Captivity of mean-field systems*

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Abstract

We investigate the exit time of some mean-field system which is related to McKean-Vlasov diffusions under convex interaction and double-well exterior force. In one hand, we show that the meta-potential which intervenes in the system admits a number of wells which tends to infinity when the number of particles tends to infinity. In the other hand, by using the convergence of the self-stabilizing processes, the phase transitions of the granular media equation, we show that there exist some traps such that the diffusion in $\mathbb{R}^N$ can not escape when N tends to infinity. This means that in high dimension and with fixed noise, the meta-potential is not the function which drives the random dynamical system.

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Introduction

We are interested in the exit times - when N tends towards infinity - of a mean-field system of the following form:

$$\begin{cases} dX_t^1 = \sqrt{\epsilon}dB_t^1 - V'\left(X_t^1\right)dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^1 - X_t^j\right)dt \\ \vdots \\ dX_t^i = \sqrt{\epsilon}dB_t^i - V'\left(X_t^i\right)dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^i - X_t^j\right)dt \\ \vdots \\ dX_t^N = \sqrt{\epsilon}dB_t^N - V'\left(X_t^N\right)dt - \frac{1}{N}\sum_{j=1}^N F'\left(X_t^N - X_t^j\right)dt \end{cases} \tag{I}$$

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where the N Brownian Motions $\left(B_t^i\right)_{t\in\mathbb{R}_+}$ are independent. We write X_t^i instead of $X_t^{i,N,\epsilon}$ for simplifying the reading. In this paper, we will make some smoothness assumptions on the interacting potential F.

There is a huge literature about mean-field systems. Let us note some applications of this kind of system: [CDPS10] about social interactions or [CX10] about the stochastic partial differential equations. Diffusion (I) is continuous but discret space have also been studied, particularly the Currie-Weiss model, see [BBI09] or [MP98] for example.

By introducing $\mathcal{X}_t := \left(X_t^1,\cdots,X_t^N\right)$ and $\mathcal{B}_t := \left(B_t^1,\cdots,B_t^N\right)$, (I) corresponds to a classical diffusion in $\mathbb{R}^N$:

$$d\mathcal{X}_t = \sqrt{\epsilon}d\mathcal{B}_t - N\,\nabla\Upsilon^N\left(\mathcal{X}_t\right)dt \tag{I}$$

where Υ^N is called the meta-potential and is defined by:

$$\Upsilon^N\left(\mathcal{X}\right) := \frac{1}{N}\sum_{i=1}^N V\left(X_i\right) + \frac{1}{2N^2}\sum_{1\leq i,j\leq N} F\left(X_i - X_j\right) \tag{II}$$

for all $\mathcal{X} = \left(X_1,\cdots,X_N\right)\in\mathbb{R}^N$. The motion of the process $\left(\mathcal{X}_t\right)_{t\in\mathbb{R}_+}$ is subject to three concurrent forces. The first one is the gradient of the so-called confining potential V which acts independently on each coordinate. The second term represents the average tension of the interacting potential F between the coordinates. The third influence is some heat process $(\sqrt{\epsilon}\,\mathcal{B}_t)_{t\in\mathbb{R}_+}$ which allows the particle to escape from the stable domains of the meta-potential Υ^N. Let us stress that ϵ is not necessary asymptotically small. The first two forces generate the meta-potential Υ^N. We divided by N in (II) for the definition of Υ^N in order to stress the influence of N in the line level. Indeed, Lemma 5.3 in [Tug10a] tells us that for all probability law μ on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure, we have:

$$\Upsilon^N\left(X^1,\cdots,X^N\right) \longrightarrow \int_{\mathbb{R}}\left\{V(x) + \frac{1}{2}F*\mu(x)\right\}\mu(x)dx \tag{III}$$

where $\left(X^i\right)_{i\in\mathbb{N}^*}$ is an iid sequence of random values with law μ. Here, $*$ denotes the convolution.

When N tends to infinity, each particle can be seen as a diffusion in $\mathbb{R}^N$ which satisfies the following non-linear stochastic differential equation:

$$\left\{\begin{array}{l} X_t = X_0 + \sqrt{\epsilon}B_t - \int_0^t V'\left(X_s\right)ds - \int_0^t F'*u_s\left(X_s\right)ds \\ u_s = \mathcal{L}\left(X_s\right)\end{array}\right. . \tag{IV}$$

The own law of this process intervenes in the equation. Consequently, it is non-markovian. We call it a self-stabilizing process. Let us give briefly some of the previous works on these diffusions (IV). For the existence problem, see [McK67, BRTV98, CGM08, Mél96, HIP08, Tug10a]. Also, in [McK67], the author proved that the law of the solution du_t admits a $\mathcal{C}^\infty$-continuous density u_t with respect to the Lebesgue measure since $t > 0$ and the density satisfies the following non-linear parabolic partial differential equation:

$$\frac{\partial}{\partial t}u_t(x) = \frac{\partial}{\partial x}\left\{\frac{\epsilon}{2}\frac{\partial}{\partial x}u_t(x) + u_t(x)\Big(V'(x) + F' * u_t(x)\Big)\right\}. \tag{V}$$

This permits to study the stationary measure(s) of the process (IV). Particularly, in [HT10a], it has been proved that there is non-uniqueness of the stationary measures for ϵ small enough. The exact number of stationary measures for ϵ non necessary small has been the subject of [HT10b, HT09, Tug11]. By knowing the exact number of stationary measures, we can study the convergence in long-time behavior. In the convex case, see [BRV98, Mal03, Ver06, BCCP98, CMV03, CGM08].

The long-time behavior of the law u_t in the more difficult non-convex case has been proved in [Tug10b]: u_t converges weakly towards a stationary measure under assumptions easy to verify. For doing this, the main tool is the following free-energy:

$$\Upsilon_\epsilon(u) := \int_{\mathbb{R}}\left\{\frac{\epsilon}{2}\log(u(x)) + V(x) + \frac{1}{2}F * u(x)\right\}u(x)dx \tag{VI}$$

for all the measures du which are absolutely continuous with respect to the Lebesgue measure. We can note that du_t - defined in (IV) - satisfies this hypothesis since $t > 0$, see [HT10a]. We can observe that the second term of Υ_ϵ is linked to the meta-potential Υ^N by the limit (III).

The link between the self-stabilizing processes and the mean-field system for N tending to $+\infty$ is called the propagation of chaos, see [Szn91] under Lipschitz properties ; [BRTV98] if V is a constant ; [Mal01, Mal03] when both potentials are convex ; [BAZ99] for a more precise result ; [BGV07], [DPdH96] or [DG87] for a sharp estimate ; [CGM08] for a uniform result in time in the non-uniformly convex case. Also, a half-uniform propagation of chaos has been proved in [Tug11] when we restrict to a symmetric initial measure and when V is a double-well potential.

