Kawasaki dynamics in continuum: micro- and mesoscopic descriptions

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Abstract

The time evolution of an infinite system of interacting point particles on $\mathbb{R}^d$ is described on both micro- and mesoscopic levels as the evolution $\mu_0 \mapsto \mu_t$ of probability measures on the configuration space Γ. The particles are supposed to hop over $\mathbb{R}^d$ and repel each other, similarly to the Kawasaki dynamics on the lattice $\mathbb{Z}^d$. The microscopic description is based on solving linear equations for the correlation functions by means of a combination of methods including an Ovcyannikov-type technique, which yields the evolution in a scale of Banach spaces. Then the evolution of the corresponding measures is obtained therefrom by a special procedure based on local approximations. The mesoscopic description is performed within the Vlasov scaling method, which yields a linear infinite chain of equations obtained from those for the correlation function of the model. Its main peculiarity is that for the initial r_0 being the correlation function of the inhomogeneous Poisson measure with density ϱ_0, the solution r_t is the correlation function of such a measure with density ϱ_t which solves a nonlinear differential equation of convolution type.

1 Introduction

Models of infinite systems of interacting point particles distributed over $\mathbb{R}^d$ are widely used in such areas of applied mathematics as mathematical

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physics, ecology, sociology, etc. Therefore, the rigorous description of various properties of such models, including their time evolution, is an actual mathematical problem, see [3, 4, 5, 6, 7, 8, 11, 12] and the literature quoted in those papers. Basically, in this area the states of the system are described in terms of probability measures on the space of the particle configurations

$$\Gamma \equiv \Gamma(\mathbb{R}^d) := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d \}, \quad (1.1)$$

where $|A|$ denotes the cardinality of A. Thus, the evolution of the states is the evolution of measures, where one might distinguish between equilibrium and non-equilibrium cases. The equilibrium evolution is built with the help of the reversible (Gibbs) measures, if such exist for the considered model, and with the corresponding Dirichlet forms. The result is a stationary Markov process, see [13]. The non-equilibrium evolution, where the initial state can be "far away" from the thermal equilibrium, is much more interesting and much more complex – for the model considered in this work, it has been constructed only for noninteracting particles [14]. Our goal in this paper is to go further in this direction.

The non-equilibrium evolution $\mu_0 \mapsto \mu_t$ of the states of a given system can be defined by means of a Kolmogorov-type (Fokker-Planck) equation

$$\frac{d}{dt}\mu_t = L^* \mu_t, \qquad \mu_t|_{t=0} = \mu_0, \quad (1.2)$$

where the 'generator' L^* is specified within the choice of the model. Often, the only possibility to approach the problem is to use the adjoint equation

$$\frac{d}{dt}F_t = LF_t, \qquad F_t|_{t=0} = F_0, \quad (1.3)$$

related to (1.2) by the duality

$$\int_\Gamma F_t d\mu_0 = \int_\Gamma F_0 d\mu_t, \quad (1.4)$$

which would yield the evolution in question in a weak sense. However, for nontrivial models, even such weak evolution cannot be obtained directly. Following classical works on the Hamiltonian dynamics one can try to study the evolution $\mu_0 \mapsto \mu_t$ via the evolution of the corresponding correlation functions obtained from the equation

$$\frac{d}{dt}k_t = L^\Delta k_t, \qquad k_t|_{t=0} = k_0, \quad (1.5)$$

where the relationship between k_t and μ_t may not be straightforward[1]. This means that the solutions of (1.5) need not be a correlation functions and

[1]In those classical works, (1.5) is known as the BBGKY hierarchy and the mentioned connection is usually not discussed.

hence do not correspond to the states of the system. Typically, one tries to solve (1.5) by constructing an operator semigroup $S(t)$ such that $k_t = S(t)k_0$, which in nontrivial cases inevitably leads to certain restrictions typical to perturbation schemes. At the same time, it is clear that such methods cannot describe many interesting features of the system behavior. An alternative approach could be based on methods which guarantee the existence of solutions on bounded time intervals only. Noteworthy, (1.5) is in fact an infinite chain of linear equations and L^{Δ} can be defined as a continuous operator only in trivial cases, which makes Peano-type arguments useless in this situation.

The equations (1.2), (1.3), (1.5) provide the microscopic description of the system dynamics, which can be redundantly detailed in some sense. The mesoscopic description allows one to ignore some 'less important' details and hence to simplify the picture. In theoretical physics, it is achieved by truncating the chain as in (1.5), in which higher order correlation functions are replaced by the products of the functions of lower orders. This leads to a closed system of finitely many equations (e.g. to a single equation), which is, however, nonlinear. A typical example is the Boltzmann equation obtained from the BBGKY hierarchy – see [4] for more details on this issue. Nowadays, one of the mathematically consistent ways of constructing the description of this kind is the Vlasov scaling. In its framework, we obtain from (1.5) a new chain of linear equations for limiting 'correlation functions' r_t, which, however, may not be correlation functions at all but have one important property. Namely, if the initial state μ_0 is the Poisson measure with density ϱ_0, i.e., the particles are placed into an external field and do not interact, then μ_t is also the Poisson measure with some density ϱ_t, at least for small enough t. This resembles the mean-field approximation – a celebrated tool of theoretical physics. The density evolution $\varrho_0 \mapsto \varrho_t$ is obtained from a nonlinear equation, see (4.15) below, which in addition is also nonlocal, c.f. [15].

In the present work, we consider an infinite system of interacting particles in continuum, which change their positions in $\mathbb{R}^d$ by hopping from site to site and which has a Gibbs measure as invariant state – the so called Kawasaki system. For this system, we manage to construct the evolution of states on both micro- and mesoscopic levels. The construction of the microscopic evolution is performed in Section 3 in the following steps. First, we consider a finite Kawasaki system, for which in Theorem 3.1 we obtain the evolution $\mu_0 \mapsto \mu_t$ described by a stochastic semigroup. Here we crucially use a version of Miyadera's theorem obtained in [19] as Theorem 2.7. Next, for an infinite Kawasaki system, we prove that the corresponding equation (1.5) has a unique (classical) solution k_t on some time interval in a scale of Banach spaces (Theorem 3.3). Here we use a version of Ovcyannikov's approach, see [16] and pp. 9–13 in [2]. This result, however, does not guarantee that k_t is the correlation function of a probability measure on Γ. To prove the latter, see Theorem 3.7, we approximate the considered infinite system by finite systems and use Theorem 3.1. Afterwards, in Section 4 we perform the Vlasov scaling and obtain the so called Vlasov hierarchy, which is a linear evolution equation of the type of $dr_t/dt = L_V r_t$ in the same scale of spaces as the equation for the correlation functions. Its main peculiarity is fact that if r_0 is the correlation function of the nonhomogeneous Poisson measure π_{ϱ_0} with density ϱ_0, then the solution r_t is the correlation function for π_{ϱ_t} provided ϱ_t lies in a certain ball in $L^\infty(\mathbb{R}^d)$ and solves a nonlinear nonlocal equation, see Lemma 4.2. In Theorem 4.4, we prove that this is the case if ϱ_0 lies in that ball. Finally, in Theorem 4.5 we obtain that the rescaled correlation functions converge in the scaling limit to the corresponding r_t.

2 The basic notions and the model

2.1 The notions

In this paper, we work in the approach of [5, 6, 7, 8, 9, 12] where all the relevant details can be found. Thus, we provide here only the necessary minimum.

By $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ we denote the sets of all Borel and all bounded Borel subsets of $\mathbb{R}^d$, respectively. For $X \in \mathcal{B}(\mathbb{R}^d)$, the set of n-particle configurations in X is

$$\Gamma_X^{(0)} = \{\emptyset\}, \qquad \Gamma_X^{(n)} = \{\eta \subset X : |\eta| = n\}, \quad n \in \mathbb{N}. \tag{2.1}$$

$\Gamma_X^{(n)}$ can be identified with the symmetrization of the set

$$\{(x_1, \dots, x_n) \in X^n : x_i \neq x_j, \text{ for } i \neq j\} \subset \mathbb{R}^{nd},$$

which allows one to introduce the corresponding topology and hence the Borel σ-algebra $\mathcal{B}(\Gamma_X^{(n)})$. The set of finite configurations in X is

$$\Gamma_{0,X} = \bigsqcup_{n\in\mathbb{N}_0} \Gamma_X^{(n)}. \tag{2.2}$$

We equip it with the topology of the disjoint union and hence with the Borel σ-algebra $\mathcal{B}(\Gamma_{0,X})$. For $X = \mathbb{R}^d$, we write $\Gamma^{(n)}$ and Γ_0 meaning the corresponding sets (2.1) and (2.2), respectively. The set of all configurations in $\mathbb{R}^d$ is defined in (1.1). We equip it with the vague topology, that is the weakest topology in which all the maps

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle = \sum_{x\in\gamma} f(x), \quad f \in C_0(\mathbb{R}^d),$$

are continuous. Here $C_0(\mathbb{R}^d)$ stands for the set of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact supports. In [10], it was shown that the vague topology on Γ admits a metrization which turns it into a complete and separable metric (Polish) space. By $\mathcal{B}(\Gamma)$ we denote the corresponding Borel σ-algebra.

