathrm{Klnj}(A).$$

As a consequence, we can stratify $\mathrm{Klnj}(A)$. The following is Theorem 7.2 of [9].

Theorem 14.3. *The S -linear tensor triangulated category $\mathrm{Klnj}(A)$ is stratified by the canonical action of S . So the maps σ and τ of Section 10 give a bijection between localising subcategories of $\mathrm{Klnj}(A)$ and subsets of $\mathrm{Spec} S$.*

Now the inclusion map $kE \rightarrow A$ induces a restriction map

$$\mathrm{Ext}_A^*(k, k) \rightarrow \mathrm{Ext}_{kE}^*(k, k).$$

The structure of $\mathrm{Ext}_{kE}^*(k, k)$ is as follows. If p is odd then it is a tensor product of an exterior algebra and a polynomial algebra

$$\mathrm{Ext}_{kE}^*(k, k) \cong \Lambda(u_1, \dots, u_r) \otimes k[x_1, \dots, x_r]$$

with $|u_i| = 1$ and $|x_i| = 2$. The elements x_i are the restrictions of the elements of the same name in $\mathrm{Ext}_A^*(k, k) \cong S$. If $p = 2$ then

$$\mathrm{Ext}_{kE}^*(k, k) = k[u_1, \dots, u_r]$$

with $|u_i| = 1$. The elements $x_i \in \mathrm{Ext}_A^*(k, k)$ restrict to u_i^2 .

In both cases, this allows us to regard $S \cong \mathrm{Ext}_A^*(k, k)$ as a subring of $H^*(E, k)$ over which it is finitely generated as a module. The restriction map from $\mathrm{Ext}_A^*(k, k)$ to $H^*(E, k) = \mathrm{Ext}_{kE}^*(k, k)$ induces a bijection

$$\mathrm{Spec} H^*(E, k) \rightarrow \mathrm{Spec} S.$$

Using the induction and restriction functors, the criterion of Lemma 11.4 is essential in deducing the following theorem from Theorem 14.3. See Theorem 4.11 of [9] for details.

Theorem 14.4. *The tensor triangulated category $\mathrm{Klnj}(kE)$ is stratified by the canonical action of $H^*(E, k)$, or equivalently of $S \subseteq H^*(E, k)$. So the maps σ and τ of Section 10 give a bijection between localising subcategories of $\mathrm{Klnj}(kE)$ and subsets of $\mathrm{Spec} H^*(E, k)$.*

There is an issue here with the tensor structure. The usual diagonal map $\Delta(g_i) = g_i \otimes g_i$ on kE is not compatible with the inclusion $kE \rightarrow A$. There is another diagonal map on kE given by $\Delta(z_i) = z_i \otimes 1 + 1 \otimes z_i$ coming from regarding kE as a restricted universal enveloping algebra, and this diagonal map is compatible with the inclusion. Each diagonal map gives rise to a canonical action of $H^*(E, k)$ on $\mathrm{Klnj}(kE)$, but the two actions are not the same. Lemma 3.10 of [9] helps us out at this point:

Lemma 14.5. *Let \mathbb{T} be a triangulated category admitting two tensor triangulated structures with the same unit $\mathbb{1}$, and assume that $\mathbb{1}$ generates \mathbb{T} . Let*

$$\phi, \phi': R \rightarrow Z^*(\mathbb{T})$$

be two actions of R on \mathbb{T} . If the maps $R \rightarrow \mathrm{End}_{\mathbb{T}}^(\mathbb{1})$ induced by ϕ and ϕ' agree then \mathbb{T} is stratified by R through ϕ if and only if it is stratified by R through ϕ' .*

15. Quillen stratification

There is a general machine for understanding cohomological properties of a general finite group from its elementary abelian subgroups, called Quillen stratification. In this section, we explain how to use this machine to compare localising subcategories of $\mathrm{Klnj}(kE)$ and $\mathrm{Klnj}(kG)$.

