

TENSOR TRIANGULAR GEOMETRY OF NON-COMMUTATIVE MOTIVES

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ABSTRACT. In this article we initiate the study of the tensor triangular geometry of the category Mot_k of non-commutative motives (over a base ring k). Since the full computation of the spectrum of Mot_k seems completely out of reach, we provide some information about the spectrum of certain subcategories of Mot_k . More precisely, we show that when k is a finite field (or its algebraic closure) the spectrum of the bootstrap subcategory of Mot_k is closely related to the Zariski spectrum of \mathbb{Z} . Moreover, we prove that if we slightly enlarge the bootstrap category to contain the non-commutative motive associated to the ring of polynomials $k[t]$, and assume that k is a field of characteristic zero, then the corresponding spectrum is richer than the Zariski spectrum of \mathbb{Z} .

INTRODUCTION

Tensor triangular geometry. Given a tensor triangulated category \mathcal{T} , such as the derived category of sheaves of modules over a scheme or the stable homotopy category of topological spectra, the classification of its objects up to isomorphism is a problem whose solution is in general out of reach. Hopkins and Smith, in their groundbreaking work [12], had the original idea of classifying objects not up to isomorphism but rather up to some elementary operations like direct sum, tensor product, (de-)suspension, retract, and mapping cone. Intuitively speaking, if one object can be reached by another one throughout these elementary operations, the two objects contain the same information and should be identified. These general ideas were developed and matured during the last decade by Balmer and his collaborators into a subject named *Tensor triangular geometry*; see Balmer's ICM address [3]. The main idea is the construction of a topological space $\text{Spc}(\mathcal{T})$ for every tensor triangulated category \mathcal{T} , called the *spectrum of \mathcal{T}* , which encodes the geometric aspects of \mathcal{T} . Concretely, $\text{Spc}(\mathcal{T})$ is given by the set of *prime* \otimes -ideals of \mathcal{T} , i.e. those \otimes -ideals $\mathcal{P} \subsetneq \mathcal{T}$ such that $\mathbf{1} \notin \mathcal{P}$ and if $a \otimes b \in \mathcal{P}$ then $a \in \mathcal{P}$ or $b \in \mathcal{P}$. The topology is the one generated by the open basis given by the complements of the support sets $\text{supp}(a) := \{\mathcal{P} \in \text{Spc}(\mathcal{T}) \mid a \notin \mathcal{P}\}$.

Non-commutative motives. A *differential graded (=dg) category*, over a base commutative ring k , is a category enriched over complexes of k -modules (morphisms sets are complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} f \circ (dg)$. Due to their remarkable properties, dg categories are nowadays widely used in algebraic geometry, representation theory, symplectic geometry, and even in mathematical physics; see Keller's ICM address [14].

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All the classical invariants such as Hochschild homology (HH), algebraic K -theory (K), and even topological cyclic homology (THH), extend naturally from k -algebras to dg categories. In order to study all these invariants simultaneously the *universal additive invariant* of dg categories was constructed in [24]. Roughly speaking, it consists of a functor $U_k : \mathbf{dgc}at_k \rightarrow \mathbf{Mot}_k$ defined on the category of dg categories and with values in a triangulated category, that preserves filtered colimits, inverts Morita equivalences (see [14, §4.6]), sends split short exact sequences (see [24, §13]) to direct sums

$$0 \longrightarrow \mathcal{A} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \mathcal{B} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \mathcal{C} \longrightarrow 0 \quad \mapsto \quad U_k(\mathcal{A}) \oplus U_k(\mathcal{C}) \simeq U_k(\mathcal{B}),$$

and which is universal¹ with respect to these properties. All the mentioned invariants are additive and so they factor uniquely through U_k . Because of this universal property, which is reminiscent of the theory of motives, \mathbf{Mot}_k is called the *triangulated category of non-commutative motives* (over the base ring k). The importance of this category should be clear by now. For instance, among other applications, \mathbf{Mot}_k allowed the first conceptual characterization of algebraic K -theory since Quillen's foundational work, a streamlined construction of the higher Chern characters, a unified and conceptual proof of the fundamental theorems in homotopy K -theory and periodic cyclic homology, the description of the fundamental isomorphism conjecture in terms of the classical Farrell-Jones conjecture, etc; see [6, 23, 24, 25, 26]. The tensor product of k -algebras extends naturally to dg categories, giving rise to a symmetric monoidal structure on $\mathbf{dgc}at$. Making use of Day's convolution product, this monoidal structure was extended (in a universal way) to \mathbf{Mot}_k ; see [10]. Hence, the triangulated category \mathbf{Mot}_k of non-commutative motives carries moreover a compatible tensor product.

1. STATEMENT OF RESULTS

The category \mathbf{Mot}_k of non-commutative motives is tensor triangulated and so we are naturally in the realm of tensor triangular geometry. Notice that although defined in terms of a very simple and precise universal property, the category \mathbf{Mot}_k is highly complex. For instance, all the information concerning additive invariants is completely encoded in \mathbf{Mot}_k . Hence, the full computation of the spectrum of \mathbf{Mot}_k is a major challenge which seems completely out of reach at the present time. In this article we give the first step towards the solution of this major challenge by providing some information about the spectrum of certain subcategories of \mathbf{Mot}_k . Let \mathbf{Boot}_k be the *bootstrap* subcategory of \mathbf{Mot}_k , i.e. the thick triangulated subcategory of \mathbf{Mot}_k generated by the \otimes -unit object, and $\mathbf{Boot}_{k;\mathbb{Q}}$ its rationalization (see §2.3).

Theorem 1.1. *Assume that the base ring k is finite or that it is the algebraic closure of a finite field. Then, if we denote by n the number of prime ideals in k , we have an equivalence $\mathbf{Boot}_{k;\mathbb{Q}} \simeq \mathcal{D}^{\text{perf}}(\mathbb{Q}^n)$ of tensor triangulated categories. As a consequence, $\text{Spc}(\mathbf{Boot}_{k;\mathbb{Q}})$ has precisely n distinct points.*

A consequence of Theorem 1.1 is the fact that over a finite field or over its algebraic closure (in which case $n = 1$) the \otimes -unit object can be reached by any non-trivial non-commutative motive in $\mathbf{Boot}_{k;\mathbb{Q}}$ throughout the elementary operations of direct sum, tensor product, (de-)suspension, and retract.

¹The formulation of the precise universal property makes use of the language of Grothendieck derivators; see [24, §10].

Recall that Hochschild homology (HH) and algebraic K -theory (K) are examples of additive invariants. Hence, they factor uniquely through Mot_k and give rise to triangulated functors $\overline{HH} : \text{Mot}_k \rightarrow \mathcal{D}(k)$ and $\overline{K} : \text{Mot}_k \rightarrow \text{Ho}(\text{Spt})$ (where $\text{Ho}(\text{Spt})$ stands for the homotopy category of spectra) such that $\overline{HH} \circ U_k = HH$ and $\overline{K} \circ U_k = K$. Periodic cyclic homology (HP), although not an additive invariant², factors also through Mot_k ; see §5.2. We obtain then a triangulated functor $\overline{HP} : \text{Mot}_k \rightarrow \mathcal{D}_2(k)$, with values in the derived category of 2-periodic complexes, such that $\overline{HP} \circ U_k = HP$. Given a tensor triangulated category \mathcal{T} , Balmer constructed a continuous comparison map

$$\rho : \text{Spc}(\mathcal{T}) \longrightarrow \text{Spec}(\text{End}_{\mathcal{T}}(\mathbf{1})) \quad \mathcal{P} \mapsto \{f \in \text{End}_{\mathcal{T}}(\mathbf{1}) \mid \text{cone}(f) \notin \mathcal{P}\}$$

towards the Zariski spectrum of the endomorphism ring of the \otimes -unit $\mathbf{1}$ of \mathcal{T} ; see [4]. As a consequence, the computation of the full spectrum $\text{Spc}(\mathcal{T})$ can be reduced to the computation of the different fibers of ρ .

Theorem 1.2. *Assume that k is a finite field \mathbb{F}_q or its algebraic closure $\overline{\mathbb{F}_q}$ (where $q = p^r$ with p a prime number and r a positive integer). Then:*

- (i) *We have a continuous surjective map $\rho : \text{Spc}(\text{Boot}_k) \twoheadrightarrow \text{Spec}(\mathbb{Z})$.*
- (ii) *The fiber $\rho^{-1}(\{0\})$ at the prime ideal $\{0\} \subset \mathbb{Z}$ has a single point given by*

$$\{M \in \text{Boot}_k \mid \overline{K}_*(M)_{\mathbb{Q}} = 0\},$$

where $\overline{K}_*(M)$ stands for the homotopy groups of the spectrum $\overline{K}(M)$.

- (iii) *The fiber $\rho^{-1}(p\mathbb{Z})$ at the prime ideal $p\mathbb{Z} \subset \mathbb{Z}$ has a single point. Moreover, this point admits the following three different descriptions:*

- (a) $\{M \in \text{Boot}_k \mid \overline{HH}(M) = 0\}$
- (b) $\{M \in \text{Boot}_k \mid \overline{HP}(M) = 0\}$
- (c) $\{M \in \text{Boot}_k \mid \overline{K}_*(M)_{(p)} = 0\},$

where $\overline{K}_*(M)_{(p)}$ stands for the localization of $\overline{K}_*(M)$ at the prime $p\mathbb{Z}$.

Theorem 1.2 suggests that over a finite field (or over its algebraic closure) the spectrum of the bootstrap category is completely controlled by the Zariski spectrum of the integers. However, if we slightly enlarge the bootstrap category and don't impose finiteness conditions on the base ring k , the above picture changes and some complexity starts to appear. Let EBoot_k be the tensor triangulated subcategory of Mot_k generated by the \otimes -unit object and by the non-commutative motive $U_k(k[t])$ associated to the k -algebra of polynomials in the variable t ; see §5.4.

Theorem 1.3. *Assume that k is a field of characteristic zero. Then,*

$$\{M \in \text{EBoot}_k \mid \overline{HH}(M) = 0\} \quad \text{and} \quad \{M \in \text{EBoot}_k \mid \overline{HP}(M) = 0\}$$

are two distinct points in $\text{Spc}(\text{EBoot}_k)$.

Making use of Balmer's general theory [4], we then prove:

Proposition 1.4. *Assume that k is a field of characteristic zero. Then:*

- (i) *We have a continuous surjective map $\rho : \text{Spc}(\text{EBoot}_k) \twoheadrightarrow \text{Spec}(\mathbb{Z})$.*
- (ii) *The two distinct points described in Theorem 1.3 correspond to two distinct points in the fiber $\rho^{-1}(\{0\})$.*

²Since it does not preserve filtered colimits.

Proposition 1.4 shows us that, in contrast with the bootstrap category, the spectrum $\mathrm{Spc}(\mathrm{EBoot}_k)$ is necessarily more complex than the Zariski spectrum of the integers.

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2. PRELIMINARIES

2.1. Triangular notations. The suspension functor of any triangulated category \mathcal{T} will be denoted by Σ . Whenever \mathcal{T} admits arbitrary coproducts, we will write \mathcal{T}^c for its full subcategory of compact objects; see [19, Def. 4.2.7]. We will use the symbols $\langle \mathcal{S} \rangle$ and $\langle \mathcal{S} \rangle_{\mathrm{loc}}$ to denote, respectively, the thick and localizing subcategories of \mathcal{T} generated by a set \mathcal{S} of objects of \mathcal{T} ; similarly in the case of a single object X of \mathcal{T} . Recall that a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ is a triangulated category \mathcal{T} equipped with a symmetric monoidal structure, with \otimes -unit object $\mathbf{1}$, in which the bifunctor $- \otimes - : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ is exact in each variable. We will write $\pi_i(-)$ for the functor $\mathrm{Hom}_{\mathcal{T}}(\Sigma^i \mathbf{1}, -)$ on \mathcal{T} co-represented by the i^{th} suspension of $\mathbf{1}$. Finally, given an additive functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we will denote by

$$\mathrm{Ker}(F) := \{X \in \mathcal{C} \mid F(X) \simeq 0\} \quad \text{and} \quad \mathrm{Im}(F) := \{Y \in \mathcal{D} \mid \exists X \in \mathcal{C}, Y \simeq F(X)\}$$

the full and replete (i.e. closed under isomorphic objects) kernel and image of F .

Remark 2.1. In this article, we will prove several results concerning Balmer's spectrum $\mathrm{Spc}(\mathcal{T})$ of tensor triangulated categories \mathcal{T} . Recall from [3, Construction 29] that $\mathrm{Spc}(\mathcal{T})$ always comes naturally equipped with a sheaf of commutative rings which turns it into a locally ringed space. At this point we would like to inform the reader that all our (iso)morphisms between Balmer's (or Zariski's) spectra are in fact (iso)morphisms of locally ringed spaces. These upgradings of our results are always trivial to check. Hence, in order to simplify the exposition, we have decided to omit them.

