

A FUNCTORIAL EQUIVARIANT K-THEORY SPECTRUM AND AN EQUIVARIANT LEFSCHETZ FORMULA

IVO DELL'AMBROGIO, HEATH EMERSON, TAMAZ KANDELAKI, AND RALF MEYER

ABSTRACT. We construct a symmetric spectrum representing the G -equivariant K-theory of C^* -algebras for a compact group or a proper groupoid G . Our spectrum is functorial for equivariant $*$ -homomorphisms. We use this to establish the additivity of the canonical traces for endomorphisms of strongly dualisable objects in the bootstrap class in KK^G , in analogy to previous results for traces in stable homotopy theory. As an application, we prove an equivariant analogue of the Lefschetz trace formula for Hodgkin Lie groups.

1. INTRODUCTION

Let \mathcal{C} be a symmetric monoidal category with tensor product \otimes and tensor unit $\mathbb{1}$. An object A of \mathcal{C} is called *dualisable* if there is an object A^* , called its *dual*, and a natural isomorphism

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, A^* \otimes C)$$

for all objects B and C of \mathcal{C} . Such duality isomorphisms exist if and only if there are two morphisms $\eta: \mathbb{1} \rightarrow A^* \otimes A$ and $\varepsilon: A \otimes A^* \rightarrow \mathbb{1}$, called unit and counit of the duality, that satisfy two appropriate conditions. Let $f: A \rightarrow A$ be an endomorphism in \mathcal{C} . Then the *trace* of f is the composite endomorphism

$$\mathbb{1} \xrightarrow{\eta} A^* \otimes A \xrightarrow{\text{braid}} A \otimes A^* \xrightarrow{f \otimes \mathrm{id}_{A^*}} A \otimes A^* \xrightarrow{\varepsilon} \mathbb{1},$$

where **braid** denotes the braiding isomorphism.

We want to compute such traces in the case where \mathcal{C} is the equivariant Kasparov category $\mathfrak{K}\mathfrak{K}^G$ of separable G - C^* -algebras for a compact group G . Here \otimes is the minimal C^* -algebra tensor product equipped with the diagonal action of G , and $\mathbb{1}$ is the C^* -algebra of complex numbers; the endomorphism ring of $\mathbb{1}$ is the representation ring of G , $\mathrm{KK}_0^G(\mathbb{1}, \mathbb{1}) \cong \mathrm{R}(G)$. More generally, our construction still works if G is a proper groupoid, in which case \otimes is the tensor product over the object space G^0 of the groupoid and $\mathbb{1} = C_0(G^0)$ (see also [6]).

If G is trivial, $A = C(X)$ for a smooth compact manifold and $f \in \mathrm{KK}(A, A)$ comes from a self-map with simple isolated fixed points, then the Kasparov product that gives its trace in $\mathrm{KK}(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$ may be computed directly as in [5], and the result is the Lefschetz number of f , expressed as a sum of contributions from the fixed points of f . By the Lefschetz Fixed Point Theorem, this is equal to the alternating sum of the maps induced by f on the rational cohomology groups of X . Here we may replace rational cohomology by K-theory because of the Chern character. Our

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goal is to establish a G -equivariant generalisation of this result for suitable compact groups G .

The trace in the sense of symmetric monoidal categories is the analogue of the local expression of the Lefschetz invariant in terms of fixed points. The analogue of the global homological formula for the Lefschetz invariant is the graded Hattori–Stallings trace of the action of f on $K_*^G(A) := \mathrm{KK}_*^G(\mathbb{1}, A)$, viewed as a module over the ring $R(G) := \mathrm{KK}_*^G(\mathbb{1}, \mathbb{1})$. This Hattori–Stallings trace is defined if $K_*^G(A)$ has a finite projective $R(G)$ -module resolution.

Theorem 1.1. *Let G be a connected compact Lie group with torsion-free fundamental group. Let A be a separable G - C^* -algebra that is non-equivariantly KK -equivalent to a commutative C^* -algebra. Assume that $K_*^G(A)$ is finitely generated as an $R(G)$ -module. Then A is dualisable and $\mathrm{tr} f = \mathrm{tr} K_*^G(f)$ for any $f \in \mathrm{KK}_0^G(A, A)$.*

This equivariant Lefschetz Fixed Point Theorem depends on the following Additivity Theorem for traces:

Theorem 1.2. *Let $A \rightarrow B \rightarrow C \rightarrow A[1]$ be an exact triangle in the thick triangulated subcategory of $\mathfrak{K}\mathfrak{K}^G$ generated by $\mathbb{1}$ (thus in particular A , B and C are dualisable). If the left square in the following diagram*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f_A & & \downarrow f_B & & \downarrow f_C & & \downarrow f_{A[1]} \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \end{array}$$

commutes, then there is an arrow $f_C \in \mathrm{KK}_0^G(C, C)$ making the whole diagram commute, and such that $\mathrm{tr}(f_C) - \mathrm{tr}(f_B) + \mathrm{tr}(f_A) = 0$.

That traces should be additive in this sense is plausible in any category that is triangulated and symmetric monoidal, provided the tensor product and the triangulated category structure are compatible in a suitable sense. The compatibility axioms needed for this were first worked out by J. Peter May in [22].

Some of the axioms in [22] are, however, amazingly complicated, and already the simplest ones, which require the category in question to be *closed* symmetric monoidal, fail for $\mathfrak{K}\mathfrak{K}^G$. This obvious problem may be circumvented by embedding $\mathfrak{K}\mathfrak{K}^G$ into a larger symmetric monoidal triangulated category that is closed and satisfies May's axioms. A promising approach to construct such an embedding would use simplicial presheaves on separable G - C^* -algebras, following ideas of Paul Arne Østvær [27]. Another approach would be to show that $\mathfrak{K}\mathfrak{K}^G$ is part of a derivator and that triangulated categories coming from derivators always satisfy additivity of traces. But since both approaches require a lot of technical work, we choose a different approach. It has the disadvantage that it only applies to the subcategory of $\mathfrak{K}\mathfrak{K}^G$ generated by the tensor unit.

Our main tool is to lift G -equivariant K -theory to a symmetric monoidal functor from G - C^* -algebras to a suitable category of module spectra. This lifting descends to a triangulated functor from $\mathfrak{K}\mathfrak{K}^G$ to the homotopy category of module spectra, which is fully faithful on the thick triangulated subcategory generated by the tensor unit. Hence we may transport additivity of traces from the category of module spectra, where it is known, to the bootstrap category in $\mathfrak{K}\mathfrak{K}^G$.

There exist several suggestions how to lift K-theory to spectra. First, Ulrich Bunke, Michael Joachim and Stephan Stolz [2] construct an orthogonal K-theory spectrum with homotopy groups $K_*(A) \cong KK_*(\mathbb{C}, A)$ using unbounded Kasparov cycles. However, their construction is only functorial for *essential* *-homomorphisms, which means for our purposes that it is *not* functorial. A later construction by Michael Joachim and Stephan Stolz [15] based on the Cuntz picture is incorrect because the iterated Cuntz algebras are not symmetric with respect to permutations.

Therefore, we provide our own symmetric K-theory spectrum, which is very close to the model of [2], but functorial for arbitrary *-homomorphisms. We use the description of K-theory for graded algebras by Jody Trout [31], which may be traced back to the thesis of Ulrich Haag.

Outline of the article. In Section 2, we recall some basic notation regarding Kasparov theory and traces in symmetric monoidal categories. In Section 3, we lift G -equivariant K-theory to a lax monoidal functor \mathbf{K}^G from $\mathfrak{K}\mathfrak{K}^G$ to the category of symmetric module spectra over the symmetric ring spectrum $\mathbf{K}^G(\mathbb{1})$. Our construction works equally well for real C^* -algebras, with no extra costs, so we will cover both cases throughout. In Section 4, we prove Theorem 1.2 on additivity of traces. In Section 5, we apply additivity of traces to equivariant Kasparov theory and establish the Lefschetz Fixed Point Theorem 1.1.

2. TRACES IN SYMMETRIC MONOIDAL CATEGORIES

Let \mathcal{C} be a symmetric monoidal category with tensor product \otimes , unit object $\mathbb{1}$, and braiding isomorphisms $\mathbf{braid}_{A,B}: A \otimes B \rightarrow B \otimes A$. We omit from our notation, and usually just ignore, the structural associativity and unit isomorphisms; this is justified by the Coherence Theorem (see [20, Chapter XI]).

We shall consider the following examples.

Example 2.1. Let G be a locally compact group or, more generally, a locally compact groupoid. Let $\mathcal{C}^*\mathbf{sep}^G$ be the category whose objects are the separable G - C^* -algebras and whose morphisms from A to B are the G -equivariant *-homomorphisms. Given two G - C^* -algebras A and B , let $A \otimes_{\min} B$ denote their spatial C^* -tensor product.

If G is a group, let $A \otimes B$ be $A \otimes_{\min} B$ equipped with the diagonal action of G . This defines a symmetric monoidal structure on $\mathcal{C}^*\mathbf{sep}^G$, where the unit object $\mathbb{1}$ is \mathbb{C} (\mathbb{R} in the case of real C^* -algebras) equipped with the trivial action of G . The braiding isomorphism is the unique extension of the obvious braiding isomorphism on the algebraic tensor product, which is dense in $A \otimes_{\min} B$; the associativity and unit isomorphisms are similarly evident maps.

If G is, more generally, a locally compact groupoid with object space X , then we modify the above definitions. Now $A \otimes_{\min} B$ is canonically a C^* -algebra over $X \times X$. It yields a C^* -algebra over X by restricting to the diagonal in $X \times X$. There is a canonical diagonal action of G on this restriction, and the resulting G - C^* -algebra is our tensor product $A \otimes B$. The tensor unit $\mathbb{1}$ is $C_0(X)$ equipped with the canonical action of G . (We also use the notation $A \otimes_X B$ to emphasise the dependence of the tensor product on X .) As before, we get a symmetric monoidal category $\mathcal{C}^*\mathbf{sep}^G$.

