FRACTIONAL CALCULUS FOR POWER FUNCTIONS

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ABSTRACT. We calculate the fractional Laplacian for functions of the form $u(x) = (1-|x|^2)^p_+$ and $v(x) = x_d u(x)$. As an application, we estimate the first eigenvalues of the fractional Laplacian in a ball.

1. MAIN RESULT AND DISCUSSION

We consider the fractional Laplacian $\Delta^{\alpha/2}$ defined by

$$(1) \qquad \Delta^{\alpha/2} u(x) = \mathcal{A}_{d,-\alpha} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \cap \{|y-x|>\varepsilon\}} \frac{u(y) - u(x)}{|x-y|^{d+\alpha}} \, dy.$$

Here $\mathcal{A}_{d,-\alpha} = \frac{2^\alpha \Gamma(\frac{\alpha+d}{2})}{\pi^{d/2}|\Gamma(-\frac{\alpha}{2})|}$. This is an important operator for probability and analysis [8, 24]. The main results of this paper are the following formulae for the fractional Laplacian of power functions.

Theorem 1. *Let $d \geq 1$, $0 < \alpha < 2$ and $p > -1$. Define*

$$(2) \qquad u_p^{(d)}(x) = (1 - |x|^2)^p_+, \quad x \in \mathbb{R}^d,$$

$$(3) \qquad v_p^{(d)}(x) = (1 - |x|^2)^p_+ x_d, \quad x \in \mathbb{R}^d,$$

$$(4) \qquad \Phi_{p,\alpha}^{(d)}(x) = \frac{\mathcal{A}_{d,-\alpha} B(-\frac{\alpha}{2}, p+1) \pi^{d/2}}{\Gamma(\frac{d}{2})} \; {}_2F_1\left(\frac{\alpha+d}{2}, -p+\frac{\alpha}{2}; \frac{d}{2}; x\right).$$

If $x \in \mathbb{R}^d$ and $|x| < 1$, then

$$(5) \qquad \Delta^{\alpha/2} u_p^{(d)}(x) = \Phi_{p,\alpha}^{(d)}(|x|^2),$$

$$(6) \qquad \Delta^{\alpha/2} v_p^{(d)}(x) = x_d \Phi_{p,\alpha}^{(d+2)}(|x|^2).$$

By B and Γ we denote Euler's beta and gamma functions, respectively, and ${}_2F_1$ is Gauss' hypergeometric function [14]. Some special cases of the above theorem were known before. A calculation of $\Delta^{\alpha/2} u_{\frac{\alpha}{2}}^{(d)}$ was done by Getoor [17], and explicit expression for $\Delta^{\alpha/2} u_{\frac{\alpha}{2}-1}^{(d)}$ and $\Delta^{\alpha/2} v_{\frac{\alpha}{2}-1}^{(d)}$ may be deduced from [19], see also [6]. A formula for $\Delta^{\alpha/2} u_p^{(d)}$ with $p \in (0,1)$ and $\alpha \in (-1,1)$ is announced in [5]. Finally, (5) was given in [12] for $p = \frac{\alpha-1}{2}$ and $d = 1$, where it was used to obtain a fractional Hardy inequality with a remainder term. For similar results on fractional derivatives we refer the reader to [25].

We note that if $p = n + \alpha/2$ and n is a nonnegative integer, then $\Delta^{\alpha/2} u_p^{(d)}$ and $\Delta^{\alpha/2} v_p^{(d)}$ are polynomials of degree n and $n+1$, respectively. For $d = 1$, every polynomial can be expressed as a linear combination of polynomials of the form $(1-x^2)^j$

and $x(1-x^2)^k$. Hence, if P is *any* polynomial of degree n, then $\Delta^{\alpha/2}(P(x)(1-x^2)^{\alpha/2})$, for $x \in (-1,1)$, is a polynomial of degree n, too. This also trivially holds for $\alpha = 2$. This may allow one for the development of an explicit integro–differential calculus for $\Delta^{\alpha/2}(\cdot 1_{|x|<1})$, which is one of the motivations for this work. It is noteworthy that $u_p^{(d)}(x) = {}_2F_1\left(\frac{d}{2}, -p; \frac{d}{2}; |x|^2\right)$ for $|x| < 1$, therefore (5) may be rewritten in the following elegant form

$$(7) \qquad \Delta^{\alpha/2}\left({}_2F_1\left(\frac{d}{2}, -p; \frac{d}{2}; |x|^2\right)1_{|x|<1}\right) = C_{d,\alpha,p}\ {}_2F_1\left(\frac{d}{2} + \frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{d}{2}; |x|^2\right),$$

where $x \in \mathbb{R}^d$, $|x| < 1$, $C_{d,\alpha,p} = \mathcal{A}_{d,-\alpha}B(-\frac{\alpha}{2}, p+1)\pi^{d/2}/\Gamma(\frac{d}{2})$, $d = 1, 2, \dots$ and $\alpha \in (0,2)$.

Let $B = B_1^{(d)} = \{x \in \mathbb{R}^d : |x| < 1\}$ be the unit ball in $\mathbb{R}^d$. We consider the quadratic form

$$\mathcal{E}(u) = \frac{\mathcal{A}_{d,-\alpha}}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dx\, dy,$$

with the domain

$$D(\mathcal{E}) = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u) < \infty,\ u = 0 \text{ on } B^c\},$$

and the corresponding symmetric bilinear form $\mathcal{E}(\cdot, \cdot)$. Then $(\mathcal{E}(\cdot, \cdot), D(\mathcal{E}))$ is the Dirichlet form of the α-stable, rotation invariant Lévy process killed upon leaving B. To point out applications of Theorem 1, we consider the spectral problem of finding $\phi \in D(\mathcal{E})$ such that

$$(8) \qquad \mathcal{E}(\phi, g) = \lambda \int \phi(x)g(x)\, dx, \quad g \in D(\mathcal{E}).$$

It is known that there exists an orthonormal basis of $L^2(B) \subset L^2(\mathbb{R}^d)$, consisting of eigenfunctions ϕ_1, ϕ_2, ϕ_3, ... with the corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_2 \le \dots$. The latter means that $\phi = \phi_n$ and $\lambda = \lambda_n$ satisfy (8). We note that these eigenfunctions are in the domain of the generator, and we have that $\Delta^{\alpha/2}\phi_n = -\lambda_n\phi_n$ on B in the sense of definition (1), see also [7].

The eigenproblem is the main motivation for this research. We should note that the eigenvalues λ_1, λ_2, ... are not known explicitly even in the case of $d = 1$ and $B = (-1,1)$. A number of methods to study this one-dimensional case, and more general cases, were developed by several authors [1, 2, 3, 4, 11, 20, 21, 22, 23, 26]. The symmetry of eigenfunctions plays an important role in these investigations. We also note that this spectral problem may be formulated in terms of the isotropic α-stable Lévy process in $\mathbb{R}^d$. For details we refer the reader to [13].

Let λ_* be the smallest number such that there exists an eigenfunction ϕ_* which is antisymmetric, $\phi_*(-x) = -\phi_*(x)$, and has eigenvalue λ_*. It is conjectured, but not yet proved in the full range of $\alpha \in (0,2)$ and d, that $\lambda_* = \lambda_2$. In the classical case ($\alpha = 2$), and also in the 1–dimensional case for $\alpha \ge 1$ [20], we do have $\lambda_* = \lambda_2$. It is natural to ask whether $\lambda_* = \lambda_2$. While we do not answer this question here, the calculus of power functions given by Theorem 1 may be used to investigate the eigenfunctions of the fractional Laplacian in the ball in this and related problems.

We should note that there always exists an antisymmetric eigenfunction. Indeed, there exists a non-symmetric eigenfunction ϕ, and we may let $\tilde\phi(x) = \phi(x) - \phi(-x)$, which is an antisymmetric eigenfunction with the same eigenvalue as ϕ. Similarly ϕ_* may and will be assumed antisymmetric with respect to the last coordinate axis.