Let $\mathcal{V}_G = \mathrm{Spec} H^*(G, k)$. If H is a subgroup of G then the restriction map $H^*(G, k) \rightarrow H^*(H, k)$ induces a map

$$\mathrm{res}_{G,H}^*: \mathcal{V}_H \rightarrow \mathcal{V}_G.$$

Quillen [27, 28] proved the following. Given a prime $\mathfrak{p} \in \mathcal{V}_G$ we say that \mathfrak{p} *originates* in an elementary abelian p -subgroup $E \leq G$ if \mathfrak{p} is in the image of $\mathrm{res}_{G,E}^*$ but not of $\mathrm{res}_{G,E'}^*$ for E' a proper subgroup of E .

Theorem 15.1. *Given a prime $\mathfrak{p} \in \mathcal{V}_G$, there exists a pair (E, \mathfrak{q}) such that \mathfrak{p} originates in E and $\mathrm{res}_{G,E}^*(\mathfrak{q}) = \mathfrak{p}$, and all such pairs are G -conjugate. This sets up a bijection between primes $\mathfrak{p} \in \mathcal{V}_G$ and conjugacy classes of pairs (E, \mathfrak{q}) where $\mathfrak{q} \in \mathcal{V}_E$ originates in E .*

In order to be able to use this, we first need a version of the ‘‘subgroup theorem’’ for elementary abelian p -groups. The following is Theorem 9.5 of [9], and its proof is a fairly straightforward consequence of the Stratification Theorem 14.4 for elementary abelian p -groups.

Theorem 15.2. *Let $E' \leq E$ be elementary abelian p -groups. If X is an object in $\mathrm{Klnj}(kE)$ then*

$$\mathcal{V}_{E'}(X \downarrow_{E'}) = (\mathrm{res}_{E,E'}^*)^{-1} \mathcal{V}_E(X).$$

Next, we need a version of Chouinard's theorem [19] for $\mathbf{Klnj}(kG)$, see Proposition 9.6 of [9]:

Theorem 15.3. *An object X in $\mathbf{Klnj}(kG)$ is zero if and only if the restriction of X to every elementary abelian p -subgroup of G is zero.*

We are now ready to outline the proof of Theorem 11.5. This amounts to showing that for $\mathfrak{p} \in \mathcal{V}_G$, the tensor ideal localising subcategory $\Gamma_{\mathfrak{p}} \mathbf{Klnj}(kG)$ is minimal. For this purpose, we use the criterion of Lemma 11.4. Let X and Y be non-zero objects in $\Gamma_{\mathfrak{p}} \mathbf{Klnj}(kG)$. By Theorem 15.3 there exists an elementary abelian subgroup E_0 such that $X \downarrow_{E_0}$ is non-zero. Choose a prime $\mathfrak{q}_0 \in \mathcal{V}_{E_0}(X \downarrow_{E_0})$. Using standard properties of support under induction and restriction, we obtain $\text{res}_{G, E_0}^*(\mathfrak{q}_0) = \mathfrak{p}$. So we can choose a pair (E, \mathfrak{q}) with $E_0 \geq E$ and $\mathfrak{q}_0 = \text{res}_{E_0, E}^*(\mathfrak{q})$, so that the conjugacy class of (E, \mathfrak{q}) corresponds to \mathfrak{p} under the bijection of Theorem 15.1. By Theorem 15.2 we have $\Gamma_{\mathfrak{q}}(X \downarrow_E) \neq 0$. Since the pair (E, \mathfrak{q}) is determined up to conjugacy by \mathfrak{p} , the object $Y \in \Gamma_{\mathfrak{p}} \mathbf{Klnj}(kG)$ also has $\Gamma_{\mathfrak{q}}(Y \downarrow_E) \neq 0$.