2.2. Non-commutative motives. Recall from [24, §15] and [10, §7] the construction of the tensor triangulated category³ Mot_k of non-commutative motives (over a base ring k). Its \otimes -unit object is the non-commutative motive $\mathbf{1} = U_k(\underline{k})$ associated to the dg category \underline{k} with one object and with k as the dg algebra of endomorphisms (concentrated in degree zero). Moreover, Mot_k admits arbitrary coproducts and is compactly generated. A set of compact generators is given by (the shifts of) the non-commutative motives $U_k(\mathcal{A})$ associated to the *homotopically finitely presented* dg categories \mathcal{A} ; see [10, §3]. Heuristically, this notion is the homotopical version of the classical notion of finite presentation. Finally, Quillen's algebraic K -theory can be recovered out of Mot_k as the functors co-represented by the shifts of the \otimes -unit; see [24, Thm. 15.10]. More precisely, we have the following computation

$$\mathrm{Hom}_{\mathrm{Mot}_k}(\Sigma^n U_k(\underline{k}), U_k(\mathcal{A})) \simeq \begin{cases} K_n(\mathcal{A}) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

³In [24] the category Mot_k was denoted by $\mathcal{M}_{\mathrm{dg}}^{\mathrm{add}}(e)$. The symmetric monoidal structure constructed in [24] concerns the localizing analogue of Mot_k . However, the arguments in the case of the category Mot_k are completely similar.

for every small dg category \mathcal{A} . In particular, we have

$$(2.2) \quad \mathrm{Hom}_{\mathrm{Mot}_k}(\Sigma^n U_k(\underline{k}), U_k(\underline{k})) \simeq \begin{cases} K_n(k) & n \geq 0 \\ 0 & n < 0, \end{cases}$$

where $K_n(k)$ is the n^{th} algebraic K -theory group of the base ring k . Moreover, under the isomorphism (2.2), composition in the triangulated category Mot_k corresponds to multiplication in the \mathbb{Z} -graded ring $K_*(k)$; see [26, Thm. 2.4].

2.3. Rationalization. Associated to every \mathbb{Z} -linear category \mathcal{C} , there is an evident rationalized version $\mathcal{C}_{\mathbb{Q}}$ with the same objects and where each Hom-group is tensored with \mathbb{Q} . In this subsection we will study the rationalization $\mathrm{Boot}_{k;\mathbb{Q}} := (\mathrm{Boot}_k)_{\mathbb{Q}}$ of the bootstrap category and of some of its appropriate enlargements.

Lemma 2.3. *The category $\mathrm{Boot}_{k;\mathbb{Q}}$ inherits from Boot_k a unique tensor triangulated structure such that the canonical functor $\mathrm{Boot}_k \rightarrow \mathrm{Boot}_{k;\mathbb{Q}}$, $f \mapsto f \otimes 1$, becomes \otimes -exact. Furthermore, this canonical functor can be identified with the Verdier quotient of Boot_k by its thick \otimes -ideal $\langle \mathrm{cone}(n \cdot \mathrm{id}_{U_k(\underline{k})}) \mid n \in \mathbb{Z} \setminus \{0\} \rangle_{\otimes}$.*

Proof. Given any \mathbb{Z} -linear category \mathcal{C} , the category $\mathcal{C}_{\mathbb{Q}}$ is canonically isomorphic to the localization $S^{-1}\mathcal{C}$ of \mathcal{C} at the set $S := \{n \cdot \mathrm{id}_X \mid n \in \mathbb{Z} \setminus \{0\}, X \in \mathcal{C}\}$. Since Boot_k is a (\mathbb{Z} -linear) tensor triangulated category with unit $U_k(\underline{k})$, $\mathrm{cone}(n \cdot \mathrm{id}_M) = \mathrm{cone}(n \cdot \mathrm{id}_{U_k(\underline{k})}) \otimes M$ for every non-commutative motive M , and so it suffices to invert the (“central”) multiplicative system $(\mathbb{Z} \setminus \{0\}) \cdot \mathrm{id}$ in the endomorphism ring of $U_k(\underline{k})$. Hence, the result follows now from the general theory of central localization; see [4, Thm. 3.6]. \square

Remark 2.4. The “ambient” category Mot_k has arbitrary coproducts, but we cannot expect them to be preserved by the canonical functor $\mathrm{Mot}_k \rightarrow \mathrm{Mot}_{k;\mathbb{Q}}$. In fact, as explained in Lemma 2.3, $\mathrm{Mot}_{k;\mathbb{Q}}$ can be identified with the Verdier quotient of Mot_k by its thick \otimes -ideal $\mathcal{J} := \langle \mathrm{cone}(n \cdot \mathrm{id}_{U_k(\underline{k})}) \mid n \in \mathbb{Z} \setminus \{0\} \rangle_{\otimes}$. Therefore, since the ideal \mathcal{J} is not necessarily localizing, the quotient functor does not necessarily preserves arbitrary coproducts.

In view of the above remark, we will perform a slightly more refined version of the rationalization which, while extending that of the bootstrap category, it is also compatible with arbitrary coproducts. We will consider then the following auxiliary category:

Definition 2.5. Let $\mathcal{T}_k := \langle U_k(\underline{k}) \rangle_{\mathrm{loc}} \subset \mathrm{Mot}_k$ be the localizing triangulated subcategory of Mot_k generated by the \otimes -unit object $U_k(\underline{k})$.

Lemma 2.6. *The tensor triangulated category \mathcal{T}_k is compactly generated (the shifts of the \otimes -unit are the compact generators). Its full triangulated \otimes -subcategory of compact objects is precisely the bootstrap category Boot_k . Moreover, the tensor product is exact in each variable.*

Proof. Consider the full subcategory $\mathcal{S} := \{M \mid M \otimes \mathcal{T}_k \subset \mathcal{T}_k\}$ of Mot_k . Since \mathcal{T}_k is localizing and the bifunctor $-\otimes-$ is exact and commutes with arbitrary coproducts in each variable, we conclude that \mathcal{S} is also a localizing subcategory of Mot_k . Moreover, \mathcal{S} contains the \otimes -unit which implies that $\mathcal{T}_k \subset \mathcal{S}$. Thus, \mathcal{T}_k is a tensor triangulated subcategory. By construction, it is compactly generated by the shifts of the \otimes -unit. Lemma [17, Lemma 2.2] (applied to $R = \{\Sigma^n U_k(\underline{k}) \mid n \in \mathbb{Z}\}$) implies that its compact objects are precisely those of the thick subcategory $\langle U_k(\underline{k}) \rangle = \mathrm{Boot}_k$.

Finally, the property that the tensor product is exact in each variable is inherited from the ambient category Mot_k . \square

For simplicity we now write $\mathbf{1} = U_k(\underline{k})$ and $\pi_* = \text{Hom}_{\mathcal{T}_k}(\Sigma^*\mathbf{1}, -)$ (cf. §2.1), where π_* is considered as a functor $\mathcal{T} \rightarrow \text{End}_{\mathcal{T}_k}(\mathbf{1})\text{-Mod}^{\mathbb{Z}}$ taking values in \mathbb{Z} -graded $\text{End}_{\mathcal{T}_k}(\mathbf{1})$ -modules. Let S be a multiplicative subset of $\text{End}_{\mathcal{T}_k}(\mathbf{1})$, i.e. $\text{id}_{\mathbf{1}} \in S$ and $S \cdot S \subset S$. In the following general proposition we collect the properties of the Bousfield localization of \mathcal{T}_k with respect to the homological functor $S^{-1}\pi_*(-) = \pi_*(-) \otimes_{\text{End}_{\mathcal{T}_k}(\mathbf{1})} S^{-1}\text{End}_{\mathcal{T}_k}(\mathbf{1})$.

Proposition 2.7. *For every object M in \mathcal{T}_k , we have a functorial distinguished triangle*

$$\Gamma_S M \longrightarrow M \xrightarrow{\eta_M} L_S M \longrightarrow \Sigma \Gamma_S M$$

with $S^{-1}\pi_*(L_S M) \simeq \pi_*(L_S M)$ and $S^{-1}\pi_*(\Gamma_S M) = 0$. Moreover, the following properties hold:

- (1) $\text{Ker}(L_S) = \{M \in \mathcal{T}_k \mid S^{-1}\pi_*(M) = 0\} = \text{Im}(\Gamma_S)$.
- (2) $L_S \mathcal{T}_k = \text{Im}(L_S) = \{M \in \mathcal{T}_k \mid S^{-1}\pi_*(M) \simeq \pi_*(M)\} = \text{Ker}(\Gamma_S)$.
- (3) η_M is initial among maps from M to an object in $L_S \mathcal{T}_k$.
- (4) η_M is final among maps from M which induce an isomorphism on $S^{-1}\pi_*(-)$.
- (5) The above functorial triangle can be obtained, up to unique canonical isomorphism, by tensoring M with the triangle $\Gamma_S \mathbf{1} \rightarrow \mathbf{1} \rightarrow L_S \mathbf{1} \rightarrow \Sigma \Gamma_S \mathbf{1}$.
- (6) $\eta_{L_S} = L_S \eta : L_S \simeq L_S \circ L_S$; in particular $L_S \mathbf{1} \otimes L_S \mathbf{1} \simeq L_S \mathbf{1}$.
- (7) The functor $L_S : \mathcal{T}_k \rightarrow L_S \mathcal{T}_k$ realizes the localization of \mathcal{T}_k with respect to the class of $S^{-1}\pi_*(-)$ -isomorphisms.
- (8) Given any $M \in \text{Boot}_k$ and $N \in \mathcal{T}_k$, the map $\eta_N : N \rightarrow L_S N$ induces an isomorphism $S^{-1}\text{Hom}_{\mathcal{T}_k}(M, N) \simeq \text{Hom}_{\mathcal{T}_k}(M, L_S N)$.

The category $L_S \mathcal{T}_k$ is tensor triangulated and generated by its tensor unit $L_S \mathbf{1}$. The functor $L_S = L_S \mathbf{1} \otimes -$ restricts to a coproduct-preserving \otimes -exact functor $\mathcal{T}_k \rightarrow L_S \mathcal{T}_k$, which is left adjoint to the inclusion $L_S \mathcal{T}_k \hookrightarrow \mathcal{T}_k$. Its restriction to compact objects can be expressed as the composition of the \otimes -exact functor $\text{Boot}_k \rightarrow S^{-1}\text{Boot}_k$ provided by central localization with the cofinal fully-faithful functor $S^{-1}\text{Boot}_k \hookrightarrow (L_S \mathcal{T}_k)^c$.

Proof. The existence of the functorial distinguished triangle as well as properties (2)-(6) and (8) can be obtained by applying [11, Thm. 2.33] to the compactly generated tensor triangulated category $\mathcal{T} := \mathcal{T}_k$. Note that the hypothesis in *loc. cit.* concerning the agreement between compact and rigid objects are automatically satisfied in \mathcal{T}_k since this category is compactly generated by the \otimes -unit.

It remains then to verify property (7), the equality

$$(2.8) \quad \text{Ker}(L_S) = \{M \in \mathcal{T}_k \mid S^{-1}\pi_*(M) = 0\}$$

in (1), and the fact that the restricted functor $L_S : \text{Boot}_k = (\mathcal{T}_k)^c \rightarrow (L_S \mathcal{T}_k)^c$ admits the claimed factorization. The latter result follows from Neeman's localization theorem [17, Thm. 2.1]. The remaining two results are actually equivalent. Thanks to Neeman's localization theorem, the functor $L_S : \mathcal{T}_k \rightarrow L_S \mathcal{T}_k$ is the Verdier quotient of \mathcal{T}_k by $\text{Ker}(L_S)$. Hence, $L_S \mathcal{T}_k$ is the localization of \mathcal{T}_k at the class of maps f such that $\text{cone}(f) \in \text{Ker}(L_S)$. Since property (7) asserts that such maps coincide with those inverted by the stable homological functor $S^{-1}\pi_*(-)$, we conclude that the equality (2.8) is equivalent to property (7). Let us then prove

equality (2.8). The inclusion “ \subset ” follows from the fact that $\pi_*(\Gamma_S M) = 0$ for all M , and from equality $\text{Ker}(L_S) = \text{Im}(\Gamma_S)$. On the other hand, by property (8) we have $S^{-1}\pi_*(M) \simeq \pi_*(L_S M)$. Thus, if $S^{-1}\pi_*(M) = 0$, then $\pi_*(L_S M) = 0$. Since $\mathbf{1}$ generates \mathcal{T}_k this latter condition is equivalent to the condition $L_S M = 0$. This shows the inclusion “ \supset ”. \square

Notation 2.9. Let \mathcal{T} be a tensor triangulated category and S the set $\text{End}_{\mathcal{T}}(\mathbf{1}) \setminus \mathfrak{p}$, where \mathfrak{p} is a prime ideal of $\text{End}_{\mathcal{T}}(\mathbf{1})$. Following the classical convention, we will write $L_{\mathfrak{p}}$ and $\Gamma_{\mathfrak{p}}$ instead of L_S and Γ_S . When $S = (\mathbb{Z} \setminus \{0\}) \cdot \text{id}_{\mathbf{1}}$ (for instance when defining the rationalization $\text{Boot}_{k;\mathbb{Q}} = S^{-1}\text{Boot}_k$) we will write $L_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ instead.