Noteworthy, for $p = n + \alpha/2$ the right-hand sides of (5) and (6) are polynomials, which gives an efficient way to explicitly estimate λ_1 and λ_* using some versions of Barta inequality [26], see Sections 4 and 5. In particular, we obtain the following corollary improving some of the estimates from [21].

Corollary 2. *Let*

$$(9) \qquad \mu_{d,\alpha} = \frac{2^{\alpha}\Gamma(\frac{\alpha}{2} + 1)\Gamma(\frac{\alpha+d}{2})(\alpha + 2)(\alpha + d)(6 - \alpha)}{\Gamma(\frac{d}{2})(12d + (16 - 2d)\alpha)}.$$

We have

$$(10) \qquad \lambda_1 \geq \mu_{d,\alpha}$$

and

$$(11) \qquad \lambda_* \geq \mu_{d+2,\alpha}.$$

We observe that the lower bound for λ_* in dimension d is the same as the lower bound for λ_1 in dimension $d + 2$. We recall that in the case of the Laplacian, we have that $\lambda_2^{(d)} = \lambda_1^{(d+2)}$, see for example [18]. This motivates the following question: Is it true that $\lambda_2^{(d)} = \lambda_1^{(d+2)}$, or at least $\lambda_2^{(d)} \leq \lambda_1^{(d+2)}$?

The paper is structured as follows. In Section 2 we prove Theorem 1 for $d = 1$, then in Section 3 we prove the d-dimensional case. In Sections 4 and 5 we derive lower and upper bounds for the eigenvalues λ_1 and λ_*. In Proposition 11 below, we give Hardy inequality for $\mathcal{E}$ with weights resulting from antisymmetric functions. The result is an unexpected analogue of [12, Lemma 2.2] and [16, Proposition 2.3] with $p = 2$.

2. One–dimensional case

The following technical result is the first step in the proof of Theorem 1.

Lemma 3. *If $p > -1$, $0 < \alpha < 2$ and $x \in (-1, 1)$, then*

$$I_m(p) := p.v. \int_{-1}^{1} \frac{(1 - tx)^{\alpha - m - 2p} - 1}{|t|^{1+\alpha}} (1 - t^2)^p \, dt$$

$$(12) \qquad = B\left(-\frac{\alpha}{2}, p + 1\right)\left({}_2F_1\left(-\frac{\alpha}{2}, p + m - \frac{1}{2} - \frac{\alpha}{2}; \frac{1}{2}; x^2\right) - 1\right),$$

where $m = 1$ or $m = 2$, and p.v. means the Cauchy principal value.

Proof. We assume that $p \neq \frac{\alpha-2}{2}$, since otherwise the beta function on the right-hand side of (12) is zero and the result is obvious. We have

$$
\begin{aligned}
I_m(p) &= p.v. \int_{-1}^1 \frac{(1-tx)^{\alpha-m-2p}-1}{|t|^{1+\alpha}}(1-t^2)^p\,dt \\
&= \int_{-1}^1 \sum_{k=2}^\infty \frac{(2p+m-\alpha)_k(tx)^k}{k!|t|^{1+\alpha}}(1-t^2)^p\,dt \\
&= 2\int_0^1 \sum_{k=1}^\infty \frac{(2p+m-\alpha)_{2k}(tx)^{2k}}{(2k)!|t|^{1+\alpha}}(1-t^2)^p\,dt \\
&= \left(\sum_{k=0}^\infty \frac{(2p+m-\alpha)_{2k}B\left(k-\frac{\alpha}{2},p+1\right)}{(2k)!}x^{2k}\right) - B\left(-\frac{\alpha}{2},p+1\right) \\
&=: S_m - B\left(-\frac{\alpha}{2},p+1\right).
\end{aligned}
$$

Here $(a)_n$ is the Pochhammer symbol, that is, $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$ for $n = 1,2,\ldots$. We observe that

$$
(13) \qquad (2p+m-\alpha)_{2k} = 2^{2k}\left(p+\frac{m}{2}-\frac{\alpha}{2}\right)_k\left(p+\frac{m+1}{2}-\frac{\alpha}{2}\right)_k,
$$

and, by the doubling formula,

$$
(14) \qquad\qquad \Gamma(2x) = \Gamma(x)\Gamma(x+1/2)2^{2x-1}/\Gamma(1/2),
$$

applied to $2x = 2k+1$, we have

$$
(15) \qquad\qquad (2k)! = 2^{2k}\left(\tfrac{1}{2}\right)_k k!.
$$

We also have,

$$
(16) \qquad\qquad B\left(k-\frac{\alpha}{2},p+1\right) = \frac{\left(-\frac{\alpha}{2}\right)_k}{\left(p+1-\frac{\alpha}{2}\right)_k}B\left(-\frac{\alpha}{2},p+1\right).
$$

Thus, by (13), (15) and (16) we obtain

$$
S_m = B\left(-\frac{\alpha}{2},p+1\right)\sum_{k=0}^\infty \frac{\left(-\frac{\alpha}{2}\right)_k\left(p+\frac{m}{2}-\frac{\alpha}{2}\right)_k\left(p+\frac{m+1}{2}-\frac{\alpha}{2}\right)_k}{\left(p+1-\frac{\alpha}{2}\right)_k\left(\frac{1}{2}\right)_k k!}x^{2k}.
$$

For $m = 1$ or $m = 2$ the factor $\left(p+1-\frac{\alpha}{2}\right)_k$ in the denominator cancels with one of the terms in the numerator, and the result follows. $\qquad\square$

In the following lemma we prove (5) for $d = 1$. Recall that

$$
(17) \qquad\qquad u_p(x) = (1-x^2)_+^p, \qquad x \in \mathbb{R}.
$$

Lemma 4. *If $0 < \alpha < 2$, $p > -1$ and $x \in (-1,1)$ then*

$$
(18) \quad \Delta^{\alpha/2}u_p(x) = \mathcal{A}_{1,-\alpha}B\left(-\frac{\alpha}{2},p+1\right){}_2F_1\left(-\frac{\alpha}{2},p+\frac{1}{2}-\frac{\alpha}{2};\frac{1}{2};x^2\right)(1-x^2)^{p-\alpha}
$$

$$
(19) \qquad\qquad = \mathcal{A}_{1,-\alpha}B\left(-\frac{\alpha}{2},p+1\right){}_2F_1\left(\frac{\alpha+1}{2},-p+\frac{\alpha}{2};\frac{1}{2};x^2\right).
$$

Proof. We recall from [12, Lemma 2.1] the following formula

$$Lu_p(x) = \frac{(1-x^2)^{p-\alpha}}{\alpha}\left((1-x)^\alpha + (1+x)^\alpha\right.$$

(20)
$$\left. - (2p+2-\alpha)B(p+1,1-\alpha/2) + \alpha I_1(p)\right),$$

where $I_1(p)$ is given by (12).

By $(2p+2-\alpha)B(p+1,1-\alpha/2) = -\alpha B(p+1,-\alpha/2)$ and Lemma 3,

$$\alpha I_1(p) - (2p+2-\alpha)B(p+1,1-\alpha/2)$$
$$= \alpha B\left(-\frac{\alpha}{2},p+1\right){}_2F_1\left(-\frac{\alpha}{2},p+\frac{1}{2}-\frac{\alpha}{2};\frac{1}{2};x^2\right).$$

This proves (18). The formula (19) follows from [14, formula 2.9(2), page 105]. □

TABLE 1. The fractional Laplacian for some functions vanishing outside of $(-1,1)$.

$u(x)$ on $(-1,1)$	$\Delta^{\alpha/2}u(x)$ for $x \in (-1,1)$
$(1-x^2)^{-1+\alpha/2}$	0
$(1-x^2)^{\alpha/2}$	$-\Gamma(\alpha+1)$
$(1-x^2)^{1+\alpha/2}$	$-\Gamma(\alpha+1)\frac{\alpha+2}{2}(1-(1+\alpha)x^2)$
$(1-x^2)^{2+\alpha/2}$	$-\Gamma(\alpha+1)\frac{\alpha+2}{2}\frac{\alpha+4}{4}\left(1-(2\alpha+2)x^2 + (\frac{\alpha}{3}+1)(\alpha+1)x^4\right)$

In the following lemma we prove (6) for $d=1$.

