BOUNDS FOR CHARACTERISTIC FUNCTIONS IN TERMS OF QUANTILES AND ENTROPY

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Abstract. Upper bounds on characteristic functions are derived in terms of the entropic distance to the class of normal distributions.

Let X be a random variable with density p and the characteristic function

$$f(t) = \mathbf{E}\, e^{itX} = \int_{-\infty}^{+\infty} e^{itx} p(x)\, dx \qquad (t \in \mathbf{R}).$$

By the Riemann-Lebesgue theorem, $f(t) \to 0$, as $t \to \infty$. So, for all $T > 0$,

$$\delta(T) = \sup_{|t| \geq T} |f(t)| < 1.$$

An important problem is how to quantify this property by giving explicit upper bounds on $\delta(T)$. The problem arises naturally in various local limit theorems for densities of sums of independent summands; see, for example, [St], [P] or [Se] for an interesting discussion. Our motivation, which, however, we do not discuss in this note, has been the problem of optimal rates of convergence in the entropic central limit theorem for non-i.i.d. summands. Let us only mention that in investigating this rate of convergence explicit bounds on $\delta(T)$ (also known as Cramer's condition (C)) in terms of the entropy of X are crucial.

A first possible answer may be given for random variables with finite variance, say $\sigma^2 = \mathrm{Var}(X)$, and which have a uniformly bounded density, say p.

Theorem 1. *Assume $p(x) \leq M$ a.e. Then, for all $\sigma|t| \geq \frac{\pi}{4}$,*

$$|f(t)| \, < \, 1 - \frac{c_1}{M^2 \sigma^2}. \tag{1}$$

Moreover, in case $0 < \sigma|t| < \frac{\pi}{4}$,

$$|f(t)| \, < \, 1 - \frac{c_2\, t^2}{M^2}. \tag{2}$$

Here, $c_1, c_2 > 0$ are certain absolute constants.

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A similar bound with a slightly different dependence on (M,σ) in the right-hand side of (1) was obtained in the mid 1960's by Statulevičius [St]. He also considered more complicated quantities reflecting the behavior of the density p on non-overlapping intervals of the real line (cf. Remark 10 at the end of this note).

The bounds (1)-(2) may be considerably generalized for classes of non-bounded densities, using other quantites of the distribution of $p(X)$ with respect to the measure $p(x)\,dx$. One of the results in this note is the following assertion.

Theorem 2. *Let m be a median of the random variable $p(X)$. If $\sigma|t| \geq \frac{\pi}{4}$, we have*

$$|f(t)| \, < \, 1 - \frac{c_1}{m^2\sigma^2}. \tag{3}$$

Moreover, in case $0 < \sigma|t| < \frac{\pi}{4}$,

$$|f(t)| \, < \, 1 - \frac{c_2\, t^2}{m^2}. \tag{4}$$

Here, $c_1, c_2 > 0$ are absolute constants. (*One may take $c_1 = 10^{-6}$ and $c_2 = 10^{-7}$*).

Since the median of $p(X)$ is majorized by the maximum $M = \text{ess sup}_x\, p(x)$, Theorem 2 immediately implies Theorem 1. Note in this case that the constants c_1 and c_2 in (1)-(2) can be improved in comparison with the constants in (3)-(4).

One may further generalize Theorem 2 by removing the requirement that the second moment of X is finite. In this case, the standard deviation σ should be replaced in (3)-(4) with quantiles of $|X - X'|$, where X' is an independent copy of X. This will be explained below in the proof of Theorem 2. Thus, quantitative estimates for the maximum of the characteristic functions on half-axes, such as (3)-(4), can be given in the class of all absolutely continuous distributions on the line.

Let us describe a few applications, where the median m of $p(X)$ can be controled explicitly in terms of more other quantities. First, assume that the characteristic function f is square integrable, i.e.,

$$\|f\|_2^2 = \int_{-\infty}^{+\infty} |f(t)|^2\, dt < +\infty. \tag{5}$$

By Chebyshev's inequality and Parseval's identity, for any $\lambda > 0$,

$$\mathbf{P}\{p(X) \geq \lambda\} \leq \frac{\mathbf{E}\, p(X)}{\lambda} = \frac{1}{\lambda} \int_{-\infty}^{+\infty} p(x)^2\, dx = \frac{\|f\|_2^2}{2\pi\, \lambda}.$$