Let us note also some works about the propagation of chaos with different hypotheses about the dynamic or the phase space: [Gra90, Gra92, Der03, JM08].

Exit-time of diffusions have already been studied when the dimension is fixed and when the coefficient diffusion $\sqrt{\epsilon}$ tends to 0. Indeed, for classical diffusions, Freidlin and Wentzell (see for example [DZ10, FW98]) proved a Kramer's type law theorem. In the case of the mean-field system (I), the exit time of an open set $\mathcal{O} \subset \mathbb{R}^N$ which contains at least one well of the meta-potential Υ^N is exponentially equivalent to $\exp\left[\frac{2N}{\epsilon}H\right]$ with $H := \sup_{x,y\in\mathcal{O}}\left(\Upsilon^N(x) - \Upsilon^N(y)\right)$. However, in this paper, we do not want ϵ to be small but N to be big and we can not apply this method even in the small-noise asymptotic since a commutation between the two limits is not possible.

The global method that we will use for studying the exit-time consist on using the propagation of chaos, the convergence established in [Tug10b] and the result about the phase transition proved in [Tug11]. Indeed, let us recall that it has been proved in [DG87] that the empirical law of the mean-field system satisfies a large deviations principle with a rate function which depends on the law of Diffusion (IV). Consequently, the long-time behavior of $\mathcal{L}(X_t)$ provides some consequences on the exit time for the particle system (I).

The paper is organized as follows. First, the assumptions and the notations are presented in first section with some of the results about self-stabilizing diffusions and two simple lemmas which will be used subsequently. The second section deals with the potential geometry, particularly the number of wells. Propagation of chaos is proved in third section then used for obtaining the main results, in particular the fact that even when there are $2^N(1-o(1))$ wells, there are at most $o(2^N)$ steady states. For concluding the introduction, we write the statements of the three main results. Set $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\alpha}{2}x^2$ with $\alpha > 0$. We have:

Big number of wells *If* $\alpha < \frac{2}{3}$. *Then* Υ^N *admits* $2^N(1-o(1))$ *wells.*

Small number of steady states *Set* $\mathcal{X} \in \mathbb{R}^N$ *and* $\mathcal{X}_t$ *the diffusion* (I) *starting with* $\mathcal{X}$. *Set* $\kappa > 0$. *There exists* $\epsilon_0 > 0$ *such that for all* $\epsilon \in]0;\epsilon_0[$, *there exists* $T_\kappa \geq 0$ *such that for all* $t > 0$, *we have:*

$$\lim_{N\to+\infty} \mathbb{P}\left[\min\left\{p(s); 1-p(s); \left|\frac{1}{2} - p(s)\right|\right\} \leq \kappa\,,\ \forall T \leq s \leq T+t\right] = 1$$

where $p(s) := \frac{1}{N}\left\{i \in [\![1;N]\!] \ \mid\ X_s^i > 0\right\}$.

Stability of the positive half-space *Set* ϵ *small enough and a law* μ_0 *such that* $\int_{\mathbb{R}} V(x)\mu_(x)dx + \frac{\alpha}{2}\mathrm{Var}(\mu_0) < -\frac{\max\{1-\alpha\,;0\}^2}{4}$ *and* $\mathbb{E}(\mu_0) > 0$. *We consider a sequence of iid random values with law* μ_0: $\left(X_0^i\right)_{i\in\mathbb{N}^*}$ *and* $\mathcal{X}_t$ *the diffusion* (I) *starting with* $\mathcal{X} := \left(X_0^1, \cdots, X_0^N\right)$. *Then, for all* $t \geq 0$, *we have:*

$$\lim_{N\to+\infty} \mathbb{P}\left\{\sum_{i=1}^N X_t^i > 0 \ \mid\ \forall\, 0 \leq s \leq t\right\} = 1\,.$$

1 Preliminaries

In this paper, we take $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \alpha\frac{x^2}{2}$ with $\alpha \in \mathbb{R}_+^*$. Let us recall that the value $\sup_{z\in\mathbb{R}} -V''(z)$ plays an important role in [HT10a, HT10b, HT09]. Here, we know directly that $\sup_{x\in\mathbb{R}} -V''(x) = 1$. We could consider more general assumptions, the same than the ones in [Tug10b] and [Tug11]. But, all the mathematical difficulties of the present problem are present in this case. Considering these two functions instead of general polynomial ones will permit to avoid some technical and tedious computations, especially in the next section. Indeed, the main results (see Section 4) only need a result of propagation of chaos (see Section 3) and a result of convergence in long-time

(see [Tug10b]).
Furthermore, by adapting Theorem 2.1 in [Tug11] to our problem, we have the following result:

Proposition 1.1. *For all $\alpha > 0$, there exists $\epsilon_c(\alpha) \in \mathbb{R}_+$ such that:*

- *For all $\epsilon \geq \epsilon_c(\alpha)$, Diffusion* (IV) *admits a unique stationary measure: u_ϵ^0.*
- *For all $\epsilon < \epsilon_c(\alpha)$, Diffusion* (IV) *admits exactly three stationary measures: u_ϵ^0, u_ϵ^+ and u_ϵ^- with $\pm \int_{\mathbb{R}} x u_\epsilon^{\pm}(x)dx > 0$.*

Moreover, the critical value $\epsilon_c(\alpha)$ is the unique solution of the equation:

$$\int_{\mathbb{R}_+} \left(x^2 - \frac{1}{2\alpha}\right) \exp\left[(1-\alpha)\, x^2 - \frac{\epsilon}{2} x^4\right] dx = 0\,. \tag{1.1}$$

We have then a simple representation of the threshold between the uniqueness and the thirdness of the stationary measure(s) of Diffusion (IV):

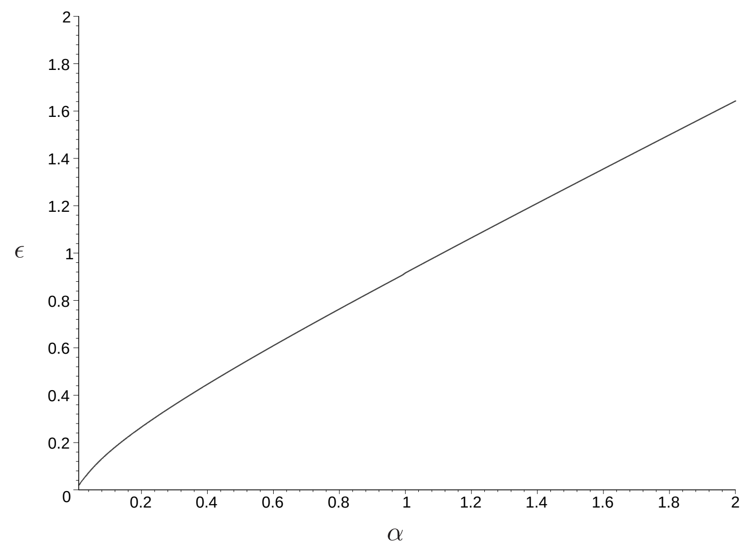

Figure 1: Critical value $\epsilon_c\,(\alpha)$

All the measure u_0 which will be considered in the following satisfy the three following hypotheses:

(AC) u_0 is absolutely continuous with respect to the Lebesgue measure. We call u_0 the density for simplicity.