Remark 2.10. The cofinal inclusion $\iota : \text{Boot}_{k;\mathbb{Q}} \hookrightarrow (L_{\mathbb{Q}}\mathcal{T}_k)^c$ (described in Proposition 2.7), and the fact that $L_{\mathbb{Q}}\mathcal{T}_k$ is idempotent complete, allow us to identify $(L_{\mathbb{Q}}\mathcal{T}_k)^c$ with the idempotent completion of $\text{Boot}_{k;\mathbb{Q}}$. Hence, thanks to [2, Prop. 3.13], we obtain the following homeomorphism of topological spaces

$$\text{Spc}(\iota) : \text{Spc}((L_{\mathbb{Q}}\mathcal{T}_k)^c) \xrightarrow{\sim} \text{Spc}(\text{Boot}_{k;\mathbb{Q}}).$$

3. PROOF OF THEOREM 1.1

Our proof is based on a general result of tensor triangulated geometry (see Theorem 3.3 and Corollary 3.4), and on the following two computations: the first one concerning the Grothendieck group and the second one higher algebraic K -theory.

Computation 1: By hypothesis, the base ring k is finite or the algebraic closure of a finite field. Hence, it has only finitely many prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_i, \dots, \mathfrak{p}_n$. As proved in [7, Thm. 3.1.4] (the case where k is the algebraic closure of a finite field is clear), there is a positive integer m and a ring isomorphism

$$k \simeq k/\mathfrak{p}_1^m \times \cdots \times k/\mathfrak{p}_i^m \times \cdots \times k/\mathfrak{p}_n^m.$$

The Grothendieck group functor commutes with direct products and so

$$K_0(k) \simeq K_0(k/\mathfrak{p}_1^m) \times \cdots \times K_0(k/\mathfrak{p}_i^m) \times \cdots \times K_0(k/\mathfrak{p}_n^m).$$

Notice that the rings k/\mathfrak{p}_i^m , $i = 1, \dots, n$, are *local*, i.e. they have a unique maximal ideal. Hence, by combining [27, Lemma 2.2] with [28, Lemma 2.1(2)], we have $K_0(k/\mathfrak{p}_i^m) = \mathbb{Z}$ and so by (2.2) we conclude that $\text{End}_{\mathcal{T}_k}(\mathbf{1}) = K_0(k) \simeq \mathbb{Z}^n$. Using Proposition 2.7, we then obtain the following computation

$$(3.1) \quad \text{End}_{L_{\mathbb{Q}}\mathcal{T}_k}(\mathbf{1}) = \text{Hom}_{L_{\mathbb{Q}}\mathcal{T}_k}(L_{\mathbb{Q}}U_k(\underline{k}), L_{\mathbb{Q}}U_k(\underline{k})) \simeq \text{End}_{\mathcal{T}_k}(\mathbf{1}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^n.$$

Computation 2: When the base ring k is finite, [29, Prop. 1.16] implies that all the abelian groups $K_n(k)$, with $n > 0$, are finite. When k is the algebraic closure $\overline{\mathbb{F}_q}$ of a finite field \mathbb{F}_q (where $q = p^r$ with p a prime number and r a positive integer) Quillen [20] performed the following computation

$$K_n(\overline{\mathbb{F}_q}) \simeq \begin{cases} \mathbb{Z} & n = 0 \\ \bigoplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l & n > 0 \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

In both cases, we observe that the groups $K_n(k)$, with $n \neq 0$, are torsion. Using Proposition 2.7 and (2.2) we then obtain the following computation

$$(3.2) \quad \text{Hom}_{L_{\mathbb{Q}}\mathcal{T}_k}(\Sigma^n \mathbf{1}, \mathbf{1}) \simeq \text{Hom}_{\mathcal{T}_k}(\Sigma^n U_k(\underline{k}), U_k(\underline{k})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq 0 \quad n \neq 0.$$

The following general result is of independent interest. Due to its generality and simplicity, we believe it will soon be part of the toolkit of tensor triangular geometry.

Theorem 3.3. *Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category with arbitrary coproducts verifying the following conditions:*

- (1) *The tensor product preserves arbitrary coproducts in each variable.*
- (2) *The \otimes -unit $\mathbf{1}$ is a compact object.*
- (3) *The graded endomorphism ring $\text{End}_{\mathcal{T}}(\mathbf{1})_* := \text{Hom}_{\mathcal{T}}(\Sigma^* \mathbf{1}, \mathbf{1})$ consists simply of a field F concentrated in degree zero.*

Then, the functor $X \mapsto \pi_(X) = \text{Hom}_{\mathcal{T}}(\Sigma^* \mathbf{1}, X)$ induces a \otimes -equivalence between \mathcal{T} and the category $F\text{-VS}^{\mathbb{Z}}$ of \mathbb{Z} -graded F -vector spaces (equipped with the usual tensor product given by $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes_F V_j$).*

Proof. Our proof relies on Meyer-Nest's relative homological algebra theory [16]. We will adopt the terminology from *loc. cit.*. Since $\pi_* : \mathcal{T} \rightarrow F\text{-VS}^{\mathbb{Z}}$ is a stable homological functor, its kernel on morphisms $\mathcal{I} := \{f \in \text{Mor}(\mathcal{T}) \mid \pi_*(f) = 0\}$ is a homological ideal in \mathcal{T} . We claim that π_* , considered as a stable homological functor to the stable abelian category of \mathbb{Z} -graded F -vector spaces and degree-preserving maps, is the *universal \mathcal{I} -exact functor on \mathcal{T}* . The category \mathcal{T} has all small coproducts and $F\text{-VS}^{\mathbb{Z}}$ has enough projectives; indeed all its objects are projective. Hence, by the recognition criterion [16, Thm. 57], it suffices to show that π_* admits a right inverse and left adjoint π^\dagger . In other words, we need to define a functor $\pi^\dagger : F\text{-VS}^{\mathbb{Z}} \rightarrow \mathcal{T}$ and two natural isomorphisms:

$$(a) (\pi_* \circ \pi^\dagger)(V) \simeq V \quad (b) \text{Hom}_{\mathcal{T}}(\pi^\dagger(V), X) \simeq \text{Hom}_{F\text{-VS}^{\mathbb{Z}}}(V, \pi_*(X)),$$

where $V \in F\text{-VS}^{\mathbb{Z}}$ and $X \in \mathcal{T}$. Every object V in $F\text{-VS}^{\mathbb{Z}}$ is naturally isomorphic to a direct sum of the form $\bigoplus_{\alpha} F[i_{\alpha}]$, where $F[i]$ denotes the \mathbb{Z} -graded vector space consisting of F in degree i and zero elsewhere. Therefore, it suffices to define the functor π^\dagger on the full subcategory $\mathcal{F} := \{F[i] \mid i \in \mathbb{Z}\}$ of $F\text{-VS}^{\mathbb{Z}}$; it will then naturally extend to a coproduct-preserving additive functor defined on $F\text{-VS}^{\mathbb{Z}}$. On objects let $\pi^\dagger(F[i]) = \Sigma^i \mathbf{1}$, and on morphisms consider the canonical identification

$$\text{Hom}(F[i], F[j]) = \begin{cases} F & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \text{Hom}_{\mathcal{T}}(\Sigma^i \mathbf{1}, \Sigma^j \mathbf{1}).$$

Due to condition (3), this gives rise to a well-defined functor. When we restrict ourselves to the full subcategory \mathcal{F} , the isomorphism (a) is clear and a canonical choice for the above isomorphism (b) is given by

$$\text{Hom}_{\mathcal{T}}(\pi^\dagger F[i], X) = \pi_i(X) \simeq \text{Hom}_{F\text{-VS}^{\mathbb{Z}}}(F[i], \pi_* X).$$

It is now straightforward, using hypothesis (2) on the compactness of the objects $\Sigma^i \mathbf{1}$, to extend these isomorphisms from \mathcal{F} to $F\text{-VS}^{\mathbb{Z}}$, thus proving (a) and (b). The conditions of the recognition criterion [16, Thm. 57] are then satisfied, and so we conclude that π_* is in fact the universal \mathcal{I} -exact functor on \mathcal{T} . As a consequence, π_* restricts to an equivalence between the full subcategory of \mathcal{I} -projective objects of \mathcal{T} and the full subcategory of projective objects of $F\text{-VS}^{\mathbb{Z}}$. Moreover, the universal \mathcal{I} -exact functor also detects \mathcal{I} -projectives, *i.e.* $X \in \mathcal{T}$ is \mathcal{I} -projective if and only if $\pi_*(X)$ is projective; see [16, Thm. 59]. It follows immediately from this and from the semi-simplicity of $F\text{-VS}^{\mathbb{Z}}$ that π_* is an equivalence between \mathcal{T}

and $F\text{-VS}^{\mathbb{Z}}$. The fact that this equivalence is moreover symmetric monoidal follows from the following natural chain of isomorphisms

$$\begin{aligned} \pi_*(X \otimes Y) &\simeq \bigoplus_{\alpha, \beta} \pi_*(\Sigma^{n_\alpha} \mathbf{1} \otimes \Sigma^{n_\beta} \mathbf{1}) \simeq \bigoplus_{\alpha, \beta} \pi_*(\Sigma^{n_\alpha + n_\beta} \mathbf{1} \otimes \mathbf{1}) \simeq \bigoplus_{\alpha, \beta} \pi_*(\Sigma^{n_\alpha + n_\beta} \mathbf{1}) \simeq \\ &\bigoplus_{\alpha, \beta} F[n_\alpha + n_\beta] \simeq \bigoplus_{\alpha, \beta} F[n_\alpha] \otimes F[n_\beta] \simeq \bigoplus_{\alpha \in A} F[n_\alpha] \otimes \bigoplus_{\beta \in B} F[n_\beta] \simeq \pi_*(X) \otimes \pi_*(Y), \end{aligned}$$

where $X = \bigoplus_{\alpha \in A} \Sigma^{n_\alpha} \mathbf{1}$ and $Y = \bigoplus_{\beta \in B} \Sigma^{n_\beta} \mathbf{1}$. \square

Corollary 3.4. *Let \mathcal{T} be as in Theorem 3.3, and let us denote by \mathcal{T}^c its tensor triangulated subcategory of compact objects. Then $\rho : \text{Spc}(\mathcal{T}^c) \xrightarrow{\sim} \text{Spec}(F)$ is a bijection and therefore a homeomorphism.*

Proof. An immediate consequence of Theorem 3.3 is the fact that every object in \mathcal{T} decomposes as a direct sum of shifts of the \otimes -unit; the triangulated category \mathcal{T} is even a semi-simple abelian category. In particular, $\{0\} \subset \mathcal{T}_c$ is the only proper thick \otimes -ideal of the category of compact objects. Since $\{0\}$ is also a prime \otimes -ideal, we conclude that it is the unique point of the spectrum. Clearly, the comparison map $\rho : \text{Spc}(\mathcal{T}^c) \rightarrow \text{Spec}(F)$ identifies this point with the unique point of $\text{Spec}(F)$. \square

Lemma 3.5. *Let \mathcal{C} be an idempotent complete \mathbb{Z} -linear tensor category with \otimes -unit $\mathbf{1}$. Assume that the identity morphism $\text{id}_{\mathbf{1}}$ can be written as the sum of the orthogonal idempotents $e_1, \dots, e_i, \dots, e_n$ of the endomorphism ring $\text{End}_{\mathcal{C}}(\mathbf{1})$. Then:*

- (1) *Every object X in \mathcal{C} splits as $X \simeq \text{Im}(e_1 \cdot \text{id}_X) \oplus \dots \oplus \text{Im}(e_n \cdot \text{id}_X)$.*
- (2) *Let $e_i \mathcal{C}$ denote the full subcategory of \mathcal{C} formed by the objects $e_i X := \text{Im}(e_i \cdot \text{id}_X)$, $X \in \mathcal{C}$. Then, $e_i \mathcal{C}$ is stable under tensor products and the assignment $X \mapsto e_i X$ extends uniquely to a well-defined \mathbb{Z} -linear \otimes -functor $\mathcal{C} \rightarrow e_i \mathcal{C}$.*
- (3) *The combined assignment $X \mapsto (e_1 X, \dots, e_i X, \dots, e_n X)$ gives rise to an equivalence of \mathbb{Z} -linear \otimes -categories*

$$\mathcal{C} \xrightarrow{\sim} e_1 \mathcal{C} \oplus \dots \oplus e_i \mathcal{C} \oplus \dots \oplus e_n \mathcal{C},$$

whose quasi-inverse functor is given by $(X_1, \dots, X_n) \mapsto X_1 \oplus \dots \oplus X_n$.