The right-hand side is smaller than $\frac{1}{2}$, whenever $\lambda > \frac{\|f\|_2^2}{\pi}$, so this ratio provides an upper bound on any median of $p(X)$. Hence, Theorem 2 implies:

Corollary 3. *If $\sigma|t| \geq \frac{\pi}{4}$, then*

$$|f(t)| \, < \, 1 - \frac{c}{\|f\|_2^2\, \sigma^2},$$

where $c > 0$ is an absolute constant. Moreover, in case $0 < \sigma|t| < \frac{\pi}{4}$,

$$|f(t)| \, < \, 1 - \frac{ct^2}{\|f\|_2^2\, \sigma^2}.$$

The integral $\int_{-\infty}^{+\infty} p(x)^2\,dx = \frac{1}{2\pi}\,\|f\|_2^2$ appears in problems of quantum mechanics and information theory, where it is referred to as the informational energy or the quadratic entropy.

However, it is infinite for many probability distributions. The condition (5), that is, $\int_{-\infty}^{+\infty} p(x)^2\,dx < +\infty$, may be relaxed in terms of the so-called entropic distance to normality, which is defined as the difference of the entropies $D(X) = h(Z) - h(X)$. Here

$$h(X) = -\int_{-\infty}^{+\infty} p(x)\log p(x)\,dx$$

denotes the (differential) entropy for a random variable with density p, and Z is a normal random variable with the same variance as X. The quantity $h(X)$ is well defined in the usual Lebesgue sense, as long as X has a finite variance $\sigma^2 = \mathrm{Var}(X)$. It is as well-known that

$$h(X) \leq h(Z), \qquad Z \sim N(a,\sigma^2),$$

where equality is possible only, when X has a normal distribution. Hence,

$$0 \leq D(X) \leq +\infty.$$

Note that this functional is translation and scale invariant with respect to X, and is not determined by the mean or variance of X. It may also be described as the Kullback-Leibler distance (or the informational divergence) from the distribution of X to the normal distribution $N(a,\sigma^2)$ with the same mean and variance as X. Hence, the closeness of $D(X)$ to zero indicates in a rather strong sense how close is the distribution of X to the class of normal distributions.

Although the value $D(X) = +\infty$ is still possible, the condition $D(X) < +\infty$, or equivalently

$$\mathbf{E}\log p(X) = \int_{-\infty}^{+\infty} p(x)\log p(x)\,dx < +\infty,$$

is much weaker than (5). From Theorem 2 we derive:

Corollary 4. *Assume a random variable X with finite variance $\sigma^2 = \mathrm{Var}(X)$ has a finite entropy. Then, for all $\sigma|t| \geq \frac{\pi}{4}$, the characteristic function satisfies*

$$|f(t)| \,<\, 1 - ce^{-4D(X)} \tag{6}$$

with some absolute constant $c > 0$. Moreover, in case $0 < \sigma|t| < \frac{\pi}{4}$,

$$|f(t)| \,<\, 1 - c\sigma^2 t^2\, e^{-4D(X)}. \tag{7}$$

Here, the coefficient 4 in the exponents can be improved at the expense of the constant c in (6)-(7), and chosen to be arbtrarily close to 2.

Let us turn to the proofs. Since the argument involves a symmetrization of the distribution of X, we need study how the median and other functionals of $p(X)$ will change under convolutions.

Notations. Given $0 < \kappa < 1$, we write $m_\kappa = m_\kappa(\xi)$ to indicate that m_κ is a κ-quantile of a random variable ξ (or, a quantile of order κ), which may be any number such that

$$\mathbf{P}\{\xi < m_\kappa\} \leq \kappa, \quad \mathbf{P}\{\xi > m_\kappa\} \leq 1 - \kappa.$$

If $\kappa = 1/2$, the value $m = m_{1/2}$ represents a median of ξ.

Lemma 5. *Let X be a random variable with density p, and let q be the density of the random variable $Y = X + X'$, where X' is independent of X. Then*

$$\mathbf{E}\, u(q(Y)) \leq \mathbf{E}\, u(p(X)), \tag{8}$$

for any function $u \geq 0$, such that $v(t) = tu(t)$ is non-decreasing and convex in $t \geq 0$.