(FM) $\int_{\mathbb{R}} x^{32} u_0(x)dx < \infty$.

(FE) $\int_{\mathbb{R}} \log\left(u_0(x)\right) u_0(x)dx < \infty$.

The assumptions (AC) and (FE) guarantee that $\Upsilon_\epsilon(u_0) < \infty$. The condition (FM) simply expresses the existence of a strong solution on $\mathbb{R}_+$ for Equation (IV). Indeed, by Theorem 2.12 in [HIP08], we know that there is a solution if $\int_{\mathbb{R}} x^{8q^2} u_0(x)dx < \infty$ where $q := \max\left\{\frac{\deg(V)}{2}; \frac{\deg(F)}{2}\right\}$ if V and F are polynomial functions.

The three assumptions imply the existence of $M_0 > 0$ such that

$$\max_{j\in[\![1;32]\!]} \sup_{t\in\mathbb{R}_+} \mathbb{E}\left[|X_t|^j\right] \leq M_0 \,. \tag{1.2}$$

We deduce immediately that the family $(u_t)_{t\in\mathbb{R}_+}$ is tight. Also, we have the convergence of u_t towards a stationary measure u_ϵ, see [Tug10b].

Let us recall the main results of Section 2 in [HT10b] (Theorem 2.1 and Theorem 2.4), about the small-noise asymptotic of the stationary measures:

Proposition 1.2. *The law u_ϵ^+ converges weakly towards δ_1 and the law u_ϵ^- converges weakly towards δ_{-1}.*
The law u_ϵ^0 converges weakly towards $\frac{1}{2}\delta_{x_\alpha} + \frac{1}{2}\delta_{-x_\alpha}$ with

$$x_\alpha := \begin{cases} \sqrt{1-\alpha} & \text{if } \alpha \leq 1 \\ 0 & \text{if } \alpha \geq 1 \end{cases} .$$

We present first some general notations. Let us note that for each $x \in \mathbb{R}$, $\mathcal{E}(x)$ denotes here the unique integer such that $x - 1 < \mathcal{E}(x) \leq x$ and C_n^p is the binomial coefficient $C_n^p := \frac{n!}{p!(n-p)!}$.

For all the measures μ_1 and μ_2, $\mathbb{W}_2(\mu_1, \mu_2)$ is the Wasserstein distance between μ_1 and μ_2:

$$\mathbb{W}_2(\mu_1, \mu_2) := \sqrt{\inf_{\{\mathcal{L}(X)=\mu_1\,,\,\mathcal{L}(Y)=\mu_2\}} \mathbb{E}\left\{|X-Y|^2\right\}} .$$

Now, we present some notations linked to the spaces $\mathbb{R}^N$. By convention, we consider that $\mathbb{R}^\infty$ is the set of the measures on $\mathbb{R}$. In the following, $\mathcal{X}$ is an arbitrary element of $\mathbb{R}^N$ and the i-th coordinate is written X_i, when $N < \infty$. Let us introduce some definitions:

Definition 1.3. **1)** *For all $\mathcal{X} := (X_1, \cdots, X_N) \in \mathbb{R}^N$, we consider the norm $||\mathcal{X}||_N := \sqrt{\frac{\sum_{i=1}^N X_i^2}{N}}$.*

2) *For all $r > 0$, we introduce $\mathbb{B}_r^N := \left\{\mathcal{X} \in \mathbb{R}^N \mid ||\mathcal{X}||_N \leq r\right\}$.*

3) *We consider also $\mathbb{E}_+^N := \left\{\mathcal{X} \in \mathbb{R}^N \mid \sum_{i=1}^N X_i > 0\right\}$.*

4) *For each $\mathcal{X}_0 \in \mathbb{R}^N$, we call $\mathcal{X}_t$ the mean-field system* (I) *starting by $\mathcal{X}_0$ and X_t^i the i-th coordinate of $\mathcal{X}_t$.*

We introduce now the notion of signature.

Definition 1.4. *Set $N \in \mathbb{N}$ and $\mathcal{X} \in \mathbb{R}^N$. We say that $\mathcal{X}$ has the signature $(p, 1-p)$ with $p \in \frac{1}{N}[\![1; N]\!]$ if $\#\{i \in [\![1; N]\!] \mid X_i > 0\} = pN$.*

Set a sequence $\mathcal{X} \in \mathbb{R}^{\mathbb{N}}$. We say that $\mathcal{X}$ has the signature $(p, 1-p)$ with $p \in [0; 1]$ if $\lim_{N\to+\infty} \frac{\#\{i\in[\![1;N]\!] \mid X_i>0\}}{N} = p$. Obviously, if $\mu \in \mathbb{R}^\infty$ is a measure, we say that it has the signature $(p, 1-p)$ if $\mu(\{0\}) = 0$ and $\mu(\mathbb{R}_+) = p$.

Immediately, if $\mathcal{X}$ is a sequence of iid random values with law μ_0 absolutely continuous with respect to the Lebesgue measure, $\mathcal{X}$ has the signature $\left(\int_{\mathbb{R}_+} \mu_0(x)dx, \int_{\mathbb{R}_-} \mu_0(x)dx\right)$.

Definition 1.5. *Set $p \in [0; 1]$ and $N \in \mathbb{N} \cup \{+\infty\}$. We call $\mathbb{S}_p^N$ the set of the elements $\mathcal{X} \in \mathbb{R}^N$ which have the signature $(p, 1-p)$.*

Remark 1.6. *The particularity of the random dynamical system that we consider is its invariance for each element $\sigma \in \mathcal{S}_N$ where $\mathcal{S}_N$ is the set of all the permutations of the set $[\![1; N]\!]$. Consequently, we will not work on $\mathbb{R}^N$ but on $\mathbb{R}^N/\mathcal{S}_N$. For simplifying the reading, we will not specify that we consider class of equivalence instead of elements of $\mathbb{R}^N$. Particularly, for $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^N$, the expression $\sum_{i=1}^N (X_i - Y_i)^2$ will be in fact a notation which corresponds to $\inf_{\sigma\in\mathcal{S}_N} \sum_{i=1}^N \left(X_{\sigma(i)} - Y_i\right)^2$.*

Let us stress that this last expression corresponds to the Wasserstein distance between two measures with support containing N elements at most. Also, we can remark that $\mathbb{E}_+^N$, $\mathbb{B}_r^N$ and $\mathbb{S}_p^N$ are invariant under the actions of any permutations so we can use these three sets without paying attention on the difference between $\mathbb{R}^N$ and $\mathbb{R}^N/\mathcal{S}_N$.