- (4) *When \mathcal{C} is moreover triangulated in such a way that the suspension functor Σ is $\text{End}_{\mathcal{C}}(\mathbf{1})$ -linear (for instance if \mathcal{C} is the homotopy category associated to a symmetric monoidal stable Quillen model category⁴) the triangulated structure restricts to each factor $e_i \mathcal{C}$ and the decomposition in (3) becomes an equivalence of tensor triangulated categories.*

Proof. Items (1)-(3) are straightforward consequences of the fact that \mathcal{C} is canonically an $\text{End}_{\mathcal{C}}(\mathbf{1})$ -linear category via the tensor product; see [4, Prop. 2.2]. In what concerns item (4), this follows from the fact that a direct sum of triangles is distinguished if and only if each summand is a distinguished triangle; see [5, Lemma 1.6]. \square

We now have all the ingredients needed for the proof of Theorem 1.1. By combining (3.1) with (3.2), we observe that the graded endomorphism ring $\text{End}_{L_{\mathbb{Q}}\mathcal{T}_k}(\mathbf{1})_*$ is just the direct sum of m copies of \mathbb{Q} (concentrated in degree zero). This fact, combined with Lemma 3.5, implies that $L_{\mathbb{Q}}\mathcal{T}_k$ decomposes as a direct sum of isomorphic categories $\mathcal{T}_i = \mathcal{T}$ ($i = 1, \dots, n$), each one satisfying all the hypotheses

⁴Recall from [24, Def. 15.1] and [10, Thm. 7.5] that this is precisely the case for the tensor triangulated category of non-commutative motives.

of Theorem 3.3 with $F = \mathbb{Q}$. We obtain in this way the following commutative diagram of \otimes -exact functors:

$$\begin{array}{ccccc} \mathrm{Boot}_{k;\mathbb{Q}} & \xrightarrow{\iota} & L_{\mathbb{Q}}\mathcal{T}_k & \xrightarrow[\simeq]{\pi_*} & \mathbb{Q}^n\text{-Mod}^{\mathbb{Z}} & \xleftarrow[\simeq]{H_*} & \mathcal{D}(\mathbb{Q}^n) \\ & & \downarrow \simeq & & \downarrow \simeq & & \\ & & \bigoplus_{i=1}^n \mathcal{T} & \xrightarrow[\simeq]{(\pi_*)^i} & \bigoplus_{i=1}^n \mathbb{Q}\text{-VS}^{\mathbb{Z}} & & \end{array} .$$

All the functors are equivalences, except the fully faithful embedding ι ; see Remark 2.10. However, when we restrict ourselves to compact objects, ι becomes also an equivalence since every $M \in (L_{\mathbb{Q}}\mathcal{T}_k)^c$ is isomorphic to a finite direct sum of shifts of the \otimes -unit and thus belongs to the essential image of $\iota : \mathrm{Boot}_{k;\mathbb{Q}} = \langle \mathbf{1} \rangle \rightarrow L_{\mathbb{Q}}\mathcal{T}_k$. Hence, by restricting the above upper horizontal composition to the full subcategory of compact objects, we obtain the searched equivalence $\mathrm{Boot}_{k;\mathbb{Q}} \simeq \mathcal{D}^{\mathrm{perf}}(\mathbb{Q}^n)$ of Theorem 1.1; recall that for any commutative ring R the compact and perfect objects in $\mathcal{D}(R)$ agree, i.e. $\mathcal{D}(R)^c = \mathcal{D}^{\mathrm{perf}}(R)$. The conclusion concerning Balmer's spectrum follows from Corollary 3.4.

4. PROOF OF THEOREM 1.2

Our proof is based on a general result of tensor triangular geometry (see Theorem 4.3) and on the following two computations of higher algebraic K -theory: the first one concerning finite fields and the second one their algebraic closures.

Computation 1: Let $k = \mathbb{F}_q$ be a finite field, where $q = p^r$ with p a prime number and r a positive integer. Quillen's computation [20] of the algebraic K -theory of \mathbb{F}_q , combined with (2.2), gives rise to the following computation:

$$\mathrm{Hom}_{\mathrm{Boot}_k}(\Sigma^n \mathbf{1}, \mathbf{1}) \simeq K_n(\mathbb{F}_q) \simeq \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/(q^m - 1)\mathbb{Z} & n = 2m - 1, m > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Computation 2: Let $k = \overline{\mathbb{F}_q}$ be the algebraic closure of a finite field \mathbb{F}_q . Quillen's computation [20] of the algebraic K -theory of $\overline{\mathbb{F}_q}$, combined with the isomorphism (2.2), gives rise to the following computation:

$$\mathrm{Hom}_{\mathrm{Boot}_k}(\Sigma^n \mathbf{1}, \mathbf{1}) \simeq K_n(\overline{\mathbb{F}_q}) \simeq \begin{cases} \mathbb{Z} & n = 0 \\ \bigoplus_{\ell \neq p} \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell} & n > 0 \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

In the above cases, since $K_n(k)$ is trivial in negative degrees, Balmer's result [4, Thm. 7.13] applied to the tensor triangulated category Boot_k implies that the comparison map

$$\rho : \mathrm{Spc}(\mathrm{Boot}_k) \rightarrow \mathrm{Spc}(\mathrm{End}_{\mathrm{Boot}_k}(\mathbf{1})) = \mathrm{Spec}(\mathbb{Z})$$

is surjective. This proves item (i) of Theorem 1.2. In what concerns the prime ideals $s\mathbb{Z} \subset \mathbb{Z}$, where s is either 0 or p , we have the computation

$$(4.1) \quad \mathrm{End}_{(\mathrm{Boot}_k)_{(s)}}(\mathbf{1})_* \simeq (\mathrm{End}_{\mathrm{Boot}_k}(\mathbf{1})_*)_{(s)} \simeq \mathbb{Z}_{(s)},$$

with $\mathbb{Z}_{(s)}$ concentrated in degree zero. The left-hand-side isomorphism follows from Proposition 2.7. The right-hand-side isomorphism follows from the fact that the groups $K_n(k)$, $n > 0$, are not only torsion (this suffices for the case $s = 0$) but moreover all their torsion is coprime to p .

The following general result, which as Theorem 3.3 we expect to soon be part of the toolkit of tensor triangular geometry, computes the spectrum of a tensor triangulated category whose graded endomorphism ring of the \otimes -unit has a special form.

Definition 4.2. Let \mathcal{T} be a triangulated category with suspension functor Σ . We say that \mathcal{T} has period n , with $n > 0$, if there exists a natural isomorphism $\eta : \Sigma^n \simeq \text{Id}_{\mathcal{T}}$ and moreover n is the minimal positive integer verifying this condition. If such an integer n does not exist, we will say that \mathcal{T} has period $n = 0$.

Theorem 4.3. Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category with period $n \geq 0$. Assume moreover that \mathcal{T} has arbitrary coproducts, that the tensor product is coproduct-preserving in each variable, and that:

- (1) \mathcal{T} is compactly generated by the shifts of the \otimes -unit $\mathbf{1}$.
- (2) The \mathbb{Z} -graded endomorphism ring of the \otimes -unit is of the form

$$\text{End}_{\mathcal{T}}(\mathbf{1})_* = R[\eta, \eta^{-1}],$$

where $R := \text{End}_{\mathcal{T}}(\mathbf{1})$ is the degree zero subring and $\eta : \Sigma^n \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is an isomorphism in degree n . When $n = 0$, we take $\eta = \text{id}_{\mathbf{1}}$.

- (3) The ring R is hereditary and noetherian.

Then, the continuous comparison map $\rho : \text{Spc}(\mathcal{T}^c) \xrightarrow{\sim} \text{Spec}(R)$ is a homeomorphism.

The proof of Theorem 4.3 furnishes us the following description of the inverse of ρ .

Corollary 4.4. Under the hypotheses of Theorem 4.3, the inverse of the homeomorphism $\rho : \text{Spc}(\mathcal{T}^c) \xrightarrow{\sim} \text{Spec}(R)$ is given by

$$\text{Spec}(R) \ni \mathfrak{p} \mapsto \rho^{-1}(\mathfrak{p}) = \{X \in \mathcal{T}^c \mid \pi_*(X)_{\mathfrak{p}} \neq 0\} \in \text{Spc}(\mathcal{T}^c).$$

Corollary 4.5. When $k = \mathbb{F}_q$ or $k = \overline{\mathbb{F}_q}$ (where $q = p^r$ with p a prime number and r an integer ≥ 1) we have the following homeomorphism

$$\rho : \text{Spc}((\text{Boot}_k)_{(p)}) \xrightarrow{\sim} \text{Spec}(\mathbb{Z}_{(p)}).$$

Proof. By Proposition 2.7 (with $S = p\mathbb{Z}$), the tensor triangulated category $L_{p\mathbb{Z}}\mathcal{T}_k$ satisfies hypothesis (1) of Theorem 4.3 (with period $n = 0$). Hypotheses (2) and (3), with $R \simeq \mathbb{Z}_{(p)}$, follow from the above computation (4.1). Hence, the proof follows from Theorem 4.3 combined with the following commutative diagram

$$\begin{array}{ccc} \text{Spc}((L_{p\mathbb{Z}}\mathcal{T}_k)^c) & \xrightarrow[\simeq]{\text{Spc}(\iota)} & \text{Spc}((\text{Boot}_k)_{(p)}) \\ \rho \downarrow \simeq & & \downarrow \rho \\ \text{Spec}(\mathbb{Z}_{(p)}) & \xlongequal{\quad} & \text{Spec}(\mathbb{Z}_{(p)}), \end{array}$$

where the upper horizontal homeomorphism is the one induced by the cofinal inclusion $\iota : (\text{Boot}_k)_{(p)} \hookrightarrow (L_{p\mathbb{Z}}\mathcal{T}_k)^c$ of Proposition 2.7. \square

Remark 4.6. Note that a tensor triangulated category \mathcal{T} as in Theorem 3.3, which is moreover compactly generated by the shifts of the \otimes -unit, satisfies all the conditions of Theorem 4.3 (with $n = 0$ and R a field). In particular, by (4.1) (with $s = 0$), this holds for the rationalized bootstrap category $\mathcal{T} = \text{Boot}_{k;\mathbb{Q}}$, with k a finite field or its algebraic closure, $R = \mathbb{Q}$, and $\pi_* = (\overline{K}_*)_{\mathbb{Q}}$ the rationalized K -theory. This

furnishes us another proof of the homeomorphism $\rho : \mathrm{Spc}(\mathrm{Boot}_{k;\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Spec}(\mathbb{Q})$ when k is a field as above.

Before proving Theorem 4.3 and Corollary 4.4, let us first derive Theorem 1.2 from them. Balmer's general results [4, Thm. 5.4] furnish us a commutative diagram

$$(4.7) \quad \begin{array}{ccccc} \mathrm{Spc}((\mathrm{Boot}_k)_{(0)}) & \longrightarrow & \mathrm{Spc}((\mathrm{Boot}_k)_{(p)}) & \longrightarrow & \mathrm{Spc}(\mathrm{Boot}_k) \\ \rho \downarrow \simeq & & \rho \downarrow \simeq & & \downarrow \rho \\ \mathrm{Spec}(\mathbb{Z}_{(0)}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}_{(p)}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

where the horizontal maps are the canonical inclusions induced by the respective localizations, and where moreover each square is a pull-back of spaces. The middle vertical map is a homeomorphism by Corollary 4.5, and the left vertical map is a homeomorphism by Theorem 1.1 (or by Remark 4.6). Since $\mathrm{Spec}(\mathbb{Z}_{(p)}) = \{\{0\}, p\mathbb{Z}\}$, we see from the diagram that the fibers of the continuous surjective map $\rho : \mathrm{Spc}(\mathrm{Boot}_k) \rightarrow \mathrm{Spec}(\mathbb{Z})$ over $\{0\}$ and over $p\mathbb{Z}$ contain each one precisely one point. This is in particular one of the claims of items (ii) and (iii) of Theorem 1.2. It remains then to describe these points. The description of the point of the fiber $\rho^{-1}(\{0\})$ follows from the above results. By Corollary 4.4 (with $\mathcal{T} = \mathrm{Boot}_{k;\mathbb{Q}} = (\mathrm{Boot}_k)_{(0)}$, $R = \mathbb{Q}$, and $\pi_* = (\overline{K}_*)_{\mathbb{Q}}$ as in Remark 4.6) we have the following equality

$$\rho^{-1}(\{0\}) = \{M \in (\mathrm{Boot}_k)_{(0)} \mid \overline{K}_*(M)_{\mathbb{Q}} = 0\}.$$

Hence, by the commutativity of the outer square in (4.7), we conclude that

$$\rho^{-1}(\{0\}) = \{M \in \mathrm{Boot}_k \mid \overline{K}_*(M)_{\mathbb{Q}} = 0\}$$

as asserted in item (ii) of Theorem 1.2. The description (c) of the point of the fiber $\rho^{-1}(p\mathbb{Z})$ follows also from the above results. By Corollary 4.4 (with $\mathcal{T} = L_{p\mathbb{Z}}\mathcal{T}_k$, $\mathcal{T}^c = (\mathrm{Boot}_k)_{(p)}$, and $R = \mathbb{Z}_{(p)}$ as in Corollary 4.5, and thus $\pi_* = (\overline{K}_*)_{(p)}$) we have the following equality

$$\rho^{-1}(p\mathbb{Z}_{(p)}) = \{M \in (\mathrm{Boot}_k)_{(p)} \mid \overline{K}_*(M)_{(p)} = 0\}.$$

Hence, by the commutativity of the right hand-side square in diagram (4.7), we conclude that

$$\rho^{-1}(p\mathbb{Z}) = \{M \in \mathrm{Boot}_k \mid \overline{K}_*(M)_{(p)} = 0\}.$$

as asserted in item (iii) of Theorem 1.2. The descriptions (a) and (b) of the fiber $\rho^{-1}(p\mathbb{Z})$ in terms of Hochschild and periodic cyclic homology will be delayed until §5.3, after we have discussed the latter theories in detail. That will conclude the proof of Theorem 1.2

Remark 4.8. The reader may wonder why we have included in Theorem 4.3 the periodicity case $n \neq 0$. The case $n = 2$ will be used in §5.3 in the conclusion of the proof of Theorem 1.2. The remaining cases are of independent interest and are used in the proof of Corollary 4.9. Although not used in this article, Corollary 4.9 is relevant in the discussion of periodic cyclic homology as a functor into 2-periodic complexes; see §5.2 for details. In a future work we intend to develop and expand this circle of ideas.