Proof. Using a simple approximation, we may assume that the function $v(t) = tu(t)$ has a finite Lipschitz constant. In particular, $v(t) \leq Ct$, for all $t \geq 0$ with some constant C. Write

$$\mathbf{E}\, u(p(X)) = \int_{-\infty}^{+\infty} u(p(x))\, p(x)\, dx = \int_{-\infty}^{+\infty} v(p(x))\, dx.$$

The density of Y is given by $q(x) = \int p(x-y)\, d\mu(y)$, where μ is the distribution of X'. Hence,

$$\mathbf{E}\, u(q(Y)) = \int_{-\infty}^{+\infty} v\bigg(\int_{-\infty}^{+\infty} p(x-y)\, d\mu(y)\bigg)\, dx \equiv I(\mu).$$

In particular, $\mathbf{E}\, u(q(Y)) = I(\delta_y) = \mathbf{E}\, u(p(X))$, for any mass point $\mu = \delta_y$, that is, when $X' = y$ a.s. Also note that $I(\mu) \leq C$ in the class of all Borel probability measures μ on the line.

Since v is convex, the functional $\mu \rightarrow I(\mu)$ is convex, as well. Hence, for μ discrete, say, $\mu = \alpha_1 \delta_{y_1} + \dots + \alpha_N \delta_{y_N}$, where $\alpha_i \geq 0$, $\alpha_1 + \dots + \alpha_N = 1$, we get by Jensen's inequality,

$$\mathbf{E}\, u(q(Y)) \leq \sum_{k=1}^N \alpha_i I(\delta_{y_i}) = \mathbf{E}\, u(p(X)). \tag{9}$$

Our next task is to extend this inequality to general μ's. First assume that p is continuous everywhere and take a sequence of discrete probability measures μ_n, weakly convergent to a given probability measure μ on the real line. Using the identity

$$\int_{-\infty}^{+\infty} p(x-y)\, d\mu(y) = \int_0^{+\infty} \mu\{y \in \mathbf{R} : p(x-y) > t\}\, dt$$

and similarly for μ_n, one may apply Fatou's lemma to conclude that, for all $x \in \mathbf{R}$,

$$\int_{-\infty}^{+\infty} p(x-y)\, d\mu(y) \leq \liminf_{n\rightarrow\infty} \int_{-\infty}^{+\infty} p(x-y)\, d\mu_n(y).$$

Since v is continuous and non-decreasing,

$$\int_{-\infty}^{+\infty} v\bigg(\int_{-\infty}^{+\infty} p(x-y)\, d\mu(y)\bigg)\, dx \leq \liminf_{n\rightarrow\infty} \int_{-\infty}^{+\infty} v\bigg(\int_{-\infty}^{+\infty} p(x-y)\, d\mu_n(y)\bigg)\, dx.$$

Due to (9), the right-hand side does not exceed $\mathbf{E}\, u(p(X))$, so we obtain the inequality (8), that is,

$$\int_{-\infty}^{+\infty} v\bigg(\int_{-\infty}^{+\infty} p(x-y)\, d\mu(y)\bigg)\, dx \leq \int_{-\infty}^{+\infty} v(p(x))\, dx = \mathbf{E}\, u(p(X)), \tag{10}$$

when p is continuous.

To extend (10) to more general densities, note that the functionals $A(p) = \int_{-\infty}^{+\infty} v(p)\,dx$ and $B(p) = \int_{-\infty}^{+\infty} v(\int p(x-y)\,d\mu(y))\,dx$, considered on the set L^1_+ of all integrable functions $p \geq 0$, are continuous with respect to L^1-norm.

Indeed, it was assumed that $|v(t) - v(s)| \leq L|t-s|$, for all $t, s \geq 0$ with some constant L. Hence, given $p_1, p_2 \in L^1_+$, we have $|A(p_1) - A(p_2)| \leq L \int_{-\infty}^{+\infty} |p_1(x) - p_2(x)|\,dx$ and

$$\bigg| v\bigg(\int_{-\infty}^{+\infty} p_1(x-y)\,d\mu(y) \bigg) - v\bigg(\int_{-\infty}^{+\infty} p_2(x-y)\,d\mu(y) \bigg) \bigg| \leq L \int_{-\infty}^{+\infty} |p_1(x-y) - p_2(x-y)|\,d\mu(y).$$

Integrating over x, we see that

$$|B(p_1) - B(p_2)| \leq L \int_{-\infty}^{+\infty} |p_1(x) - p_2(x)|\,dx.$$

This proves continuity of the functionals A and B. Therefore, the inequality (10) extends from the class of all continuous densities p to the class of all densities p.

Thus, Lemma 5 is proved.