Let Z be an injective resolution of the permutation module $k(G/E)$, as an object in $\mathbf{Klnj}(kG)$. This has the property that

$$X \otimes_k Z \cong X \otimes_k k(G/E) \cong X \downarrow_E \uparrow^G.$$

Thus using Frobenius reciprocity we have

$$\text{Hom}_{kG}^*(X \otimes_k Z, Y) \cong \text{Hom}_{kG}^*(X \downarrow_E \uparrow^G, Y) \cong \text{Hom}_{kE}^*(X \downarrow_E, Y \downarrow_E).$$

Since $\Gamma_{\mathfrak{q}} \mathbf{Klnj}(kE)$ is minimal, using the first part of Lemma 11.4 in one direction shows that the right hand side is non-zero. Using the other part in the other direction then shows that $\Gamma_{\mathfrak{p}} \mathbf{Klnj}(kG)$ is minimal. Thus $\mathbf{Klnj}(kG)$ is stratified as a tensor triangulated category by $H^*(G, k)$.

16. Applications

In this section, we give some applications of the classification of localising subcategories of $\mathbf{StMod}(kG)$ and $\mathbf{Klnj}(kG)$ (Theorem 11.5). To illustrate the methods, we give the proof in the case of the tensor product theorem. The remaining proofs can be found in Section 11 of [9].

The tensor product theorem. The tensor product theorem states that if X and Y are objects in $\mathbf{StMod}(kG)$, or more generally in the larger category $\mathbf{Klnj}(kG)$, then

$$\mathcal{V}_G(X \otimes_k Y) = \mathcal{V}_G(X) \cap \mathcal{V}_G(Y).$$

This was first proved by Benson, Carlson and Rickard [5, Theorem 10.8] for $\mathbf{StMod}(kG)$ in the case where k is algebraically closed. The method of proof was to reduce to elementary abelian subgroups and then use the version of Dade's lemma given in Theorem 4.4.

The proof of the Stratification Theorem 11.5 does not involve any form of Dade's lemma, and so we get a new proof of the tensor product theorem as follows. Since

$$\Gamma_{\mathfrak{p}}(X \otimes_k Y) \cong \Gamma_{\mathfrak{p}}(\mathbb{1}) \otimes_k X \otimes_k Y \cong \Gamma_{\mathfrak{p}}(X) \otimes_k Y \cong X \otimes \Gamma_{\mathfrak{p}}(Y),$$

if either $\Gamma_{\mathfrak{p}}(X)$ or $\Gamma_{\mathfrak{p}}(Y)$ is zero then so is $\Gamma_{\mathfrak{p}}(X \otimes_k Y)$. This shows that

$$\mathcal{V}_G(X \otimes_k Y) \subseteq \mathcal{V}_G(X) \cap \mathcal{V}_G(Y).$$

For the reverse containment, suppose that $\mathfrak{p} \in \mathcal{V}_G(X) \cap \mathcal{V}_G(Y)$. Thus $\Gamma_{\mathfrak{p}}(X) \neq 0$ and $\Gamma_{\mathfrak{p}}(Y) \neq 0$. It follows from Theorem 11.5 and $\Gamma_{\mathfrak{p}}(X) \neq 0$ that $\Gamma_{\mathfrak{p}}(\mathbb{1})$ is in $\text{Loc}^{\otimes}(\Gamma_{\mathfrak{p}}(X))$, and hence that $\Gamma_{\mathfrak{p}}(Y)$ is in $\text{Loc}^{\otimes}(\Gamma_{\mathfrak{p}}(X \otimes_k Y))$. Since $\Gamma_{\mathfrak{p}}(Y) \neq 0$, this implies that $\Gamma_{\mathfrak{p}}(X \otimes_k Y) \neq 0$.

The subgroup theorem. The Subgroup Theorem 15.2 for elementary abelian groups was proved using the stratification theorem in that context, and was used in order to prove the stratification theorem for general finite groups. The following general version of the subgroup theorem follows in the same way from the stratification theorem for finite groups.