Corollary 4.9. *Let $n \geq 2$ be an even positive integer, and $\mathcal{T} = \mathcal{D}_n(R)$ the derived tensor triangulated category of n -periodic complexes (see Definition 5.5 for the case $n = 2$; the general case is similar) over a hereditary noetherian commutative ring R . Then, we have a homeomorphism $\rho : \mathrm{Spc}(\mathcal{D}_n(R)^c) \xrightarrow{\sim} \mathrm{Spec}(R)$.*

Proof. Given any ring R , the triangulated category $\mathcal{T} := \mathcal{D}_n(R)$ is by construction n -periodic and is generated by the shifts of the complex with R concentrated in degrees $n\mathbb{Z}$. When $n \geq 2$ is even and R is commutative, it is moreover a tensor triangulated category satisfying hypotheses (1) and (2) of Theorem 4.3. If R is hereditary and noetherian, we observe that hypothesis (3) is also satisfied. \square

From now on, \mathcal{T} , R and n will be as in Theorem 4.3. The proof of this general result, which will occupy us until the end of this section, is inspired by some ideas and results from [11].

4.1. Relative homological algebra in the hereditary case. In analogy with the stable homotopy category of topological spectra, we will use homological notation; in particular we will keep using the shorthand notation $\pi_i := \mathrm{Hom}_{\mathcal{T}}(\Sigma^i \mathbf{1}, -)$, $i \in \mathbb{Z}$. The functor π_i will be considered as a homological functor from \mathcal{T} to the abelian category $R\text{-Mod}$ of R -modules. By taking into account the periodicity of \mathcal{T} , we will also consider the functor $\pi_* := \{\pi_i\}_i$ as a *stable* homological functor into the *stable* abelian category $R\text{-Mod}^{\mathbb{Z}/n}$ of \mathbb{Z}/n -graded modules. Note that $R\text{-Mod}^{\mathbb{Z}/n}$ comes equipped with a natural shift endofunctor

$$M = \{M_i\}_{i \in \mathbb{Z}/n} \mapsto M[1] := \{M_{i-1}\}_{i \in \mathbb{Z}/n}$$

such that $\pi_* \circ \Sigma \simeq [1] \circ \pi_*$. In what follows, following Meyer-Nest [16], we will develop the homological algebra of \mathcal{T} relative to the stable homological ideal $\mathcal{I} := \{f \in \mathrm{Mor}(\mathcal{T}) \mid \pi_*(f) = 0\}$. Recall that an object $X \in \mathcal{T}$ is called π_* -*projective* if $\mathrm{Hom}_{\mathcal{T}}(X, f) = 0$ for every $f \in \mathcal{I}$.

Lemma 4.10. *The functor $\pi_* : \mathcal{T} \rightarrow R\text{-Mod}^{\mathbb{Z}/n}$ is the universal \mathcal{I} -exact functor on \mathcal{T} . In particular, π_* restricts to an equivalence between the full subcategory of π_* -projective objects in \mathcal{T} and the full subcategory of projective objects in $R\text{-Mod}^{\mathbb{Z}/n}$.*

Proof. The projective objects in $R\text{-Mod}^{\mathbb{Z}/n}$ are additively generated by the free modules $R[i]$, $i \in \mathbb{Z}/n$. Hence, following [16, Thm. 57] and [16, Remark 58], it suffices to construct objects $\pi^\dagger(R[i])$ in \mathcal{T} verifying the following two conditions:

- (a) There are isomorphisms $\pi_*(\pi^\dagger(R[i])) \simeq R[i]$.
- (b) The induced map $\mathrm{Hom}_{\mathcal{T}}(\pi^\dagger(R[i]), \pi^\dagger(R[j])) \rightarrow \mathrm{Hom}_R(R[i], R[j])$ is an isomorphism for every $i, j \in \mathbb{Z}/n$.

The assignment $R[i] \mapsto \pi^\dagger(R[i])$ can then be extended to arbitrary projective modules, yielding a quasi-inverse for the restriction of π_* to π_* -projective objects. Let us set $\pi^\dagger(R[i]) := \Sigma^i \mathbf{1}$. Condition (a) holds by definition of R , and condition (b) follows from the natural isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma^i \mathbf{1}, \Sigma^j \mathbf{1}) = \begin{cases} R & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \simeq \mathrm{Hom}_R(R[i], R[j])$$

in $R\text{-Mod}^{\mathbb{Z}/n}$. This shows that π_* is the universal \mathcal{I} -exact functor. The remaining claim is proved in [16, Thm. 59]. \square

Let $F : \mathcal{T} \rightarrow \mathbf{Ab}$ and $G : \mathcal{T}^{\text{op}} \rightarrow \mathbf{Ab}$ be homological functors with values in abelian groups. Via π_* -projective resolutions in \mathcal{T} , one can define the so called \mathcal{I} -relative left and right derived functors $\mathbf{L}_\ell F$ and $\mathbf{R}^\ell G$ of F and G , respectively. The universal \mathcal{I} -exact functor π_* can then be used to compute these derived functors in terms of R -modules as follows.

Notation 4.11. Let $\text{Ext}_R^\ell(M, N)_*$ be the graded Ext, i.e. the ℓ^{th} right derived functor of the graded Hom-functor $\text{Hom}_R(M, N)_*$, where $\ell \geq 0$ and $M, N \in R\text{-Mod}^{\mathbb{Z}/n}$. We will write $\hat{\otimes}_R$ for the tensor product of \mathbb{Z}/n -graded R -modules defined by $M \hat{\otimes}_R N := \{\bigoplus_{u+v=i} M_u \otimes_R N_v\}_{i \in \mathbb{Z}/n}$. Under this notation, $\text{Tor}_\ell^R(M, N)_*$ will stand for the ℓ^{th} left derived functor of $\hat{\otimes}_R$.

Remark 4.12. In terms of the usual derived functors for ungraded R -modules, the graded-Ext and graded-Tor can be computed as follows ($u, v, i \in \mathbb{Z}/n$)

$$\text{Ext}_R^\ell(M, N)_i = \bigoplus_{u+v=i} \text{Ext}_R^\ell(M_u, N_v) \quad \text{Tor}_\ell^R(M, N)_i = \bigoplus_{u+v=i} \text{Tor}_\ell^R(M_u, N_v).$$

Thus, the sum of all the components of $\text{Ext}_R^\ell(M, N)_*$, resp. of $\text{Tor}_\ell^R(M, N)_*$, gives rise to the usual ungraded Ext, resp. ungraded Tor, of the R -modules $\bigoplus_i M_i$ and $\bigoplus_i N_i$.

Remark 4.13. Note that every functor $F : \mathcal{T} \rightarrow \mathbf{Ab}$ lifts canonically to a functor

$$F_* = \{F \circ \Sigma^{-i}\}_i : \mathcal{T} \rightarrow R\text{-Mod}^{\mathbb{Z}/n},$$

where the R -action is induced by functoriality. Similarly for contravariant functors $G : \mathcal{T}^{\text{op}} \rightarrow \mathbf{Ab}$. Moreover, it follows immediately from the above definitions that $\mathbf{L}_\ell(F_*) = (\mathbf{L}_\ell F)_*$ and $\mathbf{R}^\ell(G_*) = (\mathbf{R}^\ell G)_*$.

Lemma 4.14. *The relative derived functors admit the following computation:*

- (1) *Given a homological functor $F : \mathcal{T} \rightarrow \mathbf{Ab}$, which preserves arbitrary coproducts, there are natural isomorphisms of graded R -modules*

$$\mathbf{L}_\ell F_* \simeq \text{Tor}_\ell^R(F_*(\mathbf{1}), \pi_*(-))_* \quad (\ell = 0, 1).$$

- (2) *Given a homological functor $G : \mathcal{T}^{\text{op}} \rightarrow \mathbf{Ab}$, which sends coproducts into products, there are natural isomorphisms of graded R -modules*

$$\mathbf{R}^\ell G_* \cong \text{Ext}_R^\ell(\pi_*(-), G_*(\mathbf{1})) \quad (\ell = 0, 1).$$

Proof. The proof is, *mutatis mutanda*, the one of [16, Thm. 72]. \square

Proposition 4.15 (Künneth formula). *For any objects $X, Y \in \mathcal{T}$, there is a natural short exact sequence of \mathbb{Z}/n -graded R -modules*

$$0 \rightarrow \pi_*(X) \hat{\otimes}_R \pi_*(Y) \rightarrow \pi_*(X \otimes Y) \rightarrow \text{Tor}_1^R(\pi_* \Sigma X, \pi_* Y)_* \rightarrow 0.$$

Proof. Let $F : \mathcal{T} \rightarrow \mathbf{Ab}$ be an arbitrary homological functor and X an object of \mathcal{T} such that $\text{Hom}_{\mathcal{T}}(X, Z) = 0$ for all objects Z such that $\text{id}_Z \in \mathcal{I}$. Then, as shown in [16, Thm. 66], there exists a natural short exact sequence $0 \rightarrow \mathbf{L}_0 F_*(X) \rightarrow F_*(X) \rightarrow \mathbf{L}_1 F_*(\Sigma X) \rightarrow 0$. Since \mathcal{T} is generated by the shifts of $\mathbf{1}$, the only object Z such that $\text{id}_Z \in \mathcal{I}$ is the zero object. Hence, this applies to arbitrary objects $X \in \mathcal{T}$. Finally, by specializing ourselves to the functor $F := \pi_*(- \otimes Y) : \mathcal{T} \rightarrow \mathbf{Ab}$, we observe that Lemma 4.14(1) identifies the outer terms of $\mathbf{L}_0 F_*$ and $\mathbf{L}_1 F_*$ as required. \square

Proposition 4.16 (Universal coefficient formula). *For any objects $X, Y \in \mathcal{T}$, there is a natural short exact sequence of R -modules*

$$0 \rightarrow \mathrm{Ext}_R^1(\pi_* \Sigma X, \pi_* Y)_0 \rightarrow \mathrm{Hom}_{\mathcal{T}}(X, Y) \xrightarrow{\pi_*} \mathrm{Hom}_R(\pi_* X, \pi_* Y) \rightarrow 0.$$

Proof. Similarly to Proposition 4.15, there is a natural short exact sequence $0 \rightarrow \mathrm{R}^1 G_*(\Sigma X) \rightarrow G_*(X) \rightarrow \mathrm{R}^0 G_*(X) \rightarrow 0$. By first applying Lemma 4.14(2) (with $G = \mathrm{Hom}_{\mathcal{T}}(-, Y)$) and then passing to the degree zero components, we obtain the above short exact sequence. Note that the middle map is π_* because the isomorphism $\mathrm{R}^0 G(X) \simeq \mathrm{Ext}_R^0(\pi_* X, \pi_* Y)_0$ of Lemma 4.14(2) identifies it with the canonical map $G(X) \rightarrow \mathrm{R}^0 G(X)$ appearing in the short exact sequence. \square

Lemma 4.17. *Given any $M \in R\text{-Mod}^{\mathbb{Z}/n}$, there exists an object $X \in \mathcal{T}$ and an isomorphism $\pi_*(X) \cong M$.*

Proof. Since the ring R is hereditary, every R -module M has a length-one free resolution $0 \rightarrow F' \xrightarrow{f} F \rightarrow M \rightarrow 0$. The map f can be realized (uniquely) in \mathcal{T} as a morphism $\pi^\dagger F' \xrightarrow{\varphi} \pi^\dagger F$ between π_* -projective objects; see the proof of Lemma 4.10. Let us then define X by choosing a distinguished triangle $\pi^\dagger F' \rightarrow \pi^\dagger F \rightarrow X \rightarrow \Sigma(\pi^\dagger F')$. Applying the functor π_* to the latter triangle, we obtain an exact sequence $F' \rightarrow F \rightarrow \pi_*(X) \rightarrow F'[1] \rightarrow F[1]$ in $R\text{-Mod}^{\mathbb{Z}/n}$. The rightmost map in this sequence is the shift $f[1]$ of f and is therefore injective. These observations allow us to conclude that $\pi_*(X) \simeq M$. \square