Lemma 6. *Let X be a random variable with density p, and q be the density of $Y = X + X'$, where X' is independent of X. If $m_\kappa = m_\kappa(p(X))$, $0 < \kappa < 1$, then*

$$\mathbf{P}\big\{q(Y) \geq m_\kappa / b\big\} \leq \frac{1-\kappa}{1-b} \qquad \big(0 < b < \kappa\big).$$

For example, choosing $\kappa = 1/2$ and $b = 1/4$, we get $\mathbf{P}\{q(Y) \geq 4m\} \leq \frac{2}{3}$, where m is a median of $p(X)$.

Proof. Apply Lemma 5 to $u(t) = \frac{1}{t}\,(t - m_\kappa)^+$, $t \geq 0$. Then $u(t) \leq 1_{\{t > m_\kappa\}}(t)$, so

$$\mathbf{E}\,u(p(X)) \leq \mathbf{P}\{p(X) > m_\kappa\} \leq 1 - \kappa.$$

On the other hand, $u(t) = 1 - \frac{m_\kappa}{t} \geq 1 - b$, whenever $t \geq \frac{m_\kappa}{b}$, so $u(t) \geq (1-b) \cdot 1_{\{t \geq m_\kappa / b\}}$ and

$$\mathbf{E}\,u(q(Y)) \geq (1-b)\,\mathbf{P}\{q(Y) \geq m_\kappa / b\}.$$

It remains to insert the two bounds in (8).

Lemma 7. *If a random variable X with finite variance $\sigma^2 = \mathrm{Var}(X)$ has a density, bounded by a constant M, then $M^2\sigma^2 \geq \frac{1}{12}$.*

This elementary inequality is known. Without proof it was already mentioned and used in [St]. High dimensional variants were studied in Hensley [H] and Ball [B]. Equality in the lemma is possible, and is achieved for a uniform distribution on bounded intervals.

It should also be emphasized that, although the proof of Theorem 2 will be based on Lemma 7, modulo the constant $\frac{1}{12}$, this lemma immediately follows from Theorem 2. Indeed, the right-hand side of (3),

$$1 - \frac{c}{m^2\sigma^2},$$

must be non-negative, which implies $m^2\sigma^2 \geq c$, where $m = m(p(X))$. But $m \leq M$, so $M^2\sigma^2 \geq c$, as well.

Note as well that the inequality of Lemma 7 may be written in an equivalent form in the space of all integrable functions $q \geq 0$ on the line as the relation

$$\Big(\sup_x \, q(x)\Big)^2 \int_{-\infty}^{+\infty} x^2\, q(x)\, dx \geq \frac{1}{12}\,\Big(\int_{-\infty}^{+\infty} q(x)\, dx\Big)^3. \tag{11}$$

Proof of Theorem 2. Let q be the density of $Y = X - X'$, where X' is an independent copy of X. Obviously, Y has the characteristic function $|f(t)|^2$ (where f is the characteristic function of X), and we have the identity

$$\frac{1}{2}\,(1 - |f(2\pi t)|^2) = \int_{-\infty}^{+\infty} \sin^2(\pi t x)\, q(x)\, dx. \tag{12}$$

Our task is to bound the integral in (12) from below.

By Lemma 6, given $0 < b < \kappa_1 < 1$, we have

$$\mathbf{P}\Big\{q(Y) < \frac{1}{b}\, m_{\kappa_1}\Big\} = \int_{q(x)<\frac{1}{b}\, m_{\kappa_1}} q(x)\, dx \geq 1 - \frac{1-\kappa_1}{1-b}, \tag{13}$$

where m_{κ_1} is a quantile of $p(X)$ of order κ_1.

We start with the obvious bound

$$|\sin(\pi\alpha)| \geq 2\psi(\alpha), \quad \alpha \in \mathbf{R},$$

where $\psi(\alpha)$ denotes the shortest distance from α to the set of all integers. Here, an equality is only possible in case $\alpha = k/2$ for an integer k. Hence, (12) gives

$$\frac{1}{2}\,(1 - |f(2\pi t)|^2) > 4 \int_W \psi(tx)^2\, q(x)\, dx \tag{14}$$

for arbitrary measurable sets $W \subset \mathbf{R}$. We apply (14) to the sets of the form

$$W = \big\{x \in \mathbf{R} : -A \leq |tx| \leq A,\ q(x) \leq m_{\kappa_1}/b\big\},$$

where $A = N + \frac{1}{2}$ with $N = 0, 1, 2, \dots$ to be chosen later on.