Example 2.2. Let $\mathfrak{K}\mathfrak{K}^G$ be the category whose objects are the separable G - C^* -algebras and whose morphisms from A to B are the bivariant K-groups $KK^G(A, B)$, defined in [17] for groups and in [18] for groupoids. There is a canonical functor $\mathcal{C}^*\mathbf{sep}^G \rightarrow \mathfrak{K}\mathfrak{K}^G$, which is characterised by a universal property. Using the latter, one

checks easily that the tensor product of Example 2.1 extends along the canonical functor to a symmetric monoidal structure on $\mathfrak{K}\mathfrak{K}^G$ with the same object function $(A, B) \mapsto A \otimes B$. As explained in [25, Appendix A], the category $\mathfrak{K}\mathfrak{K}^G$ is triangulated. Moreover, C^* -direct sums provide countable coproducts in $\mathfrak{K}\mathfrak{K}^G$, and the tensor bifunctor \otimes preserves exact triangles and coproducts in each variable.

Example 2.3. Later we will also work with $\mathbb{Z}/2$ -graded C^* -algebras. They may be turned into a symmetric monoidal category using the usual tensor product \otimes_{\min} as above because a $\mathbb{Z}/2$ -grading is the same thing as a $\mathbb{Z}/2$ -action. But we will instead use the (minimal) *graded* C^* -tensor product $\widehat{\otimes}$. In $A \widehat{\otimes} B$, the copies of A and B *graded* commute, that is, the even elements commute with the other tensor factor, and the odd elements in A and B anticommute. Thus $A \widehat{\otimes} B$ and $A \otimes B$ are equal if one of the C^* -algebras involved is trivially graded. More generally, if one of the factors is *evenly graded*, then there is a C^* -algebra isomorphism $A \otimes B \cong A \widehat{\otimes} B$.

The tensor product $\widehat{\otimes}$ is part of a symmetric monoidal structure. The tensor unit is \mathbb{C} (or \mathbb{R}) with trivial $\mathbb{Z}/2$ -grading. The braiding isomorphism $\sigma : A \widehat{\otimes} B \rightarrow B \widehat{\otimes} A$ maps $a \widehat{\otimes} b \mapsto (-1)^{|a| \cdot |b|} b \widehat{\otimes} a$ for homogeneous $a \in A$, $b \in B$. This yields symmetric monoidal categories both at the $*$ -homomorphism and KK-theory level. Contrary to the case of trivially graded algebras, $\mathfrak{K}\mathfrak{K}^G$ has no natural triangulation.

Example 2.4. Finally, we combine Examples 2.2 and 2.3, considering $\mathbb{Z}/2$ -graded C^* -algebras with an action of a locally compact groupoid G . We assume that the grading and G -action are compatible in the sense that we get an action of $G \times \mathbb{Z}/2$. Then the graded tensor product over X provides a symmetric monoidal structure on $\mathbb{Z}/2$ -graded G - C^* -algebras, as well as on the Kasparov category $\mathfrak{K}\mathfrak{K}^G$.

Let us return to a general symmetric monoidal category \mathcal{C} .

Definition 2.5. An object A of \mathcal{C} is called *dualisable* if there are morphisms

$$\eta : \mathbb{1} \rightarrow A \otimes A^* \quad \text{and} \quad \varepsilon : A^* \otimes A \rightarrow \mathbb{1}$$

called *unit* (or coevaluation) and *counit* (or evaluation), respectively, which satisfy the *zigzag equations*: each of the composites

$$\begin{aligned} A &\cong \mathbb{1} \otimes A \xrightarrow{\eta \otimes \text{id}_A} (A \otimes A^*) \otimes A \cong A \otimes (A^* \otimes A) \xrightarrow{\text{id}_A \otimes \varepsilon} A \otimes \mathbb{1} \cong A, \\ A^* &\cong A^* \otimes \mathbb{1} \xrightarrow{\text{id}_{A^*} \otimes \eta} A^* \otimes (A \otimes A^*) \cong (A^* \otimes A) \otimes A^* \xrightarrow{\varepsilon \otimes \text{id}_{A^*}} \mathbb{1} \otimes A^* \cong A^* \end{aligned}$$

should be the identity. (The unnamed isomorphisms are the canonical ones.)

The data $(A, A^*, \eta, \varepsilon)$ is equivalent to natural isomorphisms

$$\mathcal{C}(B \otimes A, C) \cong \mathcal{C}(B, C \otimes A^*)$$

for all objects B and C of \mathcal{C} . The dual A^* is determined uniquely up to canonical isomorphism.

In the context of equivariant Kasparov theory, this categorical notion of duality has been considered in [30] and, more recently, in [4].

Proposition 2.6. *The full subcategory \mathcal{C}_d of dualisable objects in \mathcal{C} is closed symmetric monoidal for the tensor structure of \mathcal{C} and the internal Hom functor $A^* \otimes B$. The duality $A \mapsto A^*$ is a symmetric monoidal equivalence $\mathcal{C}_d \simeq (\mathcal{C}_d)^{\text{op}}$.*

For the proof, see [19, Chapter III.1].

Definition 2.7. Let A be a dualisable object of \mathcal{C} and let $f: A \rightarrow A$ be an endomorphism in \mathcal{C} . The *trace* of f , denoted $\text{tr}(f): \mathbb{1} \rightarrow \mathbb{1}$, is the following composition:

$$\mathbb{1} \xrightarrow{\eta} A \otimes A^* \begin{array}{l} \xrightarrow{f \otimes \text{id}_{A^*}} A \otimes A^* \\ \xrightarrow{\text{braid}} A^* \otimes A \end{array} \begin{array}{l} \xrightarrow{\text{braid}} A^* \otimes A \\ \xrightarrow{\text{id}_{A^*} \otimes f} A^* \otimes A \end{array} \xrightarrow{\varepsilon} \mathbb{1}.$$

The *Euler characteristic* of A is $\chi(A) := \text{tr}(\text{id}_A)$.

2.1. Lax monoidal functors. Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be two symmetric monoidal categories. A (*lax*) *symmetric monoidal functor* F from \mathcal{C} to \mathcal{D} is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ together with a morphism

$$i: \mathbb{1}_{\mathcal{D}} \rightarrow F(\mathbb{1}_{\mathcal{C}})$$

and a natural transformation

$$c = c_{A,B}: F(A) \otimes_{\mathcal{D}} F(B) \rightarrow F(A \otimes_{\mathcal{C}} B) \quad (A, B \in \mathcal{C}),$$

which are compatible with the associativity, unit and braiding isomorphisms in \mathcal{C} and \mathcal{D} in a suitable sense (see [20, Chapter XI.2] or [19, p. 126]). We call F (or rather, (F, c, i)) *normal* if i is an isomorphism, and *strong* if both c and i are isomorphisms.

Lemma 2.8. *Let $(F, c, i): \mathcal{C} \rightarrow \mathcal{D}$ be a normal symmetric monoidal functor, let $A \in \mathcal{C}$ be dualisable with dual A^* , and let $c: F(A) \otimes F(A^*) \rightarrow F(A \otimes A^*)$ be invertible (this happens, in particular, if F is strong). Then $F(A)$ is dualisable with dual $F(A^*)$, and $\text{tr} F(f) = i^{-1} \circ F(\text{tr} f) \circ i \in \mathcal{D}(\mathbb{1}_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ for all $f \in \mathcal{C}(A, A)$.*

Proof. Let η and ε be the unit and counit for the duality between A and A^* . Then

$$c^{-1} \circ F(\eta) \circ i: \mathbb{1}_{\mathcal{D}} \rightarrow F(A) \otimes F(A^*) \quad \text{and} \quad i^{-1} \circ F(\varepsilon) \circ c: F(A^*) \otimes F(A) \rightarrow \mathbb{1}_{\mathcal{D}}$$

are unit and counit for $F(A)$ and $F(A^*)$. The coherence assumptions for symmetric monoidal functors ensure that the zigzag equations carry over and that the traces in \mathcal{C} and \mathcal{D} agree as asserted (see [19, Chapter III, Proposition 1.9].) \square

3. EQUIVARIANT K-THEORY AS A FUNCTOR TO SYMMETRIC SPECTRA

Let G be a proper, locally compact groupoid with Haar system. For ($\mathbb{Z}/2$ -graded) G - C^* -algebras A , we construct symmetric spectra $\mathbf{K}^G(A)$ with natural isomorphisms

$$\pi_n(\mathbf{K}^G(A)) \cong \text{KK}_n^G(\mathbb{1}, A) \quad \text{for } n \in \mathbb{Z}.$$

If the orbit space of G is compact, then $\text{KK}_n^G(\mathbb{1}, A)$ is isomorphic to the K-theory of the crossed product C^* -algebra $G \ltimes A$, but not in general. In particular, if G is a compact group, we have natural isomorphisms

$$\pi_n(\mathbf{K}^G(A)) \cong \text{KK}_n^G(\mathbb{1}, A) \cong \text{K}_n(G \ltimes A) \cong \text{K}_n^G(A).$$

Our construction is close to the one of [2], but has the additional feature that it is functorial for G -equivariant $*$ -homomorphisms. Even more, $\mathbf{K}^G(\mathbb{1})$ is a ring spectrum, $\mathbf{K}^G(A)$ a module spectrum over $\mathbf{K}^G(\mathbb{1})$ for all A , and \mathbf{K}^G comes with the structure of a lax symmetric monoidal functor between the symmetric monoidal categories of G - C^* -algebras and $\mathbf{K}^G(\mathbb{1})$ -modules.

3.1. A suitable description of K-theory. First we recall the description of K-theory in [31]. This definition uses $\mathbb{Z}/2$ -graded C^* -algebras in a crucial way and, as a benefit, also works for $\mathbb{Z}/2$ -graded C^* -algebras as coefficients. We note that [31] only considers complex C^* -algebras, but all results also hold for real C^* -algebras with no additional effort.

Throughout this section, we use the *graded* minimal C^* -tensor product $\widehat{\otimes}$ defined in Example 2.3.

Let \widehat{S} be the C^* -algebra $C_0(\mathbb{R})$ with the $\mathbb{Z}/2$ -grading by reflection at the origin, so that the even and odd parts of \widehat{S} consist of the even and odd functions, respectively.

Let $\widehat{\mathcal{K}}$ be the C^* -algebra of compact operators on $\mathcal{H} := \ell^2(\mathbb{N} \times \mathbb{Z}/2)$ with the even grading corresponding to the decomposition $\mathcal{H} \cong \ell^2(\mathbb{N}) \otimes \delta_0 \oplus \ell^2(\mathbb{N}) \otimes \delta_1$.