Lemma 5. *Let $p > -1$ and $v_p(x) = x(1-x^2)_+^p$ for $x \in \mathbb{R}$. For $0 < \alpha < 2$ and $x \in (-1,1)$ we have*

(21) $$\Delta^{\alpha/2}v_p(x) = \mathcal{A}_{1,-\alpha}B\left(-\frac{\alpha}{2},p+1\right)(\alpha+1)$$

$$\times {}_2F_1\left(-\frac{\alpha}{2},p+\frac{3}{2}-\frac{\alpha}{2};\frac{3}{2};x^2\right)x(1-x^2)^{p-\alpha}$$

(22) $$= \mathcal{A}_{1,-\alpha}B\left(-\frac{\alpha}{2},p+1\right)(\alpha+1)\ {}_2F_1\left(\frac{\alpha+3}{2},-p+\frac{\alpha}{2};\frac{3}{2};x^2\right)x.$$

Proof. We write

$$\mathcal{A}_{1,-\alpha}^{-1}\Delta^{\alpha/2}v_p(x)$$

$$= p.v.\int_{-1}^{1}\frac{y(1-y^2)^p - x(1-x^2)^p}{|y-x|^{1+\alpha}}\,dy - v_p(x)\int_{\mathbb{R}\setminus(-1,1)}\frac{dy}{|y-x|^{1+\alpha}}$$

$$= p.v.\int_{-1}^{1}\frac{y(1-y^2)^p - x(1-x^2)^p}{|y-x|^{1+\alpha}}\,dy - \frac{v_p(x)}{\alpha}\left(\frac{1}{(x+1)^\alpha} + \frac{1}{(1-x)^\alpha}\right)$$

$$=: I - \frac{v_p(x)}{\alpha}\left(\frac{1}{(x+1)^\alpha} + \frac{1}{(1-x)^\alpha}\right).$$

To evaluate I, we change the variable to $t = \frac{x-y}{1-xy}$, see [12, the proof of Lemma 2.1] for more details. We obtain

$$I = (1 - x^2)^{p-\alpha} p.v. \int_{-1}^{1} \frac{(1 - t^2)^p (x - t) - x(1 - tx)^{2p+1}}{|t|^{1+\alpha}} (1 - tx)^{\alpha-2-2p} \, dt$$

$$= (1 - x^2)^{p-\alpha} \Bigg[x p.v. \int_{-1}^{1} \frac{(1 - tx)^{\alpha-2p-2} - 1}{|t|^{1+\alpha}} (1 - t^2)^p \, dt$$

$$+ x p.v. \int_{-1}^{1} \frac{(1 - t^2)^p - 1}{|t|^{1+\alpha}} \, dt + x p.v. \int_{-1}^{1} \frac{1 - (1 - tx)^{\alpha-1}}{|t|^{1+\alpha}} \, dt$$

$$- p.v. \int_{-1}^{1} \frac{(1 - t^2)^p t (1 - tx)^{\alpha-2p-2}}{|t|^{1+\alpha}} \, dt \Bigg].$$

We have by [12, the proof of Lemma 2.1]

$$p.v. \int_{-1}^{1} \frac{(1 - t^2)^p - 1}{|t|^{1+\alpha}} \, dt = \frac{2}{\alpha} \left[1 - (p + 1 - \alpha/2) B(p + 1, 1 - \alpha/2) \right],$$

$$p.v. \int_{-1}^{1} \frac{1 - (1 - tx)^{\alpha-1}}{|t|^{1+\alpha}} \, dt = \frac{1}{\alpha} \left(1 - x \right)^\alpha + \frac{1}{\alpha} \left(1 + x \right)^\alpha - \frac{2}{\alpha}.$$

By Lemma 3 we obtain

$$p.v. \int_{-1}^{1} \frac{(1 - tx)^{\alpha-2p-2} - 1}{|t|^{1+\alpha}} (1 - t^2)^p \, dt = I_2(p)$$

$$= B\left(-\frac{\alpha}{2}, p + 1 \right) \left({}_2F_1\left(-\frac{\alpha}{2}, p + \frac{3}{2} - \frac{\alpha}{2}; \frac{1}{2}; x^2 \right) - 1 \right).$$

For $p \neq \frac{\alpha-2}{2}$ we have

$$K := p.v. \int_{-1}^{1} \frac{(1 - t^2)^p t (1 - tx)^{\alpha-2p-2}}{|t|^{1+\alpha}} \, dt$$

$$= p.v. \int_{-1}^{1} \frac{(1 - t^2)^p t}{|t|^{1+\alpha}} \sum_{k=0}^{\infty} \frac{(2p + 2 - \alpha)_k}{k!} (tx)^k \, dt$$

$$= \sum_{k=0}^{\infty} 2 \int_{0}^{1} \frac{(1 - t^2)^p t}{|t|^{1+\alpha}} \frac{(2p + 2 - \alpha)_{2k+1}}{(2k + 1)!} (tx)^{2k+1} \, dt$$

$$= \sum_{k=0}^{\infty} B(p + 1, \frac{2k + 2 - \alpha}{2}) \frac{(2p + 2 - \alpha)_{2k+1}}{(2k + 1)!} x^{2k+1}.$$

Using the doubling formula (14) for $x = k + 1$, we obtain

$$(2k + 1)! = 2^{2k+1} \left(\tfrac{1}{2} \right)_{k+1} k!.$$

We also have

$$(2p + 2 - \alpha)_{2k+1} = 2^{2k+1} (p + 1 - \frac{\alpha}{2})_{k+1} (p + \frac{3}{2} - \frac{\alpha}{2})_k,$$

and

$$B(p + 1, \frac{2k + 2 - \alpha}{2}) = B(-\frac{\alpha}{2}, p + 1) \frac{(-\frac{\alpha}{2})_{k+1}}{(p + 1 - \frac{\alpha}{2})_{k+1}}.$$

Hence

$$K = B(-\frac{\alpha}{2}, p+1) \sum_{k=0}^{\infty} \frac{(-\frac{\alpha}{2})_{k+1}(p+\frac{3}{2}-\frac{\alpha}{2})_k}{(\frac{1}{2})_{k+1}k!} x^{2k+1}.$$

$$= -\alpha B(-\frac{\alpha}{2}, p+1) \sum_{k=0}^{\infty} \frac{(1-\frac{\alpha}{2})_k (p+\frac{3}{2}-\frac{\alpha}{2})_k}{(\frac{3}{2})_k k!} x^{2k+1}$$

$$= -\alpha B\left(-\frac{\alpha}{2}, p+1\right) x \cdot {}_2F_1\left(1-\frac{\alpha}{2}, p+\frac{3}{2}-\frac{\alpha}{2}; \frac{3}{2}; x^2\right).$$

This holds also for $p = \frac{\alpha-2}{2}$, since in this case we have $K = 0$. Thus,

$$\Delta^{\alpha/2} v_p(x) = \mathcal{A}_{1,-\alpha} B\left(-\frac{\alpha}{2}, p+1\right)(1-x^2)^{p-\alpha} x$$

$$\times \left({}_2F_1\left(-\frac{\alpha}{2}, p+\frac{3}{2}-\frac{\alpha}{2}; \frac{1}{2}; x^2\right) + \alpha \ {}_2F_1\left(1-\frac{\alpha}{2}, p+\frac{3}{2}-\frac{\alpha}{2}; \frac{3}{2}; x^2\right)\right).$$

Formula (21) follows by [14, formula 2.8(35), page 103], and (22) is then a consequence of [14, formula 2.9(2), page 105]. □

In Table 2 we list the fractional Laplacian of a few functions v_p. We note that the first example in Table 2 may be considered a linear combination of Martin kernels of the interval, see [10, (89)] and the references given there.

TABLE 2. The fractional Laplacian for some functions, when defined as zero outside of $(-1, 1)$.