Given $t \neq 0$, split the integral (14) into the sets $W_k = \{x \in W : k - \frac{1}{2} < |t|x < k + \frac{1}{2}\}$, that is, write

$$\int_W \psi(tx)^2\, q(x)\, dx = \sum_{k=-N}^{N} \int_{W_k} \psi(tx)^2\, q(x)\, dx = \sum_{k=-N}^{N} \int_{W_k} (|t|x - k)^2\, q(x)\, dx.$$

Changing the variable $x = y + \frac{k}{|t|}$ on each W_k, we may also write

$$\int_W \psi(tx)^2\, q(x)\, dx = t^2 \sum_{k=-N}^{N} \int_{-\frac{1}{2|t|}}^{\frac{1}{2|t|}} y^2\, q\Big(y + \frac{k}{|t|}\Big)\, 1_{\{q(y+\frac{k}{|t|})<\frac{1}{b}\, m_{\kappa_1}\}}\, dy. \tag{15}$$

Now, by the inequality (11), applied to the functions $q_k(y) = q\big(y + \frac{k}{|t|}\big)\, 1_{\{q(y+\frac{k}{|t|})<\frac{1}{b}\, m_{\kappa_1}\}}$ on the interval $(-\frac{1}{2|t|}, \frac{1}{2|t|})$, and using a uniform bound $q_k(y) \leq \frac{1}{b}\, m_{\kappa_1}$, we have

$$\int_{-\frac{1}{2|t|}}^{\frac{1}{2|t|}} y^2\, q_k(y)\, dy \ \geq\ \frac{b^2}{12\, m_{\kappa_1}^2} \bigg[\int_{-\frac{1}{2|t|}}^{\frac{1}{2|t|}} q_k(y)\, dy\bigg]^3 \ =\ \frac{b^2}{12\, m_{\kappa_1}^2} \bigg[\int_{W_k} q(x)\, dx\bigg]^3.$$

Using this estimate in (15), the inequality (14) yields

$$\frac{1}{2}\,(1 - |f(2\pi t)|^2) \ >\ \frac{bt^2}{3\, m_{\kappa_1}^2} \sum_{k=-N}^{N} q_k^3, \tag{16}$$

where $q_k = \int_{W_k} q(x)\, dx$.

Next, subject to the constraint $q_{-N} + \cdots + q_N = Q$ with $q_k \geq 0$, the sum $\sum_{k=-N}^N q_k^3$ is minimized, when all q_k are equal to each other, i.e., for $q_k = Q/(2N+1)$. So,

$$\sum_{k=-N}^{N} q_k^3 \geq \frac{Q^3}{(2N+1)^2}. \tag{17}$$

In our case,

$$Q = \sum_{k=-N}^{N} \mathbf{P}\{Y \in W_k\} = \mathbf{P}\{Y \in W\} = \mathbf{P}\big\{|tY| \leq A,\ q(Y) \leq m_{\kappa_1}/b\big\}.$$

Hence, combining (16) with (17), we arrive at

$$\frac{1}{2}\,(1 - |f(2\pi t)|^2) \ >\ \frac{t^2}{(2N+1)^2} \cdot \frac{b^2}{3\, m_{\kappa_1}^2}\ \mathbf{P}\big\{|tY| \leq A,\ q(Y) \leq m_{\kappa_1}/b\big\}^3. \tag{18}$$

To bound from below the probability in (18), fix $0 < \kappa_2 < 1$ and take a quantile $m_{\kappa_2} = m_{\kappa_2}(|Y|)$ for the random variable Y of order κ_2. Then $\mathbf{P}\{|tY| \leq A\} \geq \kappa_2$, as long as

$$m_{\kappa_2}|t| \leq A = N + \frac{1}{2}. \tag{19}$$

In this case, by (13),

$$\mathbf{P}\big\{|tY| \leq A,\ q(Y) \leq m_{\kappa_1}/b\big\} \geq \kappa_2 - \frac{1 - \kappa_1}{1 - b},$$

which makes sense, if $\kappa_2 - \frac{1-\kappa_1}{1-b} > 0$. Hence, (18) may be continued as

$$\frac{1}{2}\,(1 - |f(2\pi t)|^2) \ >\ \frac{t^2}{(2N+1)^2} \cdot \frac{b^2}{3m_{\kappa_1}^2} \left(\kappa_2 - \frac{1-\kappa_1}{1-c}\right)^3. \tag{20}$$