Since we use the classes of equivalence of $\mathbb{R}^N$ with respect to the permutations instead of $\mathbb{R}^N$ itself, we introduce some particular classes.

Definition 1.7. **1)** *Set $x \in \mathbb{R}$, we define the vector $\overline{x} \in \mathbb{R}^N$ as a pure state with coordinates equal to x: $\overline{x} := (x, \cdots, x)$.*

2) *Set $a, b \in \mathbb{R}$ and $p \in]0; 1[$. The class of vectors (a, b, p) denotes all the bi-state vectors with $\mathcal{E}(pN)$ coordinates equal to a and $N - \mathcal{E}(pN) \neq \mathcal{E}((1-p)N)$ coordinates equal to b. Let us note that this class contains exactly $C_N^{\mathcal{E}(pN)}$ elements of $\mathbb{R}^N$.*

In order to conclude the preliminaries, we put two lemmas which will be used in the next section. The first one is about linear algebra:

Lemma 1.8. *Let $a, b \in \mathbb{R}$, $c \in \mathbb{R}_-$, $N \geq 1$ and $k \in [\![1; N-1]\!]$. Set I_k the identity matrix with size k and J_k the matrix whose each coordinate is equal to 1 and with size k. We define in the same way I_{N-k} and J_{N-k}. Finally, we define the following matrix per blocks:*

$$M := \begin{pmatrix} (a-c)I_k + cJ_k & (c) \\ (c) & (b-c)I_{N-k} + cJ_{N-k} \end{pmatrix}.$$

If $a + (N-1)c > 0$ and $b + (N-1)c > 0$ then $M > 0$.

Proof. Let us assume $a+(N-1)c>0$ and $b+(N-1)c>0$. By definition of M, for all $\mathcal{X}\in\mathbb{R}^N$ with $\mathcal{X}\neq\overline{0}$, we have:

$$\begin{aligned}\langle\mathcal{X};M\mathcal{X}\rangle &= \sum_{i=1}^{k}(a+(N-1)c)X_i^2+\sum_{i=k+1}^{N}(b+(N-1)c)X_i^2\\ &\quad -N^2c\left\{\frac{1}{N}\sum_{i=1}^{N}X_i^2-\left(\frac{1}{N}\sum_{i=1}^{N}X_i\right)^2\right\}.\end{aligned}$$

It achieves the proof since $c\leq 0$. □

The second result is combinatorial and is just the direct application of Sterling formula:

Lemma 1.9. *Set $\delta\in\left]0;\frac{1}{2}\right[$. Then* $\lim_{N\to+\infty}2^{-N}\sum_{n=\left(\frac{1}{2}-\delta\right)N}^{n=\left(\frac{1}{2}+\delta\right)N}C_N^n=1.$

2 Potential geometry

This section is devoted to the geometry of the meta-potential Υ^N. We note that it is immediate that $\overline{-1}$, $\overline{1}$ and $\overline{0}$ are critical points of Υ^N. First, we will study the geometry when $\alpha\geq 1$ that is to say in the synchronized case.

Proposition 2.1. *Set $N\geq 2$ and $\alpha\geq 1$. Then, the meta-potential Υ^N admits exactly three critical points: $\overline{-1}$, $\overline{1}$ and $\overline{0}$. The first two ones are wells and if $\alpha>1$, the point $\overline{0}$ is a saddle whose the signature of the Hessian is $(N-1,1)$.*

Proof. **Step 1.** Let us prove that there are exactly three critical points if $\alpha\geq 1$. For all $1\leq i\leq N$, the derivative of the meta-potential with respect to x_i is

$$\frac{\partial}{\partial x_i}\Upsilon^N(x_1,\cdots,x_N)=\frac{1}{N}\left\{V'(x_i)+\alpha x_i-\frac{\alpha}{N}\sum_{j=1}^{N}x_j\right\}.$$

Set a critical point $\mathcal{X}$. We deduce $\rho(x_i)=\rho(x_j)$ for all the indexes i and j with $\rho(x):=V'(x)+\alpha x$. By recalling $\rho'(x)=V''(x)+\alpha\geq 0$ for all $x\in\mathbb{R}$ according to the hypothesis $\alpha\geq 1$. We deduce directly $x_i=x_j$ for all the indexes i and j. Consequently, there exists $x\in\mathbb{R}$ such that $\mathcal{X}=\overline{x}$.
However, since the derivative with respect to x_1 is equal to 0, we obtain $V'(x)+\alpha x-\frac{\alpha}{N}\sum_{j=1}^{N}x=0$ which implies $V'(x)=0$. Then $x\in\{-1,0,1\}$.

Step 2. We compute the Hessian of Υ^N on the points $\overline{x}$:

$$\frac{\partial^2}{\partial x_i^2}\Upsilon^N(x,\cdots,x)=\frac{1}{N}\left\{V''(x)+\alpha\left(1-\frac{1}{N}\right)\right\}$$

$$\text{and}\quad\frac{\partial^2}{\partial x_i\partial x_j}\Upsilon^N(x,\cdots,x)=-\frac{\alpha}{N^2}.$$

We apply Lemma 1.8 and we deduce that the Hessian is strictly positive if it is in $\overline{\pm 1}$. And, a simple algebra calculous tells us that the two eigenvalues in $\overline{0}$ are $\alpha - 1 > 0$ associated to an eigenspace of dimension $N-1$ and -1 which achieves the proof. □

Remark 2.2. *We can remark that $\overline{1}$ and $\overline{-1}$ are wells of the meta-potential Υ^N even if $\alpha < 1$.*

According to Proposition 1.3 in [Tug11], the diffusion (IV) admits an outlying stationary measure around 0 for ϵ small enough if we have

$$V(x) + \frac{\alpha}{2}x^2 > 0 \quad \text{for all} \quad x \neq 0\,. \tag{2.1}$$

If $\alpha > 1$, (2.1) holds which proves the existence of an outlying stationary measure near δ_0 for ϵ small enough. But, $\overline{0}$ is *never* a wells of Υ^N when $\alpha > 1$.
This points out the importance of the entropy and ϵ since there is no correspondance between the wells of Υ^N and the stationary measures of (IV).

Now we will study the critical points when $\alpha \leq 1$.