For two $\mathbb{Z}/2$ -graded C^* -algebras A and B , let $\text{Hom}(A, B)$ denote the pointed topological space of grading-preserving $*$ -homomorphisms from A to B with the compact-open topology and the zero map as base point.

A pointed continuous map from a pointed compact space X to $\text{Hom}(A, B)$ is equivalent to an element of $\text{Hom}(A, C_0(X, B))$, where $C_0(X, B)$ denotes the C^* -algebra of continuous functions $X \rightarrow B$ vanishing at the base point of X (see [14, Proposition 3.4]). Thus $\pi_n(\text{Hom}(A, B))$ is isomorphic to the group of homotopy classes of grading-preserving $*$ -homomorphisms $A \rightarrow C_0(\mathbb{R}^n) \otimes B$.

It is shown in [31, Theorem 4.7] that there is a natural bijection

$$(3.1) \quad \pi_0(\text{Hom}(\widehat{S}, A \widehat{\otimes} \widehat{\mathcal{K}})) \cong \text{KK}_0(\mathbb{C}, A) \cong K_0(A)$$

for any graded σ -unital C^* -algebra A . We need the following generalisation:

Proposition 3.1. *Let G be a proper locally compact groupoid with Haar system and let A be a $\mathbb{Z}/2$ -graded separable G - C^* -algebra. Let G^0 denote the object space of G and let $\mathbb{1} := C_0(G^0)$ with the canonical G -action. Then there is an isomorphism*

$$\pi_n(\text{Hom}_G(\mathbb{1} \widehat{\otimes} \widehat{S}, A \widehat{\otimes} \widehat{\mathcal{K}}_G)) \cong \text{KK}_n^G(\mathbb{1}, A).$$

for every $n \geq 0$, which is natural with respect to grading-preserving G -equivariant $*$ -homomorphisms.

Here $\text{Hom}_G(A, B) \subseteq \text{Hom}(A, B)$ denotes the subspace of G -equivariant maps in $\text{Hom}(A, B)$ and

$$\widehat{\mathcal{K}}_G := \mathcal{K}(L^2(G \times \mathbb{N} \times \mathbb{Z}/2)) \cong \mathcal{K}(L^2G) \widehat{\otimes} \widehat{\mathcal{K}}.$$

We have $\mathbb{1} \widehat{\otimes} \widehat{S} \cong \mathbb{1} \otimes \widehat{S} \cong C_0(G^0, \widehat{S})$, and there is a G -equivariant C^* -algebra isomorphism $A \widehat{\otimes} \widehat{\mathcal{K}}_G \cong A \otimes \widehat{\mathcal{K}}_G$ because $\widehat{\mathcal{K}}_G$ is evenly graded.

Proof. The argument in [31] carries over to this situation with minor modifications. If G is a compact group, then $\mathbb{1}$ is \mathbb{C} or \mathbb{R} with trivial G -action, and

$$\begin{aligned} \text{Hom}_G(\mathbb{1} \widehat{\otimes} \widehat{S}, A \widehat{\otimes} \widehat{\mathcal{K}}_G) &= \text{Hom}_G(\widehat{S}, A \widehat{\otimes} \widehat{\mathcal{K}} \otimes \mathcal{K}(L^2G)) \\ &= \text{Hom}(\widehat{S}, (A \widehat{\otimes} \widehat{\mathcal{K}} \otimes \mathcal{K}(L^2G))^G) \cong \text{Hom}(\widehat{S}, (A \widehat{\otimes} \widehat{\mathcal{K}}) \rtimes G), \end{aligned}$$

where $B^G \subseteq B$ denotes the C^* -subalgebra of G -invariant elements. A convenient reference for the isomorphism $(B \otimes \mathcal{K}(L^2G))^G \cong B \rtimes G$ is [7, Chapter 11]. Hence (3.1)

yields

$$\begin{aligned} \pi_n(\mathrm{Hom}_G(\mathbb{1} \widehat{\otimes} \widehat{S}, A \widehat{\otimes} \widehat{\mathcal{K}}_G)) &\cong \pi_0(\mathrm{Hom}(\widehat{S}, C_0(\mathbb{R}^n) \otimes (A \rtimes G) \widehat{\otimes} \widehat{\mathcal{K}})) \\ &\cong K_0(C_0(\mathbb{R}^n) \otimes A \rtimes G) \cong K_n^G(A) \cong \mathrm{KK}_n^G(\mathbb{C}, A). \end{aligned}$$

Now let G be a proper locally compact groupoid with Haar system instead. The reasoning above carries over almost literally to provide an isomorphism

$$\pi_n(\mathrm{Hom}_G(\mathbb{1} \widehat{\otimes} \widehat{S}, A \widehat{\otimes} \widehat{\mathcal{K}}_G)) \cong K_n(A \rtimes G)$$

if G is cocompact. In general, we get the group $\mathcal{RKK}_n(G^0/G; C_0(G^0/G), A \rtimes G)$. In both cases, the result is isomorphic to $\mathrm{KK}_n^G(\mathbb{1}, A)$ by [32, Proposition 6.25]. \square

Remark 3.2. We compare our model of Kasparov theory to the one used by Ulrich Bunke, Michael Joachim and Stephan Scholz [2] for trivial G . They use regular, odd, self-adjoint, unbounded operators D on $\mathcal{H}_A := A \otimes \ell^2(\mathbb{N} \times \mathbb{Z}/2)$ that satisfy $(1 + D^2)^{-1} \in \mathcal{K}(\mathcal{H}_A)$. The functional calculus for such an operator is an *essential* grading-preserving $*$ -homomorphisms from \widehat{S} to $A \widehat{\otimes} \widehat{\mathcal{K}}$ and, conversely, any such essential homomorphism is of this form for a unique D as above. Non-essential grading-preserving $*$ -homomorphisms from \widehat{S} to $A \widehat{\otimes} \widehat{\mathcal{K}}$ correspond to unbounded operators on certain hereditary subalgebras of $A \widehat{\otimes} \widehat{\mathcal{K}}$ (see [31, §3]).

Thus the space used in [2] to model K-theory is homeomorphic to the space of essential, grading-preserving $*$ -homomorphisms $\widehat{S} \rightarrow A \widehat{\otimes} \widehat{\mathcal{K}}$. The Kasparov Stabilisation Theorem and standard homotopies of isometries show that the restriction to essential homomorphisms does not change the homotopy type (see [23, 31]).

The space of essential morphisms is only functorial for essential homomorphisms $A \rightarrow A'$. The better functoriality of $\mathrm{Hom}(\widehat{S}, A \widehat{\otimes} \widehat{\mathcal{K}})$ is crucial for our purposes.

Another related reference is [8], where the Cuntz picture of Kasparov theory is carried over to the $\mathbb{Z}/2$ -graded case. It is shown in [8] (for the complex case) that

$$\mathrm{KK}_0(A, B) \cong \pi_0 \mathrm{Hom}(\chi A, B \widehat{\otimes} \widehat{\mathcal{K}})$$

for a certain C^* -algebra χA . Furthermore, $\chi \mathbb{C}$ is isomorphic as a graded C^* -algebra to $\widehat{\mathbb{M}}_2(\widehat{S})$. Since the additional stabilisation does not matter, this provides another proof of the isomorphism $\mathrm{KK}_0(\mathbb{C}, B) \cong \pi_0 \mathrm{Hom}(\widehat{S}, B \widehat{\otimes} \widehat{\mathcal{K}})$.

3.2. The coalgebra structure of \widehat{S} . It is crucial for our purposes that \widehat{S} is a counital, cocommutative, coassociative coalgebra object in the category of $\mathbb{Z}/2$ -graded C^* -algebras (we work non-equivariantly in this subsection). That is, we have a comultiplication $\Delta: \widehat{S} \rightarrow \widehat{S} \widehat{\otimes} \widehat{S}$ and a counit $\epsilon: \widehat{S} \rightarrow \mathbb{1}$ that satisfy the equations

$$(\Delta \widehat{\otimes} \mathrm{id}_{\widehat{S}}) \circ \Delta = (\mathrm{id}_{\widehat{S}} \widehat{\otimes} \Delta) \circ \Delta, \quad (\mathrm{id}_{\widehat{S}} \widehat{\otimes} \epsilon) \circ \Delta = \mathrm{id}_{\widehat{S}}, \quad \mathrm{braid} \circ \Delta = \Delta.$$

This additional structure is mentioned without proof in [10] in connection with the definition of equivariant E-theory.

In order to understand this structure, we need that essential grading-preserving $*$ -homomorphisms $\widehat{S} \rightarrow A$ correspond bijectively to odd, self-adjoint, regular unbounded multipliers D of A with $(1 + D^2)^{-1} \in A$ by the functional calculus for regular unbounded self-adjoint multipliers of C^* -algebras ([31, §3]).

The identity map on \widehat{S} corresponds to the identical function X on \mathbb{R} , viewed as an unbounded multiplier on \widehat{S} . In the graded tensor product $\widehat{S} \widehat{\otimes} \widehat{S}$, the elements

$X \widehat{\otimes} 1$ and $1 \widehat{\otimes} X$ anticommute, so that

$$(X \widehat{\otimes} 1 + 1 \widehat{\otimes} X)^2 = X^2 \widehat{\otimes} 1 + 1 \widehat{\otimes} X^2.$$

Hence $X \widehat{\otimes} 1 + 1 \widehat{\otimes} X$ induces an essential grading-preserving $*$ -homomorphism

$$\Delta: \widehat{S} \rightarrow \widehat{S} \widehat{\otimes} \widehat{S}, \quad X \mapsto X \widehat{\otimes} 1 + 1 \widehat{\otimes} X.$$

(The analogous map without gradings is the comultiplication

$$\Delta: C_0(\mathbb{R}) \rightarrow C_b(\mathbb{R} \times \mathbb{R}), \quad \Delta f(x, y) = f(x + y),$$

which induces no element in $\mathrm{KK}_0(C_0\mathbb{R}, C_0\mathbb{R} \otimes C_0\mathbb{R})$.)

The counit $\epsilon: \widehat{S} \rightarrow \mathbb{1}$ is induced by $0 \in \mathbb{1}$.