Assume that $s = m_{\kappa_2}|t| \geq \frac{1}{2}$ and take the value $N = [s - \frac{1}{2}] + 1$ to satisfy (19). Then,

$$\frac{|t|}{2N+1} = \frac{1}{m_{\kappa_2}} \cdot \frac{s}{2\,[s - \frac{1}{2}] + 3} \geq \frac{1}{3m_{\kappa_2}},$$

where the inequality becomes an equality for s approaching 1/2 from the right. Thus,

$$\frac{1}{2}\,\big(1 - |f(2\pi t)|^2\big) \ >\ \frac{b^2}{27\, m_{\kappa_1}^2 m_{\kappa_2}^2} \left(\kappa_2 - \frac{1-\kappa_1}{1-b}\right)^3,$$

where $0 < b < \frac{\kappa_1+\kappa_2-1}{\kappa_2}$, provided that $\kappa_1 + \kappa_2 > 1$. The right-hand side may be optimized over all admissible b, and we arrive at the inequality of the form

$$\frac{1}{2}\,(1 - |f(2\pi t)|^2) \ > \ \frac{c_{\kappa_1,\kappa_2}}{m_{\kappa_1}^2 m_{\kappa_2}^2}, \qquad m_{\kappa_2}\,|t| \geq \frac{1}{2}, \tag{21}$$

where

$$c_{\kappa_1,\kappa_2} \ = \sup_{0<b<\frac{\kappa_1+\kappa_2-1}{\kappa_2}} \ \frac{b^2}{27}\left(\kappa_2 - \frac{1-\kappa_1}{1-b}\right)^3, \tag{22}$$

provided that $\kappa_1 + \kappa_2 > 1$ (Note that the sup is attained, so we may keep in (21) the strict inequality sign). Replacing $2\pi t$ with t in (21) and using

$$\frac{1}{2}\,(1 - |f(t)|^2) = \frac{1}{2}\,(1 - |f(t)|)\,(1 + |f(t)|) \leq 1 - |f(t)|,$$

we arrive at

$$1 - |f(t)| \ > \ \frac{c_{\kappa_1,\kappa_2}}{m_{\kappa_1}^2 m_{\kappa_2}^2}, \qquad m_{\kappa_2}|t| \geq \pi.$$

Finally, if $m_{\kappa_2}|t| \leq \frac{1}{2}$, the optimal value in (19) will be $N = 0$, and (20) becomes

$$\frac{1}{2}\,(1 - |f(2\pi t)|^2) \ > \ \frac{b^2 t^2}{3\,m_{\kappa_1}^2}\left(\kappa_2 - \frac{1-\kappa_1}{1-b}\right)^3.$$

Again, replacing $2\pi t$ with t, we arrive at

$$1 - |f(t)| \ > \ \frac{9}{4\pi^2}\,\frac{c_{\kappa_1,\kappa_2} t^2}{m_{\kappa_1}^2}, \qquad 0 < m_{\kappa_2}|t| \leq \pi$$

with the same constants c_{κ_1,κ_2} as in (22).

Let us summarize the results obtained so far.

Theorem 8. *Let a random variable X have density p and the characteristic function f. Let m_{κ_1} and m_{κ_2} be quantiles or orders κ_1, κ_2 for random variables $p(X)$ and $|X - X'|$, where X' is an independent copy of X $(0 < \kappa_1, \kappa_2 < 1,\ \kappa_1 + \kappa_2 > 1)$. Then,*

$$|f(t)| \ < \ 1 - \frac{c_1}{m_{\kappa_1}^2 m_{\kappa_2}^2}, \qquad m_{\kappa_2}|t| \geq \pi, \tag{23}$$

$$|f(t)| \ < \ 1 - \frac{c_2 t^2}{m_{\kappa_1}^2}, \qquad 0 < m_{\kappa_2}|t| \leq \pi, \tag{24}$$

where $c_1, c_2 > 0$ are constants, depending on (κ_1, κ_2), only.