Theorem 16.1. *Let $H \leq G$ be finite groups. If X is an object in $\text{Klnj}(kG)$ then*

$$\mathcal{V}_H(X \downarrow_H) = (\text{res}_{G,H}^*)^{-1} \mathcal{V}_G(X).$$

Thick subcategories. Theorem 11.5 also gives a new proof for the classification of tensor ideal thick subcategories of $\text{stmod}(kG)$ (Theorem 2.7) and $\text{D}^b(\text{mod}(kG))$, avoiding the use of the version of Dade's lemma given in Theorem 4.4.

Localising subcategories closed under products and duality. We state the following theorem for $\text{StMod}(kG)$. A similar statement holds for $\text{Klnj}(kG)$.

Theorem 16.2. *Under the bijection given in Theorem 6.2, the following properties of a tensor ideal localising subcategory $\mathcal{C}_{\mathcal{V}}$ of $\text{StMod}(kG)$ are equivalent.*

- (i) $\mathcal{C}_{\mathcal{V}}$ is closed under products.
- (ii) $\mathcal{V}_G \setminus \mathcal{V}$ is specialisation closed.
- (iii) There exists a set \mathcal{X} of finitely generated kG -modules such that $\mathcal{C}_{\mathcal{V}}$ is the full subcategory of modules M satisfying $\underline{\text{Hom}}_{kG}(N, M)$ for all N in \mathcal{X} .
- (iv) When a kG -module M is in $\mathcal{C}_{\mathcal{V}}$ so is $\text{Hom}_k(M, k)$.

The telescope conjecture. Our final application is a statement about certain localising such categories which are defined as follows.

Definition 16.3. A localising subcategory \mathcal{C} of a triangulated category \mathcal{T} is said to be *strictly localising* if there is a localisation functor $L: \mathcal{T} \rightarrow \mathcal{T}$ such that an object X is in \mathcal{C} if and only if $L(X) = 0$.

A localising subcategory \mathcal{C} is said to be *smashing* if it is strictly localising, and the localisation functor L preserves coproducts.

The following is the analogue for $\text{StMod}(kG)$ and $\text{KInj}(kG)$ of the telescope conjecture of algebraic topology (Bousfield [17], Ravenel [29, 30]).

Theorem 16.4. *A tensor ideal localising subcategory of $\text{StMod}(kG)$ or of $\text{KInj}(kG)$ is smashing if and only if it is generated by compact objects.*

17. Costratification

Let \mathcal{T} be a compactly generated triangulated category with small products and coproducts. Recall that a *colocalising subcategory* of \mathcal{T} is a thick subcategory that is closed under products.

If \mathcal{T} is tensor triangulated then such a subcategory \mathcal{C} is closed under tensor products with simple modules if and only if it is *Hom closed*, in the sense that for all X in \mathcal{T} and all Y in \mathcal{C} , the function object (see Section 9) $\mathcal{H}om(X, Y)$ is in \mathcal{C} .

The main theorem of [11] classifies the Hom closed colocalising subcategories of $\text{StMod}(kG)$ and of $\text{KInj}(kG)$.

Theorem 17.1. *There is a one to one correspondence between Hom closed colocalising subcategories \mathcal{D} of $\text{StMod}(kG)$ (respectively $\text{KInj}(kG)$) and tensor ideal localising subcategories \mathcal{C} of $\text{StMod}(kG)$ (respectively $\text{KInj}(kG)$) given by $\mathcal{C} = {}^\perp \mathcal{D}$, the full subcategory of objects X satisfying $\text{Hom}^*(X, Y) = 0$ for all Y in \mathcal{D} .*

The proof of this theorem goes via the notions of cosupport and costratification. We explain these concepts and fix to this end an R -linear tensor triangulated category \mathcal{T} , as in Section 9. In addition to the axioms listed there, we assume also $\text{Hom}(?, Y)$ is exact for each object $Y \in \mathcal{T}$; all tensor triangulated categories encountered in this work have this property.

For each prime $\mathfrak{p} \in \text{Spec } R$ denote by $\Lambda^{\mathfrak{p}}$ the right adjoint of the functor $\Gamma_{\mathfrak{p}}$ which exists by the Brown representability theorem.