The equality $(\Delta \widehat{\otimes} \mathrm{id}_{\widehat{S}}) \circ \Delta = (\mathrm{id}_{\widehat{S}} \widehat{\otimes} \Delta) \circ \Delta$ holds because both sides are induced by the unbounded multiplier $X \widehat{\otimes} 1 \widehat{\otimes} 1 + 1 \widehat{\otimes} X \widehat{\otimes} 1 + 1 \widehat{\otimes} 1 \widehat{\otimes} X$. We have $(\mathrm{id}_{\widehat{S}} \widehat{\otimes} \epsilon) \circ \Delta = \mathrm{id}_{\widehat{S}}$ because $(\mathrm{id}_{\widehat{S}} \widehat{\otimes} \epsilon)(X \widehat{\otimes} 1 + 1 \widehat{\otimes} X) = X \cdot \epsilon(1) + 1 \cdot \epsilon(X) = X$, and $\mathrm{braid} \circ \Delta = \Delta$ because $X \widehat{\otimes} 1 + 1 \widehat{\otimes} X$ is symmetric with respect to braid .

3.3. The symmetric module spectra $\mathbf{K}^G(A)$. Our references for symmetric spectra are [28, 29]. We will work with the category \mathfrak{Sp}^Σ of symmetric spectra based on compactly generated spaces; accordingly, all relevant spaces and constructions – such as limits and colimits – take place in the category of (pointed) compactly generated spaces (see [11, §2.4]). The convenient definitions in [28, Definitions I.1.1–4] involve a minimal collection of data for symmetric spectra, symmetric ring spectra, and modules over a symmetric spectrum.

We are going to construct a symmetric spectrum $\mathbf{K}^G(A)$ for any $\mathbb{Z}/2$ -graded G - C^* -algebra A . Even more, $\mathbf{K}^G(\mathbb{1})$ will be a commutative symmetric ring spectrum and $\mathbf{K}^G(A)$ a symmetric module spectrum over $\mathbf{K}^G(\mathbb{1})$, and altogether our symmetric spectra will form a symmetric lax monoidal functor from the category of graded G - C^* -algebras to the category of $\mathbf{K}^G(\mathbb{1})$ -modules. To get all this structure, we need:

- pointed spaces $\mathbf{K}_n^G(A)$ with pointed continuous Σ_n -actions, where Σ_n denotes the symmetric group, for $n \geq 0$ and separable G - C^* -algebras A ;
- Σ_n -equivariant, pointed, continuous maps $\mathbf{K}_n^G(f): \mathbf{K}_n^G(A) \rightarrow \mathbf{K}_n^G(B)$ for G -equivariant $*$ -homomorphisms $A \rightarrow B$;
- $\Sigma_n \times \Sigma_m$ -equivariant pointed maps

$$c_{n,m}: \mathbf{K}_n^G(A_1) \wedge \mathbf{K}_m^G(A_2) \rightarrow \mathbf{K}_{n+m}^G(A_1 \widehat{\otimes} A_2)$$

for $n, m \geq 0$ and separable G - C^* -algebras A_1 and A_2 ;

- unit maps $\iota_0: \mathbb{S}^0 \rightarrow \mathbf{K}_0^G(\mathbb{1})$ (that is, $\iota_0 \in \mathbf{K}_0^G(\mathbb{1})$) and $\iota_1: \mathbb{S}^1 \rightarrow \mathbf{K}_1^G(\mathbb{1})$, where \mathbb{S}^n denotes the pointed n -sphere.

First, we construct this data and check the relevant properties only in the case where G is a compact *group*. The groupoid case is essentially the same, but notationally more complicated.

We will identify $\mathbb{R}^m \cong \mathbb{S}^m \setminus \{*\}$ where $*$ is the base point. Our construction uses the Clifford algebra $\mathrm{Cl}_\mathbb{R}$ of \mathbb{R} , which is the unital $\mathbb{Z}/2$ -graded C^* -algebra with basis $1, F$ where F is an odd, self-adjoint, involution ($F^2 = 1$). This C^* -algebra plays the role of a formal desuspension for (real or complex) K-theory: there is an invertible element in $\mathrm{KK}_0(\mathbb{1}, C_0(\mathbb{R}, \mathrm{Cl}_\mathbb{R}))$ (see [16]).

For a separable $\mathbb{Z}/2$ -graded G - C^* -algebra A and $n \geq 0$, let

$$(3.2) \quad \mathbf{K}_n^G(A) := \mathrm{Hom}_G(\widehat{S}, A \widehat{\otimes} (\mathrm{Cl}_\mathbb{R} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes} n}),$$

where $B^{\widehat{\otimes} n}$ denotes the $\mathbb{Z}/2$ -graded tensor product of n copies of B . This is a pointed, compactly generated topological space. Let Σ_n act trivially on \widehat{S} and A , and on $(\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes} n}$ by the permutation action from the braiding of $\widehat{\otimes}$ (this involves signs according to the Koszul sign rule, see Example 2.3).

Remark 3.3. If A is evenly graded, then $\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} A$ is isomorphic as a $\mathbb{Z}/2$ -graded C^* -algebra to $\mathbb{C}\ell_{\mathbb{R}} \otimes A$ with the grading coming only from $\mathbb{C}\ell_{\mathbb{R}}$. Hence we may replace $\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G$ above by $\mathbb{C}\ell_{\mathbb{R}} \otimes \mathcal{K}(\ell^2(\mathbb{N} \times G))$.

A grading-preserving G -equivariant $*$ -homomorphism $f: A_1 \rightarrow A_2$ induces pointed, continuous Σ_n -equivariant maps

$$(3.3) \quad \mathbf{K}_n^G(f): \mathbf{K}_n^G(A_1) \rightarrow \mathbf{K}_n^G(A_2), \quad \alpha \mapsto (f \widehat{\otimes} \text{id}) \circ \alpha.$$

Let A_1 and A_2 be $\mathbb{Z}/2$ -graded separable G - C^* -algebras and let $\alpha_1 \in \mathbf{K}_n^G(A_1)$, $\alpha_2 \in \mathbf{K}_m^G(A_2)$. We get a grading-preserving G -equivariant $*$ -homomorphism

$$\alpha_1 \widehat{\otimes} \alpha_2: \widehat{S} \widehat{\otimes} \widehat{S} \rightarrow A_1 \widehat{\otimes} (\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes} n} \widehat{\otimes} A_2 \widehat{\otimes} (\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes} m}.$$

Let $\pi_{n,m}$ be the braiding isomorphism that reorders the tensor factors in the target C^* -algebra to $A_1 \widehat{\otimes} A_2 \widehat{\otimes} (\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes} n+m}$, without changing the order among the factors $\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G$. We define a $\Sigma_n \times \Sigma_m$ -equivariant pointed continuous map

$$(3.4) \quad c_{n,m}^{A_1, A_2}: \mathbf{K}_n^G(A_1) \wedge \mathbf{K}_m^G(A_2) \rightarrow \mathbf{K}_{n+m}^G(A_1 \widehat{\otimes} A_2),$$

$$\alpha_1 \wedge \alpha_2 \mapsto \pi_{n,m} \circ (\alpha_1 \widehat{\otimes} \alpha_2) \circ \Delta: \widehat{S} \rightarrow A_1 \widehat{\otimes} A_2 \widehat{\otimes} (\mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes} n+m}.$$

In particular, for $A_1 = A_2 = \mathbb{1}$, this yields the maps

$$c_{n,m}: \mathbf{K}_n^G(\mathbb{1}) \wedge \mathbf{K}_m^G(\mathbb{1}) \rightarrow \mathbf{K}_{n+m}^G(\mathbb{1} \widehat{\otimes} \mathbb{1}) \cong \mathbf{K}_{n+m}^G(\mathbb{1})$$

that are needed for the structure of a symmetric ring spectrum, and for $A_1 = A$ and $A_2 = \mathbb{1}$, this yields the maps

$$c_{n,m}: \mathbf{K}_n^G(A) \wedge \mathbf{K}_m^G(\mathbb{1}) \rightarrow \mathbf{K}_{n+m}^G(A \widehat{\otimes} \mathbb{1}) \cong \mathbf{K}_{n+m}^G(A)$$

that are needed for the structure of a $\mathbf{K}^G(\mathbb{1})$ -module.

The unit $\iota_0 \in \mathbf{K}_0^G(\mathbb{1}) \cong \text{Hom}(\widehat{S}, \mathbb{1})$ is the counit ϵ of \widehat{S} .

The other unit ι_1 requires two ingredients. The function $t \mapsto t \cdot F$ defines an unbounded, regular, self-adjoint, odd multiplier D of $C_0(\mathbb{R}, \mathbb{C}\ell_{\mathbb{R}})$ with $(1 + D^2)^{-1}: t \mapsto (1 + t^2)^{-1} \in C_0(\mathbb{R}, \mathbb{C}\ell_{\mathbb{R}})$. Hence the functional calculus for D is a $*$ -homomorphism

$$\beta: \widehat{S} \rightarrow C_0(\mathbb{R}, \mathbb{C}\ell_{\mathbb{R}}).$$

This describes an invertible element in $\text{KK}(\mathbb{1}, C_0(\mathbb{R}, \mathbb{C}\ell_{\mathbb{R}}))$, see [16].

Moreover, let $\gamma: \mathbb{1} \rightarrow \widehat{\mathcal{K}}_G$ be the $*$ -homomorphism that maps the unit element in \mathbb{R} or \mathbb{C} to the rank-one-projection onto the subspace spanned by the G -invariant vector $1_G \otimes \delta_0$. Let

$$\iota_1 := \beta \widehat{\otimes} \gamma: \widehat{S} \cong \widehat{S} \widehat{\otimes} \mathbb{1} \rightarrow C_0(\mathbb{R}, \mathbb{C}\ell_{\mathbb{R}}) \widehat{\otimes} \widehat{\mathcal{K}}_G.$$

This is an element of

$$\text{Hom}_G(\widehat{S}, C_0(\mathbb{S}^1, \mathbb{C}\ell_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)) \cong \text{Hom}(\mathbb{S}^1, \mathbf{K}_1^G(\mathbb{1}))$$

Now we generalise to the case where G is a proper locally compact groupoid with Haar system. Let X denote its object space. We replace \widehat{S} by

$$\widehat{S}_X := \widehat{S} \widehat{\otimes} C_0(X) \cong C_0(X, \widehat{S}).$$

Since the tensor product in the category of graded G - C^* -algebras is taken over X , we have canonical isomorphisms

$$\widehat{S}_X^{\otimes n} \cong C_0(X, \widehat{S}^{\otimes n}) \quad \text{for all } n \geq 0.$$

Thus the coalgebra structure on \widehat{S} turns \widehat{S}_X into a coassociative, cocommutative, counital coalgebra object in the symmetric monoidal category of $\mathbb{Z}/2$ -graded G - C^* -algebras (as in Example 2.4).