Theorem 2.3. *If $\frac{2}{3} \leq \alpha < 1$, the meta-potential Υ^N admits exactly 2^N critical points but only two wells: $\overline{1}$ and $\overline{-1}$.*
If $\alpha < \frac{2}{3}$, the meta-potential Υ^N admits exactly 2^N critical points and $2^N(1 - o(1))$ of these critical points are wells of Υ^N.

Let us remark that the bifurcation $\alpha = \frac{2}{3}$ already appeared in the proof of Theorem 3.2 in [HT10a] since $\frac{2}{3} = -\frac{V'(c)}{c}$ where c is the unique positive real such that $V''(c) = 0$.

Proof. In order to stress the importance of α, we will write Υ_α^N instead of Υ^N in this proof.

Step 1. First, we study the number of critical points. The partial derivative with respect to the coordinate x_i is:

$$\frac{\partial}{\partial x_i}\Upsilon_\alpha^N(x_1, \cdots, x_N) = \frac{1}{N}x_i^3 - \frac{1}{N}(1-\alpha)x_i - \alpha\frac{1}{N^2}\sum_{j=1}^N x_j\,.$$

Since the third derivative of the application $x \mapsto x^3 - (1-\alpha)x$ is nonnegative, we deduce that for all $\mathcal{X} \in \mathbb{R}^N$ such that $\nabla\Upsilon_\alpha^N(\mathcal{X}) = \overline{0}$, $\mathcal{X}$ has the form $\overline{x}$ or (a_1, a_2, p). Let $p \in \frac{1}{N}[\![1; N]\!]$. We have now to solve

$$a_1^3 - a_2^3 + (1-\alpha)(a_2 - a_1) = 0 \tag{2.2}$$

$$a_1^3 - (1-\alpha)\,a_1 = \alpha\,(pa_1 + (1-p)a_2)\,. \tag{2.3}$$

In the case where $a_1 = a_2 = a$, we find directly $a \in \{1, 0, -1\}$ and p does not have any importance. If $a_2 \neq a_1$, we can divide (2.2) by $a_2 - a_1$ before solving it. Then we put $p := \frac{a_2 - a_2^3}{\alpha(a_2 - a_1)}$ which shall be in $\frac{1}{N}[\![1;N]\!]$. Dividing (2.2) by $a_2 - a_1$, we obtain:

$$a_2^2 + a_1a_2 + a_1^2 = 1 - \alpha \tag{2.4}$$

which implies the existence of $t \in]-\pi;\pi]$ such that

$$a_2(t) = 2\sqrt{\frac{1-\alpha}{3}}\cos\left(t - \frac{\pi}{6}\right) \quad \text{and} \quad a_1(t) = -2\sqrt{\frac{1-\alpha}{3}}\cos\left(t + \frac{\pi}{6}\right).$$

The equations (2.4) and (2.3) imply that $a_1a_2 < 0$. Indeed, let us assume that $a_1 \geq 0$ and $a_2 \geq 0$. Since $p \geq 0$, (2.3) provides $a_1 > \sqrt{1-\alpha}$ and the same holds with a_2. We obtain then $a_1^2 + a_1a_2 + a_2^2 \geq 2(1-\alpha)$ which is impossible according to (2.4). Without loss of generality, we assume $a_1(t) < 0$ and $a_2(t) > 0$. We deduce that $t \in I := \left]-\frac{\pi}{3};\frac{\pi}{3}\right[$. After computation, we find:

$$p(t) = \frac{1}{2} + \frac{\tan(t)}{2\alpha\sqrt{3}}\left\{(3\alpha - 2) + \frac{8(1-\alpha)}{3}\sin^2(t)\right\}.$$

We remark that $p\left(-\frac{\pi}{3}\right) = 0$ and $p\left(\frac{\pi}{3}\right) = 1$. Also, a simple study proves that the function $t \mapsto p(t)$ is not in $[0;1]$ if $t \in \left]-\frac{\pi}{2};-\frac{\pi}{3}\right[\bigcup\left]\frac{\pi}{3};\frac{\pi}{2}\right[$.
Then, for all $N \in \mathbb{N}^*$, for all $k \in [\![0;N]\!]$, there exists $t_k \in I$ such that $\left(a_1(t_k)\,;\,a_2(t_k)\,;\,\frac{k}{N}\right)$ is a critical point of the meta-potential Υ_α^N. Since $\overline{1}$ and $\overline{-1}$ are also critical points of Υ^N, we deduce that the meta-potential admits exactly 2^N critical points.
Step 2. Now, we study the Hessian in these critical points. We compute the second derivatives of Υ_α^N in a point $\left(a_1(t_k)\,;\,a_2(t_k)\,;\,\frac{k}{N}\right)$:

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2}\Upsilon_\alpha^N\left(a_1(t_k)\,;\,a_2(t_k)\,;\,\frac{k}{N}\right) &= \frac{V''(a_1(k))}{N} + \frac{\alpha}{N} - \frac{\alpha}{N^2} \quad \forall i \in [\![1;k]\!],\\
\frac{\partial^2}{\partial x_i^2}\Upsilon_\alpha^N\left(a_1(t_k)\,;\,a_2(t_k)\,;\,\frac{k}{N}\right) &= \frac{V''(a_2(k))}{N} + \frac{\alpha}{N} - \frac{\alpha}{N^2} \quad \forall i \in [\![k+1;N]\!]\\
\frac{\partial^2}{\partial x_i \partial x_j}\Upsilon_\alpha^N\left(a_1(t_k)\,;\,a_2(t_k)\,;\,\frac{k}{N}\right) &= -\frac{\alpha}{N^2} \quad \forall i,j \in [\![1;N]\!] \quad i \neq j\,.
\end{aligned}$$

By applying Lemma 1.8, we deduce that $V''(a_1(t_k)) > 0$ and $V''(a_2(t_k)) > 0$ implies the strict convexity of Υ_α^N in $\left(a_1(t_k)\,;\,a_2(t_k)\,;\,\frac{k}{N}\right)$. It remains then to study the values t such that $V''(a_1(t)) > 0$ and $V''(a_2(t)) > 0$. This is equivalent to

$$\cos\left(t - \frac{\pi}{6}\right) > \frac{1}{2\sqrt{1-\alpha}} \quad \text{and} \quad \cos\left(t + \frac{\pi}{6}\right) > \frac{1}{2\sqrt{1-\alpha}}.$$

These two inequalities hold if and only if $t \in I_\alpha := \bigcap\left]\frac{\pi}{6} - t(\alpha);t(\alpha) - \frac{\pi}{6}\right[$ with $t(\alpha) := \arccos\left(\frac{1}{2\sqrt{1-\alpha}}\right)$. And, $\frac{\pi}{6} < t(\alpha)$ is equivalent to $\alpha < \frac{2}{3}$. Let us remark that $\alpha > \frac{3}{4}$ would imply $1 < \frac{1}{2\sqrt{1-\alpha}}$ and then we would not be able to define