Corollary 4.18. *The functor π_* induces a bijection between the set of isomorphism classes of objects in \mathcal{T} and the set of isomorphism classes of objects in $R\text{-Mod}^{\mathbb{Z}/n}$.*

Proof. It suffices to show that the object X of Lemma 4.17 which realizes the graded module M is well-defined up to isomorphism. Take an isomorphism $f : M \xrightarrow{\sim} N$ of graded modules, and realize the corresponding modules by objects X_M and X_N . Due to Proposition 4.16, we can then lift f to a morphism $\varphi : X_M \rightarrow X_N$ in \mathcal{T} . Since f is invertible, $\pi_* \mathrm{cone}(\varphi) = 0$ and so φ is also invertible. \square

Corollary 4.19. *The short exact sequences of Propositions 4.15 and 4.16 admit (unnatural) R -linear splittings.*

Proof. By Corollary 4.18, every object $X \in \mathcal{T}$ is isomorphic to a direct sum $\bigoplus_{i \in \mathbb{Z}/n} X_i$, where each $\pi_*(X_i) = (\pi_i X)[i]$ is concentrated in degree i . Then, the naturality of the short exact sequences and the additivity of the functors imply that the exact sequence of Proposition 4.16 decomposes as a direct sum of exact sequences

$$0 \rightarrow \mathrm{Ext}_R^1(\pi_* \Sigma X_i, \pi_* X_j)_0 \rightarrow \mathrm{Hom}_{\mathcal{T}}(X_i, X_j) \rightarrow \mathrm{Hom}_R(\pi_* X_i, \pi_* X_j) \rightarrow 0,$$

where $i, j \in \mathbb{Z}/n$. For every fixed pair (i, j) , we observe that either the first or the third group in the above sequence vanishes. In both cases, this implies that the sequence splits. \square

4.2. The residue field objects.

Definition 4.20. Let $\mathfrak{p} \subset R$ be a prime ideal. The *residue field object at \mathfrak{p}* , denoted by $\kappa(\mathfrak{p})$, is the object of \mathcal{T} which realizes, as in Lemma 4.17, the residue field $k(\mathfrak{p})[0] = (R/\mathfrak{p})_{\mathfrak{p}}[0]$ of R at \mathfrak{p} in degree zero.

Lemma 4.21. *For every object $X \in \mathcal{T}$, we have an isomorphism $X \otimes \kappa(\mathfrak{p}) \simeq \bigoplus_{a \in A} \kappa(\mathfrak{p})[i_a]$, where A is a set and the i_a 's are elements of \mathbb{Z}/n .*

Proof. By Proposition 4.15, $\pi_*(X \otimes \kappa(\mathfrak{p}))$ is isomorphic to some \mathbb{Z}/n -graded $k(\mathfrak{p})$ -vector space, since this is the case for the two outer modules in the corresponding short exact sequence. Moreover, every $k(\mathfrak{p})$ -vector space is of the form

$$\bigoplus_{a \in A} k(\mathfrak{p})[i_a] \cong \pi_* \left(\bigoplus_{a \in A} \Sigma^{i_a} \kappa(\mathfrak{p}) \right).$$

Hence, thanks to Corollary 4.18, the object $X \otimes \kappa(\mathfrak{p})$ decomposes as $\bigoplus_{a \in A} \kappa(\mathfrak{p})[i_a]$. \square

Lemma 4.22. *Let \mathfrak{p}_1 and \mathfrak{p}_2 be two prime ideals of R . Then, $\kappa(\mathfrak{p}_1) \otimes \kappa(\mathfrak{p}_2) \neq 0$ if and only if $\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof. This follows simply from Proposition 4.15 and from the general fact that $\kappa(\mathfrak{p}_1) \otimes_R^L \kappa(\mathfrak{p}_2) \neq 0$ in $\mathcal{D}(R)$ if and only if $\mathfrak{p}_1 = \mathfrak{p}_2$. \square

Proposition 4.23 (Tensor product formula). *Given objects $X, Y \in \mathcal{T}$ and a prime ideal $\mathfrak{p} \in \text{Spec}(R)$, we have an isomorphism of \mathbb{Z}/n -graded $k(\mathfrak{p})$ -vector spaces*

$$(4.24) \quad \pi_*(X \otimes Y \otimes \kappa(\mathfrak{p})) \cong \pi_*(X \otimes \kappa(\mathfrak{p})) \hat{\otimes}_{k(\mathfrak{p})} \pi_*(Y \otimes \kappa(\mathfrak{p})).$$

Proof. The proof, as in [11, Prop. 5.36], is based on an explicit computation. In order to simplify the notation, let us write κ for $\kappa(\mathfrak{p})$ and $\hat{\otimes}$ for $\hat{\otimes}_{k(\mathfrak{p})}$. By Lemma 4.21, we have the following isomorphisms:

$$X \otimes \kappa \cong \bigoplus_{a \in A} \Sigma^{i_a} \kappa \quad Y \otimes \kappa \cong \bigoplus_{b \in B} \Sigma^{i_b} \kappa.$$

Hence,

$$X \otimes Y \otimes \kappa \simeq X \otimes \bigoplus_b \Sigma^{i_b} \kappa \simeq \bigoplus_b \Sigma^{i_b} (X \otimes \kappa) \simeq \bigoplus_{a,b} \Sigma^{i_a + i_b} \kappa,$$

which implies that $\pi_*(X \otimes Y \otimes \kappa) \simeq \bigoplus_{a,b} k[i_a + i_b]$. On the other hand, we have

$$\pi_*(X \otimes \kappa) \hat{\otimes} \pi_*(Y \otimes \kappa) \simeq \left(\bigoplus_a k[i_a] \right) \hat{\otimes} \left(\bigoplus_b k[i_b] \right) \simeq \bigoplus_{a,b} k[i_a + i_b].$$

These computations allow us to conclude that both sides of (4.24) are isomorphic as graded R -modules, which concludes the proof. \square

Corollary 4.25. *For every object $X \in \mathcal{T}$ and prime ideal $\mathfrak{p} \in \text{Spec}(R)$, we have $\pi_*(X \otimes \kappa(\mathfrak{p})) = 0$ if and only if $\pi_*(X) \otimes_R^L k(\mathfrak{p}) = 0$ in $\mathcal{D}(R)$; we are considering $\pi_*(X)$ as the R -module $\bigoplus_i \pi_i(X)$.*

Proof. This follows from the fact that the short exact sequence of Proposition 4.15 (applied to $Y = \kappa(\mathfrak{p})$), implies that $\pi_*(X \otimes \kappa(\mathfrak{p})) = 0$ if and only if the complex $\pi_*(X) \otimes_R^L k(\mathfrak{p})$ is acyclic. \square

4.3. Two supports. In this subsection, we will define two supports for \mathcal{T} with values in $\text{Spec}(R)$, i.e. two functions from the set of objects of \mathcal{T} to the set of subsets of $\text{Spec}(R)$. The first one makes use of the residue field object of Definition 4.20 as follows.

Definition 4.26. For every object $X \in \mathcal{T}$, let $\sigma(X) := \{\mathfrak{p} \in \text{Spec}(R) \mid X \otimes \kappa(\mathfrak{p}) \neq 0\}$.

The second one, although more straightforward, will only play an auxiliary role in our proof in relation to compact objects.

Definition 4.27. For every object $X \in \mathcal{T}$, let $\tilde{\sigma}(X) := \{\mathfrak{p} \in \text{Spec}(R) \mid (\pi_* X)_{\mathfrak{p}} \neq 0\}$.

Lemma 4.28. *An object $X \in \mathcal{T}$ is compact if and only if $\pi_*(X)$ is a finitely generated R -module.*

Proof. Clearly, $\pi_*(\mathbf{1})$ is a finitely generated R -module. Since R is noetherian, a simple inductive argument using the long exact sequence for π_* implies that $\pi_*(X)$ is also finitely generated for every object X in the thick triangulated subcategory of \mathcal{T} generated by $\mathbf{1}$. Note that by hypothesis (1) this latter category coincides with \mathcal{T}^c . Conversely, if $\pi_*(X)$ is a finitely generated R -module, then X is isomorphic to the cone of a morphism between objects of the form $\bigoplus_{a \in A} \Sigma^{i_a} \mathbf{1}$ with A a finite set. This would imply that X is compact, which achieves the proof. \square

Lemma 4.29. *If $X \in \mathcal{T}$ is a compact object, then $\sigma(X) = \tilde{\sigma}(X)$.*

Proof. Let us consider $\pi_*(X)$ as the ungraded R -module $\bigoplus_i \pi_i(X)$. By Corollary 4.25 we have $\mathfrak{p} \notin \sigma(X)$, with $\mathfrak{p} \in \text{Spec}(R)$, if and only if the complex $\pi_*(X) \otimes_{R_{\mathfrak{p}}}^L k(\mathfrak{p}) = \pi_*(X) \otimes_{R_{\mathfrak{p}}}^L k(\mathfrak{p})$ is zero in $\mathcal{D}(R)$. Let $0 \rightarrow P' \rightarrow P \rightarrow \pi_*(X) \rightarrow 0$ be a R -projective resolution of $\pi_*(X)$. Thanks to Lemma 4.28, we may assume that P and P' are finitely generated. Hence, the localized sequence $0 \rightarrow P'_{\mathfrak{p}} \rightarrow P_{\mathfrak{p}} \rightarrow \pi_*(X)_{\mathfrak{p}} \rightarrow 0$ is a length-one resolution by finite projectives over the local ring $R_{\mathfrak{p}}$. As proved in [21, (2.2.4)], there exists a quasi-isomorphism $(d : Q' \rightarrow Q) \cong (P'_{\mathfrak{p}} \rightarrow P_{\mathfrak{p}})$ of perfect complexes over $R_{\mathfrak{p}}$, where $d : Q' \rightarrow Q$ is *minimal*, i.e. $d(Q') \subset \mathfrak{p}R_{\mathfrak{p}} \cdot Q$. Note that $(d : Q' \rightarrow Q)$ is again a resolution of $\pi_*(X)_{\mathfrak{p}}$. In sum, the following isomorphism holds in $\mathcal{D}(R)$:

$$\pi_*(X) \otimes_{R_{\mathfrak{p}}}^L k(\mathfrak{p}) \cong (d : Q' \rightarrow Q) \otimes_{R_{\mathfrak{p}}} k(\mathfrak{p}) \cong (0 : Q'/\mathfrak{p}R_{\mathfrak{p}}Q' \rightarrow Q/\mathfrak{p}R_{\mathfrak{p}}Q).$$

Hence, $\pi_*(X) \otimes_{R_{\mathfrak{p}}}^L k(\mathfrak{p}) = 0 \Leftrightarrow Q'/\mathfrak{p}R_{\mathfrak{p}}Q' = Q/\mathfrak{p}R_{\mathfrak{p}}Q = 0 \Leftrightarrow Q' = Q = 0 \Leftrightarrow \pi_*(X)_{\mathfrak{p}} = 0 \Leftrightarrow \mathfrak{p} \in \tilde{\sigma}(X)$, which achieves the proof. \square

Corollary 4.30. *If $X \in \mathcal{T}$ is a compact object, the subset $\sigma(X) \subset \text{Spec}(R)$ is closed.*

Proof. When X is a compact object, the R -module $\pi_*(X)$ is finitely generated by Lemma 4.28. Lemma 4.29 then implies that $\sigma(X) = \tilde{\sigma}(X)$, which for finite modules is well-known to be a closed set in the Zariski topology; see [1, §3 Exercise 19 (vii)]. \square

4.4. Computation of the spectrum. In this subsection we will verify that the support $(\text{Spec}(R), \sigma)$ on \mathcal{T} , defined in the previous subsection, satisfies the conditions (S0)-(S9) of [11, Thm. 3.1]. As a consequence, the restriction of $(\text{Spec}(R), \sigma)$ to the compact objects of \mathcal{T}^c is a *classifying* support datum, i.e. the canonical map $\varphi : \text{Spec}(R) \rightarrow \text{Spc}(\mathcal{T}^c)$ is a homeomorphism; see [11, Thm. 2.19]. This result,

combined with Lemmas 4.29 and 4.31, implies that $\rho : \mathrm{Spc}(\mathcal{T}^c) \rightarrow \mathrm{Spec}(R)$ is a homeomorphism.