Now we may extend the definitions above almost literally, replacing \widehat{S} by \widehat{S}_X and $\widehat{\otimes}$ by $\widehat{\otimes}_X$ where necessary. That is, we let

$$\mathbf{K}_n^G(A) := \text{Hom}_G(\widehat{S}_X, A \widehat{\otimes}_X (\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes}_X n}).$$

The maps $\mathbf{K}_n^G(f)$ for a grading-preserving G -equivariant $*$ -homomorphism f and the maps $c_{n,m}^{A_1, A_2}$ are defined as above, and $\iota_0 \in \mathbf{K}_0^G(\mathbb{1})$ is the counit of \widehat{S}_X .

In the groupoid case, L^2G is a continuous field of Hilbert spaces over X , that is, a Hilbert module over $C_0(X) = \mathbb{1}$. Since G is proper, the fibrewise constant functions belong to L^2G . They form a one-dimensional subfield of L^2G and yield an embedding $C_0(X) \rightarrow \mathcal{K}(L^2G)$. Tensoring with the embedding from a rank-one-projection in $\widehat{\mathcal{K}}$, we get an embedding $\gamma: C_0(X) \rightarrow \mathcal{K}(L^2G) \widehat{\otimes} \widehat{\mathcal{K}} = \widehat{\mathcal{K}}_G$. As above, this yields

$$\iota_1 := \beta \widehat{\otimes} \gamma: \widehat{S}_X \rightarrow C_0(\mathbb{R}, \text{Cl}_{\mathbb{R}}) \widehat{\otimes} \widehat{\mathcal{K}}_G,$$

which we view as a continuous map $\iota_1: \mathbb{S}^1 \rightarrow \mathbf{K}_1^G(\mathbb{1})$.

Lemma 3.4. *The maps $c_{n,m}^{A_1, A_2}$ are natural, associative, commutative, and unital with respect to ι_0 .*

Proof. Naturality is obvious. It means that

$$c_{n,m}^{B_1, B_2} \circ (\mathbf{K}_n^G(f_1) \wedge \mathbf{K}_m^G(f_2)) = \mathbf{K}_{n+m}^G(f_1 \widehat{\otimes}_X f_2) \circ c_{n,m}^{A_1, A_2}$$

for grading-preserving G -equivariant $*$ -homomorphisms $f_j: A_j \rightarrow B_j$, $j = 1, 2$.

Associativity means that the following square commutes:

$$\begin{array}{ccc} \mathbf{K}_n^G(A_1) \wedge \mathbf{K}_m^G(A_2) \wedge \mathbf{K}_p^G(A_3) & \xrightarrow{c_{n,m}^{A_1, A_2} \wedge \text{id}} & \mathbf{K}_{n+m}^G(A_1 \widehat{\otimes}_X A_2) \wedge \mathbf{K}_p^G(A_3) \\ \text{id} \wedge c_{m,p}^{A_2, A_3} \downarrow & & \downarrow c_{n+m,p}^{A_1 \widehat{\otimes}_X A_2, A_3} \\ \mathbf{K}_n^G(A_1) \wedge \mathbf{K}_{m+p}^G(A_2 \widehat{\otimes}_X A_3) & \xrightarrow{c_{n,m+p}^{A_1, A_2 \widehat{\otimes}_X A_3}} & \mathbf{K}_{n+m+p}^G(A_1 \widehat{\otimes}_X A_2 \widehat{\otimes}_X A_3) \end{array}$$

Indeed, both compositions map $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$ to the composite map

$$\begin{aligned} \widehat{S}_X &\xrightarrow{\Delta^2} \widehat{S}_X \widehat{\otimes}_X \widehat{S}_X \widehat{\otimes}_X \widehat{S}_X \xrightarrow{\alpha_1 \widehat{\otimes}_X \alpha_2 \widehat{\otimes}_X \alpha_3} \\ &A_1 \widehat{\otimes}_X (\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes}_X n} \widehat{\otimes} A_2 \widehat{\otimes}_X (\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes}_X m} \widehat{\otimes} A_3 \widehat{\otimes}_X (\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes}_X p} \\ &\xrightarrow[\cong]{\pi} A_1 \widehat{\otimes}_X A_2 \widehat{\otimes}_X A_3 \widehat{\otimes}_X (\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes}_X n+m+p}, \end{aligned}$$

where the last map is the braiding isomorphism for the permutation that does not change the order among the factors $\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G$. This argument uses the coassociativity of Δ and that $\widehat{\otimes}_X$ is a symmetric monoidal structure.

Commutativity means that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{K}_n^G(A_1) \wedge \mathbf{K}_m^G(A_2) & \xrightarrow{\text{braid}} & \mathbf{K}_m^G(A_2) \wedge \mathbf{K}_n^G(A_1) \\ \downarrow c_{n,m} & & \downarrow c_{m,n} \\ \mathbf{K}_{n+m}^G(A_1 \widehat{\otimes}_X A_2) & \xrightarrow{\mathbf{K}_{n+m}^G(\text{braid})} & \mathbf{K}_{n+m}^G(A_2 \widehat{\otimes}_X A_1) \xrightarrow{\chi_{n,m}} \mathbf{K}_{m+n}^G(A_2 \widehat{\otimes}_X A_1), \end{array}$$

where $\chi_{n,m}$ denotes the action of the shuffle permutation that moves the first n entries to the end without changing the order among the first n and the last m entries. This follows from the cocommutativity of Δ ; the permutation $\chi_{n,m}$ appears because replacing $\alpha_1 \widehat{\otimes}_X \alpha_2$ by $\alpha_2 \widehat{\otimes}_X \alpha_1$ for $\alpha_j \in \mathbf{K}_n^G(A_j)$, $j = 1, 2$, exchanges both the factors A_1 and A_2 and $(\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes}_X n}$ and $(\text{Cl}_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^{\widehat{\otimes}_X m}$.

The right unit property for ι_0 means that the composition

$$\mathbf{K}_n^G(A) \wedge \mathbb{S}^0 \xrightarrow{\text{id} \wedge \iota_0} \mathbf{K}_n^G(A) \wedge \mathbf{K}_0^G(\mathbb{1}) \xrightarrow{c} \mathbf{K}_n^G(A \widehat{\otimes} \mathbb{1}) \cong \mathbf{K}_n^G(A)$$

is the canonical homeomorphism $\mathbf{K}_n^G(A) \wedge \mathbb{S}^0 \cong \mathbf{K}_n^G(A)$. This follows because ε is a counit for Δ . The left unit property follows from this and commutativity. \square

Lemma 3.4 implies that the Σ_n -spaces $\mathbf{K}_n^G(\mathbb{1})$ with the multiplication maps $c_{n,m}^{\mathbb{1},\mathbb{1}}$ and the units ι_0 and ι_1 form a commutative, symmetric ring spectrum $\mathbf{K}^G(\mathbb{1})$ in the sense of [28, Definitions I.1.3] (the centrality condition follows from commutativity), and that the Σ_n -spaces $\mathbf{K}_n^G(A)$ with the action maps $c_{n,m}^{A,\mathbb{1}}$ form a right $\mathbf{K}^G(\mathbb{1})$ -module $\mathbf{K}^G(A)$ in the sense of [28, Definitions I.1.4]. Furthermore, the maps $\mathbf{K}_n^G(f)$ induced by a grading-preserving $*$ -homomorphism $f: A \rightarrow B$ form a morphism of $\mathbf{K}^G(\mathbb{1})$ -modules $\mathbf{K}^G(f): \mathbf{K}^G(A) \rightarrow \mathbf{K}^G(B)$.

The right $\mathbf{K}^G(\mathbb{1})$ -modules form a symmetric monoidal category with respect to the smash product $\wedge_{\mathbf{K}^G(\mathbb{1})}$, which is defined so that a map $X \wedge_{\mathbf{K}^G(\mathbb{1})} Y \rightarrow Z$ for $\mathbf{K}^G(\mathbb{1})$ -modules X, Y and Z is a family of maps $X_n \wedge Y_m \rightarrow Z_{n+m}$ that is $\mathbf{K}^G(\mathbb{1})$ -bilinear in the sense that the following two diagrams commute:

$$\begin{array}{ccc} X_n \wedge Y_m \wedge \mathbf{K}^G(\mathbb{1})_p & \longrightarrow & X_n \wedge Y_{m+p} \\ \downarrow & & \downarrow \\ Z_{n+m} \wedge \mathbf{K}^G(\mathbb{1})_p & \longrightarrow & Z_{n+m+p} \end{array}$$

$$\begin{array}{ccccc} X_n \wedge Y_m \wedge \mathbf{K}^G(\mathbb{1})_p & \xrightarrow{\text{braid}} & X_n \wedge \mathbf{K}^G(\mathbb{1})_p \wedge Y_m & \longrightarrow & X_{n+p} \wedge Y_m \\ \downarrow & & & & \downarrow \\ Z_{n+m} \wedge \mathbf{K}^G(\mathbb{1})_p & \longrightarrow & Z_{n+m+p} & \xrightarrow[\cong]{\chi_{m,p}} & Z_{n+p+m} \end{array}$$

where $\chi_{m,p}$ is the shuffle permutation of the last m and p numbers. The tensor unit for $\wedge_{\mathbf{K}^G(\mathbb{1})}$ is $\mathbf{K}^G(\mathbb{1})$.