Given $n \in \mathbb{N}$, by $m^{(n)}$ we denote the restriction of the Lebesgue product measure $dx_1 dx_2 \cdots dx_n$ to $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$. Then the Lebesgue-Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is

$$\lambda = \delta_\emptyset + \sum_{n=1}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2.3}$$

For any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, the restriction of λ to $\Gamma_\Lambda := \Gamma_{0,\Lambda}$ will also be denoted by λ. It is worth noting that $\mathcal{B}(\Gamma_\Lambda) = \mathcal{B}(\Gamma) \cap \Gamma_\Lambda$ for all open bounded $\Lambda \subset \mathbb{R}^d$. Moreover, the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ can be viewed as the projective limit of the family $\{(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))\}_{\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)}$. Hence the Poisson measure π on $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the family $\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)}$, where

$$\pi^\Lambda = \exp(-m(\Lambda))\lambda, \tag{2.4}$$

$m(\Lambda)$ being the Lebesgue measure of Λ, see e.g. [1] and references therein. The Poisson measure π_ϱ corresponding to the density $\varrho : \mathbb{R} \to \mathbb{R}_+$ is introduced by means of the measure λ_ϱ, defined as in (2.3) with m replaced by m_ϱ, where, for $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$,

$$m_\varrho(\Lambda) = \int_\Lambda \varrho(x) dx, \tag{2.5}$$

which is supposed to be finite. Then π_ϱ is defined by its projections

$$\pi_\varrho^\Lambda = \exp(-m_\varrho(\Lambda))\lambda_\varrho^\Lambda. \tag{2.6}$$

For a measurable $f : \mathbb{R}^d \to \mathbb{R}$ and $\eta \in \Gamma_0$, the Lebesgue-Poisson exponent is

$$e(f, \eta) = \prod_{x \in \eta} f(x), \qquad e(f, \emptyset) = 1. \tag{2.7}$$

Clearly, $e(f, \cdot) \in L^1(\Gamma_0, d\lambda)$ for any $f \in L^1(\mathbb{R}^d)$, and

$$\int_{\Gamma_0} e(f, \eta)\lambda(d\eta) = \exp\left\{ \int_{\mathbb{R}^d} f(x)dx \right\}. \tag{2.8}$$

A set $M \in \mathcal{B}(\Gamma_0)$ is said to be bounded if

$$M \subset \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)} \tag{2.9}$$

for some $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $N \in \mathbb{N}$. By $B_{\rm bs}(\Gamma_0)$ we denote the set of all bounded measurable functions $G : \Gamma_0 \to \mathbb{R}$, which have bounded supports.

That is, each such G is the zero function on $\Gamma_0 \setminus M$ for some bounded M. Note that any measurable $G : \Gamma_0 \to \mathbb{R}$ is in fact a sequence of measurable symmetric functions $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$.

For $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $\gamma \in \Gamma$, by γ_Λ we denote $\gamma \cap \Lambda$; thus, $\gamma_\Lambda \in \Gamma_{0,\Lambda}$, see (1.1). We consider also the set $\mathcal{F}_{\rm cyl}(\Gamma)$ of *cylinder* functions. Each $F \in \mathcal{F}_{\rm cyl}(\Gamma)$ is characterized by the following relation: $F(\gamma) = F\restriction_{\Gamma_\Lambda}(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$. For $\gamma \in \Gamma$, by writing $\eta \Subset \gamma$ we mean that $\eta \subset \gamma$ and η is finite, i.e., $\eta \in \Gamma_0$. For $G \in B_{\rm bs}(\Gamma_0)$, we set

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \qquad \gamma \in \Gamma. \tag{2.10}$$

It is clear that K maps $B_{\rm bs}(\Gamma_0)$ into $\mathcal{F}_{\rm cyl}(\Gamma)$, and is linear and positivity preserving. This map plays an important role in the theory of configuration spaces, see [9].

By $\mathcal{M}^1_{\rm fm}(\Gamma)$ we denote the set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ which have finite local moments, that is, for which

$$\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < \infty \qquad \text{for all} \ \ n \in \mathbb{N} \ \ \text{and} \ \ \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d). \tag{2.11}$$

A measure ρ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is said to be *locally finite* if $\rho(M) < \infty$ for every bounded $M \subset \Gamma_0$. By $\mathcal{M}_{\rm lf}(\Gamma_0)$ we denote the set of all such measures. For $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, p_Λ stands for the map $\Gamma \ni \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda$. Then, for $A \subset \Gamma_\Lambda$, we write $p_\Lambda^{-1}(A) = \{\gamma \in \Gamma : p_\Lambda(\gamma) \in A\}$.

Definition 2.1. *A measure $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$ is said to be locally absolutely continuous with respect to the Poisson measure π if, for every $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, the projection*

$$\mu^\Lambda := \mu \circ p_\Lambda^{-1} \tag{2.12}$$

is absolutely continuous with respect to π^Λ and hence with respect to λ, see (2.4).

Let $M \subset \Gamma_0$ be bounded, and let $\mathbb{I}_M$ be its indicator function on Γ_0. Then $\mathbb{I}_M$ is in $B_{\rm bs}(\Gamma_0)$ and hence one can apply (2.10). For $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$, let

$$\rho_\mu(M) = \int_\Gamma (K\mathbb{I}_M)(\gamma)\mu(d\gamma), \tag{2.13}$$

which uniquely determines a measure $\rho_\mu \in \mathcal{M}_{\rm lf}(\Gamma_0))$. It is called the *correlation measure* for μ. This defines a map $K^* : \mathcal{M}^1_{\rm fm}(\Gamma) \to \mathcal{M}_{\rm lf}(\Gamma_0))$ such that $K^*\mu = \rho_\mu$. In particular, $K^*\pi = \lambda$. It is known, see Proposition 4.14 in [9], that ρ_μ is absolutely continuous with respect to λ if μ is locally absolutely continuous with respect to π. In this case, we have that for any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$,

$$\begin{aligned} k_\mu(\eta) = \frac{d\rho_\mu}{d\lambda}(\eta) &= \int_{\Gamma_{0,\Lambda}} \frac{d\mu^\Lambda}{d\pi^\Lambda}(\eta \cup \gamma)\pi^\Lambda(d\gamma) \\ &= \int_{\Gamma_{0,\Lambda}} \frac{d\mu^\Lambda}{d\lambda}(\eta \cup \xi)\lambda(d\xi). \end{aligned} \tag{2.14}$$

The Radon-Nikodym derivative k_μ is called the *correlation function* corresponding to the measure μ. The following fact is known, see Theorems 6.1, 6.2 and Remark 6.3 in [9].

Proposition 2.2. *Suppose $\rho \in \mathcal{M}_{\rm lf}(\Gamma_0))$ has the properties*

$$\rho(\{\emptyset\}) = 1, \qquad \int_{\Gamma_0} G(\eta)\rho(d\eta) \geq 0, \tag{2.15}$$

for every $G \in B_{\rm bs}(\Gamma_0)$ such that $KG \geq 0$. Then there exist $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$ such that $K^*\mu = \rho$. For the uniqueness of such μ, it is enough that the Radon-Nikodym derivative (2.14) of ρ obeys

$$k(\eta) \leq \prod_{x\in\eta} C_R(x), \tag{2.16}$$

for all $\eta \in \Gamma_0$ and for some locally integrable $C_R : \mathbb{R}^d \to \mathbb{R}_+$.

Finally, we mention the following integration rule, c.f. Lemma 2.1 in [6],

$$\int_{\Gamma_0} \sum_{\xi\subset\eta} H(\xi, \eta\setminus\xi, \eta)\lambda(d\eta) = \int_{\Gamma_0}\int_{\Gamma_0} H(\xi, \eta, \eta\cup\xi)\lambda(d\xi)\lambda(d\eta), \tag{2.17}$$

which holds for any appropriate function H.