$t(\alpha)$. The function p in this interval is decreasing when $\alpha < \frac{2}{3}$. Indeed, a computation provides:

$$p'(t) = -\frac{16(1-\alpha)\sin^4(t) - 24(1-\alpha)\sin^2(t) + 3(2-3\alpha)}{6\alpha\sqrt{3}\cos^2(t)}\,.$$

So $p'(t) \geq 0$ if and only if

$$\frac{3\sqrt{1-\alpha}-\sqrt{3}}{4\sqrt{1-\alpha}} \leq \sin^2(t) \leq \frac{3\sqrt{1-\alpha}+\sqrt{3}}{4\sqrt{1-\alpha}}\,.$$

But, for $t \in \left]\frac{\pi}{6} - t(\alpha); t(\alpha) - \frac{\pi}{6}\right[$, we have

$$\sin^2(t) \leq \sin^2\left(t(\alpha) - \frac{\pi}{6}\right) = \frac{5 - 6\alpha - \sqrt{9-12\alpha}}{8(1-\alpha)}\,.$$

However, the inequality $\frac{5-6\alpha-\sqrt{9-12\alpha}}{8(1-\alpha)} \geq \frac{3\sqrt{1-\alpha}-\sqrt{3}}{4\sqrt{1-\alpha}}$ is equivalent to $\alpha \geq \frac{2}{3}$. Then, the function p is a bijection from I_α to $\left]\underline{p(\alpha)};\overline{p(\alpha)}\right[\subset [0;1]$. We can observe that $\left]\underline{p(\alpha)};\overline{p(\alpha)}\right[\ni p(0) = \frac{1}{2}$. By applying Lemma 1.9, we deduce that the number of wells is equivalent to 2^N when N tends tends to infinity. □

We recover then a result previously obtained in [BFG07]. Even if there are $2^N(1-o(1))$ wells when $\alpha < \frac{2}{3}$ and $2^N(1-o(1))$ critical points even if $\frac{2}{3} \leq \alpha < 1$, we will state in the following that the number of classes of steady states for the dynamic in the mean-field system (I) does not depend on N.

3 Exit times

We will begin to state a weak result of propagation of chaos. In other words, we will prove on a finite interval of time $[0;T]$, when N tends towards infinity that the behavior of each particle of (I) is like the one of a self-stabilizing process (IV). We recall their definition:

$$X_t^i = X_0^i + \sqrt{\epsilon}B_t^i - \int_0^t V'(X_s^i)ds - \int_0^t \frac{1}{N}\sum_{j=1}^N F'(X_s^i - X_s^j)ds \qquad \text{(I)}$$

$$\text{and}\quad \overline{X_t^i} = X_0^i + \sqrt{\epsilon}B_t^i - \int_0^t V'(\overline{X_s^i})ds - \int_0^t F' * u_s^i(\overline{X_s^i})ds\,, \qquad \text{(IV)}$$

where $B_t^1, \cdots, B_t^N$ are N independent Brownian Motions.

Proposition 3.1. *Set a probability measure μ_0 which satisfies the hypothesis (FM). Set $X_0^1, \cdots, X_0^N$ N iid random values with law μ_0. Set $T > 0$. Then, there exists $C, K > 0$ such that:*

$$\max_{1\leq i\leq N} \mathbb{E}\left\{\sup_{t\in[0;T]} \left|X_t^i - \overline{X_t^i}\right|^2\right\} \leq \frac{C}{N}\exp\left[K\,T\right]\,. \qquad (3.1)$$

Proof. We proceed like in Step 3 in the proof of Proposition 2.2 in [Tug11] but we consider V and F instead of the small modifications of Step 2. Here $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ then Step 3.3 is modified and consequently, we obtain

$$\frac{d}{dt}\zeta_i(t) \leq 2\zeta_i(t) + \frac{2C}{\sqrt{N}}\sqrt{\zeta_i(t)}$$

where $\zeta_i(t) := \mathbb{E}\left[\left|X_t^i - \overline{X_t^i}\right|^2\right]$ and C depends only of the first 32 moments of μ_0. We obtain immediately $\frac{d}{dt}\zeta_i(t) \leq (C+2)\zeta_i(t) + \frac{C}{N}$. As $\zeta_i(0) = 0$, we deduce after applying the Grönwall lemma:

$$\zeta_i(t) \leq \frac{C}{N}\exp\left[(C+2)\,t\right] .$$

Taking the supremum on the interval $[0;T]$ achieves the proof. We can put the supremum in the expectation by conditioning with the events $\{\mathcal{T}(\omega) = t\}$ where $\mathcal{T}$ is defined by the relation $\left|X_{\mathcal{T}}^i - \overline{X_{\mathcal{T}}^i}\right| = \sup_{t\in[0;T]}\left|X_t^i - \overline{X_t^i}\right|$. □

This result is not uniform with respect to the time. Indeed, the factor $\exp\left[(C+2)T\right]$ tends to infinity when T tends to infinity. We will now prove that there is no uniform with respect to the time propagation of chaos.

Definition 3.2. *There is propagation of chaos of the system* (I) *to the system* (IV) *if there exists a positive function η which vanishes when N tends towards $+\infty$ such that*

$$\sup_{t\geq 0}\mathbb{E}\left\{\left|X_t^i - \overline{X_t^i}\right|^2\right\} \leq \eta(N) \quad \textit{for all} \quad 1 \leq i \leq N\,. \tag{3.2}$$

This uniform propagation of chaos has been stated and used in the convex case in [CGM08] for obtaining a convergence of the process. However, in this non-convex case, we will see that there is no uniformity possible.

Proposition 3.3. *Let us assume a uniform in time propagation of chaos then the diffusion* (IV) *admits at most one stationary measure.*

Proof. The proof is similar to Step 2. and Step 4. in the proof of Proposition 2.2 in [Tug11] with V and F instead of V_0 and F_0. Indeed, in the two majorations (2.10) and (2.11) in [Tug11], we did not exploit the exact higher-bound but only the fact that we have the uniform propagation of chaos with respect to the time. □

Remark 3.4. *We proved in Theorem 2.1 in [Tug11] that for ϵ which satisfies $\int_{\mathbb{R}_+}\left(x^2 - \frac{1}{2\alpha}\right)\exp\left[(1-\alpha)x^2 - \frac{\epsilon}{2}x^4\right]dx > 0$, the diffusion* (IV) *admits exactly three stationary measures. Consequently, we can not prove a general result of propagation of chaos if we do not take into account the diffusion coefficient $\sqrt{\epsilon}$.*

Definition 3.5. *Set a continuous function f from $\mathbb{R}$ to $\mathbb{R}$ and $\mathcal{X} \in \mathbb{R}^N$ with $N \in \mathbb{N}^*$. We define: $f(\mathcal{X}) := \frac{1}{N}\sum_{i=1}^N f(X_i)$. If μ is a measure absolutely continuous with respect to the Lebesgue measure, we define $f(\mu) := \int_{\mathbb{R}} f(x)\mu(x)dx$.*

We begin by providing a general theorem and after, we will derive three results from it.