Lemma 4.31. *Let $(\mathcal{K}, \otimes, \mathbf{1})$ be a tensor triangulated category and let us keep the notations $R := \mathrm{End}_{\mathcal{K}}(\mathbf{1})$, $\pi_i(X) := \mathrm{Hom}_{\mathcal{K}}(\Sigma^i \mathbf{1}, X)$ and $\tilde{\sigma}(X) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid (\pi_* X)_{\mathfrak{p}} \neq 0\}$. If the pair $(\mathrm{Spec}(R), \tilde{\sigma})$ satisfies the axioms of a support datum on \mathcal{K} , then the map $\varphi : \mathrm{Spec}(R) \rightarrow \mathrm{Spc}(\mathcal{K})$ (induced by the universal property of the latter) is a continuous section of the natural comparison map $\rho : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}(R)$. In particular, if $(\mathrm{Spec}(R), \tilde{\sigma})$ is a classifying support datum, then ρ is the inverse of φ .*

Proof. Recall that the map φ is given by $\mathfrak{p} \mapsto \varphi(\mathfrak{p}) = \{X \in \mathcal{K} \mid \mathfrak{p} \notin \tilde{\sigma}(X)\}$. For every $r \in R$ and $\mathfrak{p} \in \mathrm{Spec}(R)$, we have

$$(4.32) \quad r \notin \rho(\varphi(\mathfrak{p})) \Leftrightarrow \mathrm{cone}(r) \in \varphi(\mathfrak{p})$$

$$(4.33) \quad \Leftrightarrow \mathfrak{p} \notin \tilde{\sigma}(X)$$

$$(4.34) \quad \Leftrightarrow (\pi_*(\mathrm{cone}(r)))_{\mathfrak{p}} = 0$$

$$\Leftrightarrow r \in (R_{\mathfrak{p}})^{\times}$$

$$\Leftrightarrow r \notin \mathfrak{p},$$

where (4.32) follows from the definition of ρ , (4.33) from the definition of φ , and (4.34) from the definition of $\tilde{\sigma}$. This implies that $\rho \circ \varphi = \mathrm{Id}_{\mathrm{Spec}(R)}$, and so the proof is finished. \square

Since σ is defined via a family of coproduct-preserving exact functors, it trivially satisfies conditions (S0)-(S4) and (S6); see [11, Lemma 3.3]. Let us now recall the remaining four important conditions:

Partial Tensor Product formula:

$$(S5) \quad \sigma(X \otimes Y) = \sigma(X) \cap \sigma(Y) \text{ for every } X \in \mathcal{T}^c \text{ and every } Y \in \mathcal{T}.$$

Correspondence between quasi-compact opens and compact objects:

(S7) $\sigma(X)$ is a closed subset of $\mathrm{Spec}(R)$ with quasi-compact open complement for every compact object X .

(S8) For every closed subset $V \subset \mathrm{Spec}(R)$ with quasi-compact open complement, there exists a compact object $X \in \mathcal{T}^c$ such that $\sigma(X) = V$.

Detection of objects:

$$(S9) \quad \text{If } \sigma(X) = \emptyset, \text{ then } X = 0.$$

Condition (S5) follows from Proposition 4.23 since the tensor product $M \hat{\otimes}_{k(\mathfrak{p})} N$ of two graded $k(\mathfrak{p})$ -vector spaces is zero if and only if one of the factors is already zero. Note that since R is noetherian, every open subset of $\mathrm{Spec}(R)$ is quasi-compact. Hence, condition (S7) follows immediately from Corollary 4.30. The next lemma proves condition (S8), and the following one condition (S9).

Lemma 4.35. *For every closed subset $V \subset \mathrm{Spec}(R)$, there exists a compact object $X \in \mathcal{T}^c$ such that $\sigma(X) = V$.*

Proof. We have seen that σ satisfies condition (S5), and moreover R is noetherian by hypothesis. Since for compact objects σ coincides with $\tilde{\sigma}$, [11, Prop. 3.12 (b)] implies the claim. Recall from *loc. cit.* that the object X can be constructed as $X = \mathrm{cone}(r_1) \otimes \cdots \otimes \mathrm{cone}(r_s)$, where (r_1, \dots, r_s) is a finite set of generators for any ideal $I \subset R$ with $V = V(I)$. \square

Lemma 4.36. *If $\sigma(X) = \emptyset$ then $X = 0$.*

Proof. Let M be a module (or more generally a complex of modules) over the noetherian commutative ring R , such that $M \otimes_R^{\mathbb{L}} k(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \text{Spec}(R)$. Then, by [18, Lemma 2.12], $M = 0$ in the derived category $\mathcal{D}(R)$. If $\sigma(X) = \emptyset$, then by Corollary 4.25, the R -module $M := \pi_*(X)$ satisfies the previous condition and so $\pi_*(X) = 0$. This allow us to conclude, by hypothesis (1), that $X = 0$. \square

This ends the proof of Theorem 4.3. Now Corollary 4.4 follows immediately from Lemma 4.31, the definition of the support datum $(\text{Spec}(R), \bar{\sigma}) = (\text{Spec}(R), \sigma)|_{\mathcal{T}^c}$ on \mathcal{T}^c , and the definition of the induced map $\varphi : \text{Spec}(R) \rightarrow \text{Spc}(\mathcal{T}^c)$.

5. PROOF OF THEOREM 1.3

Let us begin this section with a few recollections and results on Hochschild and periodic cyclic homology. In §5.3, making use of these results, we will conclude the proof of Theorem 1.2. Given a k -algebra A , we will denote by \underline{A} the associated dg category with one object and with A as the dg algebra of endomorphisms (concentrated in degree zero).

5.1. Hochschild homology. Recall from [10, Example 7.9] that Hochschild homology (HH) is an example of an additive invariant which is symmetric monoidal. By [10, Thm. 7.5], we obtain then an induced tensor triangulated functor

$$(5.1) \quad \overline{HH} : \text{Mot}_k \longrightarrow \mathcal{D}(k)$$

such that $\overline{HH}(U_k(\mathcal{A})) = HH(\mathcal{A})$ for every dg category \mathcal{A} . By restricting it to the bootstrap category we obtain a tensor triangulated functor

$$(5.2) \quad \overline{HH} : \text{Boot}_k = \langle U_k(\underline{k}) \rangle \longrightarrow \langle k \rangle = \mathcal{D}^{\text{perf}}(k)$$

with values in the derived category of perfect complexes.

5.2. Periodic cyclic homology. Recall from [10, Example 7.10] that the mixed complex construction (C) gives rise to a tensor triangulated functor

$$(5.3) \quad \overline{C} : \text{Mot}_k \longrightarrow \mathcal{D}(\Lambda),$$

where Λ denotes the dg algebra $k[\epsilon]/\epsilon^2$ with ϵ of degree -1 and $d(\epsilon) = 0$. The symmetric monoidal structure on $\mathcal{D}(\Lambda)$ is induced by the tensor product on the underlying complexes of k -modules. The \otimes -unit is the complex k (concentrated in degree zero) with the trivial ϵ -action. As explained by Kassel in [13, page 210], periodic cyclic homology (HP) can be expressed as the composition of the mixed complex construction with the 2-perioditization functor (see Definition 5.6). We obtain then the triangulated functor

$$(5.4) \quad \overline{HP} : \text{Mot}_k \xrightarrow{\overline{C}} \mathcal{D}(\Lambda) \xrightarrow{\widehat{(-)}} \mathcal{D}_2(k)$$

with values in the derived category of 2-periodic complexes (see Definition 5.5).

Definition 5.5. A 2-periodic complex I is a complex of k -modules which is invariant under the shift functor $(-)[2]$. Equivalently, it consists of two k -modules and two k -linear maps

$$I_0 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{d} \end{array} I_1$$

satisfying $d^2 = 0$ (in both ways). A morphism of 2-periodic complexes is a morphism of complexes which is invariant under the shift functor $(-)[2]$. The category of 2-periodic complexes will be denoted by $\mathcal{C}_2(k)$. The derived category $\mathcal{D}_2(k)$ of 2-periodic complexes is obtained from $\mathcal{C}_2(k)$ by inverting the class of quasi-isomorphisms.

Note that $\mathcal{C}_2(k)$ is naturally endowed with a symmetric monoidal structure whose tensor product is given by

$$I \hat{\otimes} J := \left(I_0 \otimes J_0 \oplus I_1 \otimes J_1 \begin{array}{c} \xrightarrow{\begin{bmatrix} \text{id} \otimes d & d \otimes \text{id} \\ d \otimes \text{id} & -\text{id} \otimes d \end{bmatrix}} \\ \xleftarrow{\begin{bmatrix} \text{id} \otimes d & d \otimes \text{id} \\ d \otimes \text{id} & -\text{id} \otimes d \end{bmatrix}} \end{array} I_0 \otimes J_1 \oplus I_1 \otimes J_0 \right).$$

The corresponding $\hat{\otimes}$ -unit is the 2-periodic complex $k = (k \rightrightarrows 0)$. This tensor product can be naturally derived (using for instance the fact that the standard projective model structure on complexes adapts to $\mathcal{C}_2(k)$) giving rise to a symmetric monoidal structure on $\mathcal{D}_2(k)$.

Definition 5.6. The 2-perioditization functor $\widehat{(-)} : \mathcal{D}(\Lambda) \rightarrow \mathcal{D}_2(k)$ sends a mixed complex R to the 2-periodic complex

$$\widehat{R} := \left(\prod_{i \text{ even}} R_i \begin{array}{c} \xrightarrow{d+\epsilon} \\ \xleftarrow{d+\epsilon} \end{array} \prod_{i \text{ odd}} R_i \right).$$

Since the 2-perioditization functor is defined using infinite products and these do not commute with the tensor product, the functor $\widehat{(-)}$ is not symmetric monoidal. However, it is lax monoidal and the natural structure morphisms $\eta_{R,S} : \widehat{R} \hat{\otimes} \widehat{S} \rightarrow \widehat{R \otimes S}$ are isomorphisms whenever R or S satisfy Kassel's Property (P); see [13, page 211].

Lemma 5.7. *Let $F : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ be a lax symmetric monoidal functor between tensor triangulated categories and \mathcal{S} a set of objects of \mathcal{T}_1 ; recall that we have a natural structure morphism $\eta_{X,Y} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ for any two objects X and Y . If $\eta_{X,Y}$ is an isomorphism for every X and Y in \mathcal{S} , then $\eta_{X,Y}$ is also an isomorphism for every X and Y in $\langle \mathcal{S} \rangle$.*

Proof. For every fixed object Y in \mathcal{T}_1 , the subcategory

$$\mathcal{U}_Y := \{X \in \mathcal{T}_1 \mid \eta_{X,Y} \text{ is an isomorphism}\} \subset \mathcal{T}_1$$

is thick. This follows immediately from the fact that η is a natural transformation between triangulated functors $F(-) \otimes F(Y)$ and $F(- \otimes Y)$, and from the classical five lemma in homological algebra. Therefore, whenever $Y \in \mathcal{S}$, we have $\mathcal{S} \subset \mathcal{U}_Y$ by hypothesis, and so we conclude that $\langle \mathcal{S} \rangle \subset \mathcal{U}_Y$. The same arguments imply that $\{Y \in \mathcal{T}_1 \mid \mathcal{S} \subset \mathcal{U}_Y\}$ is a thick subcategory of \mathcal{T}_1 which contains \mathcal{S} . Hence, it contains $\langle \mathcal{S} \rangle$, i.e. $\eta_{X,Y}$ is an isomorphism for all $X, Y \in \langle \mathcal{S} \rangle$. \square

Corollary 5.8. *When restricted to the bootstrap category, the periodic cyclic homology functor (5.4) yields a tensor triangulated functor*

$$(5.9) \quad \overline{HP} : \text{Boot}_k \longrightarrow \mathcal{D}_2(k)^c.$$

Proof. Since the 2-perioditization functor sends the \otimes -unit of $\mathcal{D}(\Lambda)$ to the $\hat{\otimes}$ -unit of $\mathcal{D}_2(k)$, the natural structure morphism $\eta : \widehat{k \hat{\otimes} \hat{R}} \rightarrow \widehat{k \otimes R}$ is an isomorphism for any mixed complex R . The result now follows from Lemma 5.7 and the fact that the functor (5.3) is symmetric monoidal. \square

5.3. Conclusion of the proof of Theorem 1.2. Recall from §4 that it remains to prove the descriptions (a) and (b) of the unique point of the fiber $\rho^{-1}(p\mathbb{Z})$.

Lemma 5.10. *Whenever the base ring k is a field, the two kernels*

$$\text{Ker}(\overline{HH}) = \{M \in \text{Boot}_k \mid \overline{HH}(M) = 0\}$$

$$\text{Ker}(\overline{HP}) = \{M \in \text{Boot}_k \mid \overline{HP}(M) = 0\}$$

are prime \otimes -ideals of Boot_k .