2.2 The model

The Kawasaki dynamics is conservative, which means that the particles do not appear or disappear – they change their positions by hopping over the space $\mathbb{R}^d$. In terms of observables, i.e., appropriate functions $F : \Gamma \to \mathbb{R}$, the dynamics is generated by the following heuristic 'operator'

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x, y, \gamma)\,[F(\gamma\setminus x \cup y) - F(\gamma)]\,dy, \tag{2.18}$$

where the hopping rate c depends on the configuration $\gamma \in \Gamma$ and on the initial and target points x and y, respectively. This means that the evolution of states $\mu_0 \mapsto \mu_t$ is obtained as a solution of the Cauchy problem

$$\frac{d}{dt}\langle\langle F, \mu_t\rangle\rangle = \langle\langle LF, \mu_t\rangle\rangle, \quad \mu_t|_{t=0} = \mu_0, \tag{2.19}$$

where

$$\langle\langle F, \mu\rangle\rangle := \int_\Gamma F(\gamma)\mu(d\gamma). \tag{2.20}$$

In our particular model, the particles interact pairwise. For the two located at x and y, the interaction energy is $\phi(x-y)$, where the *interaction potential* $\phi : \mathbb{R} \to [0, +\infty) := \mathbb{R}_+$ is such that $\phi(x) = \phi(-x)$ and

$$c_\phi := \int_{\mathbb{R}^d} \left(1 - e^{-\phi(x)}\right) dx < \infty. \tag{2.21}$$

That is, the particles repel each other. For a configuration γ, the total energy is

$$E^{\phi}(\gamma) = \sum_{\{x,y\}\subset\gamma} \phi(x-y),$$

and the energy increment caused by the appearance of a particle at point z is

$$E^{\phi}(z,\gamma) = \sum_{x\in\gamma} \phi(x-z). \tag{2.22}$$

The hopping rate in (2.18) has the form

$$c(x,y,\gamma) = a(x-y)\exp\Big(-E^{\phi}(y,\gamma)\Big), \tag{2.23}$$

where $a : \mathbb{R} \to \mathbb{R}_+$ is such that $a(x) = a(-x)$ and

$$\alpha := \int_{\mathbb{R}^d} a(x)dx < \infty. \tag{2.24}$$

The above model described in (2.18) – (2.24) will be called the *Kawasaki system*.

3 Microscopic dynamics

The direct construction of the non-equilibrium dynamics of states of infinite systems based on (2.19), (2.20), even for very simple models of this sort, is usually unrealistic. In our case, we proceed as follows. Since the Kawasaki dynamics is conservative, one can try to construct the evolution of states of finite systems directly, which we do in Subsection 3.1 below. This allows us to obtain the evolution of the projections (2.12) $\mu_0^{\Lambda} \mapsto \mu_t^{\Lambda}$, $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, as the evolution of probability measures on Γ_0. Note that μ_t^{Λ}, $t > 0$, is no more supported on Γ_{Λ} as the particles can leave Λ. The next step is to describe the evolution of the whole infinite system as the evolution of the correlation functions, see (2.14). We refer to [8] where the derivation of the corresponding equations and a detailed description of the whole approach can be found. Thus, we consider the Cauchy problem

$$\begin{aligned} \frac{d}{dt}\langle\langle G, k_t\rangle\rangle &= \langle\langle \hat{L}G, k_t\rangle\rangle = \langle\langle G, L^{\Delta}k_t\rangle\rangle, \\ k_t|_{t=0} &= k_0, \end{aligned} \tag{3.1}$$

where

$$\langle\langle G, k\rangle\rangle = \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta), \tag{3.2}$$

and, c.f. equations (4.7) and (4.8) in [8],

$$\begin{aligned} \left(\hat{L}G\right)(\eta) &= \sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} a(x-y)e(\tau_y,\xi) \\ &\times \ e(t_y,\eta\setminus\xi)\left[G((\xi\setminus x)\cup y)-G(\xi)\right]dy, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \left(L^{\Delta}k\right)(\eta) &= \sum_{y\in\eta}\int_{\mathbb{R}^d} a(x-y)e(\tau_y,\eta\setminus y\cup x) \\ &\times \left(\int_{\Gamma_0} e(t_y,\xi)k(\xi\cup x\cup\eta\setminus y)\lambda(d\xi)\right)dx \\ &- \int_{\Gamma_0} k(\xi\cup\eta)\left(\sum_{x\in\eta}\int_{\mathbb{R}^d} a(x-y)e(\tau_y,\eta)\right. \\ &\times \left. e(t_y,\xi)dy\right)\lambda(d\xi). \end{aligned} \tag{3.4}$$

Here e is as in (2.7) and

$$t_x(y) = e^{-\phi(x-y)} - 1, \qquad \tau_x(y) = t_x(y) + 1. \tag{3.5}$$

In order for the problem (3.1) to make sense we have to place it into an appropriate Banach space setting. Here we have the following two possibilities: (a) to obtain the evolution $k_0 \mapsto k_t$ directly as the classical solution of the corresponding Cauchy problem; (b) to study the evolution of quasi-observables $G_0 \mapsto G_t$, and then to get the evolution $k_0 \mapsto k_t$ in the weak sense

$$\langle\langle G_t, k_0\rangle\rangle = \langle\langle G_0, k_t\rangle\rangle, \tag{3.6}$$

see (3.2). In Subsection 3.2 and 3.3 below we realize both mentioned possibilities and obtain $k_0 \mapsto k_t$, $t \in [0, T_*)$, by means of Ovcyannikov-type arguments. As in this setting we have no stochastic semigroups, the fact that k_0 is the correlation function for μ_0 does not imply that the obtained k_t is a correlation function of any measure $\mu_t \in \mathcal{M}^1(\Gamma)$. This means that the integrals (2.15) with $\rho_t(d\eta) = k_t(\eta)\lambda(d\eta)$ may not be positive. Then we use the evolution $\mu_0^\Lambda \mapsto \mu_t^\Lambda$, obtained in Subsection 3.1, which yields the evolution of the corresponding correlation functions $k_0^\Lambda \mapsto k_t^\Lambda$, $t \in \mathbb{R}_+$, and show that, for every $t \in [0, T_*)$, $k_t^\Lambda \to k_t$ as $\Lambda \to \mathbb{R}^d$. Here the convergence holds in the sense that $\langle\langle G, k_t^\Lambda\rangle\rangle \to \langle\langle G, k_t\rangle\rangle$ for all $G \in B_{\rm bs}(\Gamma_0)$. Then we apply Proposition 2.2 and obtain the evolution $\mu_0 \mapsto \mu_t$, $t \in [0, T_*)$.

3.1 The evolution of a finite system

Here we assume that the initial state μ_0 in (2.19) is supported on some Γ_Λ, $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$. We also assume that μ_0 is locally absolutely continuous with respect to the Poisson measure, which together with the former assumption imply that μ_0 is absolutely continuous with respect to the Lebesgue-Poisson measure λ. For such μ, we set, c.f. (2.14),

$$R_\mu(\eta) = \frac{d\mu}{d\lambda}(\eta), \qquad \eta \in \Gamma_0. \tag{3.7}$$

Obviously, R_μ is a positive element of unit norm of the Banach space $\mathcal{R} := L^1(\Gamma_0, d\lambda)$. Suppose that μ in (2.20) is of this type and let L^* be the adjoint 'operator' in the following sense

$$\langle\langle LF, \mu\rangle\rangle = \int_{\Gamma_0} F(\eta)(L^* R_\mu)(\eta)\lambda(d\eta). \tag{3.8}$$

By (2.17), we get

$$\begin{aligned}(L^* R_\mu)(\eta) &:= (A+B)R_\mu(\eta) \\ &= \sum_{y\in\eta}\int_{\mathbb{R}^d} a(x-y)e(\tau_y, \eta\setminus y\cup x)R_\mu(\eta\setminus y \cup x)dx - \Psi(\eta)R_\mu(\eta),\end{aligned} \tag{3.9}$$

that is, B is the multiplication operator by the function

$$\Psi(\eta) = \int_{\mathbb{R}^d}\sum_{x\in\eta} a(x-y)e(\tau_y, \eta)dy. \tag{3.10}$$

Clearly, $(AR_\mu)(\eta) \geq 0$ if $R_\mu(\eta) \geq 0$, $\Psi(\eta) \geq 0$ and

$$\Psi(\eta) \leq \alpha|\eta|, \tag{3.11}$$

as $e(\tau_y, \eta) \leq 1$. Moreover, by (2.17)

$$\int_{\Gamma_0}(AR_\mu)(\eta)\lambda(d\eta) = -\int_{\Gamma_0}(BR_\mu)(\eta)\lambda(d\eta). \tag{3.12}$$

Set

$$\mathcal{D} = \{R \in \mathcal{R} : \Psi R \in \mathcal{R}\}, \tag{3.13}$$

which is certainly dense in $\mathcal{R}$. Then (3.9) and (3.10) define an operator $L^* : \mathcal{D} \to \mathcal{R}$. In a similar way, we define also A and B given by the first and second summands in (3.9), respectively. Clearly, B is closed as a multiplication operator. Note that the possibility to define A on $\mathcal{D}$ follows from (3.12).