Theorem 3.6. *Set a law u_0 such that u_t tends towards a measure μ where u_t is the solution of (V) starting with u_0. We consider a sequence of iid random values with law u_0: $\left(X_0^i\right)_{i\geq 1}$. Set a $\mathcal{C}^\infty$-function f such that $|f(x)-f(y)| \leq C|x-y|\left(1+|x|+|y|\right)$ with $C>0$. For all $N\geq 1$, we call $\mathcal{X}_0^N := \left(X_0^1,\cdots,X_0^N\right)$. We introduce $\mathcal{X}_t^N$ the solution of the random dynamical system (I) starting with $\mathcal{X}_0^N$. Then, for all $\delta>0$, we have*

$$\lim_{N\to+\infty}\mathbb{P}\left\{\left|f\left(\mathcal{X}_s^N\right)-f(u_t)\right|<\delta\ ;\ \forall\, 0\leq s\leq t\right\}=1\,. \tag{3.3}$$

Furthermore, there exists $T_\delta\geq 0$ such that for all $t\geq 0$, we have

$$\lim_{N\to+\infty}\mathbb{P}\left\{\left|f\left(\mathcal{X}_s^N\right)-f(\mu)\right|<\delta\ ;\ \forall\, T_\delta\leq s\leq T_\delta+t\right\}=1\,. \tag{3.4}$$

Proof. Let us prove the inequality (3.3). For all $i\in[\![1;N]\!]$, set $\left(\overline{X_s^i}\right)_{s\in[0;t]}$ the diffusion (IV) starting with X_0^i. By definition, $\mathcal{L}\left(X_s^i\right)=u_s$. We have immediately the following inequality:

$$\mathbb{P}\left\{\exists\, s\in[0;t]\ \middle|\ \left|f\left(\mathcal{X}_s^N\right)-f(u_s)\right|\geq\delta\right\}=\mathbb{P}\left\{\sup_{s\in[0;t]}\left|f\left(\mathcal{X}_s^N\right)-f(u_s)\right|\geq\delta\right\}$$

$$\leq\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^N\sup_{s\in[0;t]}\left|f\left(X_s^i\right)-f\left(\overline{X_s^i}\right)\right|\geq\frac{\delta}{2}\right\}$$

$$+\mathbb{P}\left\{\sup_{s\in[0;t]}\left|\frac{1}{N}\sum_{i=1}^N f\left(\overline{X_s^i}\right)-f(u_s)\right|\geq\frac{\delta}{2}\right\}\,.$$

The second term tends towards 0 by applying the weak law of large numbers. Let us focus on the first one. We take $f(x):=x$ and we apply Cauchy-Schwarz inequality and Proposition 3.1:

$$\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^N\sup_{s\in[0;t]}\left|X_s^i-\overline{X_s^i}\right|\geq\frac{\delta}{2}\right\}\leq\frac{2}{\delta}\max_{i\in[\![1;N]\!]}\mathbb{E}\left\{\sup_{s\in[0;t]}\left|X_s^i-\overline{X_s^i}\right|\right\}$$

$$\leq\frac{2}{\delta}\sqrt{\frac{C}{N}}\exp\left[\frac{Kt}{2}\right]\,.$$

Since $\mathbb{E}\left[\left|\overline{X_s^i}\right|\right]\leq M_0$ defined in (1.2) for all $i\in[\![1;N]\!]$ and $s\geq 0$, we deduce that the same holds for X_s^i on any finite interval. Also, by using the same argument than the one at the end of the proof of Proposition 3.1, we obtain

$$\max_{i\in[\![1;N]\!]}\left\{\mathbb{E}\left[\sup_{s\in[0;t]}\left|X_s^i\right|\right]+\mathbb{E}\left[\sup_{s\in[0;t]}\left|\overline{X_s^i}\right|\right]\right\}<\infty\,.$$

Let us take now a general f. The condition on f and C, the Cauchy-Schwarz inequality and Proposition 3.1 imply

$$\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^N\sup_{[0;t]}\left|f\left(X_s^i\right)-f\left(\overline{X_s^i}\right)\right|\geq\frac{\delta}{2}\right\}\leq\frac{2}{\delta}\max_{[\![1;N]\!]}\mathbb{E}\left\{\sup_{[0;t]}\left|f\left(X_s^i\right)-f\left(\overline{X_s^i}\right)\right|\right\}$$

$$\leq\frac{2}{\delta}\sqrt{\frac{C}{N}}\exp\left[\frac{Kt}{2}\right]\sqrt{1+\max_{[\![1;N]\!]}\left\{\mathbb{E}\left[\sup_{[0;t]}\left|X_s^i\right|\right]+\mathbb{E}\left[\sup_{[0;t]}\left|\overline{X_s^i}\right|\right]\right\}}\longrightarrow 0\,.$$

In order to prove the second statement, it is sufficient to note that the tightness of the family $(u_t)_{t\in\mathbb{R}_+}$ and the convergence of u_t towards μ implies the convergence of $f(u_t)$ towards $f(\mu)$ so for all $\delta>0$, there exists $T_\delta\geq 0$ such that $|f(u_t)-f(\mu)|\leq\frac{\delta}{2}$ for all $t\geq T_\delta$ then we apply the first statement with $\frac{\delta}{2}$. □

The time T_δ is deterministic and linked to the rate of convergence towards the stationary measure μ so it depends on ϵ.

With this general theorem, we will obtain three corollaries. The first one states that the mean-field system is prisonner of a ball.

Corollary 3.7. *Set $\epsilon\geq\epsilon_c(\alpha)$ and a law μ_0 which satisfies (AC), (FM) and (FE). We consider a sequence of iid random values with law μ_0: $\left(X_0^i\right)_{i\geq 1}$. For all $N\geq 1$, we call $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. We introduce $\mathcal{X}_t^N$ the solution of the random dynamical system* (I) *starting with $\mathcal{X}_0^N$. Then, for all $r>\sqrt{\mathrm{Var}\left(u_\epsilon^0\right)}$, there exists $T_r\geq 0$ such that for all $t\geq 0$, we have*

$$\lim_{N\to+\infty}\mathbb{P}\left\{\mathcal{X}_s^N\in\mathbb{B}_r^N\left(\overline{0}\right)\ ;\ \forall T_r\leq s\leq t+T_r\right\}=1\,.$$

Proof. Theorem 2.1 in [Tug10b] states that μ_t converges towards a stationary measure u_ϵ when t tends to ∞. Since $\epsilon\geq\epsilon_c(\alpha)$, we know by Theorem 2.1 in [Tug11] that (IV) admits a unique stationary measure and it is u_ϵ^0 so μ_t converges towards u_ϵ^0. We conclude by applying Theorem 3.6 (more precisely we use the limit (3.4)) with $\delta:=r^2-\mathrm{Var}\left(u_\epsilon^0\right)$ and $f(x):=x^2$. □

We can not find smaller radius since $f(u_\epsilon^0)=\mathrm{Var}\left(u_\epsilon^0\right)$.