$v(x)$ on $(-1,1)$	$\Delta^{\alpha/2}v(x)$ for $x \in (-1,1)$
$x(1-x^2)^{-1+\alpha/2}$	0
$x(1-x^2)^{\alpha/2}$	$-\Gamma(\alpha+2)x$
$x(1-x^2)^{1+\alpha/2}$	$-\frac{\Gamma(\alpha+3)}{6}(3-(3+\alpha)x^2)x$
$x(1-x^2)^{2+\alpha/2}$	$-\frac{\Gamma(\alpha+3)(\alpha+4)}{120}\left(15-(10\alpha+30)x^2+(\alpha+3)(\alpha+5)x^4\right)x$

3. MULTI-DIMENSIONAL CASE

We recall the notation (2) and note that $u_p^{(d)}(x) = u_p(|x|)$ for $x \in \mathbb{R}^d$, with u_p given by (17). We let $S^{(d-1)} = \{x \in \mathbb{R}^d : |x| = 1\}$, the unit sphere in $\mathbb{R}^d$.

Lemma 6. *Let $d \geq 2$, $0 < \alpha < 2$ and $p > -1$. If $x \in \mathbb{R}^d$ and $|x| < 1$, then*
(23)
$$\Delta^{\alpha/2}u_p^{(d)}(x) = \frac{\mathcal{A}_{d,-\alpha}}{2\mathcal{A}_{1,-\alpha}} \int_{S^{d-1}} (1-|x|^2+|h_d x|^2)^{p-\alpha/2} \Delta^{\alpha/2}u_p\left(\frac{|x|h_d}{\sqrt{1-|x|^2+|h_d x|^2}}\right) dh.$$

Proof. Without loss of generality, we may assume that $x = (0,0,\ldots,0,|x|)$. For $|x| < 1$ we have,

$$\Delta^{\alpha/2}u_p^{(d)}(x) = \mathcal{A}_{d,-\alpha} p.v. \int_{\mathbb{R}^d} \frac{u_p^{(d)}(y) - (1-|x|^2)^p}{|x-y|^{d+\alpha}} dy$$

$$= \frac{\mathcal{A}_{d,-\alpha}}{2} \int_{S^{d-1}} dh \ p.v. \int_{\mathbb{R}} \frac{u_p^{(d)}(x+ht) - (1-|x|^2)^p}{|t|^{1+\alpha}} dt.$$

We calculate the (inner) principal value integral by changing the variable $t = -|x|h_d + s\sqrt{|h_d x|^2 - |x|^2 + 1}$. We obtain

$$
\begin{aligned}
g(x, h) &:= p.v. \int_{\mathbb{R}} \frac{u_p^{(d)}(x + ht) - (1 - |x|^2)^p}{|t|^{1+\alpha}}\, dt \\
&= p.v. \int_{\mathbb{R}} \frac{(1 - s^2)^p (1 - |x|^2 + |h_d x|^2)^p - (1 - |x|^2)^p}{|-|x|h_d + s\sqrt{|h_d x|^2 - |x|^2 + 1}|^{1+\alpha}} \sqrt{|h_d x|^2 - |x|^2 + 1}\, ds \\
&= (1 - |x|^2 + |h_d x|^2)^{p-\alpha/2} p.v. \int_{\mathbb{R}} \frac{u_p(s) - (1 - \frac{|h_d x|^2}{1 - |x|^2 + |h_d x|^2})^p}{|s - \frac{|x|h_d}{\sqrt{1 - |x|^2 + |h_d x|^2}}|^{1+\alpha}}\, ds \\
&= (1 - |x|^2 + |h_d x|^2)^{p-\alpha/2} \mathcal{A}_{1,-\alpha}^{-1} \Delta^{\alpha/2} u_p \left(\frac{|x|h_d}{\sqrt{1 - |x|^2 + |h_d x|^2}} \right). \qquad \square
\end{aligned}
$$

Proof of formula (5) of Theorem 1 for $d > 1$. We have by Lemmata 4 and 6,

$$
\begin{aligned}
\Delta^{\alpha/2} u_p^{(d)}(x) &= \frac{\mathcal{A}_{d,-\alpha} B(-\frac{\alpha}{2}, p+1)}{2} \int_{S^{d-1}} \frac{{}_2F_1\left(\frac{\alpha+1}{2}, -p + \frac{\alpha}{2}; \frac{1}{2}; \frac{|x|^2 h_d^2}{1 - |x|^2 + |h_d x|^2}\right)}{(1 - |x|^2 + |h_d x|^2)^{-p+\alpha/2}}\, dh \\
&=: \frac{\mathcal{A}_{d,-\alpha} B(-\frac{\alpha}{2}, p+1)}{2} I_{S^{d-1}}.
\end{aligned}
$$

We transform the integrand function using [14, formula 2.9(4), page 105],

$$
\frac{{}_2F_1\left(\frac{\alpha+1}{2}, -p + \frac{\alpha}{2}; \frac{1}{2}; \frac{|x|^2 h_d^2}{1 - |x|^2 + |h_d x|^2}\right)}{(1 - |x|^2 + |h_d x|^2)^{-p+\alpha/2}} = \frac{{}_2F_1\left(-\frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{1}{2}; \frac{|x|^2 h_d^2}{|x|^2 - 1}\right)}{(1 - |x|^2)^{-p+\alpha/2}}.
$$

Therefore,

$$
\begin{aligned}
I_{S^{d-1}} &= \int_{S^{d-1}} \frac{{}_2F_1\left(-\frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{1}{2}; \frac{|x|^2 h_d^2}{|x|^2 - 1}\right)}{(1 - |x|^2)^{-p+\alpha/2}}\, dh \\
&= \frac{2\pi^{\frac{d-1}{2}} (1 - |x|^2)^{p-\alpha/2}}{\Gamma(\frac{d-1}{2})} \int_{-1}^{1} {}_2F_1\left(-\frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{1}{2}; \frac{|x|^2 h^2}{|x|^2 - 1}\right) (1 - h^2)^{\frac{d-3}{2}}\, dh.
\end{aligned}
$$

Let

$$
\phi(z) = \int_{-1}^{1} {}_2F_1\left(-\frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{1}{2}; \frac{zh^2}{z - 1}\right) (1 - h^2)^{\frac{d-3}{2}}\, dh, \quad z \in \mathbb{C}, |z| < 1.
$$

Since $\mathrm{Re}\,\frac{zh^2}{z-1} < \frac{1}{2}$ for $|z| < 1$, the function ϕ is analytic in the unit disc $\{z : |z| < 1\}$. For $|z| < \frac{1}{2}$ we calculate the integral defining ϕ by using Taylor's expansion,

$$
\begin{aligned}
\phi(z) &= \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{\alpha}{2}+k)}{\Gamma(-\frac{\alpha}{2})} \frac{\Gamma(-p+\frac{\alpha}{2}+k)}{\Gamma(-p+\frac{\alpha}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+k)k!} \left(\frac{z}{z-1}\right)^k \int_{-1}^{1} h^{2k}(1-h^2)^{\frac{d-3}{2}}\,dh \\
&= \frac{\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{\alpha}{2}+k)}{\Gamma(-\frac{\alpha}{2})} \frac{\Gamma(-p+\frac{\alpha}{2}+k)}{\Gamma(-p+\frac{\alpha}{2})} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2}+k)k!} \left(\frac{z}{z-1}\right)^k \\
&= \frac{\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})} {}_2F_1\left(-\frac{\alpha}{2}, -p+\frac{\alpha}{2}; \frac{d}{2}; \frac{z}{z-1}\right) \\
&= \frac{\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})} {}_2F_1\left(\frac{\alpha+d}{2}, -p+\frac{\alpha}{2}; \frac{d}{2}; z\right)(1-z)^{-p+\alpha/2} =: \psi(z).
\end{aligned}
$$

In the last line we have used [14, formula 2.9(4), page 105] again. The functions ϕ and ψ are both analytic in the unit disc, hence $\phi(z) = \psi(z)$ for all $|z| < 1$. We put $z = |x|^2$ and the proof is finished. $\qquad\square$

Lemma 7. *Let $d \geq 2$, $0 < \alpha < 2$ and $p > -1$. If $x \in \mathbb{R}^d$ and $|x| < 1$, then*

(24)

$$
\begin{aligned}
\Delta^{\alpha/2} v_p^{(d)}(x) &= x_d \Delta^{\alpha/2} u_p^{(d)}(x) \\
&\quad + \frac{\mathcal{A}_{d,-\alpha}}{2\mathcal{A}_{1,-\alpha}} \int_{S^{d-1}} T^{p-\alpha/2} h_d \left(T^{1/2}\Delta^{\alpha/2} v_p - \langle h, x\rangle \Delta^{\alpha/2} u_p\right)\left(\frac{\langle h, x\rangle}{\sqrt{T}}\right) dh,
\end{aligned}
$$

where $T = T(x, h) = 1 - |x|^2 + \langle h, x\rangle^2$.