Let $\mathcal{R}^+$ denote the cone of positive elements of $\mathcal{R}$, and $\mathcal{D}^+ = \mathcal{R}^+ \cap \mathcal{D}$. The linear functional

$$\mathcal{R} \ni R \mapsto \varphi(R) = \int_{\Gamma_0} R(\eta)\lambda(d\eta), \tag{3.14}$$

has the property $\varphi(R) = \|R\|_{\mathcal{R}}$ for any $R \in \mathcal{R}^+$. Along with $\mathcal{R}$ we shall use the space $\mathcal{R}_1 = L^1(\Gamma_0, b(\cdot)d\lambda)$ with $b(\eta) = |\eta| + 1$. In this space, the functional corresponding to (3.14) is

$$\varphi_1(R) = \int_{\Gamma_0} R(\eta)(|\eta| + 1)\lambda(d\eta), \tag{3.15}$$

and hence $\varphi_1(R) = \|R\|_{\mathcal{R}_1}$ for all $R \in \mathcal{R}_1^+$. Clearly, $\mathcal{R}_1$ is continuously embedded into $\mathcal{R}$ and $\mathcal{R}_1 \cap \mathcal{R}^+$ is dense in $\mathcal{R}_1^+$. The main result of this subsection is given by the following statements.

Theorem 3.1. *The closure of the operator L^* defined in (3.9), (3.10) generates a stochastic C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{R}$, which leaves $\mathcal{R}_1$ invariant.*

The proof of Theorem 3.1 is based on Theorem 2.7 of [19], which we formulate here in the form adapted to the context.

Proposition 3.2 (Thieme-Voigt)**.** *Let A and B in (3.9) obey the following: (a) B is a generator of a positive C_0-semigroup on $\mathcal{R}$; (b) $-B$ is positive; (c) $A : \mathcal{D} \to \mathcal{R}$ is positive. Suppose also that (d) $\varphi((A + B)R) = 0$ for all $R \in \mathcal{D}^+$; (e) there exist $\varepsilon > 0$ and $C > 0$ such that, for all $R \in \mathcal{D}_+$ with $BR \in \mathcal{R}_1$, the following holds*

$$\varphi_1((A + B)R) \leq C\varphi_1(R) - \varepsilon\|BR\|_{\mathcal{R}}. \tag{3.16}$$

Then the closure of $A + B$ is the generator of a stochastic C_0-semigroup on $\mathcal{R}$, which leaves $\mathcal{R}_1$ invariant.

Proof of Theorem 3.1. Thus, we have to check that the conditions of the Thieme-Voigt theorem are satisfied. The validity of (a), (b), and (c) is immediate; (d) follows from (3.12). By (3.15) and (3.9), we have

$$\begin{aligned}\varphi_1((A + B)R) &= \int_{\Gamma_0} R(\eta) \sum_{x\in\eta} \int_{\mathbb{R}^d} a(x - y) \exp\Big(-E^{\phi}(y, \eta)\Big) \\ &\times \ [b(\eta \setminus x \cup y) - b(\eta)]\, \lambda(d\eta) = 0,\end{aligned}$$

since $b(\eta) = |\eta| + 1$, which reflects the fact that the Kawasaki dynamics is conservative. In view of the latter, the validity of (3.16) readily follows from (3.11). □

3.2 The evolution of correlation functions

We consider the problem

$$\frac{d}{dt}k_t = L^{\Delta}k_t, \qquad k_t|_{t=0} = k_0, \tag{3.17}$$

in the scale of Banach spaces

$$\mathcal{K}_\vartheta := \{k : \Gamma_0 \to \mathbb{R} : \|k\|_\vartheta < \infty\}, \tag{3.18}$$

where

$$\|k\|_\vartheta := \text{ess sup}\,\{|k(\eta)|\exp(\vartheta|\eta|) : \eta \in \Gamma_0\} \tag{3.19}$$

and ϑ varies in some interval of $\mathbb{R}$. The essential supremum in (3.19) is taken with respect to the Lebesgue-Poisson measure λ defined by (2.3). As usual, by a classical solution of (3.17) in the space $\mathcal{K}_\vartheta$ on the time interval I, we understand a map $I \ni t \mapsto k_t \in \mathcal{K}_\vartheta$, which is continuous on I, continuously differentiable on the interior of I and solves (3.17). Recall that our model is supposed to obey (2.21) and (2.24).

Theorem 3.3. *Given $\vartheta \in \mathbb{R}$ and $T > 0$, we let*

$$\vartheta_0 = \vartheta + \beta T, \qquad \beta = 2\alpha \exp(c_\phi e^{-\vartheta}). \tag{3.20}$$

Then the problem (3.17) with $k_0 \in \mathcal{K}_{\vartheta_0}$ has a unique classical solution $k_t \in \mathcal{K}_\vartheta$ on $[0, T)$.

According to the above theorem, given arbitrary $T > 0$ and ϑ, one can pick the initial space such that the evolution $k_0 \mapsto k_t$ lasts in $\mathcal{K}_\vartheta$ until $t < T$. On the other hand, if the initial space is given, the evolution is restricted in time to the interval $[0, T(\vartheta))$ with

$$T(\vartheta) = \frac{\vartheta_0 - \vartheta}{2\alpha} \exp\left(-c_\phi e^{-\vartheta}\right). \tag{3.21}$$

Clearly, $T(\vartheta_0) = 0$ and $T(\vartheta) \to 0$ as $\vartheta \to -\infty$. Hence, there exists T_*, which depends on ϑ_0, α, and c_ϕ, such that $T(\vartheta) \leq T_*$ for all $\vartheta \in (-\infty, \vartheta_0]$. Set

$$\vartheta(t) = \sup\{\vartheta \in (-\infty, \vartheta_0] : t < T(\vartheta)\}. \tag{3.22}$$

Then the alternative version of the above theorem can be formulated as follows.

Theorem 3.4. *For every $\vartheta_0 \in \mathbb{R}$, there exists $T_* = T_*(\vartheta_0, \alpha, c_\phi)$ such that the problem (3.17) with $k_0 \in \mathcal{K}_{\vartheta_0}$ has a unique classical solution $k_t \in \mathcal{K}_{\vartheta(t)}$ for $t \in [0, T_*)$.*

Proof of Theorem 3.3. We will seek the solution of (3.17) as the limit of the sequence $\{k_t^{(n)}\}_{n\in\mathbb{N}_0} \subset \mathcal{K}_\vartheta$, where $k_t^{(0)} = k_0$ and

$$k_t^{(n)} = k_0 + \int_0^t L^\Delta k_s^{(n-1)} ds, \quad n \in \mathbb{N}. \tag{3.23}$$

The latter can be iterated to yield

$$k_t^{(n)} = k_0 + \sum_{m=1}^{n} \frac{1}{m!} t^m \left(L^\Delta\right)^m k_0. \tag{3.24}$$

Given k, let $L_1^\Delta k$ and $L_2^\Delta k$ denote the first and the second summands in (3.4), respectively. Then, for $\vartheta' < \vartheta''$ and $k \in \mathcal{K}_{\vartheta''}$, we have

$$\begin{aligned} &|(L_1^\Delta k)(\eta)| e^{\vartheta'|\eta|} \\ &\leq \sum_{y\in\eta} \int_{\mathbb{R}^d} dx\ a(x-y) \int_{\Gamma_0} \lambda(d\xi) |k(\eta \backslash y \cup x \cup \xi)| \exp(\vartheta' |\eta \cup \xi|) \\ &\quad \times \exp(-\vartheta'' |\eta \cup \xi|) e(|t_y|, \xi) |e^{\vartheta'|\eta|} \\ &\leq \sum_{y\in\eta} \int_{\mathbb{R}^d} dx\ a(x-y) \int_{\Gamma_0} \lambda(d\xi) \|k\|_{\vartheta''} e^{-\vartheta''|\xi|} \\ &\quad \times e(|t_y|, \xi) | e^{-|\eta|(\vartheta'' - \vartheta')} \\ &= \|k\|_{\vartheta''} \alpha \exp\left(e^{-\vartheta''} c_\phi\right) |\eta| e^{-|\eta|(\vartheta'' - \vartheta'')} \\ &\leq \|k\|_{\vartheta''} \alpha \exp\left(e^{-\vartheta''} c_\phi\right) \frac{1}{e(\vartheta'' - \vartheta')}, \end{aligned} \tag{3.25}$$

which holds for λ-almost all $\eta \in \Gamma_0$. In the last line we have used (2.8) and

$$t e^{-\delta t} \leq 1/e\delta, \qquad t \geq 0, \quad \delta > 0.$$

The same estimate can be obtained also for $L_2^\Delta k$, which finally yields

$$\|L^\Delta k\|_{\vartheta'} \leq \frac{2\alpha}{e(\vartheta'' - \vartheta')} \exp\left(c_\phi e^{-\vartheta''}\right) \|k\|_{\vartheta''}, \qquad \vartheta'' > \vartheta'.$$