Recall that the constant $c_1 = c_{\kappa_1,\kappa_2}$ is described in (22),

$$c_{\kappa_1,\kappa_2} \ = \sup_{0<b<\frac{\kappa_1+\kappa_2-1}{\kappa_2}} \ \frac{b^2}{27}\left(\kappa_2 - \frac{1-\kappa_1}{1-b}\right)^3,$$

and one may take $c_2 = \frac{9}{4\pi^2}\,c_1$. For example, choosing $\kappa_1 = 1/2$, $\kappa_2 = 7/8$, and $b = 1/4$ in the sup, we get

$$c_{\kappa_1,\kappa_2} \geq \frac{1}{16 \cdot 27}\left(\frac{5}{24}\right)^3.$$

An inspection of the proof of Theorem 8 shows that (23)-(24) remain valid for any number $\tilde{m} \geq m_{\kappa_2}$ in place of m_{κ_2}. In case of finite variance $\sigma^2 = \mathrm{Var}(X)$, we have $\mathrm{Var}(X - X') = 2\sigma^2$, and by Chebyshev's inequality,

$$\mathbf{P}\{|X - X'| \geq 4\sigma\} \leq \frac{1}{8} = 1 - \kappa_2.$$

So $m_{\kappa_2} \leq \tilde{m} = 4\sigma$, and as a result, we arrive at the formulation of Theorem 2 with constants

$$c_1 = \frac{1}{16 \cdot 27} \Big(\frac{5}{24}\Big)^3 \frac{1}{4^2} = 1.3082... \cdot 10^{-6}, \quad c_2 = \frac{9}{4\pi^2}\, c_1 = 2.9823... \cdot 10^{-7}$$

in the inequalities (3)-(4). Theorem 2 is thus proved.

Now, let us turn to the entropic variant of Theorem 2.

Lemma 9. *Let X be a random variable with finite variance $\sigma^2 = \mathrm{Var}(X)$ and finite entropy. Then, any quantile m_κ $(0 < \kappa < 1)$ of the random variable $p(X)$ satisfies*

$$m_\kappa\, \sigma \leq \frac{1}{\sqrt{2\pi}}\, e^{(D(X)+1)/(1-\kappa)}. \tag{25}$$

Proof. Rewrite the entropic distance to normality as the Kullback-Leibler distance

$$D(X) = \int_{-\infty}^{+\infty} p(x)\, \log \frac{p(x)}{\varphi_{a,\sigma}(x)}\, dx,$$

where $\varphi_{a,\sigma}$ is the density of the normal law $N(a,\sigma^2)$ with $a = \mathbf{E}X$ and $\sigma^2 = \mathrm{Var}(X)$.

First, since $p(x) \geq m_\kappa$ on the set $V = \{x : p(x) \geq m_\kappa\}$ and $\mathbf{P}\{X \in V\} \geq 1 - \kappa$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} p(x)\, \log\Big(1 + \frac{p(x)}{\varphi_{a,\sigma}(x)}\Big)\, dx \ &\geq \ \int_V p(x)\, \log\Big(1 + \frac{p(x)}{\varphi_{a,\sigma}(x)}\Big)\, dx \\
&\geq \ \int_V p(x)\, \log \frac{m_\kappa}{\varphi_{a,\sigma}(x)}\, dx \\
&= \ \log(m_\kappa \sigma \sqrt{2\pi}) \int_V p(x)\, dx + \frac{1}{2\sigma^2} \int_V (x-a)^2 p(x)\, dx \\
&\geq \ (1-\kappa)\, \log(m_\kappa \sigma \sqrt{2\pi}).
\end{aligned}$$

On the other hand, using the following elementary inequality $t\log(1+t) - t\log t \leq 1$ $(t \geq 0)$, we get an upper bound

$$\begin{aligned}
\int_{-\infty}^{+\infty} p(x)\, \log\Big(1 + \frac{p(x)}{\varphi_{a,\sigma}(x)}\Big)\, dx \ &= \ \int_{-\infty}^{+\infty} \frac{p(x)}{\varphi_{a,\sigma}(x)} \log\Big(1 + \frac{p(x)}{\varphi_{a,\sigma}(x)}\Big)\, \varphi_{a,\sigma}(x)\, dx \\
&\leq \ \int_{-\infty}^{+\infty} \frac{p(x)}{\varphi_{a,\sigma}(x)} \log \frac{p(x)}{\varphi_{a,\sigma}(x)}\, \varphi_{a,\sigma}(x)\, dx + 1 \\
&= \ D(X) + 1.
\end{aligned}$$

Hence, $(1-\kappa)\, \log(m_\kappa \sigma \sqrt{2\pi}) \leq D(X) + 1$, and the lemma follows.