Proof. We have for $|x| < 1$,

$$
\begin{aligned}
\Delta^{\alpha/2} v_p^{(d)}(x) &= \mathcal{A}_{d,-\alpha}\,p.v. \int_{\mathbb{R}^d} \frac{v_p^{(d)}(y) - v_p^{(d)}(x)}{|x-y|^{d+\alpha}}\,dy \\
&= \frac{\mathcal{A}_{d,-\alpha}}{2} \int_{S^{d-1}} dh\, p.v. \int_{\mathbb{R}} \frac{v_p^{(d)}(x+ht) - v_p^{(d)}(x)}{|t|^{1+\alpha}}\,dt.
\end{aligned}
$$

We calculate the inner principal value integral by changing the variable $t = -\langle h, x \rangle + s\sqrt{T}$. We obtain

$$
\begin{aligned}
g(x,h) &:= p.v. \int_{\mathbb{R}} \frac{v_p^{(d)}(x+ht) - v_p^{(d)}(x)}{|t|^{1+\alpha}} \, dt \\
&= p.v. \int_{\mathbb{R}} \frac{(1-s^2)_+^p T^p (x_d - h_d\langle h,x \rangle + h_d s\sqrt{T}) - (1-|x|^2)^p x_d}{|-\langle h,x \rangle + s\sqrt{T}|^{1+\alpha}} \sqrt{T} \, ds \\
&= T^{p-\alpha/2} p.v. \int_{\mathbb{R}} \frac{(1-s^2)_+^p (x_d - h_d\langle h,x \rangle + h_d s\sqrt{T}) - (1 - \frac{\langle h,x \rangle^2}{T})^p x_d}{|s - \frac{\langle h,x \rangle}{\sqrt{T}}|^{1+\alpha}} \, ds \\
&= T^{p-\alpha/2}(x_d - h_d\langle h,x \rangle) p.v. \int_{\mathbb{R}} \frac{(1-s^2)_+^p - (1 - \frac{\langle h,x \rangle^2}{T})^p}{|s - \frac{\langle h,x \rangle}{\sqrt{T}}|^{1+\alpha}} \, ds \\
&\quad + T^{p+1/2-\alpha/2} h_d \, p.v. \int_{\mathbb{R}} \frac{(1-s^2)_+^p s - (1 - \frac{\langle h,x \rangle^2}{T})^p \frac{\langle h,x \rangle}{\sqrt{T}}}{|s - \frac{\langle h,x \rangle}{\sqrt{T}}|^{1+\alpha}} \, ds \\
&= T^{p-\alpha/2} \frac{x_d - h_d\langle h,x \rangle}{\mathcal{A}_{1,-\alpha}} \Delta^{\alpha/2} u_p \Big(\frac{\langle h,x \rangle}{\sqrt{T}} \Big) \\
&\quad + T^{p+1/2-\alpha/2} \frac{h_d}{\mathcal{A}_{1,-\alpha}} \Delta^{\alpha/2} v_p \Big(\frac{\langle h,x \rangle}{\sqrt{T}} \Big).
\end{aligned}
$$

The result follows from Lemma 6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of formula (6) of Theorem 1 for $d > 1$. We may assume that $x \neq 0$, since for $x = 0$ the formula is obvious. We denote $T = T(x,h) = 1 - |x|^2 + \langle h, x \rangle^2$. By Lemmata 5 and 7,

$$
\begin{aligned}
\Delta^{\alpha/2} v_p^{(d)}(x) &= x_d \Delta^{\alpha/2} u_p^{(d)}(x) + \frac{\mathcal{A}_{d,-\alpha} B(-\frac{\alpha}{2}, p+1)}{2} \times \\
&\quad \times \int_{S^{d-1}} h_d \langle h, x \rangle T^{p-\alpha/2} F(x,h) \, dh,
\end{aligned}
$$

where

$$
F(x,h) = (\alpha+1) {}_2F_1 \Big(\frac{\alpha+3}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{\langle h,x \rangle^2}{T} \Big) - {}_2F_1 \Big(\frac{\alpha+1}{2}, -p + \frac{\alpha}{2}; \frac{1}{2}; \frac{\langle h,x \rangle^2}{T} \Big).
$$

We transform $F(x,h)$ using [14, formula 2.8(35), page 103] and [14, formula 2.9(4), page 105],

$$
\begin{aligned}
F(x,h) &= \alpha \cdot {}_2F_1 \Big(\frac{\alpha+1}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{\langle h,x \rangle^2}{T} \Big) \\
&= \alpha \Big(\frac{1-|x|^2}{T} \Big)^{p-\alpha/2} {}_2F_1 \Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{\langle h,x \rangle^2}{|x|^2 - 1} \Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{S^{d-1}} h_d \langle h, x \rangle T^{p-\alpha/2} F(x,h) \, dh \\
&\quad = \alpha(1 - |x|^2)^{p-\alpha/2} \int_{S^{d-1}} {}_2F_1 \Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{\langle h,x \rangle^2}{|x|^2 - 1} \Big) h_d \langle h, x \rangle \, dh \\
&\quad =: \alpha(1 - |x|^2)^{p-\alpha/2} I_{S^{d-1}}.
\end{aligned}
$$

By symmetry, for any e_1, $e_2 \in \mathbb{R}^d$ with $|e_1| = |e_2| = 1$,

$$\int_{S^{d-1}} f_1(\langle h, e_1 \rangle) f_2(\langle h, e_2 \rangle) \, dh = \int_{S^{d-1}} f_1(\langle h, e_2 \rangle) f_2(\langle h, e_1 \rangle) \, dh,$$

where f_1 and f_2 are any functions for which the integrals make sense. Using this observation for $e_1 = \frac{x}{|x|}$ and $e_2 = (0, \ldots, 0, 1)$ we obtain

$$I_{S^{d-1}} = \int_{S^{d-1}} {}_2F_1\Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{h_d^2 |x|^2}{|x|^2 - 1}\Big) \langle h, x \rangle h_d \, dh$$

$$= \int_{S^{d-1}} {}_2F_1\Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{h_d^2 |x|^2}{|x|^2 - 1}\Big) h_d^2 x_d \, dh$$

$$= \frac{2\pi^{\frac{d-1}{2}} x_d}{\Gamma(\frac{d-1}{2})} \int_{-1}^{1} {}_2F_1\Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{h^2 |x|^2}{|x|^2 - 1}\Big) h^2 (1 - h^2)^{\frac{d-3}{2}} \, dh.$$

Let

$$\phi(z) = \int_{-1}^{1} {}_2F_1\Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; \frac{3}{2}; \frac{h^2 z}{z - 1}\Big) h^2 (1 - h^2)^{\frac{d-3}{2}} \, dh, \quad z \in \mathbb{C}, |z| < 1.$$