Therefore, L^Δ can be defined as a bounded linear operator $L^\Delta : \mathcal{K}_{\vartheta''} \to \mathcal{K}_{\vartheta'}$, $\vartheta' < \vartheta''$, with norm

$$\|L^\Delta\|_{\vartheta''\vartheta'} \leq \frac{2\alpha}{e(\vartheta'' - \vartheta')} \exp\left(c_\phi e^{-\vartheta''}\right). \tag{3.26}$$

For a given $m \in \mathbb{N}$ and $l = 0, \ldots, m$, set $\vartheta_l = \vartheta + (m-l)\epsilon$, $\epsilon = (\vartheta_0 - \vartheta)/m$. Then by (3.26) and (3.24), for any $k, n \in \mathbb{N}$, we get

$$\|k_t^{(n)} - k_t^{(n+k)}\|_\vartheta \leq \sum_{m=n+1}^{n+k} \frac{(m/e)^m}{m!} \left(\frac{t\beta}{\vartheta_0 - \vartheta}\right)^m, \tag{3.27}$$

with β as in (3.20). Thus, the sequence $\{k_t^{(n)}\}_{n\in\mathbb{N}}$ converges in $\mathcal{K}_\vartheta$ uniformly on compact subsets of $[0,T)$. Thereby, its limit is the classical solution of the problem under consideration. The uniqueness can be obtained from the fact that k_t in (3.17) is identically zero if so is k_0, as (3.17) is linear and homogeneous. □

Let now k_t, as a function of $\eta \in \Gamma_0$, be more regular than it is supposed in (3.19). Namely, instead of (3.18) we consider

$$\widetilde{\mathcal{K}}_\vartheta = \{k \in C(\Gamma_0 \to \mathbb{R}) : \|k\|_\vartheta < \infty\}, \tag{3.28}$$

where this time

$$\|k\|_\vartheta = \sup\left\{|k(\eta)| \exp(\vartheta|\eta|) : \eta \in \Gamma_0\right\}. \tag{3.29}$$

Corollary 3.5. *Let ϑ, T, and ϑ_0 be as in Theorem 3.3. Suppose in addition that the function ϕ is continuous. Then the problem (3.17) with $k_0 \in \widetilde{\mathcal{K}}_{\vartheta_0}$ has a unique classical solution $k_t \in \widetilde{\mathcal{K}}_\vartheta$ on $[0,T)$.*

3.3 The evolution of quasi-observables

Now, we consider the problem

$$\frac{dG_t}{dt} = \hat{L}G_t, \qquad G_t|_{t=0} = G_0, \tag{3.30}$$

in the Banach space

$$\mathcal{G}_\vartheta = L^1(\Gamma_0, e^{-\vartheta|\cdot|}d\lambda), \qquad \vartheta \in \mathbb{R}, \tag{3.31}$$

that is, $G \in \mathcal{G}_\vartheta$ if

$$\|G\|_\vartheta \stackrel{\text{def}}{=} \int_{\Gamma_0} \exp(-\vartheta|\eta|)\, |G(\eta)|\, \lambda(d\eta) < \infty. \tag{3.32}$$

Theorem 3.6. *For any $\vartheta_0 \in \mathbb{R}$ and $T > 0$, the Cauchy problem (3.30) with $G_0 \in \mathcal{G}_{\vartheta_0}$ has a unique classical solution $G_t \in \mathcal{G}_\vartheta$ on the interval $[0,T)$, where*

$$\vartheta = \vartheta_0 + \beta T, \qquad \beta{=}2\alpha \exp\left(c_\phi e^{-\vartheta_0}\right). \tag{3.33}$$

Proof. As above, we obtain the solution of (3.30) as the limit of the sequence $\{G_t^{(n)}\}_{n\in\mathbb{N}_0} \subset \mathcal{G}_\vartheta$, where $G_t^{(0)} = G_0$ and

$$G_t^{(n)} = G_0 + \sum_{m=1}^{n} \frac{1}{m!} t^m \hat{L}^m G_0. \tag{3.34}$$

For the norm (3.32), from (3.3) similarly as above by (2.17) we get

$$\|\hat{L}G\|_{\vartheta''} \leq \frac{2\alpha}{e(\vartheta'' - \vartheta')} \exp\left(c_\phi e^{-\vartheta''}\right) \|G\|_{\vartheta'}. \tag{3.35}$$

This means that $\hat{L}$ can be defined as a bounded linear operator $\hat{L} : \mathcal{G}_{\vartheta'} \to \mathcal{G}_{\vartheta''}$ with norm

$$\|\hat{L}\|_{\vartheta'\vartheta''} \leq \frac{2\alpha}{e(\vartheta'' - \vartheta')} \exp\left(c_\phi e^{-\vartheta_0}\right). \tag{3.36}$$

Then we apply the latter estimate in (3.34) and obtain for any $k, n \in \mathbb{N}$

$$\|G_t^{(n)} - G_t^{(n+k)}\|_\vartheta \leq \sum_{m=n+1}^{n+k} \frac{(m/e)^m}{m!} \left(\frac{t\beta}{\vartheta - \vartheta_0}\right)^m, \tag{3.37}$$

The latter estimate yields the proof, as in the case of Theorem 3.3. □

3.4 The evolution of states

Given $\vartheta \in \mathbb{R}$, let $\mathcal{M}_\vartheta$ stand for the set of all $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$, for which $k_\mu \in \mathcal{K}_\vartheta$, see (2.14) and (3.18). On the other hand, let $\mathcal{K}_\vartheta^+$ be the set of all $k \in \mathcal{K}_\vartheta$ such that, c.f. (2.15),

$$\int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta) \geq 0, \tag{3.38}$$

which holds for every $G \in B_{\rm bs}(\Gamma_0)$ such that $KG \geq 0$. Then in view of Proposition 2.2, the map $\mathcal{M}_\vartheta \ni \mu \mapsto k_\mu \in \mathcal{K}_\vartheta^+$ is a bijection as such k_μ certainly obeys (2.16). In what follows, the evolution of states $\mu_0 \mapsto \mu_t$ can and will be understood as the evolution of the corresponding correlation functions $k_{\mu_0} \mapsto k_{\mu_t}$ obtained by solving the problem (3.17).

Theorem 3.7. *Let $\vartheta_0 \in \mathbb{R}$, $\mu_0 \in \mathcal{M}_{\vartheta_0}$ and k_{μ_0} be the corresponding correlation function. There exist $\vartheta < \vartheta_0$ and $T > 0$ such that the evolution $k_{\mu_0} \mapsto k_t$, $t \in [0, T)$ of k_{μ_0} (given by the solution of* (3.30)*) lies in $\mathcal{K}_\vartheta^+$, i.e. there exists an evolution*

$$\mathcal{M}_{\vartheta_0} \ni \mu_0 \mapsto \mu_t \in \mathcal{M}_\vartheta, \quad t \in [0, T)$$

of states of the Kawasaki system.

Proof. The proof is based on Theorems 3.3, 3.4, and 3.6, and on the fact that the evolution described by the former theorem leaves $\mathcal{K}_\vartheta^+$ invariant, which we are going to prove by means of Theorem 3.1.

Given ϑ_0, take any $T < T_*$, see Theorem 3.4. Then there exists $\vartheta < \vartheta_0$ such that the solution of (3.17) lies in $\mathcal{K}_\vartheta$ for $t \in [0, T)$. Next, given $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $\mu_0 \in \mathcal{M}_{\vartheta_0}$, let μ_0^Λ be the projection of μ_0 onto Γ_Λ. Since μ_0 is in $\mathcal{M}^1_{\rm fm}(\Gamma)$, its density (3.7) is in $\mathcal{R}_1$ and the corresponding correlation functions satisfy

$$k_{\mu_0^\Lambda}(\eta) = k_{\mu_0}(\eta)e(\mathbb{I}_\Lambda, \eta) = k_{\mu_0}(\eta) \prod_{x\in\eta} \mathbb{I}_\Lambda(x), \quad \eta \in \Gamma_0. \tag{3.39}$$