Lemma 3.4 implies that the maps $c_{n,m}^{A_1,A_2}$ are $\mathbf{K}^G(\mathbb{1})$ -bilinear in this sense. Thus they produce $\mathbf{K}^G(\mathbb{1})$ -module homomorphisms

$$c^{A_1,A_2}: \mathbf{K}^G(A_1) \wedge_{\mathbf{K}^G(\mathbb{1})} \mathbf{K}^G(A_2) \rightarrow \mathbf{K}^G(A_1 \wedge A_2).$$

Proposition 3.5. *The functor \mathbf{K}^G with the maps c^{A_1,A_2} and the identity map on $\mathbf{K}^G(\mathbb{1})$ is a normal, symmetric lax monoidal functor from the category of graded G - C^* -algebras to the category of $\mathbf{K}^G(\mathbb{1})$ -modules.*

Proof. The compatibility conditions we need are contained in Lemma 3.4. \square

Proposition 3.1 yields

$$(3.5) \quad \pi_{k+n} \mathbf{K}_n^G(A) \cong \mathbf{KK}_{k+n}^G(\mathbb{1}, (\mathbb{C}l_{\mathbb{R}} \widehat{\otimes} \widehat{\mathcal{K}}_G)^n) \cong \mathbf{KK}_k^G(\mathbb{1}, A)$$

for all $n \geq 1$ and $k \geq -n$. Furthermore, since ι_1 is constructed from the generator of $\mathbf{KK}_0(\mathbb{1}, C_0(\mathbb{R}, \mathbb{C}l_{\mathbb{R}}))$, the map $\mathbf{K}_n^G(A) \rightarrow \Omega \mathbf{K}_{n+1}^G(A)$ that it induces is a weak equivalence for all $n \geq 1$. Thus $\mathbf{K}^G(A)$ is a positive Ω -spectrum in the sense of [28, Definition I.1.9]. This implies that $\mathbf{K}^G(A)$ is semistable, so that (3.5) computes the correct homotopy groups of the spectrum $\mathbf{K}^G(A)$, namely, the Hom groups out of the sphere spectra $\mathbb{S}[k]$ (see [29, Example 4.2], [13, Proposition 5.6.4 (2)]). Thus for all $k \in \mathbb{Z}$ we get

$$(3.6) \quad \begin{aligned} \mathrm{Ho} \mathfrak{Sp}^{\Sigma}(\mathbb{S}[k], \mathbf{K}^G(A)) &\cong \pi_k \mathbf{K}^G(A) = \mathrm{colim}_{n \rightarrow \infty} \pi_{k+n} \mathbf{K}_n^G(A) \\ &\cong \mathrm{colim}_{n \rightarrow \infty} \mathbf{KK}_{(k+n)-n}^G(\mathbb{1}, A) = \mathbf{KK}_k^G(\mathbb{1}, A). \end{aligned}$$

Remark 3.6. The functor \mathbf{K}^G is also compatible with other extra structure:

- (1) It commutes with pull-backs, that is, it maps the pull-back of a diagram $A_1 \rightarrow B \leftarrow A_2$ to the pull-back of $\mathbf{K}^G(A_1) \rightarrow \mathbf{K}^G(B) \leftarrow \mathbf{K}^G(A_2)$.
- (2) It maps suspensions to loop spaces, that is,

$$\mathbf{K}^G(C_0(\mathbb{R}) \otimes A) \cong \mathbf{K}^G(C_0(\mathbb{R}, A)) \cong \Omega \mathbf{K}^G(A)$$

because a $*$ -homomorphism to $A \rightarrow C_0(\mathbb{R}, B)$ is a pointed continuous map $\mathbb{S}^1 \rightarrow \mathrm{Hom}(A, B)$.

- (3) It maps the cone of a grading-preserving G -equivariant $*$ -homomorphism $f: A \rightarrow B$ to the homotopy fibre of $\mathbf{K}^G(f)$. Recall that the cone of f is

$$C_f := \{(a, b) \in A \oplus C_0((-\infty, \infty], B) : f(a) = b(\infty)\}.$$

An element of $\mathbf{K}_n^G(C_f)$ is a pair (a, b) , where $a \in \mathbf{K}_n^G(A)$ and b is a path in $\mathbf{K}_n^G(B)$ starting at the basepoint and ending at $\mathbf{K}_n^G(f)(a)$. This is the homotopy fibre of $\mathbf{K}^G(f)$, compare [21, Definition 6.8].

3.4. Extension to KK-theory. So far, the functor \mathbf{K}^G is defined on the level of C^* -algebras and $*$ -homomorphisms, but we need a functor defined on Kasparov theory. For this, we use the universal property of Kasparov theory. This requires that we restrict attention to trivially graded C^* -algebras. That is, we now consider the symmetric monoidal triangulated category \mathfrak{K}^G as in Example 2.2.

The universal property of Kasparov theory is formulated in the non-equivariant case in [9]. It is extended to the equivariant case for locally compact groups by Thomsen. The proof of the universal property in [23] for locally compact groups also works for locally compact Hausdorff groupoids with Haar system. We notice the following consequence:

Theorem 3.7. *Let G be a proper, locally compact, Hausdorff groupoid with Haar system. If a functor $\mathfrak{C}^* \mathfrak{sep}^G \rightarrow \mathcal{C}$ maps all \mathbf{KK}^G -equivalences to isomorphisms, then it factors through the canonical functor $\mathfrak{C}^* \mathfrak{sep}^G \rightarrow \mathfrak{K}^G$.*

Proof. The set $\mathbf{KK}^G(A, B)$ is naturally isomorphic to the set of homotopy classes of G -equivariant $*$ -homomorphisms $qA \rightarrow \mathcal{K}(L^2 G \otimes \ell^2 \mathbb{N}) \otimes B$ (we do not have to stabilise A because G is proper, compare [23, Theorem 8.7]). There are canonical

KK^G -equivalences $qA \rightarrow A$ and $B \rightarrow \mathcal{K}(L^2G \otimes \ell^2\mathbb{N}) \otimes B$. We have to compose with the inverses of the maps they induce to extend a functor from $\mathfrak{C}^*\mathfrak{sep}^G$ to $\mathfrak{K}\mathfrak{K}^G$. \square

Let $\mathrm{Ho}(\mathbf{K}^G(\mathbb{1})\text{-Mod})$ be the homotopy category of right $\mathbf{K}^G(\mathbb{1})$ -modules, obtained by inverting those $\mathbf{K}^G(\mathbb{1})$ -module maps that are stable equivalences in \mathfrak{Sp}^Σ , briefly called *weak equivalences*.

Theorem 3.8. *Let G be a proper, locally compact, Hausdorff groupoid with Haar system. The functor $\mathbf{K}^G: \mathfrak{C}^*\mathfrak{sep}^G \rightarrow \mathbf{K}^G(\mathbb{1})\text{-Mod}$ induces a symmetric lax monoidal functor $\mathbf{K}^G: \mathfrak{K}\mathfrak{K}^G \rightarrow \mathrm{Ho}(\mathbf{K}^G(\mathbb{1})\text{-Mod})$ making the following diagram commute:*

$$\begin{array}{ccc} \mathfrak{C}^*\mathfrak{sep}^G & \xrightarrow{\mathbf{K}^G} & \mathbf{K}^G(\mathbb{1})\text{-Mod} \\ \downarrow & & \downarrow \\ \mathfrak{K}\mathfrak{K}^G & \xrightarrow{\mathbf{K}^G} & \mathrm{Ho}(\mathbf{K}^G(\mathbb{1})\text{-Mod}). \end{array}$$

This functor is triangulated.

Proof. For $A \in \mathfrak{C}^*\mathfrak{sep}^G$ and $i \in \mathbb{Z}$, we compute

$$(3.7) \quad \begin{aligned} \mathrm{Ho}(\mathbf{K}^G(\mathbb{1})\text{-Mod})(\mathbf{K}^G(\mathbb{1})[i], \mathbf{K}^G(A)) &\cong \mathrm{Ho}(\mathfrak{Sp}^\Sigma)(\mathbb{S}[i], \mathbf{K}^G(A)) \\ &\cong \mathrm{KK}^G(\mathbb{1}[i], A) \end{aligned}$$

by a standard isomorphism and (3.6). A $\mathbf{K}^G(\mathbb{1})$ -module homomorphism is a weak equivalence if it induces an isomorphism on $\mathrm{Ho}(\mathfrak{Sp}^\Sigma)(\mathbb{S}[i], \mathbf{K}^G(A))$ for all i . Thus $\mathbf{K}^G(f)$ is a weak equivalence if f is a KK^G -equivalence. Now Theorem 3.7 yields the desired functor $\mathbf{K}^G: \mathfrak{K}\mathfrak{K}^G \rightarrow \mathrm{Ho}(\mathbf{K}^G(\mathbb{1})\text{-Mod})$. This functor clearly remains symmetric, normal, lax monoidal. It is also additive.

The triangulated structure on $\mathfrak{K}\mathfrak{K}^G$ is defined using $A \mapsto C_0(\mathbb{R}) \otimes A$ as desuspension and diagrams isomorphic to mapping cone triangles as exact triangles. We observed in Remark 3.6 that \mathbf{K}^G maps $C_0(\mathbb{R}) \otimes A$ to the loop space of $\mathbf{K}^G(A)$, and the cone of a map f to the homotopy fibre of $\mathbf{K}^G(f)$. Hence the functor $\mathbf{K}^G: \mathfrak{K}\mathfrak{K}^G \rightarrow \mathrm{Ho}(\mathbf{K}^G(\mathbb{1})\text{-Mod})$ is triangulated. \square

4. THE ADDITIVITY THEOREM

In this section we prove the Additivity Theorem. Let $\mathfrak{K}\mathfrak{K}^G$ be the triangulated category of G -equivariant bivariant KK -theory for a *cocompact* proper locally compact groupoid G with Haar system. Recall that this is a symmetric monoidal category with unit object $\mathbb{1} = C_0(X)$ for the object space X of G . Let $\mathcal{T} := \langle \mathbb{1} \rangle_{\mathrm{loc}}$ be the localising triangulated subcategory of $\mathfrak{K}\mathfrak{K}^G$ generated by $\mathbb{1}$, and let \mathcal{T}_d denote the closed symmetric monoidal category of its dualisable objects, as in Section 2.