Similarly as in the proof of formula (5) we observe that ϕ is analytic in the unit disc, and calculate $\phi(z)$ for $|z| < \frac{1}{2}$ by using Taylor's expansion,

$$\phi(z) = \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(1 - \frac{\alpha}{2})_k (-p + \frac{\alpha}{2})_k \Gamma(\frac{3}{2})}{\Gamma(k + \frac{3}{2}) k!} \Big(\frac{z}{z-1}\Big)^k h^{2k+2} (1 - h^2)^{\frac{d-3}{2}} \, dh$$

$$= \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2} + 1)} \sum_{k=0}^{\infty} \frac{(1 - \frac{\alpha}{2})_k (-p + \frac{\alpha}{2})_k \Gamma(\frac{d}{2} + 1)}{\Gamma(k + \frac{d}{2} + 1) k!} \Big(\frac{z}{z-1}\Big)^k \, dh$$

$$= \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2} + 1)} {}_2F_1\Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; 1 + \frac{d}{2}; \frac{z}{z-1}\Big).$$

Since the function in the last line is analytic in the unit disc (note that $\operatorname{Re} \frac{z}{z-1} < \frac{1}{2}$ if $|z| < 1$), we conclude that

$$I_{S^{d-1}} = \frac{\pi^{\frac{d}{2}} x_d}{\Gamma(\frac{d}{2} + 1)} {}_2F_1\Big(1 - \frac{\alpha}{2}, -p + \frac{\alpha}{2}; 1 + \frac{d}{2}; \frac{|x|^2}{|x|^2 - 1}\Big)$$

$$= \frac{\pi^{d/2} x_d}{\Gamma(1 + \frac{d}{2})} {}_2F_1\Big(\frac{\alpha + d}{2}, -p + \frac{\alpha}{2}; 1 + \frac{d}{2}; |x|^2\Big) (1 - |x|^2)^{-p + \alpha/2}.$$

In the last line we have used [14, formula 2.9(4), page 105]. By [14, formula 2.8(35), page 103] we get

$$\Delta^{\alpha/2} v_p^{(d)}(x) = x_d \frac{\mathcal{A}_{d,-\alpha} \pi^{d/2} B(-\frac{\alpha}{2}, p+1)}{\Gamma(d/2)} \Big({}_2F_1\Big(\frac{\alpha + d}{2}, -p + \frac{\alpha}{2}; \frac{d}{2}; |x|^2\Big)$$

$$+ \frac{\alpha}{d} {}_2F_1\Big(\frac{\alpha + d}{2}, -p + \frac{\alpha}{2}; 1 + \frac{d}{2}; |x|^2\Big) \Big)$$

$$= x_d \frac{\mathcal{A}_{d,-\alpha} \pi^{d/2} B(-\frac{\alpha}{2}, p+1)(\alpha + d)}{d \Gamma(d/2)} \times$$

$$\times {}_2F_1\Big(\frac{\alpha + d + 2}{2}, -p + \frac{\alpha}{2}; 1 + \frac{d}{2}; |x|^2\Big),$$

and the proof is finished. $\hfill\square$

In Table 3 we list the fractional Laplacian for some power functions in $\mathbb{R}^d$.

TABLE 3. Values of fractional Laplacian for some functions vanishing outside the unit ball in $\mathbb{R}^d$.

$f(x)$ in the ball	$\Delta^{\alpha/2}f(x)$ in the ball
$(1-|x|^2)^{\alpha/2}$	$-2^\alpha\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{d}{2})^{-1}$
$(1-|x|^2)^{1+\alpha/2}$	$-2^\alpha\Gamma(\frac{\alpha}{2}+2)\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{d}{2})^{-1}\left(1-(1+\frac{\alpha}{d})|x|^2\right)$
$(1-|x|^2)^{\alpha/2}x_d$	$-2^\alpha\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{d+\alpha}{2}+1)\Gamma(\frac{d}{2}+1)^{-1}x_d$
$(1-|x|^2)^{1+\alpha/2}x_d$	$-2^\alpha\Gamma(\frac{\alpha}{2}+2)\Gamma(\frac{d+\alpha}{2}+1)\Gamma(\frac{d}{2}+1)^{-1}\left(1-(1+\frac{\alpha}{d+2})|x|^2\right)$

4. LOWER BOUNDS FOR EIGENVALUES

Our approach to lower bounds is similar to that of [9]. The method is to use a suitable superharmonic function ν and the resulting Hardy inequality with the weight given by the Fitzsimmons' ratio $-\Delta^{\alpha/2}\nu/\nu$ ([15]). To estimate λ_1, the following calculation will be used.

Lemma 8. *Let* $\eta_{d,\alpha} = (6d - 4 + (4-d)\alpha - \alpha^2)/(\alpha+2)^2$ *and*

$$\psi(x) = (1-|x|^2)_+^{\alpha/2} + \eta_{d,\alpha}(1-|x|^2)_+^{1+\alpha/2}, \quad x \in \mathbb{R}^d.$$

Then

$$\frac{-\Delta^{\alpha/2}\psi(x)}{\psi(x)} \geq \mu_{d,\alpha}, \quad |x| < 1,$$

where $\mu_{d,\alpha}$ is defined in (9).

Proof. We have by Table 3

$$\frac{-\Delta^{\alpha/2}\psi(x)}{\psi(x)} = \frac{2^\alpha\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha+d}{2})}{\Gamma(\frac{d}{2})(1-x^2)^{\alpha/2}}\frac{1+\eta\frac{\alpha+2}{2}(1-(1+\frac{\alpha}{d})|x|^2)}{1+\eta(1-|x|^2)} =: f(|x|^2).$$

After elementary but tedious calculations we obtain

$$f'(x) = \frac{2^\alpha\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha+d}{2})\alpha(\alpha+d)(\alpha+2)\eta^2}{4d\Gamma(\frac{d}{2})(1-x)^{1+\alpha/2}(1+\eta(1-x))^2}\left(x - \frac{(4-d)\alpha + 6d - 8}{6d-4+(4-d)\alpha-\alpha^2}\right)^2 \geq 0$$

for $x \in [0,1)$. Hence

$$\frac{-\Delta^{\alpha/2}\psi(x)}{\psi(x)} \geq \frac{-\Delta^{\alpha/2}\psi(0)}{\psi(0)} = \frac{2^\alpha\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha+d}{2})(\alpha+2)(\alpha+d)(6-\alpha)}{\Gamma(\frac{d}{2})(12d+(16-2d)\alpha)}. \hfill\square$$

We will also consider the ratio $-\Delta^{\alpha/2}\nu/\nu$ for a suitable antisymmetric function ν. We should note that ν changes sign and so it cannot be called superharmonic. However, the Fitzsimmons ratio $-\Delta^{\alpha/2}\nu/\nu$ will prove nonnegative, and we will obtain a Hardy inequality with the weight $-\Delta^{\alpha/2}\nu/\nu$. To this end we start with the following simple lemma.

Lemma 9. *We have*

$$(25) \qquad \int_{B_1^{(d)}} x_d^2\phi(|x|)\,dx = \frac{1}{\pi}\int_{B_1^{(d+2)}} \phi(|x|)\,dx,$$

for any function ϕ for which the integrals are absolutely convergent.

Proof. We recall the notation $S^{(d-1)} = \{x \in \mathbb{R}^d : |x| = 1\}$ for the unit sphere in $\mathbb{R}^d$, and the formula for its area, $|S^{(d-1)}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. We have

$$\int_{B_1^{(d)}} x_d^2 \phi(|x|) \, dx = \frac{1}{d} \int_{B_1^{(d)}} |x|^2 \phi(|x|) \, dx = \frac{2\pi^{d/2}}{d\Gamma(d/2)} \int_0^1 r^{d+1} \phi(r) \, dr$$

$$= \frac{|S^{(d+1)}|}{\pi} \int_0^1 r^{d+1} \phi(r) \, dr = \frac{1}{\pi} \int_{B_1^{(d+2)}} \phi(|x|) \, dx.$$

$\square$

Lemma 10. *Let $\eta_{d,\alpha}$, $\mu_{d,\alpha}$ be as in Lemma 8, and let*

$$\psi(x) = x_d(1 - |x|^2)_+^{\alpha/2} + \eta_{d+2,\alpha} x_d (1 - |x|^2)_+^{1+\alpha/2}, \quad x \in \mathbb{R}^d.$$

Then

$$\frac{-\Delta^{\alpha/2}\psi(x)}{\psi(x)} \geq \mu_{d+2,\alpha}, \quad |x| < 1.$$

Proof. By formula (6) and Lemma 9, the result follows by Lemma 8 with $d + 2$ in place of d. $\square$

In order to estimate λ_* using Lemma 10 we will need the following analogue of [12, Lemma 2.2] for antisymmetric functions. For $x \in \mathbb{R}^d$ we denote $x^* = (x_1, \ldots, x_{d-1}, -x_d)$, the reflection of x with respect to the last coordinate axis.