Definition 17.2. If X is an object in \mathcal{T} then the *cosupport* of X is the subset

$$\text{cosupp}_R(X) = \{\mathfrak{p} \in \text{Spec } R \mid \Lambda^{\mathfrak{p}}(X) \neq 0\}.$$

Note that $\Lambda^{\mathfrak{p}}$ and $\Gamma_{\mathfrak{p}}$ provide mutually inverse equivalences between $\Gamma_{\mathfrak{p}}(\mathcal{T})$ and $\Lambda^{\mathfrak{p}}(\mathcal{T})$. Thus $\text{cosupp}_R(X)$ is a subset of $\text{supp}_R(\mathcal{T})$.

We say that the tensor triangulated category \mathcal{T} is *costratified* by R if for each $\mathfrak{p} \in \text{Spec } R$ the category $\Lambda^{\mathfrak{p}}(\mathcal{T})$ is either zero or minimal among all Hom closed colocalising subcategories of \mathcal{T} .

As for localising subcategories, the classification of colocalising subcategories can be achieved one prime at a time. The way to express this is via the following maps:

$$\left\{ \begin{array}{c} \text{Colocalising} \\ \text{subcategories of } \mathbb{T} \end{array} \right\} \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} \left\{ \begin{array}{c} \text{Families } (\mathbf{S}(\mathfrak{p}))_{\mathfrak{p} \in \text{supp}_R(\mathbb{T})} \text{ with } \mathbf{S}(\mathfrak{p}) \text{ a} \\ \text{colocalising subcategory of } \Lambda^{\mathfrak{p}}(\mathbb{T}) \end{array} \right\}$$

where $\sigma(\mathbf{S}) = (\mathbf{S} \cap \Lambda^{\mathfrak{p}}(\mathbb{T}))$ and $\tau(\mathbf{S}(\mathfrak{p})) = \text{Coloc}_{\mathbb{T}}(\mathbf{S}(\mathfrak{p}) \mid \mathfrak{p} \in \text{supp}_T(\mathbb{T}))$. The following is [11, Corollary 9.2].

Theorem 17.3. *Let \mathbb{T} be an R -linear tensor triangulated category. If \mathbb{T} is costratified by R then the maps σ and τ establish a bijection between the Hom closed colocalising subcategories of \mathbb{T} and the subsets of $\text{supp}_R(\mathbb{T})$.*

From this result one deduces Theorem 17.1 by proving costratification first for $\text{Klnj}(kG)$ (see [11, Theorem 11.10]) and then for $\text{StMod}(kG)$ (see [11, Theorem 11.13]).

Let us include an application which justifies the study of support and cosupport; it is [11, Corollary 9.6].

Theorem 17.4. *Suppose the tensor triangulated category \mathbb{T} is generated by its unit. Then \mathbb{T} is stratified by R if and only if for all objects X and Y in \mathbb{T} one has*

$$\text{Hom}_{\mathbb{T}}^*(X, Y) = 0 \quad \iff \quad \text{supp}_R(X) \cap \text{cosupp}_R(Y) = \emptyset.$$

Note that $\text{StMod}(kG)$ (respectively $\text{Klnj}(kG)$) is generated by its unit if and only if G is a p -group. We refer to [11] for more general results which do not depend on the fact that \mathbb{T} is generated by its unit.

References

- [1] L. L. Avramov, R.-O. Buchweitz, S. B. Iyengar, and C. Miller, *Homology of perfect complexes*, Adv. Math. **223** (2010), 1731–1781; Corrigendum: Adv. Math. **225** (2010) 3576–3578.
- [2] D. J. Benson, *Representations and Cohomology I: Basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, 1991, reprinted in paperback, 1998.
- [3] D. J. Benson, *Representations and Cohomology II: Cohomology of groups and modules*, Cambridge Studies in Advanced Mathematics, vol. 31, Cambridge University Press, 1991, reprinted in paperback, 1998.
- [4] ———, *Cohomology of modules in the principal block of a finite group*, New York Journal of Mathematics **1** (1995), 196–205.
- [5] D. J. Benson, J. F. Carlson, and J. Rickard, *Complexity and varieties for infinitely generated modules, II*, Math. Proc. Camb. Phil. Soc. **120** (1996), 597–615.
- [6] ———, *Thick subcategories of the stable module category*, Fundamenta Mathematicae **153** (1997), 59–80.