Here $\mathbb{I}_\Lambda$ is the indicator function of Λ. Since the density R_0^Λ, see (3.7) of μ_0^Λ is in $\mathcal{R}_1$, we obtain the evolution $\mu_0^\Lambda \mapsto \mu_t^\Lambda$ through the evolution of the corresponding densities as described in Theorem 3.1. For every μ_t^Λ, we have $k_{\mu_t^\Lambda}$, which solves the problem (3.17) with $k_t|_{t=0} = k_{\mu_0^\Lambda}$ and hence lies in $\mathcal{K}_\vartheta^+$ for $t \in [0, T)$. At the same time, the problem (3.17) $k_t|_{t=0} = k_{\mu_0}$ has the solution $k_t \in \mathcal{K}_\vartheta$ for the same $t \in [0, T)$. Our aim is to show that this k_t is in fact in $\mathcal{K}_\vartheta^+$. To this end, it is enough to show that, for a fixed $t \in [0, T)$ and for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ and $\varepsilon > 0$,

$$\left| \int_{\Gamma_0} G(\eta) k_t(\eta) \lambda(d\eta) - \int_{\Gamma_0} G(\eta) k_{\mu_t^\Lambda}(\eta) \lambda(d\eta) \right| < \varepsilon, \tag{3.40}$$

which holds for all sufficiently big Λ. Since our G belongs to any $\mathcal{G}_{\theta_0}$, see (3.31), in view of Theorem 3.6 we can estimate the left-hand side of (3.40) by

$$\begin{aligned} \delta_\Lambda(G) \quad &:= \quad \int_{\Gamma_0} |G_t(\eta)| \left(1 - e(\mathbb{I}_\Lambda, \eta)\right) k_{\mu_0}(\eta) \lambda(d\eta) \qquad (3.41) \\ &= \quad \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{nd}} |G_t^{(n)}(x_1, \ldots, x_n)| J_\Lambda(x_1, \ldots, x_n) \\ &\times \quad k_{\mu_0}^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n, \end{aligned}$$

where $G_t \in \mathcal{G}_\theta$ with θ which will be chosen later. Here

$$\begin{aligned} &J_\Lambda(x_1, \ldots, x_n) = 1 - \mathbb{I}_\Lambda(x_1) \cdots \mathbb{I}_\Lambda(x_n) \qquad (3.42) \\ &\quad = \mathbb{I}_{\Lambda^c}(x_1) \mathbb{I}_\Lambda(x_2) \cdots \mathbb{I}_\Lambda(x_n) + \mathbb{I}_{\Lambda^c}(x_2) \mathbb{I}_\Lambda(x_3) \cdots \mathbb{I}_\Lambda(x_n) \\ &\quad + \cdots + \mathbb{I}_{\Lambda^c}(x_{n-1}) \mathbb{I}_\Lambda(x_n) + \mathbb{I}_{\Lambda^c}(x_n), \end{aligned}$$

where $\Lambda^c = \mathbb{R}^d \setminus \Lambda$. Now we take $\theta < \vartheta_0$ and fix $\varepsilon > 0$. As G_t is in $\mathcal{G}_\theta$, one can pick $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty} \frac{e^{-n\theta}}{n!} \int_{\mathbb{R}^{nd}} |G_t^{(n)}(x_1, \ldots, x_n)| dx_1 \cdots dx_n < e(\vartheta_0 - \theta)\varepsilon/2. \tag{3.43}$$

Now one applies (3.42) in (3.41), takes into account that $k_{\mu_0} \in \mathcal{K}_{\vartheta_0}$, and obtains

$$\begin{aligned}\delta_\Lambda(G) &\le \sum_{n=1}^{\infty} \frac{n}{n!} e^{-n\vartheta_0} \int_{\Lambda^c} dx_1 \int_{\mathbb{R}^{(n-1)d}} |G_t^{(n)}(x_1,\ldots,x_n)| dx_2 \cdots dx_n \\ &\le \frac{1}{e(\vartheta_0-\theta)} \\ &\times \left(\sum_{n=1}^{n_\varepsilon} \frac{1}{n!} e^{-n\theta} \int_{\Lambda^c} dx_1 \int_{\mathbb{R}^{(n-1)d}} |G_t^{(n)}(x_1,\ldots,x_n)| dx_2 \cdots dx_n \right. \\ &\left. + \sum_{n=n_\varepsilon+1}^{\infty} \frac{1}{n!} e^{-n\theta} \int_{\mathbb{R}^{nd}} |G_t^{(n)}(x_1,\ldots,x_n)| dx_1 \cdots dx_n \right).\end{aligned} \tag{3.44}$$

Since the first sum in the latter two lines contains only a finite number of summands, it can be made arbitrarily small by picking big enough Λ. The second sum can be estimated by (3.43), which finally yields (3.40). Thus, for every $t \in [0,T)$, the solution k_t of (3.17) with $k_t|_{t=0} = k_{\mu_0}$ is in $\mathcal{K}_\vartheta^+$ and hence is the correlation function of a measure $\mu_t \in \mathcal{M}_\vartheta$. □

4 Mesoscopic dynamics

In our setting, the description of the dynamics of the considered model on the mesoscopic level is obtained by means of the Vlasov scaling, which was first introduced to describe mesoscopic properties of plasma. We refer to the monograph [18] as to the source of general concepts in this field, as well as to the recent paper [4] where the peculiarities of the Vlasov scaling applied to continuous particle systems are given together with the most updated bibliography on this item.

4.1 The Vlasov hierarchy

The main idea of the Vlasov scaling is to make the particle system more and more dense whereas the interaction (repulsion) respectively weaker. This corresponds to the so called mean field approximation widely employed in theoretical physics. The object of these manipulations will be the problem (3.17). The scaling parameter $\varepsilon > 0$ in the scaling limit will tend to zero. The first step is to assume that the initial state depends on ε in such a way that the particle density diverges as $\varepsilon \to 0$. Let $k_0^{(\varepsilon)}$ be the corresponding correlation function. In order to compensate this divergence we pass to the so called renormalized correlation function

$$k_{0,\mathrm{ren}}^{(\varepsilon)}(\eta) = \varepsilon^{|\eta|} k_0^{(\varepsilon)}, \tag{4.1}$$

and assume that in the limit $\varepsilon \to 0$, we obtain $k^{(\varepsilon)}_{0,\mathrm{ren}} \to r_0$, where r_0 might be a correlation function of a certain measure. Having this in mind, we consider

$$\frac{d}{dt} k^{(\varepsilon)}_t = L^\Delta_\varepsilon k^{(\varepsilon)}_t, \qquad k^{(\varepsilon)}_t|_{t=0} = k^{(\varepsilon)}_0, \tag{4.2}$$

where L^Δ_ε is as in (3.4) but with ϕ multiplied by ε. Next we expect that the solution $k^{(\varepsilon)}_t$, which exists in view of Theorem 3.3, diverges as $\varepsilon \to 0$. Thus, similarly as in (4.1) we pass to

$$k^{(\varepsilon)}_{t,\mathrm{ren}}(\eta) = \varepsilon^{|\eta|} k^{(\varepsilon)}_t, \tag{4.3}$$

which means that instead of (4.2) we are going to solve the following problem

$$\frac{d}{dt} k^{(\varepsilon)}_{t,\mathrm{ren}} = L_{\varepsilon,\mathrm{ren}} k^{(\varepsilon)}_{t,\mathrm{ren}} \qquad k^{(\varepsilon)}_{t,\mathrm{ren}}|_{t=0} = k^{(\varepsilon)}_{0,\mathrm{ren}}, \tag{4.4}$$

with

$$L_{\varepsilon,\mathrm{ren}} = R^{-1}_\varepsilon L^\Delta_\varepsilon R_\varepsilon, \qquad (R_\varepsilon k)(\eta) = \varepsilon^{-|\eta|} k(\eta). \tag{4.5}$$

By (3.4)

$$\begin{aligned} (L_{\varepsilon,\mathrm{ren}} k)(\eta) &= \sum_{y\in\eta} \int_{\mathbb{R}^d} a(x-y) e(\tau^{(\varepsilon)}_y, \eta \setminus y \cup x) \\ &\times \left(\int_{\Gamma_0} e(\varepsilon^{-1} t^{(\varepsilon)}_y, \xi) k(\xi \cup x \cup \eta \setminus y) \lambda(d\xi) \right) dx \\ &- \int_{\Gamma_0} k(\xi \cup \eta) \left(\sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) e(\tau^{(\varepsilon)}_y, \eta) \right. \\ &\times \left. e(\varepsilon^{-1} t^{(\varepsilon)}_y, \xi) dy \right) \lambda(d\xi), \end{aligned} \tag{4.6}$$

where, c.f. (3.5),

$$t^{(\varepsilon)}_x(y) = e^{-\varepsilon\phi(x-y)} - 1, \qquad \tau^{(\varepsilon)}_x(y) = t^{(\varepsilon)}_x(y) + 1. \tag{4.7}$$

As in (3.26), for any $\vartheta' \in \mathbb{R}$ and $\vartheta'' > \vartheta'$, we have

$$\|L_{\varepsilon,\mathrm{ren}}\|_{\vartheta''\vartheta'} \le \frac{2\alpha}{e(\vartheta'' - \vartheta')} \exp\left(c^{(\varepsilon)}_\phi e^{-\vartheta''} \right), \tag{4.8}$$

where, c.f. (2.21),

$$c^{(\varepsilon)}_\phi = \varepsilon^{-1} \int_{\mathbb{R}^d} \left(1 - e^{-\varepsilon\phi(x)}\right) dx. \tag{4.9}$$