Proposition 11. *Let $D \subset \mathbb{R}^n$ be open and satisfy $\{x^* : x \in D\} = D$. Let $\overline{w} : \mathbb{R}^d \to [0, \infty)$ be a bounded function of class $C^2(D)$, strictly positive on D and vanishing on D^c, and let $w(x) = x_d \overline{w}(x)$. For every $u \in C_c(D)$ such that $u(x^*) = -u(x)$ we have*

$$\mathcal{E}(u) = \int_D u^2(x) \frac{-\Delta^{\alpha/2}w(x)}{w(x)} \, dx$$

$$(26) \qquad + \mathcal{A}_{d,-\alpha} \int_{D^+} \int_{D^+} \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)} \right)^2 \left(\frac{w(x)w(y)}{|x - y|^{d+\alpha}} - \frac{w(x)w(y)}{|x^* - y|^{d+\alpha}} \right) dy \, dx$$

$$(27) \qquad \geq \int_D u^2(x) \frac{-\Delta^{\alpha/2}w(x)}{w(x)} \, dx,$$

where $D^+ = \{x \in D : x_d > 0\}$.

We give the proof in Appendix. We note that for functions u in the domain of the generator $\Delta^{\alpha/2}$ we have

$$(28) \qquad \mathcal{E}(u) = -\int_B u\Delta^{\alpha/2} u \, dx,$$

in particular (28) holds for $u(x) = (1 - |x|^2)_+^p$ with $p \geq \alpha/2$. We are now ready to prove our estimates of λ_1 and λ_*.

Proof of Corollary 2. Let ψ be as in Lemma 8. From that lemma, (28) and [12, Lemma 2.2] (or [16, Proposition 2.3]) we obtain

$$\mathcal{E}(u) \geq \int_B u^2(x) \frac{-\Delta^{\alpha/2}\psi(x)}{\psi(x)} \, dx \geq \mu_{d,\alpha} \int_B u^2(x) \, dx,$$

and hence $\lambda_1 \geq \mu_{d,\alpha}$.

To prove the second part, we observe that

$$\lambda_* = \frac{\mathcal{E}(\phi_*)}{\int_B \phi_*^2(x)\,dx}$$

and we proceed as before, using Lemma 10 and Proposition 11. $\qquad\square$

5. UPPER BOUNDS FOR EIGENVALUES

For functions $u \in D(\mathcal{E})$, by the variational formula, we have

$$(29) \qquad \lambda_1 \le \frac{\mathcal{E}(u)}{\int_B u^2\,dx}.$$

For u being a linear combinations of functions $u^{(d)}_{j+\alpha/2}$, it is easy to compute the right hand side of (29) by using (28), Theorem 1 and the formula

$$\int_B |x|^s (1-|x|^2)^t\,dx = \frac{\pi^{d/2}\Gamma(\frac{s+d}{2})\Gamma(t+1)}{\Gamma(\frac{d}{2})\Gamma(\frac{s+d}{2}+t+1)}, \quad s > -d,\ t > -1.$$

In particular, for functions of the form

$$(30) \qquad u(x) = (1-|x|^2)_+^{\alpha/2} + \eta(1-|x|^2)_+^{1+\alpha/2},$$

we can explicitly find η minimising the right hand side of (29). A calculation yields,

$$(31) \qquad \eta_{min} = \frac{\sqrt{w} + d^2 + 2d - 2a^2 - 6a - 4}{4a^2 + 12a + 8},$$

where

$$w = d^4 + 4ad^3 + 8d^3 + 8a^2 d^2 + 32ad^2 + 28d^2 + 8a^3 d + 48a^2 d$$
$$+ 88ad + 48d + 4a^4 + 24a^3 + 52a^2 + 48a + 16.$$

It is possible to further improve the estimates by taking more functions $u^{(d)}_{j+\alpha/2}$ to define u. Then, however, the ratio (29) should be minimised numerically.

Since we have $\mathcal{E}(u,v) = 0$ whenever u, $v \in D(\mathcal{E})$ and $u(x) = u(-x)$, $v(x) = -v(-x)$, it follows from the min–max formula ([18]) that

$$(32) \qquad \lambda_* \le \frac{\mathcal{E}(v)}{\int_B v^2(x)\,dx}, \quad \text{if } v \in D(\mathcal{E}),\ v(x) = -v(-x),\ v \not\equiv 0.$$

By plugging in linear combinations of the functions $v^{(d)}_{j+\alpha/2}$ we obtain upper bounds for λ_*.

Remark 12. *Linear combinations $v = \sum_{j=0}^k a_j v^{(d)}_{j+\alpha/2}$ yield the same upper bound for λ_* in dimension d, as the function $u = \sum_{j=0}^k a_j u^{(d+2)}_{j+\alpha/2}$ does for λ_1 in dimension $d+2$.*

Indeed, this follows from formulae (6), (5), (28) and Lemma 9. Hence the upper bounds for λ_* resulting from such test functions v are the same as the upper bounds obtained for $\lambda_1^{(d+2)}$. We omit the details.

In the Table 4, we give bounds for λ_1 and λ_* obtained by (29), (32) and Corollary 2.

TABLE 4. Lower and upper bounds for λ_1 and λ_*. Lower bounds of Corollary 2 are in the top row. In the middle row, in italic, there are the upper bounds obtained from (29), by plugging in a certain linear combination of functions $u_{j+\alpha/2}$ (or $v_{j+\alpha/2}$) with $j = 0, 1, \ldots, 12$. The (near optimal) linear combinations were found numerically. In the bottom row, we give upper bounds obtained by considering functions u or $x_d u(x)$, with u given by (30), and η given by (31).

α	λ_1 for $d=1$	λ_1 for $d=2$	λ_1 for $d=3$ λ_* for $d=1$	λ_1 for $d=4$ λ_* for $d=2$	λ_1 for $d=5$ λ_* for $d=3$
	0.9676	1.04874	1.08633	1.1102	1.12756
0.1	*0.97261*	*1.05096*	*1.09221*	*1.12082*	*1.14301*
	0.97273	1.05103	1.09225	1.12093	1.14327
	0.94993	1.10549	1.18391	1.23565	1.27419
0.2	*0.95747*	*1.10993*	*1.19655*	*1.25903*	*1.30877*
	0.95764	1.11001	1.19663	1.25927	1.30934
	0.96202	1.3313	1.56035	1.72814	1.86169
0.5	*0.97017*	*1.34374*	*1.60155*	*1.80843*	*1.98572*
	0.97029	1.3438	1.60173	1.8092	1.98766
	1.15384	1.96349	2.60869	3.15561	3.63636
1	*1.15778*	*2.00612*	*2.75476*	*3.45334*	*4.12131*
	1.1578	2.00618	2.75548	3.45616	4.12824
	1.58614	3.13569	4.61848	6.03622	7.39626
1.5	*1.59751*	*3.27594*	*5.05977*	*6.94732*	*8.93319*
	1.59751	3.27624	5.06201	6.95522	8.95256
	2.01395	4.28394	6.65946	9.07867	11.51297
1.8	*2.04874*	*4.56719*	*7.50312*	*10.82218*	*14.49989*
	2.04876	4.56781	7.50715	10.83601	14.53414
	2.19524	4.77496	7.54923	10.43088	13.37504
1.9	*2.24406*	*5.13213*	*8.59576*	*12.5934*	*17.09653*
	2.24409	5.1329	8.60059	12.60997	17.13776

6. APPENDIX

For reader's convenience, we give the proof of Proposition 11, consisting mostly of a justification of the interchange of limit and integral, which is delicate, because the functions w and $\Delta^{\alpha/2}w$ change sign.