Proposition 4.1. *The category \mathcal{T}_d coincides with \mathcal{T}_c , the full triangulated subcategory of $\mathrm{compact}_{\mathbb{N}_1}$ objects, and both are also equal to $\langle \mathbb{1} \rangle$, the thick triangulated subcategory of \mathcal{T} (or of $\mathfrak{K}\mathfrak{K}^G$) generated by the unit object $\mathbb{1}$.*

Proof. Since G is cocompact, the $R(G)$ -modules $\mathrm{KK}^G(\mathbb{1}[i], A)$ identify naturally with the topological G -equivariant K -theory $K_i^G(A) \cong K_i(G \rtimes A)$. Since ordinary K -theory of separable C^* -algebras yields countable abelian groups and commutes with countable coproducts in $\mathfrak{K}\mathfrak{K}^G$, and since $G \rtimes _$ commutes with coproducts and preserves separability, we conclude that the \otimes -unit $\mathbb{1}$ is a $\mathrm{compact}_{\mathbb{N}_1}$ object

of $\mathfrak{K}\mathfrak{K}^G$. (See [3, §2.1] for this countable version on the more usual notions of compact objects in triangulated categories; this notion is necessary here because $\mathfrak{K}\mathfrak{K}^G$ only has countable, not arbitrary, coproducts.)

Thus $\mathcal{T} = \langle \mathbf{1} \rangle_{\text{loc}} \subseteq \mathfrak{K}\mathfrak{K}^G$ is compactly $_{\aleph_1}$ generated, from which it follows that $\mathcal{T}_c = \langle \mathbf{1} \rangle$ ([3, Corollary 2.4]). Moreover, since \mathcal{T} is generated by the \otimes -unit, its compact and dualisable objects coincide. Indeed, it is not difficult to see that \mathcal{T}_d is a thick triangulated subcategory of \mathcal{T} (this uses that the contravariant Hom functors are sufficiently exact, see [3, §2.3]). Since the \otimes -unit is always dualisable, it follows that $\mathcal{T}_c = \langle \mathbf{1} \rangle \subseteq \mathcal{T}_d$. On the other hand, we have seen that $\mathbf{1} \in \mathcal{T}_c$, and it follows by an easy computation (as in the proof of [12, Theorem 2.1.3 (a)]) that $\mathcal{T}_d \subseteq \mathcal{T}_c$. Thus $\mathcal{T}_d = \mathcal{T}_c$. \square

Proposition 4.2. *The restriction on $\langle \mathbf{1} \rangle$ of the functor \mathbf{K}^G of Theorem 3.8 is a fully faithful, strong monoidal exact functor $\mathbf{K}^G: \langle \mathbf{1} \rangle \rightarrow \text{Ho}(\mathbf{K}^G(\mathbf{1})\text{-}\mathfrak{Mod})$.*

Proof. To show that the restriction of \mathbf{K}^G on $\langle \mathbf{1} \rangle$ is a strong monoidal functor, consider in $\text{Ho}(\mathbf{K}^G(\mathbf{1})\text{-}\mathfrak{Mod})$ the natural transformation

$$c: \mathbf{K}^G(_) \wedge_{\mathbf{K}^G(\mathbf{1})} \mathbf{K}^G(B) \rightarrow \mathbf{K}^G(_ \otimes B),$$

where B is a fixed separable G - C^* -algebra. Both the left and the right functors are exact. By construction, c is an isomorphism on suspensions of $\mathbf{1}$. It follows by a standard triangulated argument that c is an isomorphism on the thick triangulated subcategory of $\mathfrak{K}\mathfrak{K}^G$ generated by $\mathbf{1}$, which by Proposition 4.1 is precisely $\langle \mathbf{1} \rangle$.

Thus \mathbf{K}^G is strong monoidal on $\langle \mathbf{1} \rangle$. Similarly, it is fully faithful on $\langle \mathbf{1} \rangle$ because on the full subcategory of the generators $\mathbf{1}[i]$ it is an isomorphism

$$\begin{aligned} \mathbf{K}^G: \text{KK}^G(\mathbf{1}[i], \mathbf{1}[j]) &\xrightarrow{\cong} \text{Ho}(\mathbf{K}^G(\mathbf{1})\text{-}\mathfrak{Mod})(\mathbf{K}^G(\mathbf{1})[i], \mathbf{K}^G(\mathbf{1})[j]) \\ &\cong \text{Ho}(\mathfrak{S}\mathfrak{p}^\Sigma)(\mathfrak{S}[i], \mathbf{K}^G(\mathbf{1})[j]) \end{aligned}$$

by (3.6) ($i, j \in \mathbb{Z}$). \square

Proof of Theorem 1.2. We know from Proposition 4.1 that the G - C^* -algebras under considerations are dualisable. Given a diagram in $\langle \mathbf{1} \rangle \subseteq \mathfrak{K}\mathfrak{K}^G$ as in the statement of the theorem, by Proposition 4.2 we may apply the fully faithful exact functor \mathbf{K}^G to it to transfer the problem to the tensor triangulated category $\text{Ho}(\mathbf{K}^G(\mathbf{1})\text{-}\mathfrak{Mod})$, where the analogous statement is known to hold. Indeed, by [21, Theorem 12.1 (i),(iii)] the module category over any commutative ring spectrum R carries a symmetric monoidal model structure (with respect to the smash product \wedge_R , and with stable equivalences of the underlying symmetric spectra as weak equivalences); by [22, Theorem 1.9 and §5–7]), the induced tensor-triangulated structure in the homotopy category of any such model category satisfies additivity of traces as required by Theorem 1.2, and even in a more general form.

Since \mathbf{K}^G is also strong monoidal on $\langle \mathbf{1} \rangle \subseteq \mathfrak{K}\mathfrak{K}^G$, by Lemma 2.8 it identifies the traces computed in $\langle \mathbf{1} \rangle$ with those computed in $\text{Ho}(\mathbf{K}^G(\mathbf{1})\text{-}\mathfrak{Mod})$. \square

5. APPLICATION TO TRACE COMPUTATIONS

Now we relate traces in triangulated categories to the Hattori–Stallings traces in certain module categories. We work in the general setting of a tensor triangulated

category $(\mathcal{T}, \otimes, \mathbb{1})$, assuming additivity of traces. Let

$$R := \mathcal{T}_*(\mathbb{1}, \mathbb{1}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}_n(\mathbb{1}, \mathbb{1})$$

be the graded endomorphism ring of the tensor unit. It is graded-commutative.

If A is any object of \mathcal{T} , then $M(A) := \mathcal{T}_*(\mathbb{1}, A) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}_n(\mathbb{1}, A)$ is an R -module in a canonical way, and an endomorphism $f \in \mathcal{T}_n(A, A)$ yields a degree- n endomorphism $M(f)$ of $M(A)$. We will prove in Theorem 5.2 below that, under some assumptions, the trace of f equals the Hattori–Stallings trace of $M(f)$ and, in particular, depends only on $M(f)$.

Example 5.1. First we consider the example where \mathcal{T} is the category $\mathfrak{K}\mathfrak{K}$ of complex separable C^* -algebras, $A = C(X)$ for a compact smooth manifold, and $f \in \text{KK}_0(A, A)$ is the class of the $*$ -homomorphism induced by a smooth self-map of X whose graph is transverse to the diagonal. On the one hand, the trace of f in $\mathfrak{K}\mathfrak{K}$ may be computed as a Kasparov product, and the result reduces to the usual expression for the Lefschetz invariant of f in terms of fixed points (see [5] for this and some equivariant generalisations). On the other hand, $\text{KK}_*(\mathbb{1}, A) = K^*(X)$, and the Hattori–Stallings trace of the map induced by f agrees with the global cohomological formula for the Lefschetz invariant. Thus our result generalises the Lefschetz Fixed Point Theorem.

Before we can state our theorem, we must define the Hattori–Stallings trace for endomorphisms of graded modules over graded rings. This is well-known for ungraded rings (see [1]). The grading causes some notational overhead here. Let R be a (unital) graded-commutative graded ring. A finitely generated free R -module is a direct sum of copies of $R[n]$, where $R[n]$ denotes R with degree shifted by n . Let $F: P \rightarrow P$ be a module endomorphism of such a free module, let us assume that F is homogeneous of degree d . We use an isomorphism

$$(5.1) \quad P \cong \bigoplus_{i=1}^r R[n_i]$$

to rewrite F as a matrix $(f_{ij})_{1 \leq i, j \leq r}$ with R -module homomorphisms $f_{ij}: R[n_j] \rightarrow R[n_i]$ of degree d . The entry f_{ij} is given by right multiplication by some element of R of degree $n_i - n_j + d$. The (super)trace $\text{tr } F$ is defined as

$$\text{tr } F := \sum_{i=1}^r (-1)^{n_i} \text{tr } f_{ii},$$

this is an element of R of degree d .

It is straightforward to check that $\text{tr } F$ is well-defined, that is, independent of the choice of the isomorphism in (5.1). Here we use that the degree-zero part of R is central in R (otherwise, we still get a well-defined element in the commutator quotient $R_d/[R_d, R_0]$). Furthermore, if we shift the grading on P by n , then the trace is multiplied by the sign $(-1)^n$ – it is a supertrace.

If P is a finitely generated projective graded R -module, then $P \oplus Q$ is finitely generated and free for some Q , and for an endomorphism F of P we let

$$\text{tr } F := \text{tr}(F \oplus 0: P \oplus Q \rightarrow P \oplus Q).$$

This does not depend on the choice of Q .

A *finite projective resolution* of a graded R -module M is a resolution

$$(5.2) \quad \cdots \rightarrow P_\ell \xrightarrow{d_\ell} P_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M$$

of finite length by finitely generated projective graded R -modules P_j . We assume that the maps d_j have degree one (or at least odd degree). Assume that M has such a resolution and let $f: M \rightarrow M$ be a module homomorphism. Lift f to a chain map $f_j: P_j \rightarrow P_j$, $j = 0, \dots, \ell$. We define the *Hattori–Stallings trace* of f as

$$\mathrm{tr}(f) = \sum_{j=0}^{\ell} \mathrm{tr}(f_j).$$

It may be shown that this trace does not depend on the choice of resolution. It is important for this that we choose d_j of degree one. Since shifting the degree by one alters the sign of the trace of an endomorphism, the sum in the definition of the trace becomes an *alternating* sum when we change conventions to have even-degree boundary maps d_j . Still the trace changes sign when we shift the degree of M .

Recall that $(\mathcal{T}, \otimes, \mathbf{1})$ is a tensor triangulated category that satisfies additivity of traces and that $R := \mathcal{T}_*(\mathbf{1}, \mathbf{1})$ is a graded-commutative ring.