Suppose now that ϕ is in $L^1(\mathbb{R}^d)$ and set

$$\langle \phi \rangle = \int_{\mathbb{R}^d} \phi(x) dx. \tag{4.10}$$

Recall that we still assume $\phi \geq 0$. Then

$$\|L_{\varepsilon,\mathrm{ren}}\|_{\vartheta''\vartheta'} \leq \sup_{\varepsilon>0}\{\mathrm{RHS}(4.8)\} = \frac{2\alpha}{e(\vartheta''-\vartheta')} \exp\left(\langle \phi \rangle e^{-\vartheta''}\right). \tag{4.11}$$

Let us now, informally, pass in (4.6) to the limit $\varepsilon \to 0$. Then we get the following operator

$$\begin{aligned} (L_V k)(\eta) &= \sum_{y\in\eta} \int_{\mathbb{R}^d} a(x-y) \int_{\Gamma_0} e(-\phi(y-\cdot),\xi) \\ &\times \ k(\xi \cup x \cup \eta \setminus y) \lambda(d\xi) dx \\ &- \ \int_{\Gamma_0} k(\xi\cup\eta) \sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) \\ &\times \ e(-\phi(y-\cdot),\xi) dy \lambda(d\xi). \end{aligned} \tag{4.12}$$

It certainly obeys

$$\|L_V\|_{\vartheta''\vartheta'} \leq \frac{2\alpha}{e(\vartheta''-\vartheta')} \exp\left(\langle \phi \rangle e^{-\vartheta''}\right), \tag{4.13}$$

and hence along with (4.2) we can consider the problem

$$\frac{d}{dt} r_t = L_V r_t, \qquad r_t|_{t=0} = r_0, \tag{4.14}$$

which can be called the *Vlasov hierarchy* for the Kawasaki system which we consider. Repeating the arguments used in the proof of Theorem 3.3 we obtain the following

Proposition 4.1. *For every $\vartheta_0 \in \mathbb{R}$, there exists $T_* = T_*(\vartheta_0, \alpha, \langle\phi\rangle)$ such that the problem (4.4) (resp. (4.13)) with any $\varepsilon > 0$ and $k_0^{(\varepsilon)} \in \mathcal{K}_{\vartheta_0}$ (resp. $r_0 \in \mathcal{K}_{\vartheta_0}$) has a unique classical solution $k_t^{(\varepsilon)} \in \mathcal{K}_{\vartheta(t)}$ (resp. $r_t \in \mathcal{K}_{\vartheta(t)}$) for $t \in [0, T_*)$.*

Suppose now that, for a fixed $\varepsilon > 0$, $k_0^{(\varepsilon)}$ is the correlation function for a certain $\mu_0^{(\varepsilon)} \in \mathcal{M}_{\nu_0}$. In view of Theorem 3.7 we then have that the solution $k_t^{(\varepsilon)}$ is also a correlation function. However, this could not be the case for r_t, even if $r_0 = k_0^{(\varepsilon)}$. The second observation concerning r_t is that we still do not know how 'close' it is to $k_t^{(\varepsilon)}$, as the passage from $L_{\varepsilon,\mathrm{ren}}$ to L_V was informal. In the remaining part of the article we give answers to both these questions.

4.2 The Vlasov equation

Here we are going to show that the problem (4.13) has a very particular solution, which gives sense to the whole construction. For the kernel a and an appropriate $g : \mathbb{R}^d \to \mathbb{R}$, we write

$$(a * g)(x) = \int_{\mathbb{R}^d} a(x-y)g(y)dy,$$

and similarly for $g * \phi$. Then let us consider in $L^\infty(\mathbb{R}^d)$ the following Cauchy problem

$$\begin{aligned} \frac{d}{dt}\varrho_t(x) &= \left(a * \varrho_t\right)(x) \exp\left[-(\varrho_t * \phi)(x)\right] \qquad (4.15) \\ &- \varrho_t(x) \left(a * \exp\left(-\varrho_t * \phi\right)\right)(x), \\ \varrho_t|_{t=0} &= \varrho_0. \end{aligned}$$

Given $\vartheta \in \mathbb{R}$, we denote

$$\begin{aligned} \Delta_\vartheta &= \{\varrho \in L^\infty(\mathbb{R}^d) : \|\varrho\|_{L^\infty(\mathbb{R}^d)} \le e^{-\vartheta}\}, \qquad (4.16) \\ \Delta_\vartheta^+ &= \{\varrho \in \Delta_\vartheta : \varrho(x) \ge 0 \quad \text{a.e}\}. \end{aligned}$$

Lemma 4.2. *Suppose that, for some $T > 0$, the problem (4.15) with $\varrho_0 \in \Delta_\vartheta^+$ has a unique classical solution $\varrho_t \in \Delta_\vartheta^+$ on the time interval $[0, T)$. Then the solution $r_t \in \mathcal{K}_\vartheta$ of the problem (4.13) as in Proposition 4.1 with $r_0(\eta) = e(\varrho_0, \eta)$ has the form*

$$r_t(\eta) = e(\varrho_t, \eta) = \prod_{x\in\eta} \varrho_t(x). \qquad (4.17)$$

Proof. First of all we note that $e(\varrho, \cdot) \in \mathcal{K}_\vartheta$ if and only if $\varrho \in \Delta_\vartheta$, see (3.19). Now set $\tilde{r}_t = e(\varrho_t, \cdot)$ with ϱ_t solving (4.15). This $\tilde{r}_t$ solves (4.14), which can easily be checked by computing d/dt and employing (4.15). In view of the uniqueness as in Proposition 4.1, we then have $\tilde{r}_t = r_t$ on the time interval where both solutions exist, from which it can be continued to the one mentioned in Proposition 4.1. □

Remark 4.3. *As (4.17) is the correlation function for the Poisson measure π_{ϱ_t}, see (2.5) and (2.6), the result of the above lemma can be called the chaos preservation. Indeed, the most chaotic state of the system is the free state described by a Poisson measure.*

Let us show now that the problem (4.15) does have the solution we need. In a standard way (4.15) can be transformed into the following integral equation

$$
\begin{aligned}
\varrho_t(x) &= F\big((\varrho_s)_{s\le t}\big) := \varrho_0(x)e^{-\alpha t} \qquad (4.18)\\
&+ \int_0^t \exp\left(-\alpha(t-s)\right)\left(a*\varrho_s\right)(x)\exp\left[-(\varrho_s*\phi)(x)\right]ds \\
&+ \int_0^t \exp\left(-\alpha(t-s)\right)\varrho_s(x)\left[a*\left(1-\exp\left(-\varrho_s*\phi\right)\right)\right](x)ds.
\end{aligned}
$$

Now for $\varrho_0 \in \varDelta_\vartheta^+$ and some $t>0$, we consider the sequence

$$
\varrho_t^{(0)} = \varrho_0, \qquad \varrho_t^{(n)} := F\big((\varrho^{(n-1)})_{s\le t}\big), \quad n\in\mathbb{N}.
$$

It can easily be checked that $\varrho_t^{(n)} \in \varDelta_\vartheta^+$ for all $n\in\mathbb{N}$. Thus, what we need is to show that it is a Cauchy sequence. For $\varrho_s^{(n-1)}, \varrho_s^{(n-2)} \in \varDelta_\vartheta$, we have

$$
\begin{aligned}
&\left|1-\exp\left(\phi*(\varrho_s^{(n-1)}-\varrho_s^{(n-2)})\right)\right| \qquad (4.19)\\
&\le \left|\phi*(\varrho_s^{(n-1)}-\varrho_s^{(n-2)})\right| \sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{m+1}\left|\phi*(\varrho_s^{(n-1)}-\varrho_s^{(n-2)})\right|^m \\
&\le \langle\phi\rangle\|\varrho_s^{(n-1)}-\varrho_s^{(n-2)}\|_{L^\infty(\mathbb{R}^d)}\exp\left(2\langle\phi\rangle e^{-\vartheta}\right).
\end{aligned}
$$

By means of this estimate, we obtain from (4.18)

$$
\|\varrho_t^{(n)}-\varrho_t^{(n-1)}\|_{L^\infty(\mathbb{R}^d)} \le q(t)\sup_{s\in[0,t]}\|\varrho_s^{(n-1)}-\varrho_s^{(n-2)}\|_{L^\infty(\mathbb{R}^d)},
$$

$$
q(t) := 2\left(1+\langle\phi\rangle\exp\left(-\vartheta+2\langle\phi\rangle e^{-\vartheta}\right)\right)(1-e^{-\alpha t}).
$$

Then, for some $T>0$, we have from the latter

$$
\sup_{t\in[0,T]}\|\varrho_t^{(n)}-\varrho_t^{(n-1)}\|_{L^\infty(\mathbb{R}^d)} \le q(T)\sup_{t\in[0,T]}\|\varrho_t^{(n-1)}-\varrho_t^{(n-2)}\|_{L^\infty(\mathbb{R}^d)}. \qquad (4.20)
$$

Thus, the sequence $\{\varrho_t^{(n)}\}_{n\in\mathbb{N}_0}$ converges to some $\varrho_t \in \varDelta_\vartheta^+$ uniformly on compact subsets of $[0,T)$, for T such that $q(T)<1$. Clearly, this ϱ_t solves (4.15). Since the latter remains in the same set as the initial ϱ_0, the evolution $\varrho_0 \mapsto \varrho_t$ can be continued. Taking into account Lemma 4.2 we come to the following conclusion.