- [7] D. J. Benson, J. F. Carlson, and G. R. Robinson, *On the vanishing of group cohomology*, J. Algebra **131** (1990), 40–73.
- [8] D. J. Benson, S. B. Iyengar, and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 575–621.
- [9] ———, *Stratifying modular representations of finite groups*, Preprint, 2008.
- [10] ———, *Stratifying triangulated categories*, Preprint, 2009.
- [11] ———, *Cosupport and colocalising subcategories for finite groups*, J. Reine & Angew. Math., To appear.
- [12] ———, *Representations of finite groups: Local cohomology and support*, Oberwolfach Seminars, In preparation.
- [13] D. J. Benson and H. Krause, *Complexes of injective kG -modules*, Algebra & Number Theory **2** (2008), 1–30.
- [14] D. J. Benson, H. Krause, and S. Schwede, *Realizability of modules over Tate cohomology*, Trans. Amer. Math. Soc. **356** (2004), 3621–3668.
- [15] ———, *Introduction to realizability of modules over Tate cohomology*, Fields Inst. Comm., vol. 45, American Math. Society, 2005, pp. 81–97.
- [16] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand, *Algebraic vector bundles on \mathbb{P}^n and problems of linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 66–67.
- [17] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.
- [18] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, preprint, Univ. Hannover 1986; <http://hdl.handle.net/1807/16682>.
- [19] L. Chouinard, *Projectivity and relative projectivity over group rings*, J. Pure & Applied Algebra **7** (1976), 278–302.
- [20] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Note Series, vol. 119, Cambridge University Press, 1988.
- [21] M. J. Hopkins, *Global methods in homotopy theory*, Homotopy Theory, Durham 1985, Lecture Notes in Mathematics, vol. 117, Cambridge University Press, 1987.
- [22] M. Hovey, J. H. Palmieri, and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. AMS, vol. 128, American Math. Society, 1997.
- [23] B. Keller, *Deriving DG categories*, Ann. Scient. Éc. Norm. Sup. (4) **27** (1994), 63–102.
- [24] H. Krause, *The stable derived category of a noetherian scheme*, Compositio Math. **141** (2005), 1128–1162.
- [25] A. Neeman, *The chromatic tower for $D(R)$* , Topology **31** (1992), 519–532.
- [26] ———, *Colocalizing subcategories of $D(R)$* , J. Reine & Angew. Math., To appear.
- [27] D. G. Quillen, *The spectrum of an equivariant cohomology ring, I*, Ann. of Math. **94** (1971), 549–572.
- [28] ———, *The spectrum of an equivariant cohomology ring, II*, Ann. of Math. **94** (1971), 573–602.
- [29] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351–414.

- [30] ———, *Some variations on the telescope conjecture*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, American Math. Society, 1995, pp. 391–403.
- [31] J. Rickard, *Derived categories and stable equivalence*, J. Pure & Applied Algebra **61** (1989), 303–317.
- [32] Ø. Solberg, *Support varieties for modules and complexes*, Trends in representation theory of algebras and related topics, Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 239–270.

David J. Benson, Institute of Mathematics, Fraser Noble Building, University of Aberdeen, Aberdeen AB24 3UE, UK
E-mail: benson dj@maths.abdn.ac.uk

Srikanth B. Iyengar, Department of Mathematics, University of Nebraska, Lincoln NE 68588-0130, USA
E-mail: iyengar@math.unl.edu

Henning Krause, Universität Bielefeld, Fakultät für Mathematik, Postfach 100 131, D-33 501 Bielefeld, Germany
E-mail: hkrause@math.uni-bielefeld.de