Proof of Corollary 4. This Corollary can be derived, using Lemma 9 with $\kappa = 1/2$ directly from Theorem 2. However, to get a sharper statement, let $0 < \kappa_1, \kappa_2 < 1$, $\kappa_1 + \kappa_2 > 1$, as in Theorem 8. Combining the inequality (23) with (25), where $\kappa = \kappa_1$, for any $\tilde{m} \geq m_{\kappa_2}$, we get

$$|f(t)| < 1 - \frac{c_1}{m_{\kappa_1}^2 \tilde{m}^2} \leq 1 - \frac{2c_1 \pi\, \sigma^2}{\tilde{m}^2}\, e^{-2(D(X)+1)/(1-\kappa_1)}$$

in the region $\tilde{m}|t| \geq \pi$. As we have already noted, $m_{\kappa_2} \leq \tilde{m} = \sigma\sqrt{\frac{2}{1-\kappa_2}}$, so, for this value $\tilde{m}$,

$$|f(t)| < 1 - c_1\pi\,(1-\kappa_2)\, e^{-2(D(X)+1)/(1-\kappa_1)}, \qquad \tilde{m}|t| \geq \pi.$$

Similarly, by (24),

$$|f(t)| < 1 - 2c_2\,\pi\, t^2\sigma^2\, e^{-2(D(X)+1)/(1-\kappa_1)}, \qquad 0 < \tilde{m}|t| < \pi.$$

In both inequalities, the coefficient in front of $D(X)$ can be made as close to 2, as we wish, and with the constants c_1 and c_2, depending on (κ_1, κ_2), as in Theorem 8.

In particular, for $\kappa_2 = 7/8$, we have $\tilde{m} = 4\sigma$, so, whenever $\kappa_1 > 1/8$,

$$\begin{aligned} |f(t)| \ &< \ 1 - \frac{c_1\pi}{8}\, e^{-2(D(X)+1)/(1-\kappa_1)}, & \sigma|t| \geq \frac{\pi}{4}, \\ |f(t)| \ &< \ 1 - 2c_2\,\pi\, t^2\sigma^2\, e^{-2(D(X)+1)/(1-\kappa_1)}, & 0 < \sigma|t| < \frac{\pi}{4}, \end{aligned}$$

In case $\kappa_1 = 1/2$, we arrive at the desired inequalities (6)-(7).

Remark 10. Lemma 1 in the paper of Statulevičius [St] states the following. Let X be a random variable with density $p(x)$ and characteristic function $f(t)$. Let $\widetilde{p}(x)$ denote the density of the random variable $X - X'$, where X' is an independent copy of X. Then for any sequence $\{\Delta_i\}$ of non-overlapping intervals on the line with lengths $|\Delta_i|$, for all constants $0 \leq M_i \leq \infty$, and for all $t \in \mathbf{R}$, one has

$$|f(t)| \leq \exp\bigg\{-\frac{t^2}{3}\sum_{i=1}^\infty \frac{Q_i^3}{(|\Delta_i|\,|t| + 2\pi)^2\, M_i^2}\bigg\}, \tag{26}$$

where

$$Q_i = \int_{\Delta_i} \min\{\widetilde{p}(x), M_i\}\, dx.$$

In particular, as emphasized in [St], if $\mathrm{Var}(X) = \sigma^2$ and $p(x) \leq M$, one may take just one interval $\Delta = [-2\sigma, 2\sigma]$. Then $Q_1 \geq \frac{1}{2}$, so the inequality (26) leads to

$$|f(t)| \leq \exp\bigg\{-\frac{t^2}{96}\,\frac{1}{(\sigma|t| + 2\pi)^2\, M^2}\bigg\}.$$

This may be viewed as a variant of Theorem 1.

References

[B] Ball, K. Logarithmically concave functions and sections of convex sets in R^n. Studia Math. 88 (1988), no. 1, 69–84.

[H] Hensley, D. Slicing convex bodies—bounds for slice area in terms of the body's covariance. Proc. Amer. Math. Soc. 79 (1980), no. 4, 619–625.

[P] Petrov, V.V. Sums of independent random variables. Springer-Verlag, 1975.

[Se] Senatov, V.V. Central limit theorem. Exactness of approximation and asymptotic expansions. (In Russian) TVP Science Publishers, Moscow, 2009.

[St] Statulevičius, V.A. Limit theorems for densities and the asymptotic expansions for distributions of sums of independent random variables. (Russian) Teor. Verojatnost. i Primenen. 10 (1965), 645–659.

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