Proof. Since by hypothesis k is a field, the derived category $\mathcal{D}_2(k)$ of 2-periodic complexes is abelian and even semi-simple. Moreover, it has the property that $M \hat{\otimes} N = 0$ if and only if $M = 0$ or $N = 0$. In other words, $\{0\}$ is a prime \otimes -ideal of $\mathcal{D}_2(k)$. Consequently, the kernel $\text{Ker}(\overline{HP})$ of the tensor triangulated functor (5.9) is a prime \otimes -ideal of Boot_k , i.e. a point of $\text{Spc}(\text{Boot}_k)$. Similarly, the category $\mathcal{D}(k)^c$ of perfect complexes is abelian semi-simple, and $\{0\}$ is a prime \otimes -ideal of $\mathcal{D}(k)^c$. Hence, the kernel $\text{Ker}(\overline{HH})$ of the tensor triangulated functor (5.2) is a prime \otimes -ideal of Boot_k . \square

Let k be either \mathbb{F}_q or $\overline{\mathbb{F}_q}$, as in Theorem 1.2. We have proved in §4 that in this case the fiber $\rho^{-1}(p\mathbb{Z})$ contains precisely one point. Thus it remains only to show that the points $\text{Ker}(\overline{HH})$ and $\text{Ker}(\overline{HP})$ of $\text{Spc}(\text{Boot}_k)$ (see Lemma 5.10) are both sent to $p\mathbb{Z} \in \text{Spec}(\mathbb{Z})$ by the comparison map ρ . Consider the two commutative diagrams

$$\begin{array}{ccc} \text{Spc}(\mathcal{D}(k)^c) & \xrightarrow{\text{Spc}(\overline{HH})} & \text{Spc}(\text{Boot}_k) \\ \rho \downarrow \simeq & & \downarrow \rho \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(\mathbb{Z}) \end{array} \quad \begin{array}{ccc} \text{Spc}(\mathcal{D}_2(k)^c) & \xrightarrow{\text{Spc}(\overline{HP})} & \text{Spc}(\text{Boot}_k) \\ \rho \downarrow \simeq & & \downarrow \rho \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(\mathbb{Z}) \end{array}$$

obtained by applying Balmer's contravariant functor $\text{Spc}(-)$ to the tensor triangulated functors (5.2) and (5.9), respectively, and using the naturality of ρ (see [4, Thm. 5.3(c)]). The left hand-side vertical homeomorphisms are instantiations of Theorem 4.3 and Corollary 4.9, respectively. Now, since the field k has characteristic p and $\text{Spec}(k) = \{0\}$, we conclude that $\rho(\text{Ker}(\overline{HH})) = \rho(\text{Ker}(\overline{HP})) = p\mathbb{Z}$. This ends the proof of Theorem 1.2.

5.4. Enlarging the bootstrap category. Let us start with some general results.

Lemma 5.11. *Let \mathcal{T} be a tensor triangulated category and \mathcal{S} a set of objects in \mathcal{T} . If $\mathcal{S} \otimes \mathcal{S} \subset \mathcal{S}$, then $\langle \mathcal{S} \rangle \otimes \langle \mathcal{S} \rangle \subset \langle \mathcal{S} \rangle$.*

Proof. Consider the full subcategory $\mathcal{W} := \{X \in \mathcal{T} \mid X \otimes \langle \mathcal{S} \rangle \subset \langle \mathcal{S} \rangle\}$ of \mathcal{T} . The exactness of the tensor product implies that \mathcal{W} is a thick triangulated subcategory of \mathcal{T} . Since $\mathcal{S} \otimes \mathcal{S} \subset \mathcal{S}$, we conclude (using again the exactness of the tensor product) that \mathcal{S} , and hence $\langle \mathcal{S} \rangle$, is contained in \mathcal{W} . This achieves the proof. \square

Corollary 5.12. *Let \mathcal{U} be a set of objects in a tensor triangulated category \mathcal{T} . Then, the smallest thick tensor triangulated subcategory of \mathcal{T} containing \mathcal{U} is given by*

$$\langle \mathbf{1} \rangle \cup \{X_1 \otimes \cdots \otimes X_n \mid X_i \in \mathcal{U}, n \geq 1\}.$$

In other words, it is the thick triangulated subcategory generated by the tensor multiplicative system generated by \mathcal{U} .

Proof. Let \mathcal{S} be the set $\langle \mathbf{1} \rangle \cup \{X_1 \otimes \cdots \otimes X_n \mid X_i \in \mathcal{U}, n \geq 1\}$. Since $\mathcal{S} \otimes \mathcal{S} \subset \mathcal{S}$, Lemma 5.11 implies that $\langle \mathcal{S} \rangle \otimes \langle \mathcal{S} \rangle \subset \langle \mathcal{S} \rangle$. The proof now follows from the fact that $\langle \mathcal{S} \rangle$ is the smallest thick triangulated subcategory of \mathcal{T} which contains \mathcal{S} and which satisfies the property $\langle \mathcal{S} \rangle \otimes \langle \mathcal{S} \rangle \subset \langle \mathcal{S} \rangle$. \square

Now, recall from §2.2 that the triangulated category Mot_k of non-commutative motives is compactly generated and that an (essentially small) set of compact generators is given by (the shifts of) the non-commutative motives

$$(5.13) \quad \{U_k(\mathcal{A}) \mid \mathcal{A} \text{ homotopically finitely presented}\}.$$

Hence, the full subcategory $(\text{Mot}_k)^c$ of compact objects can be described as the thick triangulated subcategory of Mot_k generated by the set (5.13).

Notation 5.14. We will denote by EBoot_k the thick triangulated subcategory

$$\text{EBoot}_k := \langle U_k(\underline{k}), U_k(\underline{k[t]}), \dots, U_k(\underline{k[t_1, \dots, t_n]}), \dots \rangle \subset \text{Mot}_k$$

generated by the \otimes -unit $U_k(\underline{k})$ and by the non-commutative motives associated to the polynomial algebras $k[t_1, \dots, t_n]$, $n \geq 1$. The letter E stands for ‘‘Enlarged’’.

Lemma 5.15. *The symmetric monoidal structure on Mot_k restricts not only to $(\text{Mot}_k)^c$ but moreover to EBoot_k . We have then natural inclusions*

$$\text{Boot}_k \subset \text{EBoot}_k \subset (\text{Mot}_k)^c$$

of tensor triangulated categories. Moreover, EBoot_k is the smallest tensor triangulated subcategory of Mot_k containing the non-commutative motive $U_k(\underline{k[t]})$.

Proof. The dg categories $\underline{k[t_1, \dots, t_n]}$, $n \geq 1$, associated to the polynomial algebras, are homotopically finitely presented. Hence, the first assertion follows from Lemma 5.11 applied to the triangulated category Mot_k and to the sets

$$\mathcal{S} := (5.13) \quad \text{and} \quad \mathcal{S}' := \{U_k(\underline{k})\} \cup \{U_k(\underline{k[t_1, \dots, t_n]}) \mid n \geq 1\}.$$

In both cases the inclusion $\mathcal{S} \otimes \mathcal{S} \subset \mathcal{S}$ follows from the fact that the functor U_k is symmetric monoidal and from the stability of homotopically finitely presented dg categories under (derived) tensor product; see [10, Thm. 3.4]. The second assertion follows from Corollary 5.12 and from the natural isomorphism

$$U_k(\underline{k[t_1]}) \otimes \cdots \otimes U_k(\underline{k[t_n]}) \simeq U_k(\underline{k[t_1, \dots, t_n]}).$$

\square

5.5. **Two distinct prime \otimes -ideals of EBoot_k .** The Hochschild homology functor (5.1) restricts to a tensor triangulated functor

$$(5.16) \quad \overline{HH} : \text{EBoot}_k \longrightarrow \mathcal{D}(k).$$

Note that, in contrast with (5.2), this functor does not take values in $\mathcal{D}(k)^c = \mathcal{D}^{\text{perf}}(k)$. The reason for this is that $\overline{HH}(U_k(k[t])) = HH(k[t])$ and, as explained in [22, Example 6.1.7], we have the following computation

$$(5.17) \quad HH_n(k[t]) \simeq \begin{cases} k[t] & n = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.18. *Assume that k is a field of characteristic zero. Then, the periodic cyclic homology functor (5.4) yields a tensor triangulated functor*

$$(5.19) \quad \overline{HP} : \text{EBoot}_k \longrightarrow \mathcal{D}_2(k).$$

Proof. Since k is a field of characteristic zero, the k -algebra $k[t]$ enjoys Kassel's Property (P); see [13, Example (3.3), page 208]. This implies that the natural structure morphism

$$\eta_{C(k[t]), S} : \widehat{C(k[t])} \otimes \widehat{S} \xrightarrow{\sim} (\widehat{C(k[t])} \otimes S)$$

of the 2-perioditization functor (see Definition 5.6) is an isomorphism for all mixed complexes S . Throughout a recursive argument, we conclude that $\eta_{R,S}$ is an isomorphism for all mixed complexes R and S belonging to the set $\{C(k[t_1, \dots, t_n]) \mid n \geq 0\}$. Lemma 5.7 allow us then to conclude the proof. \square

We now have all the ingredients needed for the proof of Theorem 1.3. The same argument as in Lemma 5.10 (applied to the tensor triangulated functors (5.16) and (5.19) instead of (5.2) and (5.9)) shows that

$$\text{Ker}(\overline{HH}) = \{M \in \text{Boot}_k \mid \overline{HH}(M) = 0\}$$

$$\text{Ker}(\overline{HP}) = \{M \in \text{Boot}_k \mid \overline{HP}(M) = 0\}$$

are two prime \otimes -ideals of EBoot_k , i.e. two points of $\text{Spc}(\text{EBoot}_k)$. Hence, it remains only to prove that they are different. Consider the natural inclusion of k -algebras $k \rightarrow k[t]$ and the associated map $U_k(\underline{k}) \rightarrow U_k(k[t])$ in EBoot_k . Since EBoot_k is a triangulated category, this latter map can be completed into a distinguished triangle

$$(5.20) \quad U_k(\underline{k}) \longrightarrow U_k(k[t]) \longrightarrow \mathcal{O} \longrightarrow \Sigma(U_k(\underline{k})).$$

We now show that the non-commutative motive \mathcal{O} belongs to $\text{Ker}(\overline{HP})$ but not to $\text{Ker}(\overline{HH})$ which implies that $\text{Ker}(\overline{HP}) \neq \text{Ker}(\overline{HH})$. By applying the triangulated functors (5.16) and (5.19) to (5.20) we obtain a distinguished triangle

$$HH(k) \longrightarrow HH(k[t]) \longrightarrow \overline{HH}(\mathcal{O}) \longrightarrow \Sigma(HH(k))$$

in $\mathcal{D}(k)$ and a distinguished triangle

$$HP(k) \longrightarrow HP(k[t]) \longrightarrow \overline{HP}(\mathcal{O}) \longrightarrow \Sigma(HP(k))$$

in $\mathcal{D}_2(k)$. Since $HH_n(k) = k$ if $n = 0$ and is zero otherwise, the above computation (5.17) allows us to conclude that the map $HH(k) \rightarrow HH(k[t])$ is not an isomorphism. This implies that $\overline{HH}(\mathcal{O}) \neq 0$ in $\mathcal{D}(k)$ and so $\mathcal{O} \notin \text{Ker}(\overline{HH})$. On the other hand, since by hypothesis k is of characteristic zero, we have an isomorphism $HP(k) \xrightarrow{\sim} HP(k[t])$; see [13, §3] or [15, Exercise E.5.1.4, page 167]. This

implies that $\overline{HP}(\mathcal{O}) = 0$ and so $\mathcal{O} \in \text{Ker}(\overline{HP})$. The proof of Theorem 1.3 is then concluded.

5.6. Proof of Proposition 1.4. Item (i) follows from Balmer's connectivity criterion [4, Thm. 7.13]. In what concerns item (ii), note that if we restrict the tensor triangulated functors (5.16) and (5.19) to the \otimes -unit we obtain in both cases (up to isomorphism) the following unique ring homomorphism

$$\mathbb{Z} \simeq \text{End}_{\text{EBoot}_k}(\mathbf{1}) \longrightarrow \text{End}_{\mathcal{D}(k)}(\mathbf{1}) \simeq \text{End}_{\mathcal{D}_2(k)}(\mathbf{1}) \simeq k.$$

Since by hypothesis k is of characteristic zero, this ring homomorphism is injective. Hence, if f is a non-zero element of $\text{End}_{\text{EBoot}_k}(\mathbf{1})$, its images $\overline{HH}(f)$ and $\overline{HP}(f)$ are invertible elements of $\text{End}_{\mathcal{D}(k)}(\mathbf{1})$ and $\text{End}_{\mathcal{D}_2(k)}(\mathbf{1})$, respectively. This implies that $\overline{HH}(\text{cone}(f)) = 0$ and that $\overline{HP}(\text{cone}(f)) = 0$. By definition of the comparison map $\rho : \text{Spc}(\text{EBoot}_k) \rightarrow \text{Spc}(\mathbb{Z})$ we obtain then

$$\begin{aligned} \rho(\text{Ker}(\overline{HH})) &:= \{f \in \text{End}_{\text{EBoot}_k}(\mathbf{1}) \mid \text{cone}(f) \notin \text{Ker}(\overline{HH})\} = \{0\} \\ \rho(\text{Ker}(\overline{HP})) &:= \{f \in \text{End}_{\text{EBoot}_k}(\mathbf{1}) \mid \text{cone}(f) \notin \text{Ker}(\overline{HP})\} = \{0\}. \end{aligned}$$

This concludes the proof of Proposition 1.4.

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