Proof of Proposition 11. We have

$$(u(x) - u(y))^2 + u^2(x)\frac{w(y) - w(x)}{w(x)} + u^2(y)\frac{w(x) - w(y)}{w(y)}$$
$$= \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 w(x)w(y).$$

Let us denote

$$G_\varepsilon = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| > \varepsilon, \ |x^* - y| > \varepsilon, \ |y_d| > \varepsilon^\delta, \ |x_d| > \varepsilon^\delta\},$$

where $\delta \in (0,1)$ will be determined later. We integrate against $1_{G_\varepsilon}(x,y)|x-y|^{-d-\alpha}\,dy\,dx$ and obtain

$$\iint_{G_\varepsilon} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx + \iint_{G_\varepsilon} u^2(x)\frac{w(y)-w(x)}{w(x)}|x-y|^{-d-\alpha}\,dy\,dx$$

$$(33) \qquad = \iint_{G_\varepsilon} \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 w(x)w(y)|x-y|^{-d-\alpha}\,dy\,dx.$$

The first and third term in (33) are easy to deal with. Indeed, we have

$$\frac{\mathcal{A}_{d,-\alpha}}{2} \iint_{G_\varepsilon} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx \to \mathcal{E}(u), \quad \text{when } \varepsilon \to 0,$$

and

$$\iint_{G_\varepsilon} \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 w(x)w(y)|x-y|^{-d-\alpha}\,dy\,dx$$

$$= 2\left(\iint_{G_\varepsilon^+} - \iint_{G_\varepsilon \setminus G_\varepsilon^+}\right)\left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 |w(x)w(y)||x-y|^{-d-\alpha}\,dy\,dx$$

$$= 2\iint_{G_\varepsilon^+} \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 \left(\frac{w(x)w(y)}{|x-y|^{d+\alpha}} - \frac{w(x)w(y)}{|x^*-y|^{d+\alpha}}\right)\,dy\,dx$$

$$\to 2\int_{D^+}\int_{D^+} \left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 \left(\frac{w(x)w(y)}{|x-y|^{d+\alpha}} - \frac{w(x)w(y)}{|x^*-y|^{d+\alpha}}\right)\,dy\,dx,$$

as $\varepsilon \to 0$. Hence we are left with the second term in (33). We note that by our assumptions,

$$\Delta^{\alpha/2}w(x) = \mathcal{A}_{d,-\alpha}\int_{\mathbb{R}^d} \frac{w(y)-w(x)-(y-x)\nabla w(x)1_{|y-x|<\varepsilon^\eta}}{|x-y|^{d+\alpha}}\,dy$$

exists for all $x \in D$, where $\eta \in (0,1)$ will be determined later. Denote $G_\varepsilon(x) = \{y \in \mathbb{R}^d : (x,y) \in G_\varepsilon\}$. We have

$$(34) \qquad \iint_{G_\varepsilon} u^2(x)\frac{w(y)-w(x)}{w(x)}|x-y|^{-d-\alpha}\,dy\,dx$$

$$= \int_{\{|x_d|>\varepsilon^\delta\}} \frac{u^2(x)}{w(x)} \int_{G_\varepsilon(x)} \frac{w(y)-w(x)}{|x-y|^{d+\alpha}}\,dy\,dx.$$

We write the inner integral in the following form

$$(35) \qquad \int_{G_\varepsilon(x)} \frac{w(y)-w(x)}{|x-y|^{d+\alpha}}\,dy$$

$$= \mathcal{A}_{d,-\alpha}^{-1}\Delta^{\alpha/2}w(x) - \left(\int_{|x-y|<\varepsilon} + \int_{\varepsilon\le|x-y|<\varepsilon^\eta,\,|y_d|<\varepsilon^\delta} + \int_{|x-y|\ge\varepsilon^\eta,\,|y_d|<\varepsilon^\delta}\right.$$

$$\left. + \int_{|x-y|\ge\varepsilon,\,|y_d|\ge\varepsilon^\delta,\,|x^*-y|<\varepsilon}\right) \frac{w(y)-w(x)-(y-x)\nabla w(x)1_{|y-x|<\varepsilon^\eta}}{|x-y|^{d+\alpha}}\,dy$$

$$=: \mathcal{A}_{d,-\alpha}^{-1}\Delta^{\alpha/2}w(x) - I_1 - I_2 - I_3 - I_4.$$

Let $K \subset D$ be a compact set such that $\text{dist}(\text{supp}\,u, K^c) > 0$. Before we estimate the integrals I_j, we observe that there exists a constant $C > 0$, depending on K and

w, but not on ε, such that

$$(36) \qquad |\overline{w}(y)| \leq C, \quad y \in \mathbb{R}^d,$$

$$(37) \qquad |\overline{w}(x)| \geq C^{-1}, \quad x \in K.$$

$$(38) \qquad |w(y) - w(x) - (y - x)\nabla w(x)| \leq C|x - y|^2, \quad \text{if } y \in B(x, \varepsilon^\eta) \subset K,$$

We take $x \in \operatorname{supp} u$ with $|x_d| > \varepsilon^\delta$ and consider $\varepsilon > 0$ small enough to ensure that $B(x, \varepsilon^\eta) \subset K$. By (38) we have

$$|I_1| \leq C \int_{|x-y|<\varepsilon} |x - y|^{2-d-\alpha} \, dy = c\varepsilon^{2-\alpha},$$

$$|I_2| \leq C \int_{|x-y|<\varepsilon^\eta, \, |y_d|<\varepsilon^\delta} |x - y|^{2-d-\alpha} \, dy \leq c\varepsilon^{\delta+\eta(2-\alpha)}.$$

From (36) and (37) we conclude that

$$|w(y)| \leq C|y_d| \leq C^2 \frac{|y_d|}{|x_d|}|w(x)| \leq C^2|w(x)|, \quad \text{provided } |y_d| < |x_d|,$$

hence

$$|I_3| \leq c|w(x)| \int_{|x-y|\geq\varepsilon^\eta, \, |y_d|<\varepsilon^\delta} |x - y|^{2-d-\alpha} \, dy \leq c'|w(x)|\varepsilon^{\delta-\eta\alpha}.$$

From (38) and (36) we get

$$|w(y) - w(x) - (y - x)\nabla w(x)1_{|y-x|<\varepsilon^\eta}| \leq C.$$

Hence

$$|I_4| \leq C \int_{|x-y|\geq\varepsilon, \, |y_d|\geq\varepsilon^\delta, \, |x^*-y|<\varepsilon} |x - y|^{-d-\alpha} \, dy.$$

When $\varepsilon^\delta \leq |y_d|$ and $|x^* - y| < \varepsilon$, then

$$|y - x| \geq |x - x^*| - |y - x^*| \geq 2\varepsilon^\delta - \varepsilon \geq \varepsilon^\delta,$$

and therefore

$$|I_4| \leq C\varepsilon^{d-\delta(d+\alpha)}.$$

Finally, using estimates for I_j and (37) we obtain

$$|1_{\{|x_d|>\varepsilon^\delta\}} \frac{u^2(x)}{w(x)}(I_1 + I_2 + I_3 + I_4)|$$

$$\leq cu^2(x)(\varepsilon^{2-\alpha-\delta} + \varepsilon^{\eta(2-\alpha)} + \varepsilon^{\delta-\eta\alpha} + \varepsilon^{d-\delta(d+\alpha+1)}).$$

We choose $\delta, \eta \in (0, 1)$ so that $2 - \alpha - \delta > 0$, $d - \delta(d + \alpha + 1) > 0$ and $\delta - \eta\alpha > 0$. From (34), (35) and the fact that $\Delta^{\alpha/2}w(x)/w(x)$ has a constant sign, we conclude that

$$\mathcal{A}_{d,-\alpha} \iint_{G_\varepsilon} u^2(x)\frac{w(y) - w(x)}{w(x)}|x - y|^{-d-\alpha} \, dy \, dx \to \int_D u^2(x)\frac{\Delta^{\alpha/2}w(x)}{w(x)} \, dx,$$

and the lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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