Theorem 5.2. *Let $F \in \mathcal{T}_d(A, A)$ be an endomorphism of some object A of \mathcal{T} . Assume that A belongs to the localising subcategory of \mathcal{T} generated by $\mathbf{1}$. If the graded R -module $M(A) := \mathcal{T}_*(\mathbf{1}, A)$ has a finite projective resolution, then A is dualisable in \mathcal{T} and the trace of F is equal to the Hattori–Stallings trace of the induced module endomorphism $\mathcal{T}_*(\mathbf{1}, f)$ of $M(A)$.*

Proof. Our main tool is the phantom tower over A , which is constructed in [24]. We recall some details of this construction.

Let M^\perp be the functor from finitely generated projective R -modules to \mathcal{T} defined by the adjointness property $\mathcal{T}(M^\perp(P), B) \cong \mathcal{T}(P, M(B))$ for all $B \in \mathcal{T}$. The functor M^\perp maps the free rank-one module R to $\mathbf{1}$, is additive, and commutes with suspensions; this determines M^\perp on objects. Since $R = \mathcal{T}_*(\mathbf{1}, \mathbf{1})$, $\mathcal{T}_*(M^\perp(P_1), M^\perp(P_2))$ is isomorphic (as a graded Abelian group) to the space of R -module homomorphisms $P_1 \rightarrow P_2$. Furthermore, we have canonical isomorphisms $M(M^\perp(P)) \cong P$ for all finitely generated projective R -modules P .

By assumption, $M(A)$ has a finite projective resolution as in (5.2). Using M^\perp , we lift it to a chain complex in \mathcal{T} , with entries $\hat{P}_j := M^\perp(P_j)$ and boundary maps $\hat{d}_j := M^\perp(d_j)$ for $j \geq 1$. The map $\hat{d}_0: \hat{P}_0 \rightarrow A$ is the pre-image of d_0 under the adjointness isomorphism $\mathcal{T}(M^\perp(P), B) \cong \mathcal{T}(P, M(B))$. We get back the resolution of modules by applying M to the chain complex (\hat{P}_j, \hat{d}_j) .

Next, it is shown in [24] that we may embed this chain complex into a diagram

$$(5.3) \quad \begin{array}{ccccccc} A = N_0 & \xrightarrow{\iota_0^1} & N_1 & \xrightarrow{\iota_1^2} & N_2 & \xrightarrow{\iota_2^3} & N_3 & \longrightarrow & \cdots \\ & \swarrow \pi_0 & \circ \searrow \epsilon_0 & \swarrow \pi_1 & \circ \searrow \epsilon_1 & \swarrow \pi_2 & \circ \searrow \epsilon_2 & \swarrow \pi_3 & \circ \searrow \epsilon_3 & \cdots \\ & & \hat{P}_0 & \xleftarrow{\hat{d}_1} & \hat{P}_1 & \xleftarrow{\hat{d}_2} & \hat{P}_2 & \xleftarrow{\hat{d}_3} & \hat{P}_3 & \xleftarrow{\hat{d}_4} & \cdots \end{array}$$

where the triangles involving \hat{d}_j commute and the others are exact. This diagram is called the *phantom tower* in [24]. The circles indicate maps of degree one.

Since $\hat{P}_j = 0$ for $j > \ell$, the maps ι_j^{j+1} are invertible for $j > \ell$. Furthermore, a crucial property of the phantom tower is that these maps ι_j^{j+1} are *phantom maps*, that is, they induce the zero map on $\mathcal{T}_*(\mathbb{1}, _)$. Together, these facts imply that $M(N_j) = 0$ for $j > \ell$. Since we assumed $\mathbb{1}$ to be a generator of \mathcal{T} , this further implies $N_j = 0$ for $j > \ell$.

Next we recursively extend the endomorphism F of $A = N_0$ to an endomorphism of the phantom tower. We start with $F_0 = F: N_0 \rightarrow N_0$. Assume $F_j: N_j \rightarrow N_j$ has been constructed. As in [24], we may then lift F_j to a map $\hat{F}_j: \hat{P}_j \rightarrow \hat{P}_j$ such that the square

$$\begin{array}{ccc} \hat{P}_j & \xrightarrow{\pi_j} & N_j \\ \hat{F}_j \downarrow & & \downarrow F_j \\ \hat{P}_j & \xrightarrow{\pi_j} & N_j \end{array}$$

commutes. Now we apply the Additivity Theorem 1.2 to construct an endomorphism $F_{j+1}: N_{j+1} \rightarrow N_{j+1}$ such that $(\hat{F}_j, F_j, F_{j+1})$ is a triangle morphism and $\text{tr}(F_j) = \text{tr}(\hat{F}_j) + \text{tr}(F_{j+1})$. Then we repeat the recursion step with F_{j+1} and thus construct a sequence of maps F_j . We get

$$\text{tr}(F) = \text{tr}(F_0) = \text{tr}(\hat{F}_0) + \text{tr}(F_1) = \cdots = \text{tr}(\hat{F}_0) + \cdots + \text{tr}(\hat{F}_\ell) + \text{tr}(F_{\ell+1}).$$

Since $N_{\ell+1} = 0$, we may leave out the last term.

Finally, it remains to observe that the trace of \hat{F}_j as an endomorphism of \hat{P}_j agrees with the trace of the induced map on the projective module P_j . Since both traces are additive with respect to direct sums of maps, the case of general finitely generated projective modules reduces first to free modules and then to free modules of rank one. Both traces change by a sign if we suspend or desuspend once, hence we reduce to the case of endomorphisms of $\mathbb{1}$, which is trivial. Hence the computation above does indeed yield the Hattori–Stallings trace of $M(A)$ as asserted. \square

Remark 5.3. Note that if a module has a finite projective resolution, then it must be finitely generated. Conversely, if the graded ring R is *coherent* and *regular*, then any finitely generated module has a finite projective resolution. (Regular means that every finitely generated module has a finite *length* projective resolution; coherent means that every finitely generated homogeneous ideal is finitely presented – for instance, this holds if R is (graded) Noetherian; coherence implies that any finitely generated graded module has a resolution by finitely generated projectives.)

Moreover, if R is coherent, an induction argument shows (as in the proof of Proposition 4.1) that for every $A \in \langle \mathbb{1} \rangle = (\langle \mathbb{1} \rangle_{\text{loc}})_d$ the module $M(A)$ is finitely generated; and if R is also regular, each such $M(A)$ has a finite projective resolution.

In conclusion: if R is regular and coherent, an object $A \in \langle \mathbb{1} \rangle_{\text{loc}}$ is dualisable if and only if $M(A)$ has a finite projective resolution.

Proof of Theorem 1.1. We specialise to equivariant Kasparov theory for a compact group G and complex C^* -algebras. In this case, the tensor unit is $\mathbb{1} = \mathbb{C}$ and the ring $R = \text{KK}_*^G(\mathbb{C}, \mathbb{C})$ becomes the tensor product of the representation ring $R(G)$ of G with the ring of Laurent polynomials $\mathbb{Z}[\beta, \beta^{-1}]$ in one variable β of degree 2, which generates Bott periodicity. As a result, the category of graded R -modules is equivalent to the category of $\mathbb{Z}/2$ -graded $R(G)$ -modules. Such an object is projective or finitely generated if and only if its even and odd part are projective or finitely

generated $R(G)$ -modules, respectively. Hence we are essentially dealing with pairs of ungraded modules over the ungraded ring $R(G)$.

The Hattori–Stallings trace of an even-degree endomorphism of a $\mathbb{Z}/2$ -graded $R(G)$ -module (M_+, M_-) is the *difference* of the Hattori–Stallings traces on the $R(G)$ -modules M_+ and M_- , respectively, where we are now talking about ungraded modules over ungraded rings.

The modules we need are of the form $\mathcal{T}_*(\mathbb{1}, A)$ for a separable G - C^* -algebra A , and this is isomorphic to the G -equivariant K -theory $K_*^G(A) \cong KK_*^G(\mathbb{C}, A)$. Theorem 5.2 only applies if A belongs to the localising subcategory generated by $\mathbb{1}$. For a finite group G , this is a rather unnatural assumption. Furthermore, in that case there are many modules without finite length projective resolutions.

Under the assumption that G is a connected Lie group with torsion-free fundamental group (also called *Hodgkin Lie group*), results of [26] imply that the localising subcategory $\langle \mathbb{1} \rangle_{\text{loc}}$ of $\mathfrak{K}\mathfrak{K}^G$ contains a separable G - C^* -algebra A if and only if A belongs to the non-equivariant bootstrap class in $\mathfrak{K}\mathfrak{K}$, that is, there is an invertible element in $KK(A, B)$ for a commutative C^* -algebra B (no G -equivariance is required). Furthermore, the ring $R(G)$ is regular and Noetherian. Thus a $(\mathbb{Z}/2$ -graded) $R(G)$ -module has a finite projective resolution if and only if it is finitely generated as a $R(G)$ -module. Thus Theorem 1.1 is a special case of Theorem 5.2. \square

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UNIVERSITÄT BIELEFELD, FAKULTÄT FÜR MATHEMATIK, BIREP GRUPPE, POSTFACH 10 01 31,
33501 BIELEFELD, GERMANY

E-mail address: ambrogio@math.uni-bielefeld.de

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF VICTORIA, PO BOX 3045 STN CSC,
VICTORIA, B. C., CANADA V8W 3P4

E-mail address: hemerson@math.uvic.ca

A. RAZMADZE MATHEMATICAL INSTITUTE, TBILISI STATE UNIVERSITY, UNIVERSITY STREET 2,
TBILISI 380043, GEORGIA; TBILISI CENTRE FOR MATHEMATICAL SCIENCES, CHAVCHAVADZE AVE. 75,
3/35, TBILISI 0168, REPUBLIC OF GEORGIA

E-mail address: tam.kandel@gmail.com

MATHEMATISCHES INSTITUT AND COURANT CENTRE "HIGHER ORDER STRUCTURES", GEORG-
AUGUST UNIVERSITÄT GÖTTINGEN, BUNSENSTRASSE 3-5, 37073 GÖTTINGEN, GERMANY

E-mail address: rameyer@uni-math.gwdg.de