Remark 3.8. *We can relax the assumption $\epsilon\geq\epsilon_c(\alpha)$ if μ_0 is symmetric. Moreover, Theorem 3.1 in [HT09] implies that* $\mathrm{Var}\left(u_\epsilon^0\right)=\frac{1}{4(\alpha-1)}\epsilon+o(\epsilon)$ *if $\alpha>1$ and Theorem 4.5 in [HT09] implies that* $\mathrm{Var}\left(u_\epsilon^0\right)=1-\alpha-\frac{\epsilon}{2(1-\alpha)}+o(\epsilon)$ *if $\alpha<1$. A simple computation provides also* $\mathrm{Var}\left(u_\epsilon^0\right)=\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{\epsilon}$ *if $\alpha=1$.*

The second corollary provides sufficient condition for forbiding to cross the hyperplan $\left\{\mathcal{X}\in\mathbb{R}^N \;\mid\; \sum_{i=1}^N X_i=0\right\}$.

Corollary 3.9. *Set $\epsilon<\epsilon_c(\alpha)$ small enough and a law μ_0 which satisfies (AC), (FM) and (FE). Let us assume that $\Upsilon_\epsilon(\mu_0)<\inf_{\{u\mid\int_{\mathbb{R}}xu(x)dx=0\}}\Upsilon_\epsilon(u)$ and $\mathbb{E}(\mu_0)>0$. We consider a sequence of iid random values with law μ_0: $\left(X_0^i\right)_{i\geq1}$. For all $N\geq1$, we call $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. We introduce $\mathcal{X}_t^N$ the solution of the random dynamical system* (I) *starting with $\mathcal{X}_0^N$. Then for all $t\geq0$, we have:*

$$\lim_{N\to+\infty}\mathbb{P}\left\{\mathcal{X}_s^N\in\mathbb{E}_+^N\;;\;\forall\,0\leq s\leq t\right\}=1\,.$$

Proof. We recall that the free-energy is nonincreasing. Consequently, $\Upsilon_\epsilon\left(u_s\right)<\inf_{\{u\mid\int_{\mathbb{R}}xu(x)dx=0\}}\Upsilon_\epsilon(u)$ for all $s\in[0;t]$. This implies $\int_{\mathbb{R}}xu_s(x)dx>0$ for all $s\in[0;t]$. We conclude by applying Theorem 3.6, more precisely the limit (3.3) with $f(x):=x$ and $\delta:=\inf_{s\in[0;t]}\int_{\mathbb{R}}xu_s(x)dx>0$. □

Finally, the last corollary stresses the fact that the steady states do not correspond to the wells of Υ^N.

Corollary 3.10. *Set $p\in]0;1[$ with $p\neq\frac12$ and a law μ_0 which satisfies (AC), (FM) and (FE). We assume that μ_0 has the signature $(p,1-p)$. We consider a sequence of iid random values with law μ_0: $\left(X_0^i\right)_{i\geq1}$. For all $N\geq1$, we call $\mathcal{X}_0^N:=\left(X_0^1,\cdots,X_0^N\right)$. We introduce $\mathcal{X}_t^N$ the solution of the random dynamical system* (I) *starting with $\mathcal{X}_0^N$. Set $\kappa>0$. There exists $\epsilon_0>0$ such that for all $\epsilon\in]0;\epsilon_0[$, there exists $T_\kappa\geq0$ such that for all $t>0$, we have:*

$$\lim_{N\to+\infty}\mathbb{P}\left[\mathcal{X}_s\in\bigcup_{\rho=0}^{\kappa}\left(\mathbb{S}_\rho^N\cup\mathbb{S}_{1-\rho}^N\cup\mathbb{S}_{\frac12+\rho}^N\cup\mathbb{S}_{\frac12-\rho}^N\right)\;,\;\forall\,T_\kappa\leq s\leq T_\kappa+t\right]=1\,.$$

Proof. The law u_t converges towards u_ϵ^0, u_ϵ^+ or u_ϵ^- according to Theorem 2.1 in [Tug10b]. Since u_ϵ^0 is symmetric, $\int_{\mathbb{R}}\mathbb{1}_{]0;+\infty[}(x)u_\epsilon^0(x)dx=\frac12$. Theorem 2.4 in [HT10b] proves that $u_\epsilon^\pm$ converges towards $\delta_{\pm1}$ when ϵ tends to 0. So, for ϵ small enough, we have $\int_{\mathbb{R}}\mathbb{1}_{]0;+\infty[}(x)u_\epsilon^+(x)dx\geq1-\frac{\kappa}{3}$ and $\int_{\mathbb{R}}\mathbb{1}_{]0;+\infty[}(x)u_\epsilon^-(x)dx\leq\frac{\kappa}{3}$. We apply Theorem 3.6, more precisely (3.4) with $\delta:=\frac{\kappa}{3}$ and with

$$f(x):=\mathbb{1}_{]\frac{\eta}{3};+\infty[}(x)+\mathbb{1}_{[0;\frac{\eta}{3}]}(x)Z^{-1}\int_0^x\exp\left[-\frac{1}{y^2}-\frac{1}{\left(y-\frac{\kappa}{3}\right)^2}\right]dy$$

where $Z:=\int_0^{\eta/3}\exp\left[-\frac{1}{y^2}-\frac{1}{(y-\kappa/3)^2}\right]dy$ is such that $f\left(\frac{\eta}{3}\right)=1$. We take η sufficiently small for having $\left|f(u_\epsilon^0)-\frac12\right|\leq\frac{\kappa}{3}$, $\left|f(u_\epsilon^\pm)-(\pm1)\right|\leq\frac{\kappa}{3}$. The proof is achieved by applying Theorem 3.6 with f and $\delta:=\frac{\kappa}{3}$. □

This theorem proves that the wells with signature $(p,1-p)$ are not stable and even if it is possible to have $2^N(1-o(1))$ wells, these points do not intervene in the dynamic that achieves to prove that the meta-potential is not sufficient for understanding the behavior of the mean-field system (I).

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