Theorem 4.4. *The unique classical solution of (4.14) with $r_0 = e(\varrho_0,\cdot)$, $\varrho_0\in\varDelta_\vartheta^+$, exists for all $t>0$ and is given by (4.17) with $\varrho_t\in\varDelta_\vartheta^+$ being the solution of (4.15).*

4.3 The scaling limit $\varepsilon \to 0$

Our final task in this work is to show that the solution of (4.4) $k_t^{(\varepsilon)}$ converges in $\mathcal{K}_\vartheta$ uniformly on compact subsets of $[0, T_*)$ to that of (4.14), see Proposition 4.1. Here we should impose an additional condition on the potential ϕ, which, however, seems quite natural. Recall that in this section we suppose $\phi \in L^1(\mathbb{R}^d)$.

Theorem 4.5. *Let ϑ_0 and T_* be as in Proposition 4.1, and for $T \in [0, T_*)$, take ϑ such that $T < T(\vartheta)$, see (3.21). Assume also that $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and consider the problems (4.4) and (4.14) with $k_{0,\mathrm{ren}}^{(\varepsilon)} = r_0 \in \mathcal{K}_{\vartheta_0}$. For their solutions $k_{t,\mathrm{ren}}^{(\varepsilon)}$ and r_t, it follows that $k_{t,\mathrm{ren}}^{(\varepsilon)} \to r_t$ in $\mathcal{K}_\vartheta$, as $\varepsilon \to 0$, uniformly on $[0, T]$.*

Proof. For $n \in \mathbb{N}$, let $k_{t,n}^{(\varepsilon)}$ and $r_{t,n}$ be defined as in (3.24) with $L_{\varepsilon,\mathrm{ren}}$ and L_V, respectively. As in the proof of Theorem 3.3, one can show that the sequences of $k_{t,n}^{(\varepsilon)}$ and $r_{t,n}$ converge in $\mathcal{K}_\vartheta$ to $k_{t,\mathrm{ren}}^{(\varepsilon)}$ and r_t, respectively, uniformly on $[0, T]$. Then, for $\delta > 0$, one finds $n \in \mathbb{N}$ such that, for all $t \in [0, T]$,

$$\|k_{t,n}^{(\varepsilon)} - k_{t,\mathrm{ren}}^{(\varepsilon)}\|_\vartheta + \|r_{t,n} - r_t\|_\vartheta < \delta/2. \tag{4.21}$$

From (3.24) we then have

$$\begin{aligned} \|k_{t,\mathrm{ren}}^{(\varepsilon)} - r_t\|_\vartheta &\leq \left\| \sum_{m=1}^{n} \frac{1}{m!} t^m \left(L_{\varepsilon,\mathrm{ren}}^m - L_V^m \right) r_0 \right\|_\vartheta + \frac{\delta}{2} \\ &\leq \|L_{\varepsilon,\mathrm{ren}} - L_V\|_{\vartheta_0\vartheta} \|r_0\|_{\vartheta_0} T \exp\left(T b(\vartheta)\right) + \frac{\delta}{2}, \end{aligned} \tag{4.22}$$

where, see (4.11) and (4.13),

$$b(\vartheta) := \frac{2\alpha}{e(\vartheta_0 - \vartheta)} \exp\left(\langle \phi \rangle e^{-\vartheta} \right).$$

Here we used the following identity

$$\begin{aligned} L_{\varepsilon,\mathrm{ren}}^m - L_V^m &= \left(L_{\varepsilon,\mathrm{ren}} - L_V\right) L_{\varepsilon,\mathrm{ren}}^{m-1} + L_V \left(L_{\varepsilon,\mathrm{ren}} - L_V\right) L_{\varepsilon,\mathrm{ren}}^{m-2} \\ &+ \cdots + L_V^{m-2} \left(L_{\varepsilon,\mathrm{ren}} - L_V\right) L_{\varepsilon,\mathrm{ren}} + L_V^{m-1} \left(L_{\varepsilon,\mathrm{ren}} - L_V\right). \end{aligned} \tag{4.23}$$

Thus, we have to show that

$$\|L_{\varepsilon,\mathrm{ren}} - L_V\|_{\vartheta_0\vartheta} \to 0, \quad \text{as } \varepsilon \to 0, \tag{4.24}$$

which will allow us to make the first summand in the right-hand side of (4.22) also smaller than $\delta/2$ and thereby to complete the proof.

Subtracting (4.12) from (4.6) we get

$$\begin{aligned}(L_{\varepsilon,\mathrm{ren}} - L_V)\, k(\eta) &= \sum_{y\in\eta}\int_{\mathbb{R}^d}\int_{\Gamma_0} a(x-y)k(\xi\cup x\cup\eta\setminus y) \\ &\times\ Q_\varepsilon(y,\eta\setminus y\cup x,\xi)\lambda(d\xi)dx \\ &-\ \sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\Gamma_0} a(x-y)k(\xi\cup\eta) \\ &\times\ Q_\varepsilon(y,\eta,\xi)\lambda(d\xi)dy\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}Q_\varepsilon(y,\zeta,\xi) &:= e(\tau_y^{(\varepsilon)},\zeta)e(\varepsilon^{-1}t_y^{(\varepsilon)},\xi) - e(-\phi(y-\cdot),\xi) \\ &= e(\varepsilon^{-1}t_y^{(\varepsilon)},\xi) - e(-\phi(y-\cdot) \\ &- \left[1-e(\tau_y^{(\varepsilon)},\zeta)\right] e(\varepsilon^{-1}t_y^{(\varepsilon)},\xi).\end{aligned} \tag{4.26}$$

For $t>0$, the function $e^{-t}-1+t$ takes positive values only; hence

$$\Psi(t) := (e^{-t}-1+t)/t^2, \quad t>0,$$

is positive and bounded, say by $C>0$. Then by means of the following elementary analog of (4.23)

$$b_1\cdots b_n - a_1\cdots a_n \le \sum_{i=1}^n (b_i-a_i)b_1\cdots b_{i-1}b_{i+1}\cdots b_n, \quad b_i\ge a_1>0,$$

we obtain

$$\begin{aligned}\left|e(\varepsilon^{-1}t_y^{(\varepsilon)},\xi) - e(-\phi(y-\cdot),\xi)\right| &\le \sum_{z\in\xi}\varepsilon[\phi(y-z)]^2\Psi\left(\varepsilon\phi(y-z)\right) \\ &\times \prod_{u\in\xi\setminus z}\phi(y-u) \\ &\le \varepsilon C\sum_{z\in\xi}[\phi(y-z)]^2 e(\phi(y-\cdot),\xi\setminus z),\end{aligned}$$

and

$$\left|\left[1-e(\tau_y^{(\varepsilon)},\zeta)\right] e(\varepsilon^{-1}t_y^{(\varepsilon)},\xi)\right| \le \varepsilon\sum_{z\in\zeta}\phi(y-z)e(\phi(y-\cdot),\xi).$$

Then from (4.25) for λ-almost all η we have, see (3.19),

$$|(L_{\varepsilon,\mathrm{ren}} - L_V)\, k(\eta)| \le \varepsilon\|k\|_{\vartheta_0}\left(\widetilde{C}|\eta|e^{-\vartheta_0|\eta|} + \Upsilon(\eta)e^{-\vartheta_0|\eta|}\right), \tag{4.27}$$

with

$$\widetilde{C} = 2C\alpha\|\phi\|_{L^\infty(\mathbb{R}^d)}\langle\phi\rangle e^{-\vartheta_0}$$

and

$$\Upsilon(\eta) \;\; = \;\; 2\alpha \exp\left(\langle\phi\rangle e^{-\vartheta_0}\right) \|\phi\|_{L^\infty(\mathbb{R}^d)}|\eta|(|\eta|+1).$$

Thus, we conclude that the expression in $(\cdot)$ in the right-hand side of (4.27) is in $\mathcal{K}_\vartheta$, which yields (4.24) and hence completes the proof. □

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